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ON STABILITY OF JUMP PROCESSES WITH UNIFORMLY FINITE RANGE UPWARD

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Abstract. Here, we obtained the explicit representation and criteria for some stability results of jump processes with uniformly finite range upward based on the theory of minimal nonnegative solution and approximation procedure. In this setting, several new recursive sequences were put forward, which play a crucial role in all aspects of our results. Moreover, we give some examples to verify the validity of our results.

1. **Introduction.** Consider a continuous-time and homogeneous Markov chain $\{X(t) : t \geqslant 0\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the transition probability matrix $P(t) = (p_{ij}(t))$ on the countable state space $\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$. In this paper, we assume that the considered transition rate Q-matrix $Q = (q_{ij})$ is totally stable and conservative, which means $q_i := -q_{ii} = \sum_{j\neq i} q_{ij} < \infty$ for all $i \in \mathbb{Z}_+$.

Given a Q-matrix, if there exists such a positive integer m that for all $i \geqslant 0$,

$$q_{i,i+m} > 0, \quad q_{ij} = 0, \quad j > i + m,$$

we call it a Q-matrix with uniformly finite range upward; by contrast, if for all $i \geqslant m$,

$$q_{i,i-m} > 0, \quad q_{ij} = 0, \quad 0 \leqslant j < i - m,$$

it is called a Q-matrix with uniformly finite range downward. In particular, in the case of $m = 1$, the Q-matrix is also called a single birth Q-matrix and single death Q-matrix, or skip-free upwardly one and skip-free downward Q-matrix, respectively. For simplicity, we call them m-birth Q-matrix and m-death Q-matrix respectively. For systematic results on single birth and single death processes, refer to [1–3, 12, 13, 17, 18, 20]. In applications, the structure of m-birth processes is similar to that of level-dependent GI/M/1-type Markov chains with m phases in each level, where matrix-analytic methods usually play a crucial role. For this model and related ones, see [4, 5, 8, 10, 11, 14, 15].

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Over the past few decades, single birth processes, due to them being single exit, have become one of the largest class of Markov processes for which explicit criteria on stability can be expected. The m-birth process (where m is a positive integer), as a natural extension of single birth process, is widely used in probability models and has thus attracted significant attention. The stability theory mainly involves uniqueness and recurrence, moments of the first hitting time, ergodicity, and strong ergodicity, among others. Recently, [19] investigated the uniqueness and recurrence of m-birth and m-death processes and obtained their explicit criteria. For moments of the first hitting time and ergodicity, recall that in the single birth case, the minimal nonnegative solution theory plays a central role in the proof; see [3] for reference. However, under generalized circumstances, unlike in the single birth case, it is hard to achieve our goal since the linear equations corresponding to the moments of the first hitting time may not be easy to explicitly solve.

To overcome this difficulty, we turn another way. Considering that for m-birth processes evolving in finite state space the explicit representation is easily obtained, we choose to start with a finite state space case and then gradually transfer to the countable case. The construction of the Q-process on the finite state space and the method of converting its result into what we expect are two key aspects throughout this paper. The main method remains the minimal nonnegative solution theory with a limiting approximation procedure. We also have to mention that [9] and [18] provided some methods for constructing augmented truncation Q-matrices, whose ideas have significantly benefited us. Moreover, the same approximation has been used in [16] to investigate inverse problems for ergodicity. As a byproduct, several recursive sequences are produced during this work.

Throughout the paper, we consider only the m-birth Q-matrix $Q = (q_{ij})$. We begin with some basic notations. Define the first hitting time

$$\tau_i := \inf\{t > 0 : X(t) = i\}, \quad i \geqslant 0,$$

and the first returning time

$$\sigma_i := \inf\{t > \eta_1 : X(t) = i\}, \quad i \geqslant 0,$$

where η_1 is the first jumping time.

For a given function $b = (b_{ij})_{i,j\in\mathbb{Z}_+}$ (to be fixed in all sections, and then to be specified case by case), the following sequences are used the paper:

$$\bar{q}_n^{(k)} = q_n^{(k)} - b_{nk} := \sum_{j=0}^{k} q_{nj} - b_{nk}, \quad 0 \leqslant k \leqslant n+m-1,$$

$$\bar{F}_k^{(k)} = 1, \quad \bar{F}_n^{(k)} = \frac{1}{q_{n,n+m}} \sum_{j=k}^{n-1} \bar{q}_n^{(j+m-1)} \bar{F}_j^{(k)}, \quad n > k \geqslant 0,$$

$$c_0 = 0, \quad c_i^{(\ell)} = \frac{1}{q_{i,i+m}} \left(-q_{i\ell} + \sum_{k=0}^{i-1} q_i^{(k+m-1)} c_k^{(\ell)} \right), \quad i \geqslant 1,\ 1 \leqslant \ell \leqslant m-1,$$

$$d_0 = 0, \quad d_i = \frac{1}{q_{i,i+m}} \left(1 + \sum_{k=1}^{i-1} q_i^{(k+m-1)} d_k \right), \quad i \geqslant 1.$$

In what follows, we omit the superscript '$^-$' in $\bar{F}$ and $\bar{q}$ once $b \equiv 0$, and use the convention that $\sum_{\emptyset} = 0$. In addition, we define $c_i^{(m)} := F_i^{(0)}$ for all $i \geqslant 0$. For any given matrix A, let $\det(A)$ be the determinant of A throughout the paper. All vectors and matrices used in this paper are of adaptive size. We also denote by $\mathbb{1}_{\{\cdot\}}$ the indicator function of the set $A \subset \mathbb{Z}_+$.

The paper is organized as follows: Section 2 consists of the main results of the paper. The proofs of Theorem 2.1 and Theorem 2.3 are given in Section 3. In Section 4, some examples are given to illustrate the validity of our results.

2. **Main results.** Define an $m \times m$ matrix $A^{(n)} = (a_{i\ell}^{(n)} : 1 \leqslant i, \ell \leqslant m)$ and an m-dimensional column vector $\mathbf{b}^{(n)} = (b_1^{(n)}, b_2^{(n)}, \cdots, b_m^{(n)})^{\mathrm{T}}$, with elements given by

$$a_{i\ell}^{(n)} = \begin{cases} \sum_{k=0}^{n-m+1} c_k^{(\ell)}, & \text{if } i = 1, 1 \leqslant \ell \leqslant m; \\ c_{n-m+2}^{(\ell)}, & \text{if } i = 2, 1 \leqslant \ell \leqslant m; \\ -q_{n-m+i,\ell} + \sum_{k=0}^{n-m+1} q_{n-m+i}^{(k+m-1)} c_k^{(\ell)}, & \text{if } 3 \leqslant i \leqslant m, 1 \leqslant \ell \leqslant m-1; \\ \sum_{k=0}^{n-m+1} q_{n-m+i}^{(k+m-1)} c_k^{(m)}, & \text{if } 3 \leqslant i \leqslant m, \ell = m, \end{cases} \tag{1}$$

and

$$b_1^{(n)} = \sum_{k=0}^{n-m+1} d_k, \quad b_2^{(n)} = d_{n-m+2}, \quad b_i^{(n)} = 1 + \sum_{k=0}^{n-m+1} q_{n-m+i}^{(k+m-1)} d_k, \quad 3 \leqslant i \leqslant m. \tag{2}$$

We also denote the matrix that replaces the k-th column of $A^{(n)}$ with the column vector $\mathbf{b}^{(n)}$ by $A_k^{(n)}$ for each $1 \leqslant k \leqslant m$. Here is the first main result, which presents the explicit formula for the first moment of the first hitting time.

Theorem 2.1. *Assume that the m-birth Q-matrix $Q = (q_{ij})$ on $\mathbb{Z}_+$ is irreducible and the corresponding process is recurrent. The following statements hold.*

- *It follows that as $n \to \infty$,*

$$\frac{\det(A_k^{(n)})}{\det(A^{(n)})} \uparrow \mathbb{E}_k \tau_0 := B_k, \quad k = 1, 2, \cdots, m,$$

and

$$\mathbb{E}_j \tau_0 = \sum_{k=1}^{m} \left(\sum_{i=0}^{j-m} c_i^{(k)} \right) \mathbb{E}_k \tau_0 - \sum_{i=0}^{j-m} d_i, \quad j \geqslant m+1. \tag{3}$$

- *The process is ergodic if and only if $B_i < +\infty$ for all $i = 1, 2, \cdots, m$.*
- *The process is strongly ergodic if and only if*

$$\sup_{n \geqslant 1} \left(\sum_{k=1}^{m} \left(\sum_{i=0}^{n} c_i^{(k)} \right) \mathbb{E}_k \tau_0 - \sum_{i=0}^{n} d_i \right) < +\infty. \tag{4}$$

Actually, for the last conclusion, the recurrence assumption can be replaced by the uniqueness assumption.

Remark 2.2. Let V be a non negative function and not identically equal to zero on $\mathbb{Z}_+$. The associated integral-type functionals for m-birth processes are defined as follows:

$$\xi_0 = \int_0^{\tau_0} V(X(t)) dt. \tag{5}$$

If the starting state is 0, sometimes the upper limit of the above integral may be replaced by σ_0 according to the context. If we alter the definition of the sequence $(d_i)_{i\geqslant 0}$ slightly in the following way:

$$d_0=0,\quad d_i=\frac{1}{q_{i,i+m}}\left(V(i)+\sum_{k=1}^{i-1}q_i^{(k+m-1)}d_k\right),\quad i\geqslant 1,$$

then the results for the moments of integral-type functionals are also obtained in Theorem 2.1 with $\mathbb{E}_i\tau_0$ replaced by $\mathbb{E}_i\xi_0$ for all $i\geqslant 1$. The proof is similar to that of Theorem 2.1.

For any given $\lambda>0$, let $\tilde{q}_i^{(j)}:=q_i^{(j)}+\lambda$ for $0\leqslant j<i$ and let $\tilde{q}_i^{(j)}:=q_i^{(j)}$ for $i\leqslant j\leqslant i+m-1$. Correspondingly, we define the following notations.

$$\tilde{F}_n^{(n)}=1,\quad \tilde{F}_i^{(n)}=\frac{1}{q_{i,i+m}}\sum_{k=n}^{i-1}\tilde{q}_i^{(k+m-1)}\tilde{F}_k^{(n)},\quad 0\leqslant n<i,$$

$$\tilde{c}_0=0,\quad \tilde{c}_i^{(\ell)}=\frac{1}{q_{i,i+m}}\left(-q_{i\ell}+\sum_{k=0}^{i-1}\tilde{q}_i^{(k+m-1)}\tilde{c}_k\right),\quad i\geqslant 1,\ 1\leqslant\ell\leqslant m-1.$$

$$\tilde{d}_0=0,\quad \tilde{d}_i=\frac{1}{q_{i,i+m}}\left(\lambda+\sum_{k=0}^{i-1}\tilde{q}_i^{(k+m-1)}\tilde{d}_k\right),\quad i\geqslant 1.$$

We also define $\tilde{c}_m:=\tilde{F}_i^{(0)}$ for all $i\geqslant 0$ and one $m\times m$ matrix as follows: $H^{(n)}=(h_{i\ell}^{(n)}:1\leqslant i,\ell\leqslant m)$, where

$$h_{i\ell}^{(n)}=\begin{cases}\sum\limits_{k=0}^{n-m+1}\tilde{c}_k^{(\ell)}, & \text{if } i=1,1\leqslant\ell\leqslant m;\\ \tilde{c}_{n-m+2}^{(\ell)}, & \text{if } i=2,1\leqslant\ell\leqslant m;\\ -q_{n-m+i,\ell}+\sum\limits_{k=0}^{n-m+1}\tilde{q}_{n-m+i}^{(k+m-1)}\tilde{c}_k^{(\ell)}, & \text{if } 3\leqslant i\leqslant m,1\leqslant\ell\leqslant m-1;\\ \sum\limits_{k=0}^{n-m+1}\tilde{q}_{n-m+i}^{(k+m-1)}\tilde{c}_k^{(m)}, & \text{if } 3\leqslant i\leqslant m,\ell=m,\end{cases}\tag{6}$$

and define one m-dimensional column vector $\mathbf{s}^{(n)}=(s_1^{(n)},s_2^{(n)},\cdots,s_m^{(n)})^{\mathrm{T}}$:

$$s_1^{(n)}=\sum_{k=0}^{n-m+1}\tilde{d}_k,\quad s_2^{(n)}=\tilde{d}_{n-m+2},\quad s_i^{(n)}=\lambda+\sum_{k=0}^{n-m+1}\tilde{q}_{n-m+i}^{(k+m-1)}\tilde{d}_k,\quad 3\leqslant i\leqslant m.\tag{7}$$

We also denote the matrix that replaces the k-th column of $H^{(n)}$ with the column vector $\mathbf{s}^{(n)}$ by $H_k^{(n)}$ for $1\leqslant k\leqslant m$. The Laplace transform of the first hitting time is also obtained in the next theorem.

Theorem 2.3. *Assume that the m-birth Q-matrix $Q=(q_{ij})$ is irreducible and the corresponding process is recurrent. Then the Laplace transform of τ_0 has the representation that as $n\to\infty$,*

$$\frac{\det(H_i^{(n)})}{\det(H^{(n)})}\ \uparrow\ 1-\mathbb{E}_i\mathrm{e}^{-\lambda\tau_0}=:\psi_{i0}(\lambda),\quad 1\leqslant i\leqslant m,$$

and

$$\psi_{i0}(\lambda):=1-\mathbb{E}_i\mathrm{e}^{-\lambda\tau_0}=\sum_{\ell=1}^{m}\left(\sum_{k=0}^{i-m}\tilde{c}_k^{(\ell)}\right)\psi_{\ell 0}(\lambda)-\sum_{\ell=0}^{i-m}\tilde{d}_\ell,\quad i\geqslant m+1.$$

Remark 2.4. Similarly, as in Remark 2.2, the results for the Laplace transforms of integral-type functionals are also obtained in Theorem 2.3 with $\mathbb{E}_i e^{-\lambda\tau_0}$ replaced by $\mathbb{E}_i e^{-\lambda\xi_0}$ for all $i \geqslant 1$ if we alter the definition of the sequence $(\tilde{d}_i)_{i\geqslant 0}$ slightly in the following way:

$$\tilde{d}_0 = 0, \quad \tilde{d}_i = \frac{1}{q_{i,i+m}}\left(\lambda V(i) + \sum_{k=1}^{i-1} \tilde{q}_i^{(k+m-1)}\tilde{d}_k\right), \quad i \geqslant 1.$$

3. **Proof of the main results.** First, the following simple result for the solution to a class of linear equations can be inductively proven.

Lemma 3.1. *The solution $(h_i)_{i\geqslant 0}$ to the recursive equations*

$$h_i = \frac{1}{q_{i,i+m}}\left(f_i + \sum_{\ell\leqslant k<i} \bar{q}_i^{(k+m-1)}h_k\right), \quad i \geqslant \ell \geqslant 0,$$

can be represented as

$$h_i = \sum_{k=\ell}^{i} \frac{\bar{F}_i^{(k)} f_k}{q_{k,k+m}}, \quad i \geqslant \ell \geqslant 0. \tag{8}$$

Proof. In fact,

$$h_\ell = \frac{f_\ell}{q_{\ell,\ell+m}} = \sum_{k=\ell}^{\ell} \frac{\bar{F}_\ell^{(k)} f_k}{q_{k,k+m}}.$$

Assume that the assertion holds until $i-1 \geqslant \ell$. Then

$$\begin{aligned}
h_i &= \frac{1}{q_{i,i+m}}\left(f_i + \sum_{j=\ell}^{i-1} \bar{q}_i^{(j+m-1)}h_j\right)\\
&= \frac{1}{q_{i,i+m}}\left(f_i + \sum_{j=\ell}^{i-1} \bar{q}_i^{(j+m-1)}\sum_{k=\ell}^{j}\frac{\bar{F}_j^{(k)} f_k}{q_{k,k+m}}\right)\\
&= \frac{f_i}{q_{i,i+m}} + \sum_{k=\ell}^{i-1}\frac{f_k}{q_{k,k+m}}\cdot\frac{1}{q_{i,i+m}}\sum_{j=k}^{i-1}\bar{q}_i^{(j+m-1)}\bar{F}_j^{(k)}\\
&= \frac{\bar{F}_i^{(i)} f_i}{q_{i,i+m}} + \sum_{k=\ell}^{i-1}\frac{\bar{F}_i^{(k)} f_k}{q_{k,k+m}}\\
&= \sum_{k=\ell}^{i}\frac{\bar{F}_i^{(k)} f_k}{q_{k,k+m}}.
\end{aligned}$$

By induction on i, the assertion holds. □

Remark 3.2. In particular,

$$F_i^{(n)} = \frac{1}{q_{i,i+m}}\sum_{k=n}^{i-1} q_i^{(k+m-1)}F_k^{(n)} = \frac{1}{q_{i,i+m}}\left(q_i^{(n+m-1)} + \sum_{n+1\leqslant k\leqslant i-1} q_i^{(k+m-1)}F_k^{(n)}\right)$$

for all $i \geqslant n+1 \geqslant 1$. Then from Lemma 3.1, by letting $b \equiv 0$ and $f_i = q_i^{(n+m-1)}$ for all

$i \geqslant n+1$, it follows that

$$F_i^{(n)} = \sum_{k=n+1}^{i} \frac{F_i^{(k)} q_k^{(n+m-1)}}{q_{k,k+m}}, \quad i \geqslant n+1 \geqslant 1.$$

Meanwhile, note that for all $i \geqslant 1$ and $1 \leqslant \ell \leqslant m-1$,

$$c_i^{(\ell)} = \frac{1}{q_{i,i+m}} \left(-q_{i\ell} + \sum_{k=1}^{i-1} q_i^{(k+m-1)} c_k^{(\ell)} \right), \quad d_i = \frac{1}{q_{i,i+m}} \left(1 + \sum_{k=1}^{i-1} q_i^{(k+m-1)} d_k \right).$$

So we get that

$$c_i^{(\ell)} = -\sum_{k=1}^{i} \frac{F_i^{(k)} q_{k\ell}}{q_{k,k+m}}, \quad d_i = \sum_{k=1}^{i} \frac{F_i^{(k)}}{q_{k,k+m}}, \quad i \geqslant 1,\ 1 \leqslant \ell \leqslant m-1.$$

Similarly, by letting $b_{ik} = -\lambda \mathbb{1}_{\{k \leqslant i-m\}}$ and $f_i = \tilde{q}_i^{(n+m-1)}$ for all $i \geqslant n+1$, then $\bar{q}_i^{(k+m-1)} = \tilde{q}_i^{(k+m-1)}$ and we derive via Lemma 3.1 that

$$\tilde{F}_i^{(n)} = \sum_{k=n+1}^{i} \frac{\tilde{F}_i^{(k)} \tilde{q}_k^{(n+m-1)}}{q_{k,k+m}}, \quad i \geqslant n+1 \geqslant 1.$$

The next two equations hold in the same way.

$$\tilde{c}_i^{(\ell)} = -\sum_{k=1}^{i} \frac{\tilde{F}_i^{(k)} q_{k\ell}}{q_{k,k+m}}, \quad \tilde{d}_i = \lambda \sum_{k=1}^{i} \frac{\tilde{F}_i^{(k)}}{q_{k,k+m}}, \quad i \geqslant 1,\ 1 \leqslant \ell \leqslant m-1.$$

Before proving our first result, we need the following crucial lemma.

Lemma 3.3. *Assume that the m-birth Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Then the following relation holds:*

$$\mathbb{E}_j \tau_0 = \sum_{k=1}^{m} \left(\sum_{i=0}^{j-m} c_i^{(k)} \right) \mathbb{E}_k \tau_0 - \sum_{i=0}^{j-m} d_i, \quad j \geqslant m+1,$$

with the assumption that at most one of $\mathbb{E}_i \tau_0$ $(i = 1, 2, \cdots, m)$ is infinite.

Proof. The proof is decomposed into two steps. First, let $H = \{0\}$. By [2, Theorem 4.48], $(\mathbb{E}_i \sigma_0 : i \in \mathbb{Z}_+)$ is the minimal nonnegative solution to the equation

$$x_i = \frac{1}{q_i} \sum_{k \notin \{0,i\}} q_{ik} x_k + \frac{1}{q_i}, \quad i \in \mathbb{Z}_+. \tag{9}$$

By [2, Theorem 2.13] (localization theorem), $(\mathbb{E}_i \tau_0 : i \geqslant 1)$ is the minimal nonnegative solution (x_i^*) to the equation

$$x_i = \frac{1}{q_i} \sum_{k \notin \{0,i\}} q_{ik} x_k + \frac{1}{q_i}, \quad i \geqslant 1. \tag{10}$$

Let $(x_i)_{i \geqslant 1}$ be a finite solution to (10). Define $w_i := x_{i+1} - x_i$ for $i \geqslant 1$ and $w_0 := x_1$, then it is not difficult to derive

$$\begin{aligned} w_{i+m-1} &= \frac{1}{q_{i,i+m}} \left(-1 + q_{i0} x_1 + \sum_{k=1}^{i+m-2} q_i^{(k)} w_k \right) \\ &= \frac{1}{q_{i,i+m}} \left(-1 + q_{i0} x_1 + \sum_{k=1}^{m-1} q_i^{(k)} w_k + \sum_{k=1}^{i-1} q_i^{(k+m-1)} w_{k+m-1} \right), \quad i \geqslant 1. \end{aligned}$$

Let $h_i = w_{i+m-1}$ for all $i \geqslant 1$. Then

$$h_i = \frac{1}{q_{i,i+m}} \left(f_i + \sum_{k=1}^{i-1} q_i^{(k+m-1)} h_k \right),$$

where $f_i = -1 + q_{i0}x_1 + \sum_{k=1}^{m-1} q_i^{(k)} w_k$ for all $i \geqslant 1$. Hence by (8), one has

$$w_{i+m-1} = h_i = \sum_{k=1}^{i} \frac{F_i^{(k)} \left(-1 + q_k^{(m-1)} x_m - \sum_{\ell=1}^{m-1} q_{k\ell} x_\ell \right)}{q_{k,k+m}}, \qquad 1 \leqslant i \leqslant n-m. \tag{11}$$

Hence for all $j \geqslant m+1$, by Remark 3.2, it is obtained that

$$\begin{aligned} x_j &= \sum_{i=m}^{j-1} w_i + x_m = \sum_{i=m}^{j-1} \sum_{k=1}^{i-m+1} \frac{F_{i-m+1}^{(k)} \Big(-1 + q_k^{(m-1)} x_m - \sum_{\ell=1}^{m-1} q_{k\ell} x_\ell \Big)}{q_{k,k+m}} + x_m \\ &= \left(\sum_{i=1}^{j-m} F_i^{(0)} + 1 \right) x_m + \sum_{\ell=1}^{m-1} x_\ell \sum_{i=1}^{j-m} c_i^{(\ell)} - \sum_{i=0}^{j-m} d_i = \sum_{\ell=1}^{m} \sum_{i=0}^{j-m} c_i^{(\ell)} \cdot x_\ell - \sum_{i=0}^{j-m} d_i. \end{aligned}$$

Thus the proof is completed. □

Next we turn to the proof of our first result.

Proof of Theorem 2.1.

Denote the $(n+1) \times (n+1)$ northwest corner truncation of Q on $\{0, 1, 2, \ldots, n\}$ by $Q^{(n)}$. Now we augment the truncated transition elements to some column. Specifically, the augmentation Q-matrix $\tilde{Q}^{(n)} = (\tilde{q}_{ij}^{(n)} : 0 \leqslant i, j \leqslant n)$ is given by $\tilde{Q}^{(n)} = Q^{(n)} + (-Q^{(n)})\mathbf{1}^{\mathrm{T}}\mathbf{e}_0$, where $\mathbf{1} = (1, 1, \cdots, 1)$ and $\mathbf{e}_0 = (1, 0, \cdots, 0)$ are $(n+1)$-dimensional row vectors. In details, by the m-birth property, we have

$$\tilde{q}_{ij}^{(n)} = \begin{cases} q_{ij}, & \text{if } 0 \leqslant i \leqslant n-m,\ 0 \leqslant j \leqslant n, \\ q_{ij}, & \text{if } n-m+1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n, \\ q_{i0} + \sum_{k=n+1}^{i+m} q_{ik}, & \text{if } n-m+1 \leqslant i \leqslant n,\ j = 0. \end{cases} \tag{12}$$

We start with the following equations named approximating equations corresponding to the Q-matrix $\tilde{Q}^{(n)} = (\tilde{q}_{ij}^{(n)},\ 0 \leqslant i, j \leqslant n)$:

$$x_i = \sum_{k=1, k \neq i}^{n} \frac{\tilde{q}_{ik}^{(n)}}{-\tilde{q}_{ii}^{(n)}} x_k + \frac{1}{-\tilde{q}_{ii}^{(n)}}, \quad 1 \leqslant i \leqslant n, \tag{13}$$

Now we assert that there indeed exists a unique finite solution to equation (13). In fact, we may rewrite (13) into the form $(I - \Pi)\mathbf{x} = \mathbf{r}$, where I is the $n \times n$ identity matrix, $\mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ and $\mathbf{r} = (\frac{1}{q_1}, \frac{1}{q_2}, \cdots, \frac{1}{q_n})^{\mathrm{T}}$. The $n \times n$ matrix $\Pi = (\Pi_{ij} : 1 \leqslant i, j \leqslant n)$ with $\Pi_{ij} = \frac{q_{ij}}{q_i} \mathbb{1}_{\{i \neq j\}}$ for all $1 \leqslant i, j \leqslant n$ is called local embedding chain corrseponding to $\tilde{Q}^{(n)}$. It may be clearly seen that Π is transient via the m-birth property. Therefore, the inverse of the matrix $I - \Pi$ exists pointwisely. So there exists a unique solution that could be expressed as

$$\mathbf{x} = (I - \Pi)^{-1}\mathbf{r} = \sum_{n=0}^{+\infty} \Pi^n \mathbf{r} < +\infty.$$

So it can be shown that $(\mathbb{E}_i\tau_0^{(n)} < \infty,\ 1 \leqslant i \leqslant n)$ is the unique finite solution to (13) for the process conditioned on the finite state space, where $\tau_0^{(n)}$ is the first hitting time of 0 for the process associated with $\tilde{Q}^{(n)}$.

As for equations (13), as in the proof of Lemma 3.3, it follows that

$$x_i^{(n)} = \sum_{\ell=1}^{m}\sum_{k=0}^{i-m} c_k^{(\ell)} \cdot x_\ell^{(n)} - \sum_{k=0}^{i-m} d_k, \quad m+1 \leqslant i \leqslant n.$$

Substituting the results above into the last m equations of (13), we derive

$$C^{(n)}\mathbf{x} = \mathbf{h}^{(n)}, \quad \text{i.e.,} \quad \sum_{j=1}^{m} c_{ij}^{(n)} x_j = h_i^{(n)}, \quad 1 \leqslant i \leqslant m, \tag{14}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_m)^{\mathrm{T}}$, the $m \times m$ matrix $C^{(n)} = (c_{ij}^{(n)}, 1 \leqslant i, j \leqslant m)$ is defined as: for all $1 \leqslant i, j \leqslant m$,

$$c_{ij}^{(n)} = \begin{cases} -q_{n-m+i,j} - \sum\limits_{k=1}^{n-m} \sum\limits_{\ell=k+m}^{n} q_{n-m+i,\ell} c_k^{(j)}, & \text{if } 1 \leqslant j \leqslant m-1, \\ -\sum\limits_{k=0}^{n-m} \sum\limits_{\ell=k+m}^{n} q_{n-m+i,\ell} c_k^{(m)}, & \text{if } j = m, \end{cases} \tag{15}$$

and the m-dimensional column vector $\mathbf{h}^{(n)} = (h_1^{(n)}, h_2^{(n)}, \cdots, h_m^{(n)})^{\mathrm{T}}$ is defined as

$$h_i^{(n)} = 1 - \sum_{k=0}^{n-m} \sum_{\ell=k+m}^{n} q_{n-m+i,\ell} d_k, \qquad 1 \leqslant i \leqslant m.$$

Note that for $1 \leqslant j \leqslant m-1$,

$$c_{1j}^{(n)} = -q_{n-m+1,j} - q_{n-m+1}^{(n)} \sum_{k=0}^{n-m} c_k^{(j)} + \sum_{k=0}^{n-m} q_{n-m+1}^{(k+m-1)} c_k^{(j)} = q_{n-m+1,n+1} \sum_{k=0}^{n-m+1} c_k^{(j)},$$

$$c_{1m}^{(n)} = -q_{n-m+1}^{(n)} \sum_{k=0}^{n-m} q_{n-m+1}^{(k+m-1)} c_k^{(m)} + \sum_{k=0}^{n-m} q_{n-m+1}^{(k+m-1)} c_k^{(m)} = q_{n-m+1,n+1} \sum_{k=0}^{n-m+1} F_k^{(0)},$$

$$h_1^{(n)} = 1 - q_{n-m+1}^{(n)} \sum_{k=0}^{n-m} d_k + \sum_{k=0}^{n-m} q_{n-m+1}^{(k+m-1)} d_k = q_{n-m+1,n+1} \sum_{k=0}^{n-m+1} d_k.$$

Hence equations (14) together with the m-birth property (i.e., $q_{n-m+1,n+1} > 0$) are equivalent to equations $A^{(n)}\mathbf{x} = \mathbf{b}^{(n)}$, where the matrix $A^{(n)}$ and the column vector $\mathbf{b}^{(n)}$ are defined in (1) and (2).

Therefore, it is obtained by Cramer's rule that

$$\mathbb{E}_i\tau_0^{(n)} = \frac{\det(A_i^{(n)})}{\det(A^{(n)})}, \qquad 1 \leqslant i \leqslant m. \tag{16}$$

Next, rewrite equations (13) as $x^{(n)} = R^{(n)}x^{(n)} + f^{(n)}$, where the elements of $R^{(n)} = (r_{ij}^{(n)} : 1 \leqslant i, j \leqslant n)$ and $f^{(n)} = (f_i^{(n)} : 1 \leqslant i \leqslant n)$ have the form

$$r_{ij}^{(n)} = \frac{q_{ij}}{q_i} \mathbb{1}_{\{1 \leqslant i \neq j \leqslant n\}}, \quad f_i^{(n)} = \frac{1}{q_i} \mathbb{1}_{\{1 \leqslant i \leqslant n\}}, \quad i, j \geqslant 0.$$

Then it is obvious that $R^{(n)}$ and $f^{(n)}$ are increasing in n in the element-wise sense. Since for each n, k, $\frac{q_{nk}}{q_n} \geqslant 0$ and $\frac{1}{q_n} > 0$, by the Monotone Convergence Theorem [2, Theorem 2.7], we know that $(x_i^{(n)})_{1 \leqslant i \leqslant n}$ is increasing to the minimal non negative solution (i.e., $(\mathbb{E}_i\tau_0)_{i\geqslant 1}$) of (10) as $n\to\infty$. Therefore, $\mathbb{E}_i\tau_0^{(n)}\uparrow\mathbb{E}_i\tau_0$, as $n\to\infty$. So taking limits in (16) gives us that as $n\to\infty$,

$$\frac{\det(A_i^{(n)})}{\det(A^{(n)})}\quad\Big\uparrow\quad\mathbb{E}_i\tau_0,\quad 1\leqslant i\leqslant m.$$

Equation (3) is easily obtained from Lemma 3.3.

From [2, Theorem 4.44(1)], it follows immediately that the m-birth process is ergodic if and only if $\mathbb{E}_0\sigma_0<+\infty$, which is now equivalent to $B_i<+\infty$ for all $i=1,2,\cdots,m$. By [2, Theorem 4.44(3)] or [7], the process is strongly ergodic if and only if $\sup_{k\geqslant 0}\mathbb{E}_k\sigma_0<+\infty$, which is equivalent to (4). As mentioned in the proof of the cited book, for ergodicity, the uniqueness assumption suffices instead of the recurrence assumption. The proof is now finished. □

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3.

By [6, Theorem 9.5.1] and [2, Theorem 2.13] (localization theorem), we know that $(\psi_{i0}(\lambda):i\geqslant 1)$ is the minimal nonnegative solution (x_i^*) to the equation

$$x_i=\frac{1}{q_i+\lambda}\sum_{k\notin\{0,i\}}q_{ik}x_k+\frac{\lambda}{q_i+\lambda},\quad i\geqslant 1.\tag{17}$$

The augmented truncation Q-matrix $\tilde{Q}^{(n)}$ is defined similarly to (12). We still have to deal with the following equations, named approximating equations, corresponding to $\tilde{Q}^{(n)}$:

$$x_i=\sum_{k=1,k\neq i}^{n}\frac{\tilde{q}_{ik}^{(n)}}{\lambda-\tilde{q}_{ii}^{(n)}}x_k+\frac{\lambda}{\lambda-\tilde{q}_{ii}^{(n)}},\quad 1\leqslant i\leqslant n.$$

Then following the strategy as in the proof of Theorem 2.1 gives us that

$$H^{(n)}\mathbf{x}=\mathbf{s}^{(n)},\qquad\text{i.e.,}\qquad\sum_{j=1}^{m}h_{ij}^{(n)}x_j=s_i^{(n)},\qquad 1\leqslant i\leqslant m,$$

where the matrix $H^{(n)}$ and the column vector $\mathbf{s}^{(n)}$ are defined in (6) and (7), and it also holds that

$$x_i^{(n)}=\sum_{\ell=1}^{m}\left(\sum_{k=0}^{i-m}\tilde{c}_k^{(\ell)}\right)x_\ell^{(n)}-\sum_{\ell=0}^{i-m}\tilde{d}_\ell,\quad m+1\leqslant i\leqslant n.$$

Therefore, applying Cramer's rule and the Monotone Convergence Theorem [2, Theorem 2.7] provides all our final formulas in Theorem 2.3. Thus, the proof of Theorem 2.3 is completed. □

4. **Examples.** In order to verify the effectiveness of our results in this paper, in this section we provide some examples. The first one is the single birth process, which is the $m=1$ case.

Example 4.1. (single birth process)

Let the regular and irreducible Q-matrix $Q = (q_{ij})$ be of the following form: $q_{i,i+1} > 0$ for all $i \geqslant 0$ and $q_{i,i+j} = 0$ for all $j \geqslant 2$ with $i \geqslant 0$. Assume the corresponding process is recurrent, then it could be obtained that

$$\mathbb{E}_i\sigma_0 = \sum_{k=0}^{i-1}\left(F_k^{(0)}d - d_k\right), \quad \mathbb{E}_i \mathrm{e}^{-\lambda\sigma_0} = 1 - \lambda\sum_{k=0}^{i-1}\left(\widetilde{F}_k^{(0)}\tilde{d} - \tilde{d}_k\right), \quad i \geqslant 1,\ \lambda > 0,$$

where

$$d := \lim_{k\to\infty}\frac{\sum_{n=0}^k d_n}{\sum_{n=0}^k F_n^{(0)}}, \quad \tilde{d} = \lim_{k\to\infty}\frac{\sum_{n=0}^k \tilde{d}_n}{\sum_{n=0}^k \widetilde{F}_n^{(0)}}.$$

Furthermore, the process is ergodic (i.e., positive recurrent) if and only if $d < \infty$; and it is strongly ergodic if and only if

$$\sup_{k\geqslant 0}\sum_{n=0}^{k}\left(F_n^{(0)}d - d_n\right) < \infty.$$

Proof. This example is for the $m = 1$ case of our results. At this stage, we have for $i \geqslant 0$,

$$F_i^{(i)} = 1, \quad F_k^{(i)} = \frac{1}{q_{k,k+1}}\sum_{j=i}^{k-1} q_k^{(j)}F_j^{(i)}, \quad k > i,$$

$$d_0 = 0, \quad d_i = \frac{1}{q_{i,i+1}}\left(1 + \sum_{k=1}^{i-1} q_i^{(k)}d_k\right), \quad i \geqslant 1,$$

$c_i^{(1)} = F_i^{(0)}$ and $a_{11}^{(n)} = \sum_{k=0}^n c_k^{(1)} = \sum_{k=0}^n F_k^{(0)}$. By applying Theorem 2.1, it is derived that $\det(A_1^{(n)}) = \sum_{k=0}^n d_k$ and $\det(A^{(n)}) = \sum_{k=0}^n F_k^{(0)}$. Thus we have

$$\mathbb{E}_1\tau_0 = \lim_{n\to\infty}\frac{\det(B_1^{(n)})}{\det(D^{(n)})} = \lim_{n\to\infty}\frac{\sum_{k=1}^n d_k}{\sum_{k=0}^n F_k^{(0)}} = d,$$

$$\mathbb{E}_i\tau_0 = \sum_{k=0}^{i-1} c_k^{(1)}\mathbb{E}_1\tau_0 - \sum_{k=0}^{i-1} d_k = \sum_{k=0}^{i-1}\left(F_k^{(0)}d - d_k\right), \quad i \geqslant 2.$$

Therefore the process is strongly ergodic if and only if

$$\sup_{i\geqslant 1}\mathbb{E}_i\tau_0 = \sup_{k\geqslant 0}\sum_{n=0}^{k}\left(F_n^{(0)}d - d_n\right) < \infty.$$

Similarly, we also know $\det(H_1^{(n)}) = \lambda\sum_{k=1}^n \tilde{d}_k$ and $H^{(n)} = \sum_{k=0}^n \tilde{F}_k^{(0)}$. By applying Theorem 2.3, it is obtained that for $\lambda > 0$,

$$\mathbb{E}_i \mathrm{e}^{-\lambda\sigma_0} = 1 - \lambda\sum_{k=0}^{i-1}\left(\widetilde{F}_k^{(0)}\tilde{d} - \tilde{d}_k\right), \quad i \geqslant 1.$$

Thus we have completed the proof of the example. □

Remark 4.2. Example 4.1 shows the same results as those from [3], which justifies the validity of our results when m is reduced to one.

Next we analyze one concrete example that belongs to the class of both 2-birth and single death processes.

Example 4.3. Given a regular and irreducible single death Q-matrix $Q = (q_{ij})$ satisfying:

$$q_{i,i-1} = a > 0, \quad i \geqslant 1, \quad q_{i,i+1} = b \geqslant 0, \quad q_{i,i+2} = d > 0, \quad i \geqslant 0,$$

and $q_{ij} = 0$ for other $i, j \geqslant 0$, $i \neq j$, then the process is recurrent if and only if $a \geqslant b+2d$ and it is ergodic if and only if $a > b+2d$. Moreover, starting from $j \geqslant 1$, the moment of the hitting time of 0 has the form:

$$\mathbb{E}_j \tau_0 = \frac{j}{a-b-2d}, \quad j \geqslant 1.$$

Therefore, the process could not be strongly ergodic.

Proof. In fact, it can be seen that

$$F_i^{(i)} = 1, \quad F_{i+1}^{(i)} = -\frac{b+d}{d}, \quad i \geqslant 0, \quad F_i^{(k)} = \frac{a}{d} F_{i-2}^{(k)} - \frac{b+d}{d} F_{i-1}^{(k)}, \quad 0 \leqslant k \leqslant i-2.$$

By the basic theory of difference equations, it is known that

$$F_n^{(k)} = \frac{\lambda_2^{n-k+1} - \lambda_1^{n-k+1}}{\lambda_2 - \lambda_1}, \qquad 0 \leqslant k \leqslant n,$$

where

$$\lambda_1 = \frac{-(b+d) - \sqrt{(b+d)^2 + 4ad}}{2d}, \quad \lambda_2 = \frac{-(b+d) + \sqrt{(b+d)^2 + 4ad}}{2d}.$$

Note that $\lambda_1 < -1$ and $-\lambda_1 > \lambda_2 > 0$. For this given process, we have $m = 2$ and thus

$$c_1^{(1)} = \frac{a+b+d}{d}, \quad c_k^{(1)} = -\sum_{\ell=1}^{k} \frac{F_k^{(\ell)} q_{\ell 1}}{q_{\ell,\ell+2}} = \frac{a+b+d}{d} F_k^{(1)} - \frac{a}{d} F_k^{(2)}, \quad k \geqslant 2,$$

$$c_k^{(2)} = F_k^{(0)} = \frac{1}{\lambda_2 - \lambda_1} (\lambda_2^{k+1} - \lambda_1^{k+1}), \quad k \geqslant 0.$$

And we also have for $i \geqslant 0$,

$$d_0 = 0, \quad d_i = \sum_{k=1}^{i} \frac{F_i^{(k)}}{q_{k,k+2}} = \frac{1}{d(\lambda_2 - \lambda_1)} \left(\frac{\lambda_2 (1 - \lambda_2^i)}{1 - \lambda_2} - \frac{\lambda_1 (1 - \lambda_1^i)}{1 - \lambda_1} \right),$$

and $\mathbf{b}^{(n)} = (b_1^{(n)}, b_2^{(n)})^{\mathrm{T}}$ with $b_1^{(n)} = \sum_{k=0}^{n-1} d_k$ and $b_2^{(n)} = d_n$. Define the matrix $A^{(n)} = (a_{ij}^{(n)})_{1 \leqslant i,j \leqslant 2}$ as in (1). Again, denote $A_1^{(n)}$ and $A_2^{(n)}$ by replacing the first and second column of $A^{(n)}$ with the column vector $\mathbf{b}^{(n)}$, repectively. Now we obtain

$$\begin{aligned}
\mathbb{E}_1 \tau_0 = \lim_{n \to \infty} \frac{\det(A_1^{(n)})}{\det(A^{(n)})} &= \frac{\lambda_1^2 \lambda_2^2}{(\lambda_1 - 1)(\lambda_2 - 1)(a\lambda_1(\lambda_2 - 1) - a\lambda_2 + (b+d)\lambda_1\lambda_2)} \\
&= \frac{1}{a-b-2d}, \\
\mathbb{E}_2 \tau_0 = \lim_{n \to \infty} \frac{\det(A_2^{(n)})}{\det(A^{(n)})} &= \frac{-(b+d)\lambda_1\lambda_2 + a(\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2)}{d(\lambda_1 - 1)(\lambda_2 - 1)(a\lambda_1(\lambda_2 - 1) - a\lambda_2 + (b+d)\lambda_1\lambda_2)} \\
&= \frac{2}{a-b-2d}.
\end{aligned}$$

From Lemma 3.3, we know that for $n \geqslant 3$,

$$\mathbb{E}_n \tau_0 = \sum_{k=1}^{2} \left(\sum_{i=0}^{n-2} c_i^{(k)} \right) \mathbb{E}_k \tau_0 - \sum_{i=0}^{n-2} d_i = a_{11}^{(n-1)} \mathbb{E}_1 \tau_0 + a_{12}^{(n-1)} \mathbb{E}_2 \tau_0 - b_1^{(n-1)} = \frac{n}{a-b-2d}.$$

Thus we obtain $\sup_{n\geqslant 1}\mathbb{E}_n\tau_0 = \sup_{n\geqslant 3}\frac{n}{a-b-2d} = \infty$. So in this case, the process is ergodic but not strongly ergodic. Hence the proof is completed by Theorem 2.1 and Lemma 3.3. □

Remark 4.4. In the Example 4.3, the process could either be regarded as the single death process or the 2-birth process. Recall that via the formulas of the corresponding quantities as in example 4.3 for single death processes, it is easily checked that the results are consistent with the counterparts by using expressions obtained before.

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