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The secondary periodic element β_{p^2/p^2-1} and its applications

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Abstract Let $p \geqslant 7$ be a prime. We prove that β_{p^2/p^2-1} survives to E_∞ in the Adams-Novikov spectral sequence. Additionally, using the Thom map $\Phi : \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*) \to \mathrm{Ext}^{*,*}_A(\mathbb{Z}/p, \mathbb{Z}/p)$, we can see that h_0h_3 also survives to E_∞ in the classical Adams spectral sequence. As an application of these results, we prove that $\beta^p_{p/p}$ is divisible by β_1.

Keywords stable homotopy groups of spheres, Adams-Novikov spectral sequence, infinite descent method

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1 Introduction

Let p be an odd prime. The Adams-Novikov spectral sequence (ANSS) based on the Brown-Peterson spectrum BP is one of the most powerful tools to compute the p-component of the stable homotopy groups of spheres $\pi_*(S^0)$ (see [1, 8, 11, 22]).

The E_2-term of the ANSS is $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$, which has been extensively studied in low dimensions. For $s = 1$, $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\alpha_{kp^n/n+1}$ for $n \geqslant 0$, and $p \nmid k$ with $k \geqslant 1$, where $\alpha_{kp^n/n+1}$ has order p^{n+1} (see [11, 14]). For $s = 2$, $\mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$ is the direct sum of cyclic groups generated by $\beta_{kp^n/j,i+1}$ for suitable (n, k, j, i) (see [11, 22, 24]), and $\beta_{kp^n/j,i+1}$ has order p^{i+1}. For $s \geqslant 3$, only partial results of $\mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, BP_*)$ are known (see [13]).

To compute the stable homotopy groups of the sphere, we still need to know which elements of the E_2-page could survive to the E_∞-page of the ANSS. It is known that each element $\alpha_{kp^n/n+1}$ is a permanent cycle in the ANSS which represents an element of $\mathrm{Im}J$ with the same order. Moreover, Behrens [4] showed that for l a prime which generates $\mathbb{Z}_p^\times$, the spectrum $Q(l)$ introduced in [2, 3] detects the α and β families in the stable stems. However, we are still far from fully determining which elements of the $\beta_{kp^n/j,i+1}$ family could survive to E_∞.

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Let $\beta_{kp^n/j}$ denote $\beta_{kp^n/j,1}$. Toda [26, 27] proved that $\alpha_1\beta_1^p$ is zero in $\pi_*(S^0)$. This relation supports a non-trivial Adams-Novikov differential called the Toda differential, i.e.,

$$d_{2p-1}(\beta_{p/p}) = a \cdot \alpha_1\beta_1^p \neq 0, \tag{1.1}$$

where a is a non-zero scalar *mod* p. Hence $\beta_{p/p}$ could not survive the ANSS.

Based on the Toda differential (1.1), Ravenel [19] generalized the result and proved that there are non-trivial differentials $d_{2p-1}(\beta_{p^n/p^n}) \equiv a \cdot \alpha_1\beta^p_{p^{n-1}/p^{n-1}}$ mod ker $\beta_1^{p(p^{n-1}-1)/(p-1)}$ for $n \geqslant 1$. Consequently, β_{p^n/p^n} also cannot survive to E_∞ in the ANSS. From this, one can see that only $\beta_{kp^n/j} \in H^2(BP_*)$ for $k \geqslant 2$, $1 \leqslant j \leqslant p^n$ or $k = 1$, $1 \leqslant j \leqslant p^n - 1$ might survive to E_∞ in the ANSS. The following are some known results in this area:

Let $p \geqslant 5$. Oka proved that (a) for $k = 1$, $1 \leqslant j \leqslant p - 1$ or $k \geqslant 2$, $1 \leqslant j \leqslant p$, $\beta_{kp/j}$ are permanent cycles in the ANSS (see [15]); (b) for $k = 1$, $1 \leqslant j \leqslant 2p - 2$ or $k \geqslant 2$, $1 \leqslant j \leqslant 2p$, $\beta_{kp^2/j}$ are permanent cycles in the ANSS (see [16]); (c) for $n \geqslant 2$, $k = 1$, $1 \leqslant j \leqslant 2^{n-1}(p - 1)$ or $n \geqslant 2$, $k \geqslant 2$, $1 \leqslant j \leqslant 2^{n-1}p$, $\beta_{kp^n/j}$ are permanent cycles in the ANSS (see [17, 18]).

Let $p \geqslant 7$. Shimomura [25] proved that for $k \geqslant 1$, $1 \leqslant j \leqslant p^2 - 2$, $\beta_{kp^2/j}$ are permanent cycles in the ANSS.

In this paper, we prove the following theorem.

Theorem A. *Let* $p \geqslant 7$ *be a prime. Then* β_{p^2/p^2-1} *is a permanent cycle in the Adams-Novikov spectral sequence.*

We can briefly summarize our strategy to prove Theorem A as follows. Inspection of degrees shows that β_{p^2/p^2-1} has too low a dimension to be the target of an Adams-Novikov differential. Hence it suffices to prove that β_{p^2/p^2-1} does not support any non-trivial differential. We work with the small descent spectral sequence (SDSS), which converges to the E_2-page of the ANSS. Computations show that in dimension one less than that of β_{p^2/p^2-1}, the SDSS has 8 elements listed in Lemma 3.1, and each must be eliminated as a possible target of a differential on β_{p^2/p^2-1}. Two of them are removed by $d_2's$ in the SDSS as shown in Figure 1, leaving the six listed in Theorem 3.3. Four of them are removed by $d_{2p-1}'s$ in the ANSS as shown in Figure 2. This leaves only $\mathfrak{g}_7$ and $\mathfrak{g}_8$. They each lie in filtration 3, so they cannot be the target of an ANSS differential on β_{p^2/p^2-1}.

Assumption on prime *p*. Henceforth, in this paper, it is always implicitly assumed that $p > 5$, unless stated otherwise.

Let M be the mod p Moore spectrum and $M(1, p^n - 1)$ be the cofiber of the map $v_1^{p^n-1}$, i.e.,

$$\Sigma^* M \xrightarrow{v_1^{p^n-1}} M \longrightarrow M(1, p^n - 1).$$

Ravenel [20, Theorem 7.12] claimed that if $M(1, p^n - 1)$ is a ring spectrum and β_{p^n/p^n-1} is a permanent cycle, then $\beta_{kp^n/j}$ is a permanent cycle for all $k \geqslant 1$ and $j \leqslant p^n - 1$.

Between the ANSS and the classical Adams spectral sequence (ASS), there is the Thom reduction map

$$\Phi : \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*) \longrightarrow \mathrm{Ext}^*_A(\mathbb{Z}/p, \mathbb{Z}/p)$$

such that $\Phi(\beta_{p^n/p^n-1}) = h_0h_{n+1}$. Thus we obtain the following corollary.

Corollary B. *Let* $p \geqslant 7$ *be a prime. Then* h_0h_3 *is a permanent cycle in the classical Adams spectral sequence.*

In [5], Cohen and Goerss claimed the existence of h_0h_{n+1} in the classical ASS. One can see that the existence of h_0h_{n+1} in ASS is equivalent to the existence of β_{p^n/p^n-1} in the Adams-Novikov spectral sequence. However, Minami [12] found a fatal error in their proof, so it is still an open problem in odd primary stable homotopy theory. Due to its extreme importance, Hovey [6] listed the convergence of h_0h_{n+1} as one of the major open problems in algebraic topology.

Consider the ANSS for the Moore spectrum $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*(M)) \Rightarrow \pi_*(M)$. From the Toda differential, one can see that in the ANSS for the Moore spectrum

$$d_{2p-1}(h_{n+2}) = v_1\beta^p_{p^n/p^n}, \quad d_{2p-1}(v_1h_{n+2}) = v_1^2\beta^p_{p^n/p^n}.$$

Applying the connecting homomorphism $\delta: \mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*(M)) \to \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$ induced by the cofiber sequence

$$S^0 \xrightarrow{p} S^0 \longrightarrow M,$$

one gets an Adams differential in the ANSS for the sphere, i.e.,

$$d_{2p-1}(\beta_{p^{n+1}/p^{n+1}-1}) = \alpha_2 \beta^p_{p^n/p^n}.$$

In Section 6, we prove that $\beta^p_{p/p}$ is divisible by β_1, i.e., $\beta^p_{p/p} = \beta_1 \mathfrak{g}$. Note $\alpha_2\beta_1 = 0$, and this provides another perspective for understanding why we could have

$$d_{2p-1}(\beta_{p^2/p^2-1}) = \alpha_2 \beta^p_{p/p} = 0 \quad \text{in } \mathrm{Ext}^{2p+1,*}_{BP_*BP}(BP_*, BP_*)$$

in Theorem A.

Based on the analysis of $\beta^p_{p/p}$, we conjecture the following.

Conjecture C. *For $n < p-1$, $\beta^p_{p^n/p^n}$ is divisible by β_1 and*

$$\begin{aligned}
\beta^p_{p/p} &= \beta_1 h_{11} b_{20}^{p-3} \gamma_2, \\
\beta^p_{p^2/p^2} &= \beta_1 h_{21} h_{11} b_{30}^{p-4} \delta_3, \\
&\vdots \\
\beta^p_{p^n/p^n} &= \beta_1 h_{n,1} h_{n-1,1} \cdots h_{11} b_{n+1,0}^{p-n-2} \alpha^{(n+2)}_{n+1}, \\
&\vdots \\
\beta^p_{p^{p-2}/p^{p-2}} &= \beta_1 h_{p-2,1} h_{p-3,1} \cdots h_{11} \alpha^{(p)}_{p-1},
\end{aligned}$$

where $\alpha^{(n+2)}$ is the $(n+2)$-th letter of the Greek alphabet, and $\alpha^{(n+2)}_{n+1} \in \mathrm{Ext}^{n+2,*}_{BP_*BP}(BP_*, BP_*)$ is one of the $(n+2)$-th Greek letter family elements. These equations imply $\alpha_2\beta^p_{p^n/p^n} = \alpha_2\beta_1\mathfrak{g} = 0$ for $n < p-1$.

For $n \geqslant p-1$, we conjecture that $\beta^p_{p^n/p^n}$ is not divisible by β_1 and $\alpha_2\beta^p_{p^n/p^n}$ might be non-zero. This implies that $\beta_{p^{n+1}/p^{n+1}-1}$ does not survive to E_∞ in the ANSS when $n \geqslant p-1$.

The rest of this paper is organized as follows. In Section 2, we recall the construction of the topological small descent spectral sequence (TSDSS) and the small descent spectral sequence (SDSS), where the SDSS is a spectral sequence that converges to $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$ started from the Ext groups of a complex with p-cells. We list all the generators of the E_1-page of the SDSS along with their corresponding $t-s$ values. This allows us to compute the E_2-terms of the ANSS for specific $t-s$ values. In Section 3, we compute the Adams-Novikov E_2-term $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$ subject to $t-s = q(p^3+1)-3$ by the SDSS. In Section 4, a non-trivial Adams-Novikov differential $d_{2p-1}(h_{20}b_{11}\gamma_s) = \alpha_1\beta_1^p h_{20}\gamma_s$ is proved. We prove our main theorem by showing that $d_r(\beta_{p^2/p^2-1}) = 0$ in Section 5. At last, in Section 6, we prove that $\beta^p_{p/p}$ is divisible by β_1 and give our conjecture.

2 The small descent spectral sequence and the ABC theorem

In 1985, Ravenel [21–24] introduced the *method of infinite descent* and used it to compute the first thousand stems of the stable homotopy groups of spheres at the prime 5. This method applies a so-called small descent spectral sequence (SDSS) to identify the E_2-terms of the ANSS.

Hereafter, we set $q = 2p-2$. As mentioned in Section 1, we assume that $p > 5$ is a prime number throughout this paper. Let $T(n)$ be the Ranevel spectrum (see [22, Section 5, Chapter 6]) characterized by $BP_*(T(n)) = BP_*[t_1, t_2, \ldots, t_n]$. Then we have the following diagram:

$$S^0 = T(0) \longrightarrow T(1) \longrightarrow T(2) \longrightarrow \cdots \longrightarrow T(n) \longrightarrow \cdots \longrightarrow BP,$$

where S^0 denotes the sphere spectrum localized at p. Let $T(0)_{p-1}$ and $T(0)_{p-2}$ denote the $q(p-1)$ and $q(p-2)$ skeletons of $T(1)$, respectively, and they are denoted by X and $\overline{X}$ for simplicity. Then

$$X = S^0 \cup_{\alpha_1} e^q \cup \cdots \cup_{\alpha_1} e^{(p-2)q} \cup_{\alpha_1} e^{(p-1)q} \quad \text{and} \quad \overline{X} = S^0 \cup_{\alpha_1} e^q \cup \cdots \cup_{\alpha_1} e^{(p-2)q}.$$

The BP-homologies of them are

$$BP_*(X) = BP_*[t_1]/\langle t_1^p\rangle \quad \text{and} \quad BP_*(\overline{X}) = BP_*[t_1]/\langle t_1^{p-1}\rangle.$$

From the definition above, we get the following cofiber sequences:

$$S^0 \xrightarrow{i'} X \xrightarrow{j'} \Sigma^q\overline{X} \xrightarrow{k'} S^1, \tag{2.1}$$

$$\overline{X} \xrightarrow{i''} X \xrightarrow{j''} S^{(p-1)q} \xrightarrow{k''} \Sigma\overline{X}, \tag{2.2}$$

and the short exact sequences of BP-homologies

$$0 \longrightarrow BP_*(S^0) \xrightarrow{i'_*} BP_*(X) \xrightarrow{j'_*} BP_*(\Sigma^q\overline{X}) \longrightarrow 0, \tag{2.3}$$

$$0 \longrightarrow BP_*(\overline{X}) \xrightarrow{i''_*} BP_*(X) \xrightarrow{j''_*} BP_*(S^{(p-1)q}) \longrightarrow 0. \tag{2.4}$$

Putting (2.3) and (2.4) together, one has the following long exact sequence:

$$0 \longrightarrow BP_*(S^0) \longrightarrow BP_*(X) \longrightarrow BP_*(\Sigma^q X) \longrightarrow BP_*(\Sigma^{pq} X) \longrightarrow \cdots. \tag{2.5}$$

Putting (2.1) and (2.2) together, one has the following Adams diagram of cofibers:

$$\begin{array}{ccccccccc} S^0 & \longleftarrow & \Sigma^{q-1}\overline{X} & \longleftarrow & S^{pq-2} & \longleftarrow & \Sigma^{(p+1)q-3}\overline{X} & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X & & \Sigma^{q-1}X & & \Sigma^{pq-2}X & & \Sigma^{(p+1)q-3}X. & & \end{array} \tag{2.6}$$

Proposition 2.1 (See [22, Proposition 7.4.2]). *Let X be as above. Then*

(a) *There is a spectral sequence converging to* $\mathrm{Ext}^{s+u,*}_{BP_*BP}(BP_*, BP_*(S^0))$ *with the E_1-term*

$$E_1^{s,t,u} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X)) \otimes E[\alpha_1] \otimes P[\beta_1],$$

where $E_1^{s,t,0} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X))$, $\alpha_1 \in E_1^{0,q,1}$, $\beta_1 \in E_1^{0,qp,2}$, *and* $d_r : E_r^{s,t,u} \to E_r^{s-r+1,t,u+r}$. *Here, $E[-]$ denotes the exterior algebra and $P[-]$ denotes the polynomial algebra on the indicated generators. This spectral sequence is referred to as the small descent spectral sequence (SDSS).*

(b) *There is a spectral sequence converging to $\pi_*(S^0)$ with the E_1-term*

$$E_1^{s,t} = \pi_*(X) \otimes E[\alpha_1] \otimes P[\beta_1],$$

where $E_1^{0,t} = \pi_t(X)$, $\alpha_1 \in E_1^{1,q}$, $\beta_1 \in E_1^{2,pq}$, *and* $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$. *This spectral sequence is referred to as the topological small descent spectral sequence (TSDSS).*

The two spectral sequences mentioned above could determine the 0-line and the 1-line (i.e., $\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*(S^0))$, $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*(S^0))$) or the corresponding elements in $\pi_*(S^0)$ by $\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*(X))$ and $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*(X))$. Additionally, for $s \geqslant 2$, the s-line $\mathrm{Ext}^{s,*}_{BP_*BP}$ $(BP_*, BP_*(S^0))$ or the corresponding elements in $\pi_*(S^0)$ are produced by the corresponding elements in $\mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, BP_*(X))$ with $s \geqslant 2$ as described in the following ABC theorem [24, Theorem 7.5.1].

Theorem 2.2 (The ABC theorem). *For $t - s < q(p^3 + p - 1) - 3$ and $s \geqslant 2$,*

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X)) = A \oplus B \oplus C.$$

Here, A is the $\mathbb{Z}/p$-vector space spanned by

$$\{\beta_{ip}, \beta_{ip+1} \mid i \leqslant p-1\} \cup \{\beta_{p^2/p^2-j} \mid 0 \leqslant j \leqslant p-1\}.$$

Meanwhile, $B = R \otimes \{\gamma_i \mid i \geqslant 2\}$, where

$$R = P[b_{20}^p] \otimes E[h_{20}] \otimes \mathbb{Z}/p\{\{b_{11}^k \mid 0 \leqslant k \leqslant p-1\} \cup \{h_{11}b_{20}^k \mid 0 \leqslant k \leqslant p-2\}\}.$$

Finally,

$$C^{s,t} = \bigoplus_{i\geqslant 0} R^{s+2i,t+i(p^2-1)q}.$$

We list the bidegrees of the various elements appearing in the ABC theorem as follows:

$$\beta_{ip} \in \mathrm{Ext}^{2,q[ip^2+ip-1]}, \quad \beta_{ip+1} \in \mathrm{Ext}^{2,q[ip^2+(i+1)p]}, \quad \beta_{p^2/p^2-j} \in \mathrm{Ext}^{2,q[p^3+j]},$$
$$\gamma_i \in \mathrm{Ext}^{3,q[i(p^2+p+1)-p-2]}, \quad h_{11} \in \mathrm{Ext}^{1,qp}, \quad h_{20} \in \mathrm{Ext}^{1,q(p+1)},$$
$$b_{11} \in \mathrm{Ext}^{2,qp^2}, \quad b_{20} \in \mathrm{Ext}^{2,qp(p+1)}.$$

From the ABC theorem above, we can find all the generators of $\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*(X))$ for $s \geqslant 2$ and $t-s < q(p^3+p-1)-3$. Table 1 summarizes the first class of generators, i.e., the generators of A.

Here, $pq-2 = 2p^2-2p-2$ is the total degree of $\beta_1 \in E_1^{0,qp,2}$ in the SDSS. The reason for computing $t-s \bmod pq-2$ and the purpose of underlining certain values will become clear in Lemma 3.1.

The generators of B are summarized in Table 2.

Let us take $h_{11}b_{20}^k\gamma_i$ from the B-family as an example to illustrate the calculation.

The total degree of $h_{11}b_{20}^k\gamma_i$ is

$$q[(i+k)p^2 + (i+k)p + i - 2] - 2k - 4 = 2(i+k)p^3 - 2(k+2)p - 2(i+k)$$

for $2 \leqslant i \leqslant p-1$ and $0 \leqslant k \leqslant p-2$. To ensure that the total degree of $h_{11}b_{20}^k\gamma_i$ is less than $q(p^3+p-1)-3$, we need $i+k < p$. A straightforward computation shows

$$2(i+k)p^3 - 2(k+2)p - 2(i+k) \equiv 2(k+2i-2)p \bmod pq-2.$$

Notice that $2(k+2i-2)p > pq-2$ if $k+2i > p$, and the total degree of $h_{11}b_{20}^k\gamma_i$ is

$$2(k+2i-2)p - (pq-2) = 2(k+2i-p-1)p + 2 \bmod pq-2$$

if $k+2i > p$.

Table 1 Generators of A

Generators of A	$t-s$ and $t-s \bmod pq-2$	Index ranges
β_{ip}	$q[ip^2+ip-1]-2$	
	$\equiv 2(i-1)p+2i$	if $i \leqslant p-2$
	$\equiv 0$	if $i = p-1$
β_{ip+1}	$q[ip^2+(i+1)p]-2$	
	$\equiv 2ip+2i$	if $i \leqslant p-2$
	$\equiv \underline{2p}_{2p}$	if $i = p-1$
β_{p^2/p^2-j}	$q[p^3+j]-2$	
	$\equiv \underline{2(j+1)p-2j}_{2p}$	if $j \leqslant p-2$
	$\equiv 4$	if $j = p-1$

Table 2 Generators of B

Generators of B	$t-s$ and $t-s$ mod $pq-2$	Index ranges
$h_{11}b_{20}^k\gamma_i$	$q[(i+k)p^2+(i+k)p+i-2]$	for $2\leqslant i\leqslant p-1, 0\leqslant k\leqslant p-2,$
	$-2k-4$	and $2\leqslant i+k\leqslant p-1$
	$\equiv 2(k+2i-2)p$	if $k+2i\leqslant p$
	$\equiv 2(k+2i-p-1)p+2$	if $k+2i>p$
$h_{20}h_{11}b_{20}^k\gamma_i$	$q[(i+k)p^2+(i+k+1)p+i-1]$	for $2\leqslant i\leqslant p-1, 0\leqslant k\leqslant p-2,$
	$-2k-5$	and $2\leqslant i+k\leqslant p-1$
	$\equiv 2(k+2i-1)p-1$	if $k+2i<p$
	$\equiv 2(k+2i-p)p+1$	if $k+2i\geqslant p$
$b_{11}^k\gamma_i$	$q[(i+k)p^2+(i-1)p+i-2]$	for $2\leqslant i\leqslant p-1, 0\leqslant k\leqslant p-1,$
	$-2k-3$	and $2\leqslant i+k\leqslant p-1$
	$\equiv 2(k+2i-2)p-2k-1$	if $k+2i\leqslant p+1$
	$\equiv 1$	if $k=0$, $2i=p+1$
	$\equiv \underline{2(k+2i-p-1)p-2k+1}_{4p-3}$	if $k+2i\geqslant p+2$
$h_{20}b_{11}^k\gamma_i$	$q[(i+k)p^2+ip+i-1]$	for $2\leqslant i\leqslant p-1, 0\leqslant k\leqslant p-1,$
	$-2k-4$	and $2\leqslant i+k\leqslant p-1$
	$\equiv 2(k+2i-1)p-2(k+1)$	if $k+2i\leqslant p$
	$\equiv \underline{2(k+2i-p)p-2k}_{2p}$	if $k+2i>p$

One might have noticed that although R contains the $P[b_{20}^p]$ part, $P[b_{20}^p]$ does not show up in the B-family generators. This is because the total degree of b_{20}^p is

$$p(qp(p+1)-2)>q(p^3+p-1)-3.$$

Hence, suppose that a generator of B is a multiple of b_{20}^p, and its total degree would exceed the range of interest.

On the other hand, the $P[b_{20}^p]$ part does show up in the C-family generators. The key difference is that C is the direct sum of shifted copies of R. Based on [21, Theorems 4.11 and 4.12], we could determine all the generators of C.

In more detail, let us write $i=jp+m$ with $0\leqslant m\leqslant p-1$. Considering the i-th shifted copy $R^{s+2\underline{i},t+\underline{i}(p^2-1)q}\subset C^{s,t}$ we have the following:

(1) $b_{20}^{(j+1)p}\in R^{2(p-m)+2(\underline{jp+m}),t+(\underline{jp+m})(p^2-1)q}\subset C^{2(p-m),t}$, which is represented by $b_{20}^{p-m-1}u_{jp+m}$ for $p-1\geqslant m\geqslant 1$, where

$$u_{jp+m}\in C^{2,q[(j+1)p^2+(j+m+1)p+m]}.$$

From this, we get generators of the form

$$b_{20}^{p-m-1}u_{jp+m}\otimes E[h_{20}]\otimes\{b_{11}^k\mid 0\leqslant k\leqslant p-1\}\cup\{h_{11}b_{20}^k\mid 0\leqslant k\leqslant p-2\}.$$

(2) $b_{11}^kb_{20}^{jp}\in R^{2(k-m)+2(\underline{jp+m}),t+(\underline{jp+m})(p^2-1)q}\subset C^{2(k-m),t}$, which is represented by $b_{11}^{k-m-1}\beta_{(j+1)p/p-m}$ for $p-1\geqslant k\geqslant m+1\geqslant 1$, where

$$\beta_{(j+1)p/p-m}\in C^{2,q[(j+1)p^2+jp+m]}.$$

From this, we get generators of the form

$$b_{11}^{k-m-1}\beta_{(j+1)p/p-m}\otimes E[h_{20}].$$

- Especially, $h_{20}b_{11}^{p-1}b_{20}^{jp}\in R^{3+2(\underline{jp+p-2}),t+(\underline{jp+p-2})(p^2-1)q}\subset C^{3,t}$ is represented by $h_{11}\beta_{(j+1)p/1,2}$, which is an element of order p^2.

(3) $h_{11}b_{20}^{k}b_{20}^{jp} \in R^{2(k-m)+1+2(\underline{jp+m}),t+(\underline{jp+m})(p^2-1)q} \subset C^{2(k-m)+1,t}$, which is represented by

$$b_{20}^{k-m-1}\eta_{jp+m+1}$$

for $p-2 \geqslant k \geqslant m+1 \geqslant 1$, where

$$\eta_{jp+m+1} = h_{11}u_{jp+m} \in C^{3,q[(j+1)p^2+(j+m+2)p+m]}.$$

(4) $h_{20}h_{11}b_{20}^{k}b_{20}^{jp} \in R^{2(k-m+1)+2(\underline{jp+m}),t+(\underline{jp+m})(p^2-1)q} \subset C^{2(k-m+1)t}$, which is represented by

$$b_{20}^{k-m}\beta_{jp+m+2}$$

for $p-2 \geqslant k \geqslant m \geqslant 0$, where

$$\beta_{jp+m+2} \in C^{2,q[jp^2+(j+m+2)p+m+1]}.$$

• Especially, $h_{20}h_{11}b_{20}^{p-2}b_{20}^{jp} \in R^{2+2(\underline{jp+p-2}),t+(\underline{jp+p-2})(p^2-1)q} \subset C^{2,t}$ is represented by $\beta_{(j+1)p/1,2}$, which is an element of order p^2.

The generators of C are summarized in Table 3.

Remark 2.3. The Adams-Novikov spectral sequence for the spectrum X collapses from the E_2-term $\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*(X))$ in the range $t-s < q(p^3+p-1)-3$, since there are no elements with filtration $> 2p$. Thus we actually get the homotopy groups $\pi_{t-s}(X)$ in this range.

3 The ANSS E_2-term $\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*)$ at $t-s = q(p^3+1)-3$

Consider the Adams-Novikov differential $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ in the ANSS. From the total degree of β_{p^2/p^2-1}, we know that

$$d_r(\beta_{p^2/p^2-1}) \in \mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*)$$

such that $t-s = q(p^3+1)-3$. The SDSS $E_1^{s,t,u}$ converges to $\mathrm{Ext}_{BP_*BP}^{s+u,t}(BP_*, BP_*)$. Fixing $t-s-u = q(p^3+1)-3$, we have the following lemma.

Lemma 3.1. *When restricted to $t-s-u = q(p^3+1)-3$, the E_1-page $E_1^{s,t,u}$ of the SDSS is the $\mathbb{Z}/p$-module generated by the following $\frac{p+15}{2}$ generators*:

$$\mathfrak{g}_1 = \alpha_1\beta_1^{p^2-1}\beta_2 \in E_1^{2,*,2p^2-1}, \quad \mathfrak{g}_2 = \beta_1^{p^2-p}h_{20}\beta_{p/p} \in E_1^{3,*,2p^2-2p},$$
$$\mathfrak{g}_3 = \alpha_1\beta_1^{\frac{p^2-2p-1}{2}}h_{20}\gamma_{\frac{p+1}{2}} \in E_1^{4,*,p^2-2p}, \quad \mathfrak{g}_4 = \beta_1^{\frac{p^2-6p+1}{2}}b_{11}^2\gamma_{\frac{p+1}{2}} \in E_1^{7,*,p^2-6p+1},$$
$$\mathfrak{g}_{5,m} = \alpha_1\beta_1^{mp-\frac{p-1}{2}}b_{11}^{\frac{p-1}{2}-m}\beta_{(\frac{p+1}{2})p/p-m} \in E_1^{p+1-2m,*,*}, \quad \mathfrak{g}_6 = \beta_1^{p-1}\eta_{(p-3)p+3} \in E_1^{3,*,2p-2},$$
$$\mathfrak{g}_7 = \alpha_1\beta_{(p-1)p+1} \in E_1^{2,q(p^3+1),1}, \quad \mathfrak{g}_8 = \alpha_1\beta_{p^2/p^2} \in E_1^{2,q(p^3+1),1}.$$

The index range for m in $\mathfrak{g}_{5,m}$ is $0 \leqslant m \leqslant \frac{p-1}{2}$.

Proof. Fix $t-s-u = q(p^3+1)-3$. From the ABC theorem, we know that the generators of the E_1-terms in the SDSS are of the form $W = \beta_1^k w$ or $W = \alpha_1\beta_1^k w$, where w is an element listed in the ABC theorem.

(1) If a generator of $E_1^{s,t,u}$ is of the form $W = \beta_1^k w$, then the total degree of $\beta_1^p w$ is $q(p^3+1)-3$ and the total degree of w is $q(p^3+1)-3$ modulo the total degree of β_1 which is $t-u = qp-2$. Noting that

$$q(p^3+1)-3 \equiv 4p-3 \bmod qp-2,$$

we list all the generators whose total degree might be $4p-3$ mod $qp-2$, which are marked with an underline and the subscript $4p-3$ in Tables 1–3.

Table 3 Generators of C

Generators of C	$t-s$ and $t-s$ mod $pq-2$	Index ranges
$b_{11}^{k}b_{20}^{p-m-1}u_{jp+m}$	$q[(p-m+j+k+1)p^2+jp+m]$ $-2(p-m+k)$ $\equiv 2(j+k+1)p+2(j-k+1)$	for $1\leqslant m<p, 0\leqslant j\leqslant p-2$, and $0\leqslant k<p, j+k<m$
$h_{20}b_{11}^{k}b_{20}^{p-m-1}u_{jp+m}$	$q[(p-m+j+k+1)p^2+(j+1)p$ $+m+1]-2(p-m+k)-1$ $\equiv 2(j+k+2)p+2(j-k+1)-1$ $\equiv 2(j-k+2)p-1$	for $1\leqslant m<p, 0\leqslant j\leqslant p-2$, and $0\leqslant k<p, j+k<m$, and $j+k\leqslant p-3$ if $j+k\leqslant p-4$ or $j+k=p-3, 2j<p-5$ if $j+k=p-3, 2j\geqslant p-5$
$h_{11}b_{20}^{k+p-m-1}u_{jp+m}$	$q[(p-m+j+k+1)p^2+(j+k$ $+1)p+m]-2(p-m+k)-1$ $\equiv 2(j+k+2)p+2(j-p)+3$	for $1\leqslant m<p, 0\leqslant j\leqslant p-2$, and $0\leqslant k\leqslant p-2, j+k<m$, and $j+k\leqslant p-3$
$h_{20}h_{11}b_{20}^{k+p-m-1}u_{jp+m}$	$q[(p-m+j+k+1)p^2+(j+k$ $+2)p+m]+2(m-k-2)$ $\equiv 2(j+k+2)p+2j+2$ $\equiv 2j+4$	for $1\leqslant m<p, 0\leqslant j\leqslant p-2$, and $0\leqslant k\leqslant p-2, j+k<m$, and $j+k\leqslant p-3$ if $j+k\leqslant p-4$ if $j+k=p-3$
$b_{11}^{k-m-1}\beta_{(j+1)p/p-m}$	$q[(j+k-m)p^2+jp+m]$ $-(2k-2m)$ $\equiv 2(j+k)p+2(j-k)$ $\underline{\equiv 2(j+k-p+1)p+2(j-k+1)}_{2p}$	for $1\leqslant m+1\leqslant k<p$, and $0\leqslant j\leqslant p-2$ if $j+k\leqslant p-2$ or $j+k=p-1$, $2j<p-1$ if $j+k\geqslant p$ or $j+k=p-1$, $2j\geqslant p-1$
$h_{20}b_{11}^{k-m-1}\beta_{(j+1)p/p-m}$	$q[(j+k-m)p^2+(j+1)p+m$ $+1]-(2k-2m+1)$ $\underline{\equiv 2(j+k+1)p+2(j-k)-1}_{4p-3}$ $\equiv 2(j+k-p+2)p+2(j-k)+1$	for $1\leqslant m+1\leqslant k<p$, and $0\leqslant j\leqslant p-2$ if $j+k\leqslant p-3$ or $j+k=p-2$, $2j\leqslant p-3$ if $j+k>p-2$ or $j+k=p-2$, $2j>p-3$
$h_{11}\beta_{(j+1)p/1,2}$	$q[(j+1)p^2+(j+2)p-1]-3$ $\equiv 2jp+2(j+1)+1$ $\equiv 1$	for $0\leqslant j\leqslant p-2$ if $j\leqslant p-3$ if $j=p-2$
$b_{20}^{k-m-1}\eta_{jp+m+1}$	$q[(j+k-m)p^2+(j+k+1)p$ $+m]-(2k-2m+1)$ $\equiv 2(j+k)p+2j+1$ $\underline{\equiv 2(j+k-p+2)p+2(j-p)+3}_{4p-3}$	for $1\leqslant m+1\leqslant k\leqslant p-2$, and $0\leqslant j\leqslant p-2$ if $j+k\leqslant p-2$ if $j+k>p-2$
$b_{20}^{k-m}\beta_{jp+m+2}$	$q[(j+k-m)p^2+(j+k+2)p$ $+m+1]-2(k-m+1)$ $\underline{\equiv 2(j+k+1)p+2j}_{2p}$ $\equiv 2(j+k-p+3)p+2(j-p)+2$ $\equiv 0$	for $0\leqslant m\leqslant k\leqslant p-2$, and $0\leqslant j\leqslant p-2$ if $j+k\leqslant p-3$ if $j+k>p-3$ if $j=k=p-2$
$\beta_{(j+1)p/1,2}$	$q[(j+1)p^2+(j+1)p-1]-2$ $\equiv 2jp+2(j+1)$ $\equiv 0$	for $0\leqslant j\leqslant p-2$ if $j\leqslant p-3$ if $j=p-2$

We have

$$
\begin{aligned}
&b_{11}^{k}\gamma_i \quad \text{at } k=2 \text{ and } i=(p+1)/2,\\
&h_{20}b_{11}^{k-m-1}\beta_{(j+1)p/p-m} \quad \text{at } k=1 \text{ and } j=0,\\
&b_{20}^{k-m-1}\eta_{jp+m+1} \quad \text{at } k=3 \text{ and } j=p-3,
\end{aligned}
$$

from which we get the following generators in $E_1^{s,t,u}$:

$$b_{11}^2\gamma_{\frac{p+1}{2}} \Rightarrow \mathfrak{g}_4 = \beta_1^{\frac{p^2-6p+1}{2}} b_{11}^2\gamma_{\frac{p+1}{2}} \in E_1^{7,*,p^2-6p+1},$$
$$h_{20}\beta_{p/p} \Rightarrow \mathfrak{g}_2 = \beta_1^{p^2-p} h_{20}\beta_{p/p} \in E_1^{3,*,2p^2-2p},$$
$$\eta_{(p-3)p+3} \Rightarrow \mathfrak{g}_6 = \beta_1^{p-1}\eta_{(p-3)p+3} \in E_1^{3,*,2p-2}.$$

(2) If a generator of $E_1^{s,t,u}$ is of the form $W = \alpha_1\beta_1^k w_1$, then from the total degree of α_1 being $t-u = 2p-3$, we see that the total degree of w_1 is $2p$ modulo $qp-2$. Similarly, we can find all such w_1's, which are marked with an underline and the subscript $2p$ in Tables 1–3. We have

$$\beta_{(p-1)p+1}, \quad \beta_{p^2/p^2}, \quad h_{20}\gamma_{\frac{p+1}{2}}, \quad b_{11}^{\frac{p-1}{2}-m}\beta_{(\frac{p+1}{2})p/p-m}, \quad \beta_2,$$

from which we get the following generators in $E_1^{s,t,u}$:

$$\mathfrak{g}_7 = \alpha_1\beta_{(p-1)p+1}, \quad \mathfrak{g}_8 = \alpha_1\beta_{p^2/p^2},$$
$$\mathfrak{g}_3 = \alpha_1\beta_1^{\frac{p^2-2p-1}{2}} h_{20}\gamma_{\frac{p+1}{2}}, \quad \mathfrak{g}_{5,m} = \alpha_1\beta_1^{mp-\frac{p-1}{2}} b_{11}^{\frac{p-1}{2}-m}\beta_{(\frac{p+1}{2})p/p-m}, \quad 0 \leqslant m \leqslant \frac{p-1}{2},$$
$$\mathfrak{g}_1 = \alpha_1\beta_1^{p^2-1}\beta_2.$$

Computing the filtration of the corresponding generators, we get the lemma. □

Remark 3.2. The method in proving Lemma 3.1 is a general method in computing the E_1-term $E_1^{s,t,u}$ of the SDSS with specialized $t-s-u$.

Theorem 3.3. *When restricted to $t-s = q(p^3+1)-3$, the Adams-Novikov E_2-page $\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*)$ is the $\mathbb{Z}/p$-module generated by the following* 6 *elements*:

$$\mathfrak{g}_1 = \alpha_1\beta_1^{p^2-1}\beta_2 \in \mathrm{Ext}_{BP_*BP}^{2p^2+1,*},$$
$$\mathfrak{g}_3 = \alpha_1\beta_1^{\frac{p^2-2p-1}{2}} h_{20}\gamma_{\frac{p+1}{2}} \in \mathrm{Ext}_{BP_*BP}^{p^2-2p+4,*}, \quad \mathfrak{g}_4 = \beta_1^{\frac{p^2-6p+1}{2}} b_{11}^2\gamma_{\frac{p+1}{2}} \in \mathrm{Ext}_{BP_*BP}^{p^2-6p+8,*},$$
$$\mathfrak{g}_6 = \beta_1^{p-1}\eta_{(p-3)p+3} \in \mathrm{Ext}_{BP_*BP}^{2p+1,*},$$
$$\mathfrak{g}_7 = \alpha_1\beta_{(p-1)p+1} \in \mathrm{Ext}^{3,q(p^3+1)}, \quad \mathfrak{g}_8 = \alpha_1\beta_{p^2/p^2} \in \mathrm{Ext}^{3,q(p^3+1)}.$$

Proof. Following Ravenel [22, p. 287], we compute in the cobar complex of $N_0^2 = BP_*/(p^\infty, v_1^\infty)$:

$$d\left(\frac{v_2^{jp}}{pv_1^p}(t_2 - t_1^{p+1})\right) = \frac{v_2^{jp}}{pv_1^p}t_1^p \otimes t_1 + \frac{v_2^{jp}}{pv_1^{p-1}}b_{10},$$
$$-d\left(\frac{v_2^{jp+1}}{pv_1^{p+1}}t_1\right) = -\frac{v_2^{jp}}{pv_1^p}t_1^p \otimes t_1 - j\frac{v_2^{(j-1)p+1}}{pv_1}t_1^{p^2} \otimes t_1 + \frac{v_2^{jp}}{pv_1}t_1 \otimes t_1,$$
$$d\left(j\frac{v_2^{(j-1)p}v_3}{pv_1}t_1\right) = j\frac{v_2^{(j-1)p+1}}{pv_1}t_1^{p^2} \otimes t_1 - j\frac{v_2^{jp}}{pv_1}t_1 \otimes t_1,$$
$$-(j-1)/2d\left(\frac{v_2^{jp}}{pv_1}t_1^2\right) = (j-1)\frac{v_2^{jp}}{pv_1}t_1 \otimes t_1.$$

A straightforward calculation shows that the coboundary of

$$\frac{v_2^{jp}}{pv_1^p}t_2 - \frac{v_2^{jp}}{pv_1^p}t_1^{p+1} - \frac{v_2^{jp+1}}{pv_1^{p+1}}t_1 + j\frac{v_2^{(j-1)p}v_3}{pv_1}t_1 - (j-1)/2\frac{v_2^{jp}}{pv_1}t_1^2$$

is $\frac{v_2^{jp}}{pv_1^{p-1}}b_{10}$. Then from $\delta\delta(\frac{v_2^{jp}}{pv_1^p}) = \beta_{jp/p}$, we get a differential in the SDSS:

$$d_2(h_{20}\beta_{jp/p}) = \beta_1\beta_{jp/p-1}.$$

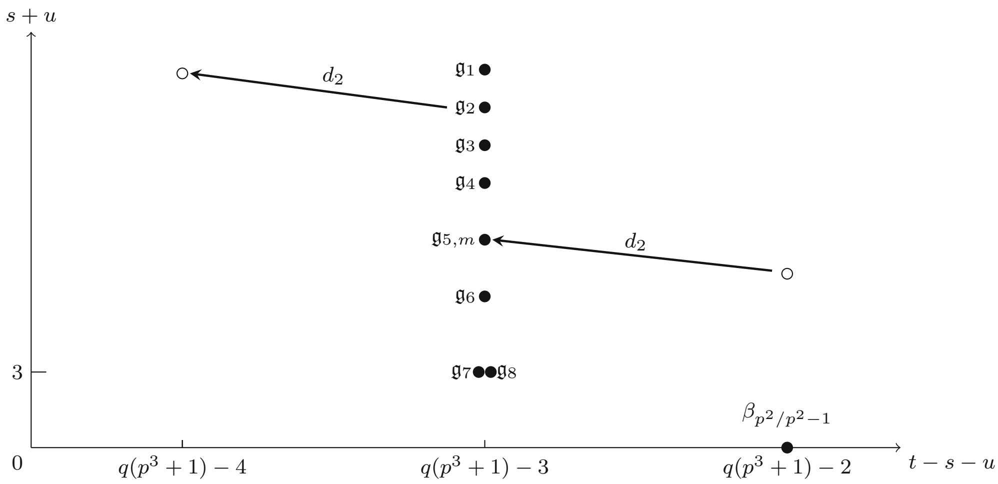

Figure 1 Two SDSS d_2 differentials

Similarly, we have

$$d_2(h_{20}\beta_{jp/i}) = \beta_1\beta_{jp/i-1} \quad \text{for } 2 \leqslant i \leqslant p. \tag{3.1}$$

Applying (3.1), we get the following differentials in the SDSS:

$$d_2(\mathfrak{g}_2) = d_2(\beta_1^{p^2-p}h_{20}\beta_{p/p}) = \beta_1^{p^2-p+1}\beta_{p/p-1},$$
$$d_2(\alpha_1\beta_1^{mp-\frac{p-1}{2}-1}b_{11}^{\frac{p-1}{2}-m}h_{20}\beta_{(\frac{p+1}{2})p/p-m+1}) = \alpha_1\beta_1^{mp-\frac{p-1}{2}}b_{11}^{\frac{p-1}{2}-m}\beta_{(\frac{p+1}{2})p/p-m} = \mathfrak{g}_{5,m},$$

which are illustrated in Figure 1. Then the theorem follows. □

4 A differential in the ANSS

This section is aimed at showing that

$$d_{2p-1}(h_{20}b_{11}\gamma_s) = \alpha_1\beta_1^p h_{20}\gamma_s \tag{4.1}$$

in the Adams-Novikov spectral sequence. This differential could imply the vanishing of $\mathfrak{g}_3$.

We begin from showing that $\pi_{q(p^2+2p+2)-2}(V(2)) = 0$, from which we show that the Toda bracket $\langle \alpha_1\beta_1, p, \gamma_s\rangle = 0$ and the Toda bracket $\langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle$ is well-defined. Then from the relation

$$\langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle = \alpha_1\beta_1^{p-1}h_{20}\gamma_s = \beta_{p/p-1}\gamma_s$$

in $\pi_*(S^0)$ and $d(h_{20}b_{11}) = \beta_1\beta_{p/p-1}$, we get the desired differential in the ANSS.

Let $p \geqslant 7$ and $V(2)$ be the Smith-Toda spectrum characterized by $BP_*(V(2)) = BP_*/I_3$, where I_3 is the invariant ideal of $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_i, \ldots]$ generated by p, v_1, and v_2. To compute the homotopy groups of $V(2)$, one has the ANSS $\{E_r^{s,t}V(2), d_r\}$ that converges to $\pi_*(V(2))$. The E_2-page of this spectral sequence is

$$E_2^{s,t}V(2) = \mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*(V(2))).$$

Let

$$\Gamma = BP_*/I_3 \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP_*/I_3 = BP_*/I_3[t_1, t_2, \ldots].$$

Then $(BP_*/I_3, \Gamma)$ is a Hopf algebroid, and its structure map is deduced from that of $(BP_*, BP_*(BP))$. By a change of ring theorem, one sees that

$$\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*(V(2))) = \mathrm{Ext}_{\Gamma}^{s,t}(BP_*, BP_*/I_3) \Rightarrow \pi_*(V(2)).$$

Lemma 4.1. *The $(q(p^2+2p+2)-2)$-dimensional stable homology group of $V(2)$ is trivial, i.e.,*

$$\pi_{q(p^2+2p+2)-2}(V(2))=0.$$

Proof. Fixing $t-s=q(p^2+2p+2)-2$, we know that the Adams-Novikov E_2-term

$$\mathrm{Ext}_{BP_*BP}^{s,s+q(p^2+2p+2)-2}(BP_*,BP_*(V(2)))=\mathrm{Ext}_{\Gamma}^{s,s+q(p^2+2p+2)-2}(BP_*,BP_*/I_3)$$

converges to $\pi_{q(p^2+2p+2)-2}(V(2))$. We prove that $\pi_{q(p^2+2p+2)-2}(V(2))=0$ by showing that

$$\mathrm{Ext}_{BP_*BP}^{s,s+q(p^2+2p+2)-2}(BP_*,BP_*(V(2)))=0.$$

In the cobar complex $C_\Gamma^s BP_*/I_3$, the inner degree of v_i, $|v_i|=|t_i|\geqslant q(p^3+p^2+p+1)$ for $i\geqslant 4$. It follows that in the range $t-s\leqslant q(p^3+p^2+p+1)-1$,

$$\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*,BP_*/I_3)=\mathrm{Ext}_{\Gamma}^{s,t}(BP_*,BP_*/I_3)=\mathrm{Ext}_{\Gamma'}^{s,t}(BP_*,BP_*/I_3),$$

where $\Gamma'=\mathbb{Z}/p[v_3][t_1,t_2,t_3]$. From $\eta_R(v_3)\equiv v_3 \bmod I_3$, we see that

$$\mathrm{Ext}_{\mathbb{Z}/p[v_3][t_1,t_2,t_3]}^{s,*}(BP_*,BP_*/I_3)\cong \mathrm{Ext}_{\mathbb{Z}/p[t_1,t_2,t_3]}^{s,*}(\mathbb{Z}/p,\mathbb{Z}/p)\otimes\mathbb{Z}/p[v_3].$$

To compute the Ext groups $\mathrm{Ext}_{\mathbb{Z}/p[t_1,t_2,t_3]}^{*}(\mathbb{Z}/p,\mathbb{Z}/p)$, we can use the modified May spectral sequence (MSS) introduced in [7,9,10,24].

There is the May spectral sequence $\{E_r^{s,t,*},\delta_r\}$ that converges to $\mathrm{Ext}_{\mathbb{Z}/p[t_1,t_2,t_3]}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p)$. The E_1-term of this spectral sequence is

$$E_1^{*,*,*}=E[h_{ij}\mid 0\leqslant j,i=1,2,3]\otimes P[b_{ij}\mid 0\leqslant j,i=1,2,3], \tag{4.2}$$

where

$$h_{ij}\in E_1^{1,q(1+p+\cdots+p^{i-1})p^j,2i-1}\quad\text{and}\quad b_{ij}\in E_1^{2,q(1+p+\cdots+p^{i-1})p^{j+1},p(2i-1)}.$$

The first May differential is given by

$$\delta_1(h_{ij})=\sum_{0<k<i}h_{i-k,k+j}h_{k,j}\quad\text{and}\quad\delta_1(b_{ij})=0. \tag{4.3}$$

For the reason of the total degree, to compute $\mathrm{Ext}_{BP_*BP}^{s,s+(q(p^2+2p+2)-2)}(BP_*,BP_*/I_3)$, we only need to consider the sub-algebra generated by $h_{30},h_{20},h_{10},h_{21},h_{11},h_{12}$, and b_{20},b_{10},b_{11}, i.e., the subcomplex

$$E[h_{ij}\mid 1\leqslant i,i+j\leqslant 3]\otimes E[b_{20},b_{11}]\otimes P[b_{10}].$$

From (4.3), we know that within $t-s\leqslant q(p^2+2p+2)-2$, May's E_2-term is

$$E_2^{s,*,*}=H^{s,*,*}(E_1^{s,*,*},\delta_1)=H^{*,*,*}(E[h_{ij}\mid 0\leqslant j,i+j\leqslant 3],\delta_1)\otimes E[b_{20},b_{11}]\otimes P[b_{10}].$$

Toda [28] computed the cohomology of $(E[h_{ij}\mid 0\leqslant j,i+j\leqslant 3],\delta_1)$. Here, we only jot down the even-dimensional elements within that range, i.e.,

$$\begin{aligned}&h_{20}h_{10},\quad q(p+2)-2,\quad h_{20}h_{11},\quad q(2p+1)-2,\\&h_{12}h_{10},\quad q(p^2+1)-2,\quad h_{21}h_{11},\quad q(p^2+2p)-2.\end{aligned}$$

Thus within $t-s\leqslant q(p^2+2p+2)-2$, even-dimensional May's E_2-term $E_2^{s,t,*}$ is a sub-algebra of

$$\mathbb{Z}/p\{1,h_{20}h_{10},h_{20}h_{11},h_{12}h_{10},h_{21}h_{11}\}\otimes E[b_{20},b_{11}]\otimes P[b_{10}].$$

Suppose that we have a generator y in $\mathrm{Ext}_{\mathbb{Z}/p[v_3][t_1,t_2,t_3]}^{s,s+q(p^2+2p+2)-2}(BP_*,BP_*/I_3)$. Then y is the form of x or v_3x, where x is an even-dimensional generator in $H^*(E[h_{ij}\mid i+j\leqslant 3])\otimes E[b_{20},b_{11}]\otimes P[b_{10}]$.

(1) If $y = v_3 x$, then $x \in E_2^{s,t,*}$ subject to $t - s = q(p+1) - 2$. An easy computation shows that the corresponding E_2-term is zero.

(2) If $y = x$, then $x \in E_2^{s,t,*}$ subject to $t - s = q(p^2 + 2p + 2) - 2$. Similarly, from

$$q(p^2 + 2p + 2) - 2 \equiv 6p - 2 \bmod qp - 2,$$

we compute the total degree $t - s \bmod qp - 2$ of the generators in

$$\mathbb{Z}/p\{1, h_{20}h_{10}, h_{20}h_{11}, h_{12}h_{10}, h_{21}h_{11}\} \otimes [b_{20}, b_{11}]$$

and find that none of them is $6p - 2$. Thus the corresponding E_2-term is zero.

The lemma then follows. □

It is easily shown that the following theorem holds from the lemma above.

Theorem 4.2. *For $p \geqslant 7$ and $s \geqslant 1$, the Toda bracket is $\langle \alpha_1\beta_1, p, \gamma_s \rangle = 0$.*

Proof. Let $\widetilde{v}_3$ be the composition of the following maps:

$$S^{q(p^2+p+1)} \xrightarrow{\widetilde{i}} \Sigma^{q(p^2+p+1)} V(2) \xrightarrow{v_3} V(2),$$

where the first map is the inclusion map to the bottom cell.

It is known that $\widetilde{v}_3$ is an order p element in $\pi_{q(p^2+p+1)}(V(2))$. Thus the Toda bracket $\langle \alpha_1\beta_1, p, \widetilde{v}_3 \rangle$ is well-defined and $\langle \alpha_1\beta_1, p, \widetilde{v}_3 \rangle \in \pi_{q(p^2+2p+2)-2}(V(2)) = 0$. It follows that the Toda bracket $\langle \alpha_1\beta_1, p, \widetilde{v}_3 \rangle = 0$.

Let $\widetilde{j} : V(2) \to S^{q(p+2)+3}$ be the map that collapses all lower-dimensional cells in $V(2)$. Then $\gamma_s = \widetilde{v}_3 \cdot v_3^{s-1} \cdot \widetilde{j}$. As a result,

$$\langle \alpha_1\beta_1, p, \gamma_s \rangle = \langle \alpha_1\beta_1, p, \widetilde{v}_3 \cdot v_3^{s-1} \cdot \widetilde{j} \rangle = \langle \alpha_1\beta_1, p, \widetilde{v}_3 \rangle \cdot v_3^{s-1} \cdot \widetilde{j} = 0.$$

This completes the proof. □

Proposition 4.3 (See [22, Proposition 7.5.11])**.** *For $p \geqslant 7$ and $s \geqslant 1$, the Toda bracket $\langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s \rangle$ is well-defined in $\pi_*(S^0)$, and*

$$\alpha_1\beta_1^{p-1}h_{20}\gamma_s = \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s \rangle = \beta_{p/p-1}\gamma_s.$$

Proof. From $\langle \beta_1^{p-1}, \alpha_1\beta_1, p \rangle = 0$, $\langle \alpha_1\beta_1, p, \alpha_1 \rangle = 0$, $\langle \alpha_1, \alpha_1\beta_1, p \rangle = 0$, and $\langle \alpha_1\beta_1, p, \gamma_s \rangle = 0$, we know that the following 4-fold Toda brackets are well-defined and

$$\beta_{p/p-1} = \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1 \rangle, \quad \alpha_1 h_{20}\gamma_s = \langle \alpha_1, \alpha_1\beta_1, p, \gamma_s \rangle.$$

On the other hand, one has

$$\begin{aligned}
\beta_1^{p-1}\alpha_1 h_{20}\gamma_s &= \beta_1^{p-1}\langle \alpha_1, \alpha_1\beta_1, p, \gamma_s \rangle \\
&= \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s \rangle \\
&= \alpha_1\langle \beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s \rangle \\
&= \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1\gamma_s \rangle \\
&= \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1 \rangle \cdot \gamma_s \\
&= \beta_{p/p-1}\gamma_s.
\end{aligned}$$

The proposition follows. □

Theorem 4.4. *For $p \geqslant 7$ and $2 \leqslant s \leqslant p - 2$, we have the following Adams-Novikov differentials:*

$$d_{2p-1}(h_{20}b_{11}\gamma_s) = \alpha_1\beta_1^p h_{20}\gamma_s.$$

Proof. Note that $b_{11} = \beta_{p/p}$. Then from (3.1), one has the differential in the small descent spectral sequence $d_2(h_{20}b_{11}) = \beta_1\beta_{p/p-1}$, which could be read as $d(h_{20}\beta_{p/p}) = \beta_1\beta_{p/p-1}$ and $d(h_{20}\beta_{p/p}\gamma_s) = \beta_1\beta_{p/p-1}\gamma_s$ in the cobar complex of BP_*, or equivalently the first Adams-Novikov differential in the ANSS. Then from the relation $\beta_{p/p-1}\gamma_s = \alpha_1\beta_1^{p-1}h_{20}\gamma_s$ in $\pi_*(S^0)$ and $\beta_{p/p-1}\gamma_s = 0$ in $\mathrm{Ext}^{5,*}_{BP_*BP}(BP_*, BP_*)$, we get the Adams differential in the ANSS, i.e.,

$$d_{2p-1}(h_{20}b_{11}\gamma_s) = \beta_1 \cdot \beta_1^{p-1}\alpha_1 h_{20}\gamma_s = \alpha_1\beta_1^p h_{20}\gamma_s.$$

The theorem follows. □

5 The proof of Theorem A

In this section, we prove our main theorem which states that β_{p^2/p^2-1} survives to E_∞ in the ANSS. Note that β_{p^2/p^2-1} has too low a dimension to be the target of an Adams-Novikov differential, and we will do this by showing that all the Adams-Novikov differentials $d_r(\beta_{p^2/p^2-1})$ are trivial.

Lemma 5.1. *Let $i \not\equiv 0 \bmod p$. In the ANSS, one has the following Adams-Novikov differential*:

$$d_{2p-1}(\eta_i) = \beta_1^p\beta_{i+1}.$$

Proof. Recall from [22, Theorem 7.3.8] that there is a spectral sequence of the following form:

$$E_1 = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X^{p^2-1})) \otimes E[h_{11}] \otimes P[b_{11}] \Rightarrow \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X)),$$

where $BP_*(X^{p^2-1}) = BP_*[t_1]/\langle t_1^{p^2}\rangle$. Moreover, it is proved in [22, Theorem 7.3.11(e)] that the above spectral sequence has non-trivial differentials $d_2(h_{20}\mu_{i-1}) = ib_{11}\beta_{i+1}$. From its definition, we know that $\eta_i = h_{11}\mu_{i-1}$ is represented by

$$\delta\delta\bigg(\frac{v_2^{p+i-1}t_2 + v_2^i t_2^p - v_2^i t_1^{p^2+p} - v_2^{i-1}v_3 t_1^p}{pv_1}\bigg)$$

(see [22, p. 288]), which is also denoted by $\delta\delta(\frac{v_2^{p+i}}{pv_1}\zeta_2)$ in [11, 29]. In the cobar complex of $N_0^2 = BP_*/(p^\infty, v_1^\infty)$, a straightforward computation shows that the coboundary of

$$\frac{v_2^i(t_3 - t_1t_2^p - t_2t_1^{p^2} + t_1^{p^2+p+1}) + v_2^{p+i-1}(t_1t_2 - t_1^{p+2}) - v_2^{i-1}v_3(t_2 - t_1^{p+1})}{pv_1} + \frac{2v_2^{p+i}}{(p+i)p^2v_1}t_1 - \frac{v_2^{p+i}}{(p+i)pv_1^2}t_1^2$$

is

$$\frac{(v_2^{p+i-1}t_2 + v_2^i t_2^p - v_2^i t_1^{p^2+p} - v_2^{i-1}v_3 t_1^p) \otimes t_1}{pv_1} + \frac{v_2^{i+1}}{pv_1}b_{11}.$$

This shows that in $\mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, N_0^2)$, we have the following relation between cohomology classes:

$$\bigg[\frac{(v_2^{p+i-1}t_2 + v_2^i t_2^p - v_2^i t_1^{p^2+p} - v_2^{i-1}v_3 t_1^p) \otimes t_1}{pv_1}\bigg] = -\bigg[\frac{v_2^{i+1}}{pv_1}b_{11}\bigg].$$

Applying the connecting homomorphism $\delta\delta$, we get $\alpha_1\eta_i = \beta_{i+1}\beta_{p/p}$.

From $\alpha_1\eta_i = \beta_{i+1}\beta_{p/p}$ and the Toda differential, one has

$$\alpha_1 d_{2p-1}(\eta_i) = d_{2p-1}(\alpha_1\eta_i) = d_{2p-1}(\beta_{i+1}\beta_{p/p}) = \alpha_1\beta_1^p\beta_{i+1}.$$

The lemma follows from $\alpha_1 d_{2p-1}(\eta_i) = \alpha_1\beta_1^p\beta_{i+1}$. □

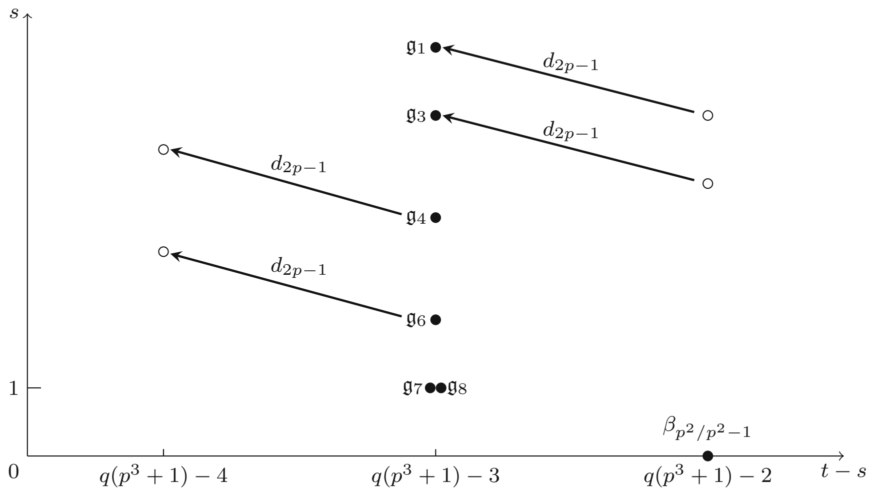

Figure 2 Four ANSS d_{2p-1} differentials

Proof of Theorem A. From $\beta_{p^2/p^2-1} \in \mathrm{Ext}^{2,q(p^3+1)}_{BP_*BP}(BP_*, BP_*)$, we know that $d_r(\beta_{p^2/p^2-1}) \in \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$ subject to $t-s=q(p^3+1)-3$. From Theorem 3.3, we know that the corresponding $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$ is the $\mathbb{Z}/p$-module generated by $\mathfrak{g}_1, \mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_6$, and $\mathfrak{g}_7, \mathfrak{g}_8$.

Since $\mathfrak{g}_7 = \alpha_1\beta_{(p-1)p+1}$ and $\mathfrak{g}_8 = \alpha_1\beta_{p^2/p^2}$ have dimensions that are too low, they lie outside the image of $d_r(\beta_{p^2/p^2-1})$. Moreover, we will show that $\mathfrak{g}_1, \mathfrak{g}_3, \mathfrak{g}_4$, and $\mathfrak{g}_6$ either support or receive a nontrivial differential (see Figure 2). Therefore, they cannot lie in the image of $d_r(\beta_{p^2/p^2-1})$.

From the Toda differential $d_{2p-1}(b_{11}) = \alpha_1\beta_1^p$, we have

$$d_{2p-1}(\beta_1^{p^2-p-1}b_{11}\beta_2) = \alpha_1\beta_1^{p^2-1}\beta_2 = \mathfrak{g}_1,$$
$$d_{2p-1}(\mathfrak{g}_4) = d_{2p-1}(\beta_1^{\frac{p^2-6p+1}{2}}b_{11}^2\gamma_{\frac{p+1}{2}}) = 2\alpha_1\beta_1^{\frac{p^2-4p+1}{2}}b_{11}\gamma_{\frac{p+1}{2}}.$$

From $d_{2p-1}(h_{20}b_{11}\gamma_s) = \alpha_1\beta_1^p h_{20}\gamma_s$ (see Theorem 4.4), we have

$$d_{2p-1}(\beta_1^{\frac{p^2-4p-1}{2}}h_{20}b_{11}\gamma_{\frac{p+1}{2}}) = \alpha_1\beta_1^{\frac{p^2-2p-1}{2}}h_{20}\gamma_{\frac{p+1}{2}} = \mathfrak{g}_3.$$

From Lemma 5.1, we have

$$d_{2p-1}(\mathfrak{g}_6) = d_{2p-1}(\beta_1^{p-1}\eta_{(p-3)p+3}) = \beta_1^{2p-1}\beta_{(p-3)p+4}.$$

Then Theorem A follows. □

6 A conjecture

Consider the cofiber sequence

$$S^0 \xrightarrow{p} S^0 \longrightarrow M$$

and the induced short exact sequence of BP-homologies

$$0 \longrightarrow BP_*(S^0) \xrightarrow{p} BP_*(S^0) \longrightarrow BP_*(M) \longrightarrow 0,$$

which induces a long exact sequence of Ext groups, i.e.,

$$\begin{array}{ccccccccc}
\cdots \to & \mathrm{Ext}^{1,t}(BP_*(S^0)) & \to & \mathrm{Ext}^{1,t}(BP_*(S^0)) & \to & \mathrm{Ext}^{1,t}(BP_*(M)) & \xrightarrow{\delta} & \mathrm{Ext}^{2,t}(BP_*(S^0)) & \to \cdots \\
 & \downarrow d_{2p-1} & & \downarrow d_{2p-1} & & \downarrow d_{2p-1} & & \downarrow d_{2p-1} & \\
\cdots \to & \mathrm{Ext}^{2p,*}(BP_*(S^0)) & \to & \mathrm{Ext}^{2p,*}(BP_*(S^0)) & \to & \mathrm{Ext}^{2p,*}(BP_*(M)) & \xrightarrow{\delta} & \mathrm{Ext}^{2p+1,*}(BP_*(S^0)) & \to \cdots .
\end{array}$$

For the connecting homomorphism δ, one has

$$\delta(h_{i+2}) = \beta_{p^{i+1}/p^{i+1}}, \quad \delta(v_1 h_{i+2}) = \beta_{p^{i+1}/p^{i+1}-1}, \quad \delta(v_1^i) = i\alpha_i.$$

From the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$, one can get a non-trivial differential in the ANSS for the Moore spectrum M:

$$d_{2p-1}(h_2) = v_1\beta_1^p.$$

Then from the relation $h_{i+1}\beta_{p/p}^{p^i} = h_{i+2}\beta_1^{p^i}$ (see [19] and [22, p. 246]), we get the following Adams-Novikov differential by induction:

$$\begin{aligned} d_{2p-1}(h_{i+2})\beta_1^{p^i} &= d_{2p-1}(h_{i+2}\beta_1^{p^i}) \\ &= d_{2p-1}(h_{i+1}\beta_{p/p}^{p^i}) \\ &= d_{2p-1}(h_{i+1})\beta_{p/p}^{p^i} \\ &= v_1\beta_{p^{i-1}/p^{i-1}}^p\beta_{p/p}^{p^i} \\ &= v_1(\beta_{p^{i-1}/p^{i-1}}\beta_{p/p}^{p^{i-1}})^p \\ &= v_1\beta_{p^i/p^i}^p\beta_1^{p^i}, \end{aligned}$$

which implies $d_{2p-1}(h_{i+2}) = v_1\beta_{p^i/p^i}^p$ in the ANSS for the Moore spectrum M. Then from the convergence of v_1 in the ANSS for the Moore spectrum, one has

$$d_{2p-1}(v_1 h_{i+2}) = v_1^2\beta_{p^i/p^i}^p.$$

Applying the connecting homomorphism δ, we have the Adams-Novikov differential for the sphere

$$d_{2p-1}(\beta_{p^{i+1}/p^{i+1}-1}) = d_{2p-1}(\delta(v_1h_{i+2})) = \delta(d_{2p-1}(v_1h_{i+2})) = \delta(v_1^2\beta_{p^i/p^i}^p) = 2\alpha_2\beta_{p^i/p^i}^p.$$

So one can prove the non-existence of $\beta_{p^{i+1}/p^{i+1}-1}$ from the non-triviality of

$$\alpha_2\beta_{p^i/p^i}^p \neq 0 \in \mathrm{Ext}_{BP_*BP}^{2p+1,*}(BP_*, BP_*).$$

The behavior for small i is already known:

(1) $\beta_{p/p-1}$ exists and $\alpha_2\beta_1^p = 0$ because $\alpha_2\beta_1 = 0$.

(2) β_{p^2/p^2-1} exists, and this implies $\alpha_2\beta_{p/p}^p = 0$.

We know that $\beta_{p/p}^p \neq 0$ in $\mathrm{Ext}_{BP_*BP}^{2p,qp^3}(BP_*, BP_*)$ (see [19, 22]). However, we could not find its representative element b_{11}^p in $\mathrm{Ext}_{BP_*BP}^{2p,qp^3}(BP_*, BP_*(X))$ due to the non-trivial differential $d(h_{11}b_{20}^{p-1}) = b_{11}^p$ in the SDSS (see [22, Theorem 7.3.12(b)] and the ABC theorem). Nonetheless, we could prove that $\beta_{p/p}^p$ is divisible by β_1.

(1) At the prime $p = 5$, $\beta_1 x_{952}$ converges to $\beta_{5/5}^5$, where $x_{952} = h_{11}b_{20}^{p-3}\gamma_2$. This implies $\alpha_2\beta_{5/5}^5 = \alpha_2\beta_1 x_{952} = 0$ (see [22, Subsection 7.5]) because $\alpha_2\beta_1 = 0$.

(2) At the prime $p \geqslant 7$, we compute $\mathrm{Ext}_{BP_*BP}^{2p,qp^3}(BP_*, BP_*)$ by the SDSS. The E_1-term

$$E_1^{s,t,u} = \mathrm{Ext}_{BP_*BP}^{s,*}(BP_*, BP_*(X)) \otimes E[\alpha_1] \otimes P[\beta_1]$$

subject to $s + u = 2p$, $t = qp^3$ is the $\mathbb{Z}/p$ module generated by $\beta_1 h_{11} b_{20}^{p-3}\gamma_2$, $\alpha_1\beta_1 b_{20}^{p-3}\eta_p$, and $\alpha_1\beta_1^{\frac{p-1}{2}} h_{20} b_{11}^{\frac{p-5}{2}} b_{20}\mu_{\frac{p-3}{2}p+p-2}$. Since each generator is divisible by β_1, we conclude that $\beta_{p/p}^p$ must also be divisible by β_1. We further conjecture that $\beta_{p/p}^p = \beta_1 h_{11} b_{20}^{p-3}\gamma_2$. Additionally, we have other conjectures regarding β_{p^n/p^n}^p for general n, which are summarized in Conjecture C.

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