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On finiteness of stationary configurations of the planar five-vortex problem

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Abstract

The finiteness problem of stationary configurations for the planar five-vortex problem is considered in this paper. The numbers of equilibria and rigidly translating configurations are shown to be at most 6 and 24 respectively. The numbers of relative equilibria and collapse configurations are shown to be finite, except perhaps if the 5-tuple of vorticities belongs to a given codimension 2 subvariety of the vorticity space. In particular, if the vorticities are of the same sign, the number of stationary configurations is finite.

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1 Introduction

The *planar N-vortex problem* which originated from Helmholtz's work in 1858 [9], considers the motion of point vortices in a fluid plane. It was given a Hamiltonian formulation by Kirchhoff as follows:

$$\Gamma_n \dot{\mathbf{r}}_n = J\frac{\partial H}{\partial \mathbf{r}_n} = J\sum_{1\le j\le N, j\neq n} \Gamma_j\Gamma_n \frac{\mathbf{r}_j-\mathbf{r}_n}{|\mathbf{r}_j-\mathbf{r}_n|^2}, \quad n=1,\ldots,N.$$

Here, $J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $\mathbf{r}_n=(x_n, y_n)\in\mathbb{R}^2$, and Γ_n $(n=1,\ldots,N)$ are the positions and vortex strengths (or vorticities) of the vortices, and the Hamiltonian is $H=-\sum_{1\le j<k\le N}\Gamma_j\Gamma_k \ln|\mathbf{r}_j-\mathbf{r}_k|$, where $|\cdot|$ denotes the Euclidean norm in

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$\mathbb{R}^2$. The N-vortex problem is a widely used model for providing finite-dimensional approximations to vorticity evolution in fluid dynamics, especially when the focus is on the trajectories of the vorticity centers rather than the internal structure of the vorticity distribution [10].

An interesting set of special solutions of the dynamical system are homographic solutions, where the relative shape of the configuration remains constant during the motion. An excellent review of these solutions can be found in [3, 10]. Following O'Neil, we refer to the corresponding configurations as *stationary*. The only stationary configurations are equilibria, rigidly translating configurations (where the vortices move with a common velocity), relative equilibria (where the vortices rotate uniformly), and collapse configurations (where the vortices collide in finite time) [13].

Many results on stationary configurations have been obtained by focusing on systems with a small number of vortices and exploring symmetric cases (cf. [2, 5, 7, 8, 11–15, 17, 18, 20–22] and the references therein). Notably, we emphasize works whose underlying concepts have proven fruitful in understanding central configurations in celestial mechanics. The equations governing stationary configurations are similar to those describing central configurations in celestial mechanics. By applying the topological methods first introduced for central configurations in celestial mechanics [16], Palmore gave without proof a lower bound on the number of relative equilibria for N vorticities of the same sign [17]. O'Neil categorized stationary configurations into the four mentioned classes and initiated the systematic examination of their numbers. Roberts constructed a continuum of five-body central configurations in celestial mechanics, where a negative mass is included, and then this construction naturally led to an extension for the five-vortex relative equilibria [18]. With the algebraic method introduced in [6], Hampton and Moeckel showed that the number of four-vortex relative equilibria is at most 74, provided that no subcollection of the four vortices has vanishing total vorticities [7]. Roberts applied Morse theoretical ideas to the study of relative equilibria in the planar n-vortex problem [19].

Albouy and Kaloshin introduced a novel method to study the finiteness of five-body central configurations in celestial mechanics [1]. The first author successfully extended this approach to fluid mechanics. Using this new method, the first author established not only the finiteness of four-vortex relative equilibria for any four nonzero vorticities but also the finiteness of four-vortex collapse configurations for a fixed angular velocity. This represents the first result on the finiteness of collapse configurations for $N \geq 4$ [22].

In this paper, we focus on the finiteness of five-vortex stationary configurations. For equilibria and rigidly translating configurations, O'Neil showed that for generic vorticities, the upper bounds are $(N-2)!$ and $(N-1)!$, respectively [13, 14]. We confirm these upper bounds for the all five-vortex system. We apply the singular sequence method developed by the first author in [22] to investigate the finiteness of relative equilibria and collapse configurations.

Because of the continuum of five-vortex relative equilibria constructed by Roberts [18] (see also Sect. 4) and the continuum of five-vortex collapse configurations constructed by Novikov and Sedov [12], the finiteness of relative equilibria and collapse configurations can only be expected for generic vorticities, and, in case of collapse configurations, fixed Λ.

Two configurations are called equivalent if they are related by rotations, translations and dilations in the plane. In the following, all the counts are made for the equivalence classes. We have proven the following results:

Theorem 1.1 *For the planar five-vortex problem with nonzero vorticities* Γ_n $(n \in \{1,2,3,4,5\})$*,*

1. *There are at most 6 equilibria;*
2. *There are at most 24 rigidly translating configurations;*
3. *There are finitely many relative equilibria provided that none of 14 polynomial systems of* (18) *holds;*
4. *For any given* $\Lambda \in \mathbb{C}^*$*, there are finitely many collapse configurations provided that none of 7 polynomial systems of* (19) *holds.*

Please see Definition 2.3 for the meaning of Λ. We refer readers to Sect. 8 for Systems (18) and (19), and to Definition 2.1 for the meaning of $L_{j_1,\ldots,j_n}$.

Denote by $\mathbb{R}^*$ the set of nonzero real numbers. By Theorem 1.1, those vorticities that may admit infinitely many relative equilibria form a subvariety of $(\mathbb{R}^*)^5$, and those vorticities that may admit infinitely many collapse configurations form a subvariety of $\{(\Gamma_1,\ldots,\Gamma_5) : \Gamma_i \in \mathbb{R}^*, i=1,\ldots,5, \sum_{1\le i<j\le 5}\Gamma_i\Gamma_j = 0\}$. We further show that the codimension of the subvariety is at least 2.

Theorem 1.2 *For any choice of five vorticities* $(\Gamma_1,\ldots,\Gamma_5) \in (\mathbb{R}^*)^5 \backslash \mathcal{A}$*, where* $\mathcal{A}$ *is a closed algebraic subset of codimension 2, there are finitely many relative equilibria of the five-vortex problem.*

For any choice of five vorticities in $\{(\Gamma_1,\ldots,\Gamma_5) : \Gamma_i \in \mathbb{R}^*, i=1,\ldots,5, \sum_{1\le i<j\le 5}\Gamma_i\Gamma_j = 0\}\backslash\mathcal{B}$*, where* $\mathcal{B}$ *is a closed algebraic subset of codimension 2, there are finitely many collapse configurations of the five-vortex problem for any given* $\Lambda \in \mathbb{C}^*$*.*

The following result on finiteness is also proved.

Theorem 1.3 *Given five vorticities, if* $\sum_{j\in J}\Gamma_j \neq 0$ *and* $\sum_{j,k\in J, j\neq k}\Gamma_j\Gamma_k \neq 0$ *for any nonempty subset* J *of* $\{1,2,3,4,5\}$*, there are finitely many complex central configurations of the five-vortex problem for any fixed* $\Lambda \in \mathbb{C}^*$*.*

Please see Definition 4.1 and 4.2 for the meaning of complex central configurations.

If all vorticities are of the same sign, relative equilibria are the only possible stationary configurations. Theorem 1.3 leads to the following result.

Theorem 1.4 *Consider the planar five-vortex problem with all positive or all negative vorticities* Γ_n $(n \in \{1,2,3,4,5\})$*. There are finitely many stationary configurations.*

The paper is structured as follows. In Sect. 2, we introduce notations and definitions. In Sect. 3, we prove results on equilibria and rigidly translating configurations. In Sect. 4, we briefly review the singular sequence method and the two-colored diagrams. In Sect. 5, we identify constraints when some particular sub-diagrams appear. In Sect. 6, we construct the problematic diagrams for the 5-vortex problem. We derive constraints on vorticities corresponding to each of the 22 diagrams in Sect. 7 and prove the main results in Sect. 8.

2 Basic notations

We recall some basic notations on stationary configurations and direct readers to a more comprehensive introduction provided by O'Neil [13] and Yu [22].

We represent vortex positions $\mathbf{r}_n \in \mathbb{R}^2$ as complex numbers $z_n \in \mathbb{C}$. The equations of motion are $\dot{z}_n = \mathbf{i}V_n$, where

$$V_n = \sum_{1\le j\le N, j\ne n} \frac{\Gamma_j z_{jn}}{r_{jn}^2} = \sum_{j\ne n} \frac{\Gamma_j}{\overline{z}_{jn}}. \tag{1}$$

Here, $z_{jn} = z_n - z_j$, $r_{jn} = |z_{jn}| = \sqrt{z_{jn}\overline{z}_{jn}}$, $\mathbf{i} = \sqrt{-1}$, and the overbar denotes complex conjugation.

Let $\mathbb{C}^N = \{z = (z_1, \dots, z_N) : z_j \in \mathbb{C}, j = 1, \dots, N\}$ denote the space of configurations for N point vortex. The collision set is defined as $\Delta = \{z \in \mathbb{C}^N : z_j = z_k \textit{ for some } j \ne k\}$. The space of collision-free configurations is given by $\mathbb{C}^N \backslash \Delta$.

Definition 2.1 The following quantities and notations are defined:

Total vorticity	$\Gamma = \sum_{j=1}^N \Gamma_j$
Total vortex angular momentum	$L = \sum_{1\le j<k\le N} \Gamma_j\Gamma_k$
Moment of vorticity	$M = \sum_{j=1}^N \Gamma_j z_j$
Angular impulse	$I = \sum_{j=1}^N \Gamma_j \lvert z_j\rvert^2 = \sum_{j=1}^N \Gamma_j z_j \overline{z}_j$
Size	$S = \sum_{1\le j<k\le N} \Gamma_j\Gamma_k r_{jk}^2.$

For $J = \{j_1, \dots, j_n\} \subset \{1, \dots, N\}$, we also define

$$\Gamma_J = \Gamma_{j_1,\dots,j_n} = \sum_{j\in J} \Gamma_j, \;\; L_J = L_{j_1,\dots,j_n} = \sum_{j<k, j,k\in J} \Gamma_j\Gamma_k.$$

A motion is called homographic if the relative shape remains constant. Following O'Neil [13], we term a corresponding configuration as a *stationary configuration*. Equivalently,

Definition 2.2 A configuration $z \in \mathbb{C}^N \backslash \Delta$ is stationary if there exists a constant $\Lambda \in \mathbb{C}$ such that

$$V_j - V_k = \Lambda(z_j - z_k), \qquad 1 \le j, k \le N. \tag{2}$$

There are only four kinds of homographic motions, equilibria, translating with a common velocity, uniformly rotating, and homographic motions that collapse in finite time. Following [7, 13, 22], we term the stationary configurations corresponding to these four classes of homographic motions as equilibria, rigidly translating configurations, relative equilibria and collapse configurations. Equivalently,

Definition 2.3 i. $z \in \mathbb{C}^N \backslash \Delta$ is an *equilibrium* if $V_1 = \cdots = V_N = 0$.

ii. $z \in \mathbb{C}^N \backslash \Delta$ is *rigidly translating* if $V_1 = \cdots = V_N = c$ for some $c \in \mathbb{C}\backslash\{0\}$.

iii. $z \in \mathbb{C}^N \backslash \Delta$ is a *relative equilibrium* if there exist constants $\lambda \in \mathbb{R}\backslash\{0\}$, $z_0 \in \mathbb{C}$ such that $V_n = \lambda(z_n - z_0)$, $\qquad 1 \le n \le N$.

iv. $z \in \mathbb{C}^N \backslash \Delta$ is a *collapse configuration* if there exist constants $\Lambda, z_0 \in \mathbb{C}$ with $Im(\Lambda) \neq 0$ such that $V_n = \Lambda(z_n - z_0), \qquad 1 \leq n \leq N$.

Proposition 2.1 [13] *Every equilibrium has vorticities satisfying* $L = 0$*; every rigidly translating configuration has vorticities satisfying* $\Gamma = 0$

Definition 2.4 A configuration z is equivalent to z' if there exist $a, b \in \mathbb{C}$ with $b \neq 0$ such that $z'_n = b(z_n + a)$ for $1 \leq n \leq N$.

A configuration is called translation-normalized if its translation freedom is removed, rotation-normalized if its rotation freedom is removed, and dilation-normalized if its dilation freedom is removed. A configuration normalized in translation, rotation, and dilation is termed a *normalized configuration*.

We count the stationary configurations according to the equivalence classes. Counting equivalence classes is the same as counting normalized configurations. Note that the removal of any of these three freedoms can be performed in various ways.

3 Equilibria and rigidly translating configurations

In this section, we investigate the number of equilibria and rigidly translating configurations of five vorticities, via the minimal polynomial system introduced by O'Neil [14]. In particular, we will prove the corresponding results in Theorem 1.3.

Recall that the equilibria and the rigidly translating configurations are solutions of the system

$$\sum_{k \neq j} \frac{\Gamma_k}{z_j - z_k} = c, \tag{3}$$

where $c = 0$ corresponds to equilibria and $c \neq 0$ corresponds to the rigidly translating configurations. If the configuration $(z_1, ..., z_N)$ is an equilibrium or rigidly translating configuration, so is each member of its equivalent class $(b(z_1 + a), \ldots, b(z_N + a))$ with $a, b \in \mathbb{C}$ and $a \neq 0$. Given a linear form $A(z_1, \ldots, z_N) = a_1 z_1 + \cdots + a_N z_N$, with $a_1 + \cdots + a_N \neq 0$, there is one and only one value of a for which the members of the class satisfy $A = 0$. Hence, it is convenient to identify the translation-normalized equilibria and the rigidly translating configurations with points of $\mathbb{C}^N$, satisfying

$$A(z_1, \ldots, z_N) = 0, \ \sum_{k \neq j} \frac{\Gamma_k}{z_j - z_k} = c, \ \ j = 1, \ldots, N, \tag{4}$$

Let ζ be a complex variable. O'Neil [14] observed that system (4) is equivalent to

$$A(z_1, \ldots, z_N) = 0, \ \sum_{j<k} \frac{\Gamma_j \Gamma_k}{(\zeta - z_j)(\zeta - z_k)} = c \sum_j \frac{\Gamma_j}{\zeta - z_j}, \ \text{for } \forall \zeta \neq z_1, \ldots, z_N. \tag{5}$$

3.1 Equilibria

The second equation of system (5) becomes $\sum_{j<k} \frac{\Gamma_j \Gamma_k}{(\zeta - z_j)(\zeta - z_k)} = 0$. Multiplying it with $(\zeta - z_1)\cdots(\zeta - z_N)$ leads to

$$\zeta^{N-2}L + \zeta^{N-3}f_1 + \zeta^{N-4}f_2 ... + \zeta f_{N-3} + f_{N-2} = 0,\ \zeta \in \mathbb{C},$$

where $f_k = f_k(z_1, \ldots, z_N)$ is a homogeneous polynomial of degree k, $1 \le k \le N-2$. Then $L = 0$, and system (5) with $c = 0$ is equivalent to

$$A(z_1, \ldots, z_N) = 0,\ f_1 = f_2 = \cdots = f_{N-2} = 0,\ (z_1, \ldots, z_N) \notin \Delta. \quad (6)$$

The $(N-1)$ homogeneous polynomials define a projective variety V_0 in $\mathbb{P}^{N-1}$, and each point of this variety represents an equivalence class of equilibria. If the variety V_0 is zero-dimensional, Bézout's theorem implies that the number of points in V_0 is $(N-2)!$, counted by multiplicity.

Proposition 3.1 [14] *Let the nonzero vorticities $\Gamma_1, \ldots, \Gamma_N$ satisfy the relation $L = 0$, and let V_0 be defined as above, so that $V_0 \setminus \Delta$ is the set of all equilibria. Suppose there are two indices p, q such that for all proper subsets J of $\{1, \ldots, N\}$, $\{p, q\} \subset J$ implies $L_J \neq 0$. Then V_0 contains exactly $(N-2)!$ points counted according to multiplicity, and there are no more than $(N-2)!$ equilibria.*

Please see Definition 2.1 for the meaning of Γ_J, $\Gamma_{j_1,\ldots,j_n}$, L_J and $L_{j_1,\ldots,j_n}$.

Remark 1 There are some facts on L_J. If $L_J = 0$, then the cardinality of J, denoted by $\mathrm{card}(J)$, can not be two, since $\Gamma_i \neq 0$ for all $i = 1, \ldots, N$; If $L_J = 0$, then $\Gamma_J \neq 0$, since $\Gamma_J^2 > 2L_J$; If $L_J = 0$, then for any subset K of J with $\mathrm{card}(K) + 1 = \mathrm{card}(J)$, we have $L_K \neq 0$. Suppose that $K \cup \{1\} = J$, $L_J = L_K = 0$. Then the identity $L_J = L_K + \Gamma_K \Gamma_1$ implies that $\Gamma_K = 0$, which contradicts with $L_K = 0$.

Consider the case of $N = 4$. Since $L = 0$, Remark 1 implies that $L_J \neq 0$ for all subsets J of $\{1, 2, 3, 4\}$. Hence, Proposition 3.1 implies that any four vorticities satisfying $L = 0$ have at most 2 distinct equilibria. In fact, there are always exactly 2 distinct equilibria, as shown by O'Neil [13] and Hampton and Moeckel [7]. We now utilize Proposition 3.1 to study the case of $N = 5$.

Proposition 3.2 *Let the nonzero vorticities $\Gamma_1, \ldots, \Gamma_5$ satisfy the relation $L = 0$. Then there are at most 6 equilibria.*

Proof We assume that for any pair of indices, there is some proper subset J of $\{1, 2, 3, 4, 5\}$ containing them such that $L_J = 0$, and arrive at a contradiction. Since $L = 0$, then $L_J = 0$ holds only if the cardinality of J is three, as noted in Remark 1.

Since $L = 0$, the vorticities cannot all have the same sign. We divide the discussion into two cases: one with a single negative vorticity and the other with two negative vorticities.

Case I: The signs of the five vorticities are $(+, +, +, +, -)$. Consider the pair $\{1, 2\}$. It must hold that $L_{125} = 0$. Similarly, for the pairs $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, and $\{3, 4\}$, we have:

$$L_{125} = L_{135} = L_{145} = L_{235} = L_{245} = L_{345} = 0.$$

It is straightforward to find the solution: $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = -2\Gamma_5$, which contradicts the equation $L = 0$.

Case II: The signs of the five vorticities are $(+, +, +, -, -)$. Consider the three pairs $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$. The following system holds:

$$L_{124}L_{125} = 0, \quad L_{134}L_{135} = 0, \quad L_{234}L_{235} = 0.$$

It is sufficient to consider two sub-cases: when there are three or two 5's in the above system.

Sub-case II-1: $L_{125} = L_{135} = L_{235} = 0$. It is straightforward to find the solution: $\Gamma_1 = \Gamma_2 = \Gamma_3 = -2\Gamma_5$. Now consider the pair $\{4, 5\}$. It must hold that $L_{145} = 0$. However,

$$L_{145} = \Gamma_1\Gamma_5 + \Gamma_4(\Gamma_1 + \Gamma_5) = -2\Gamma_5^2 - \Gamma_4\Gamma_5 < 0,$$

which is a contradiction.

Sub-case II-2: $L_{125} = L_{135} = L_{234} = 0$. It is straightforward to find that $\Gamma_2 = \Gamma_3 = -2\Gamma_4$. Now consider the pair $\{4, 5\}$, then $L_{145}L_{245}L_{345} = 0$. Since

$$L_{245} = \Gamma_2\Gamma_4 + \Gamma_5(\Gamma_2 + \Gamma_4) = -2\Gamma_4^2 - \Gamma_4\Gamma_5 < 0,$$

it must hold that $L_{145} = 0$. If $L_{145} = 0$, then by $L_{125} = L_{145}$, we obtain $\Gamma_4 = \Gamma_2$, which is a contradiction.

By Proposition 3.1, for any group of five vorticities with $L = 0$, the variety V_0 is zero-dimensional and there are at most 6 equilibria. □

The above result has also been proved differently by Tsai [21], where a complete bifurcation diagram is provided. The actual number of equilibria could be fewer or even zero, as some solutions may have multiplicity greater than one, and some points of V_0 may lie on Δ. If $V_0 \cap \Delta \neq \emptyset$, then there exists a proper subset J such that $L_J = 0$ [14].

For instance, if $L_{123} = 0$, then $\Gamma_{123} \neq 0$, we can construct point on $V_0 \bigcap \Delta$ using (5). Let

$$J = \{1, 2, 3\}, \; d_j = \zeta - z_j, \; A(z_1, \ldots, z_5) = z_1, \; z_1 = z_2 = z_3.$$

Then

$$0 = \sum_{j<k} \frac{\Gamma_j \Gamma_k}{d_j d_k} = \sum_{j<k, j,k \in J} \frac{\Gamma_j \Gamma_k}{\zeta^2} + \sum_{k=4}^{5} \frac{\Gamma_J \Gamma_k}{\zeta d_k} + \frac{\Gamma_4 \Gamma_5}{d_4 d_5}$$
$$= \frac{\Gamma_4}{d_4} \left(\frac{\Gamma_5}{z_4 - z_5} + \frac{\Gamma_J}{z_4} \right) + \frac{\Gamma_5}{d_5} \left(\frac{\Gamma_4}{z_5 - z_4} + \frac{\Gamma_J}{z_5} \right)$$
$$- \frac{\Gamma_J}{\zeta} \left(\frac{\Gamma_4}{z_4} + \frac{\Gamma_5}{z_5} \right).$$

The above equation holds if and only if $z_4 = -\frac{\Gamma_4 z_5}{\Gamma_5}$ and $L = 0$. Hence we find the following point of V_0 on Δ $(0 : 0 : 0 : -\frac{\Gamma_4}{\Gamma_5} : 1)$, with multiplicity 2. Thus, there are at most 4 equilibria for five vorticities satisfying $L = L_{123} = 0$. Similar solutions can be constructed for N vorticities satisfying $L = L_J = 0$, where the cardinality of J is $N - 2$.

Assume there are four vorticities that satisfy $L_{123} = L_{124} = L_{134} = 0$. We can choose Γ_5 such that $L = 0$. The example above shows that there are three distinct solutions of (5) on Δ, each with multiplicity 2. Hence, there is no equilibrium for this group of vorticities. Up to renumbering, the vorticities are:

$$\Gamma_1 = 1,\ \Gamma_2 = -2,\ \Gamma_3 = -2,\ \Gamma_4 = -2,\ \Gamma_5 = \frac{6}{5}.$$

Examples of groups of vorticities that do not admit any equilibrium exist for any $N \geq 5$ (see [21]).

3.2 Rigidly translating configurations

Multiplying $(\zeta - z_1) \cdots (\zeta - z_N)$ to the second equation of system (5) leads to $\Gamma = 0$ and $L = c \sum_{j=1}^{N} \Gamma_j z_j$. Then we have $(\sum_{j=1}^{N} \Gamma_j z_j) \sum_{j<k} \frac{\Gamma_j \Gamma_k}{(\zeta - z_j)(\zeta - z_k)} = L \sum_j \frac{\Gamma_j}{\zeta - z_j}$, which leads to

$$\zeta^{N-1} L\Gamma + \zeta^{N-3} g_1 \cdots + \zeta g_{N-3} + g_{N-2} = 0,\ \zeta \in \mathbb{C},$$

where $g_k = g_k(z_1, \ldots, z_N)$ is a homogeneous polynomial of degree k, $1 \leq k \leq N-2$. Note that $\sum_{j=1}^{N} \Gamma_j z_j \neq 0$, as otherwise $L = c \sum_{j=1}^{N} \Gamma_j z_j = 0$, which contradicts $\Gamma = 0$. Then system (5) with $c \neq 0$ is equivalent to

$$A(z_1, \ldots, z_N) = 0,\ g_1 = g_2 = \cdots = g_{N-2} = 0,\ (z_1, \ldots, z_N) \notin \Delta. \tag{7}$$

The $N - 1$ homogeneous polynomials define a projective variety V_c in $\mathbb{P}^{N-1}$ and each point of this variety represents an equivalent class of rigidly translating configurations. If the variety V_c is zero dimensional, Bézout theorem implies that the number of points in V_c is $(N - 1)!$, counted by multiplicity.

Proposition 3.3 [14] *Let the nonzero vorticities* $\Gamma_1, \ldots, \Gamma_N$ *satisfy the relation* $\Gamma = 0$, *and let* V_c *be defined as above so that* $V_c \setminus \Delta$ *is the set of all rigidly translating configurations. Suppose that there are two indices* p, q *such that for all proper subsets* J *of* $\{1, ..., N\}$, $\{p, q\} \subset J$ *implies* $L_J \neq 0$ *and* $\Gamma_J \neq 0$. *Then* V_c *contains exactly* $(N-1)!$ *points, counted according to multiplicity, and there are no more than* $(N-1)!$ *rigidly translating configurations.*

For $N = 4$, if the nonzero vorticities $\Gamma_1, \ldots, \Gamma_4$ satisfy the relation $\Gamma = 0$, then there are at most 6 rigidly translating configurations. This result has already been proved Hampton and Moeckel with resultant theory in [7]. It is easy to apply the above criterion to obtain an alternative proof. However, it would be similar to the one of Proposition 3.4 and simpler, so we omit it.

Note that for the vorticities $(1, -1, a, -a)$, there is always one solution in Δ with multiplicity three. Then there are at most three rigidly translating configurations. In particular, for $a = 1$, there are two solutions in Δ with multiplicity three, so there are no rigidly translating configurations [14, 21].

Proposition 3.4 *Let the nonzero vorticities* $\Gamma_1, \ldots, \Gamma_5$ *satisfy the relation* $\Gamma = 0$. *Then the dimension of* V_c *is zero, so there are at most 24 rigidly translating configurations.*

Proof We assume that for any pair of indices, there exist some proper subsets J_1, J_2 of $\{1, 2, 3, 4, 5\}$ containing them such that $L_{J_1}\Gamma_{J_2} = 0$, and we derive contradictions. Note that $\Gamma_J = 0$ holds only if the cardinality of J is two or three, and that $L_J = 0$ holds only if the cardinality of J is three or four, as noted in Remark 1.

Since $\Gamma = 0$, the vorticities can not be of the same sign. We can simplify the discussion by dividing it into two cases: one with a single negative vorticity and the other with two negative vorticities.

Case I: the signs of the five vorticities be $(+, +, +, +, -)$. Then $-\Gamma_5 = \Gamma_{1234}$ and $\Gamma_J \neq 0$ for any proper subset J. Consider the pair $\{1, 2\}$. Without loss of generality, we obtain $L_{125}L_{1235} = 0$.

If $L_{125} = L_{12} + \Gamma_5\Gamma_{12} = 0$, then $-\Gamma_5 = \frac{L_{12}}{\Gamma_{12}} < \Gamma_{12}$, which is a contradiction.

If $L_{1235} = L_{123} + \Gamma_5\Gamma_{123} = 0$, then $-\Gamma_5 = \frac{L_{123}}{\Gamma_{123}} < \Gamma_{123}$, which is a contradiction.

Case II: the signs of the five vorticities be $(+, +, +, -, -)$. There are two sub-cases: there exists some proper subset J such that $\Gamma_J = 0$ or not.

Sub-case II-1: there are proper subsets J *such that* $\Gamma_J = 0$. Then we may assume that the vorticities are $(1, b, c, -1, -(b + c))$ where $b, c > 0$. Then $\Gamma_{14} = \Gamma_{235} = 0$. Note that $\Gamma_{124} = b$, $\Gamma_{134} = c$, and

$$\begin{aligned} &L_{125} = -c - b(b + c) < 0, L_{134} = -1, \quad L_{135} = -b - c(b + c) < 0, \\ &L_{1235} = -b^2 - bc - c^2 < 0, L_{145} = -1, \quad L_{1245} = -1 - b(b + c) < 0, \\ &L_{1345} = -1 - c(b + c) < 0, L_{124} = -1, \quad L_{2345} = -b^2 - bc - c^2 < 0. \end{aligned}$$

Consider the pair of indices $\{4, 5\}$. Since $\Gamma_J \neq 0$ for any proper subset containing the two indices, then the following system holds:

$$L_{245}L_{345} = 0,$$

since it is already known that $L_{145}, L_{1245}, L_{1345}, L_{2345}$ are all negative. Similarly, consider the four pairs $\{1,2\}, \{1,3\}, \{1,5\}$. Then the following system holds

$$\Gamma_{125}L_{1234}=0,\ \Gamma_{135}L_{1234}=0,\ \Gamma_{125}\Gamma_{135}=0.$$

It is enough to divide the discussion into two sub-cases: $L_{1234}=0$ or not.

Sub-case II-1-1: $L_{1234}=bc-1=0$. Then by $\Gamma_{125}\Gamma_{135}=(1-b)(1-c)=0$, we have $b=1$ or $c=1$. By $L_{1234}=0$, it follows that $b=1$ and $c=1$. However, this contradicts with $L_{245}L_{345}=[c-b(b+c)][b-c(b+c)]=0$.

Sub-case II-1-2: $L_{1234}\neq 0$. Then $\Gamma_{125}=0$ and $\Gamma_{135}=0$. It follows that $b=1$ and $c=1$. However, this contradicts with $L_{245}L_{345}=0$.

Sub-case II-2: $\Gamma_J\neq 0$ *for any proper subset* J. We assume that $\Gamma_{123}=1$, $\Gamma_1\leq\Gamma_2\leq\Gamma_3$, $\Gamma_{45}=-1$ and $|\Gamma_4|\leq|\Gamma_5|$. Then

$$L_{123}\leq\frac{1}{3},\ L_{23}\leq\frac{1}{4}\Gamma_{23}^2<\frac{1}{4}\Gamma_{23},\ L_{45}\leq\frac{1}{4},\Gamma_3\geq\frac{1}{3},\Gamma_5\leq-\frac{1}{2}.$$

So,

$$\begin{aligned}
L_{1235}&=L_{123}+\Gamma_5\Gamma_{123}=L_{123}-\frac{1}{2}<0,\\
L_{345}&=L_{45}+\Gamma_3\Gamma_{45}=L_{45}-\Gamma_3<0,\\
L_{235}&=L_{23}+\Gamma_5\Gamma_{23}\leq L_{23}-\frac{1}{2}\Gamma_{23}<0.
\end{aligned}$$

Similarly, it holds that $L_{125}<0$, $L_{135}<0$.

Consider the pair $\{3,5\}$. All possible $L_J=0$ with J containing the pair are L_{1345}, L_{2345}. Then it must hold $L_{1345}L_{2345}=0$. Consider also the two other pairs $\{1,5\}$ and $\{2,5\}$. Then the following system must hold,

$$L_{1345}L_{2345}=0,\ L_{145}L_{1245}L_{1345}=0,\ L_{245}L_{1245}L_{2345}=0. \tag{8}$$

There are five sub-cases.

Sub-case II-2-1: $L_{1345}=L_{1245}=0$. Then $\Gamma_2=\Gamma_3$ and $\Gamma_{13}>\frac{1}{2}$. Thus,

$$L_{1345}=-\Gamma_{13}+L_{13}+L_{45}<-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}<0,$$

which is a contradiction.

Sub-case II-2-2: $L_{1345}=L_{245}=0$. The second equation implies that $\Gamma_2=L_{45}\leq\frac{1}{4}$. Then $\Gamma_{13}\geq\frac{3}{4}$, and

$$L_{1345}=L_{13}+L_{45}-\Gamma_{13}\leq\frac{1}{4}+\frac{1}{4}-\frac{3}{4}<0,$$

which is a contradiction.

Sub-case II-2-3: $L_{1345} = L_{2345}$. Then $\Gamma_1 = \Gamma_2$ and $\Gamma_{13} \geq \frac{2}{3}$. Thus,

$$L_{1345} = -\Gamma_{13} + L_{13} + L_{45} \leq -\frac{2}{3} + \frac{1}{4} + \frac{1}{4} < 0,$$

which is a contradiction.

Sub-case II-2-4: $L_{2345} = L_{145} = 0$. The second equation implies that $\Gamma_1 = L_{45} \leq \frac{1}{4}$. Then $\Gamma_{23} \geq \frac{3}{4}$, and

$$L_{2345} = L_{23} + L_{45} - \Gamma_{23} \leq \frac{1}{4} + \frac{1}{4} - \frac{3}{4} < 0,$$

which is a contradiction.

Sub-case II-2-5: $L_{2345} = L_{1245} = 0$. Then $\Gamma_1 = \Gamma_3$ and $\Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{1}{3}$. Thus,

$$L_{2345} = -\Gamma_{23} + L_{23} + L_{45} \leq -\frac{2}{3} + \frac{1}{9} + \frac{1}{4} < 0,$$

which is a contradiction.

By Proposition 3.3, for any group of five vorticities with $\Gamma = 0$, the variety V_c is zero-dimensional and there are at most 24 rigidly translating configurations. □

4 Singular sequences for central configurations and coloring rules

In this section, we briefly review the basic elements of the Albouy-Kaloshin approach developed by Yu [22] for the finiteness of relative equilibria and collapse configurations, including, among others, the notation of central configurations, the extended system, the notation of singular sequences, the two-colored diagrams, and the rules for the two-colored diagrams. For a more comprehensive introduction, please refer to [22].

4.1 Central configurations of the planar N-vortex problem

Recall Definition 2.3. Equations of relative equilibria and collapse configurations share the form:

$$V_n = \Lambda(z_n - z_0), \qquad 1 \leq n \leq N, \tag{9}$$

where $\Lambda \in \mathbb{R}\backslash\{0\}$ indicates relative equilibria and $\Lambda \in \mathbb{C}\backslash\mathbb{R}$ indicates collapse configurations.

Definition 4.1 Relative equilibria and collapse configurations are both called *central configurations*.

The equations (9) read

$$\Lambda z_n = V_n, \qquad 1 \leq n \leq N, \tag{10}$$

if the translation freedom is removed, i.e., we substitute z_n with $z_n + z_0$ in equations (10). The solutions then satisfy:

$$M = 0, \ \Lambda I = L. \tag{11}$$

To remove dilation freedom, we enforce $|\Lambda| = 1$.

Introduce a new set of variables w_n and a “conjugate" relation:

$$\Lambda z_n = \sum_{j \neq n} \frac{\Gamma_j}{w_{jn}}, \quad \overline{\Lambda} w_n = \sum_{j \neq n} \frac{\Gamma_j}{z_{jn}}, \quad 1 \leq n \leq N, \tag{12}$$

where $z_{jn} = z_n - z_j$ and $w_{jn} = w_n - w_j$.

The rotation symmetry of (10) leads to the invariance of (12) under the map

$$R_a : (z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (az_1, \ldots, az_N, a^{-1}w_1, \ldots, a^{-1}w_N)$$

for any $a \in \mathbb{C}\backslash\{0\}$.

Introduce the variables $Z_{jk}, W_{jk} \in \mathbb{C}$ $(1 \leq j < k \leq N)$ such that $Z_{jk} = 1/w_{jk}, W_{jk} = 1/z_{jk}$. For $1 \leq k < j \leq N$ we set $Z_{jk} = -Z_{kj}, W_{jk} = -W_{kj}$. Then equations (10) together with the condition $z_{12} \in \mathbb{R}$ and $|\Lambda| = 1$ are embedded into the following extended system

$$\begin{array}{rl} \Lambda z_n = \sum_{j \neq n} \Gamma_j Z_{jn}, & 1 \leq n \leq N, \\ \overline{\Lambda} w_n = \Lambda^{-1} w_n = \sum_{j \neq n} \Gamma_j W_{jn}, & 1 \leq n \leq N, \\ Z_{jk} w_{jk} = 1, & 1 \leq j < k \leq N, \\ W_{jk} z_{jk} = 1, & 1 \leq j < k \leq N, \\ z_{jk} = z_k - z_j, \quad w_{jk} = w_k - w_j, & 1 \leq j, k \leq N, \\ Z_{jk} = -Z_{kj}, \quad W_{jk} = -W_{kj}, & 1 \leq k < j \leq N, \\ z_{12} = w_{12}. & \end{array} \tag{13}$$

This is a polynomial system in the variables $\mathcal{Q} = (\mathcal{Z}, \mathcal{W}) \in \mathbb{C}^{2\mathfrak{N}}$, here

$$\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_{\mathfrak{N}}) = (z_1, z_2, \ldots, z_N, Z_{12}, Z_{13}, \ldots, Z_{(N-1)N}),$$
$$\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{\mathfrak{N}}) = (w_1, w_2, \ldots, w_N, W_{12}, W_{13}, \ldots, W_{(N-1)N}).$$

and $\mathfrak{N} = N(N+1)/2$.

Definition 4.2 A *complex normalized* central configuration of the planar N-vortex problem is a solution of (13). A *real normalized* central configuration of the planar N-vortex problem is a complex normalized central configuration satisfying $z_n = \overline{w}_n$ for any $n = 1, \ldots, N$.

Note that a real normalized central configuration of Definition 4.2 is exactly a central configuration of Definition 4.1. We will use the name “distance" for the $r_{jk} = \sqrt{z_{jk} w_{jk}}$. Strictly speaking, the distances $r_{jk} = \sqrt{z_{jk} w_{jk}}$ are now bi-valued. However, only the squared distances appear in the system, so we shall understand r_{jk}^2 as $z_{jk} w_{jk}$ from now on.

4.2 Singular sequences

Let $\|\mathcal{Z}\| = \max_{j=1,2,\ldots,\mathfrak{N}} |\mathcal{Z}_j|$ be the modulus of the maximal component of the vector $\mathcal{Z} \in \mathbb{C}^{\mathfrak{N}}$. Similarly, set $\|\mathcal{W}\| = \max_{k=1,2,\ldots,\mathfrak{N}} |\mathcal{W}_k|$.

One important feature of System (13) is the symmetry: if $\mathcal{Z}, \mathcal{W}$ is a solution, so is $a\mathcal{Z}, a^{-1}\mathcal{W}$ for any $a \in \mathbb{C}\backslash\{0\}$. Thus, we can replace the normalization $z_{12} = w_{12}$ in System (13) by $\|\mathcal{Z}\| = \|\mathcal{W}\|$. From now on, we consider System (13) with this new normalization.

Consider a sequence $\mathcal{Q}^{(n)}$, $n = 1, 2, \ldots$, of solutions of (13). Take a sub-sequence such that the maximal component of $\mathcal{Z}^{(n)}$ is fixed, i.e., there is a $j \in \{1, 2, \ldots, \mathcal{N}\}$ that is independent of n such that $\|\mathcal{Z}^{(n)}\| = |\mathcal{Z}_j^{(n)}|$. Extract again in such a way that the sequence $\mathcal{Z}^{(n)}/\|\mathcal{Z}^{(n)}\|$ converges. Extract again in such a way that the maximal component of $\mathcal{W}^{(n)}$ is fixed. Finally, extract in such a way that the sequence $\mathcal{W}^{(n)}/\|\mathcal{W}^{(n)}\|$ converges.

Definition 4.3 **(Singular sequence)** Consider a sequence of complex normalized central configurations with the property that $\mathcal{Z}^{(n)}$ is unbounded. A sub-sequence extracted by the above process is called a *singular sequence*.

Lemma 4.1 [1] *Let $\mathcal{X}$ be a closed algebraic subset of $\mathbb{C}^m$ and $f : \mathbb{C}^m \to \mathbb{C}$ be a polynomial. Either the image $F(\mathcal{X}) \subset \mathbb{C}$ is a finite set, or it is the complement of a finite set. In the second case one says that f is dominating.*

4.3 The two-colored diagrams

For two sequences of non-zero numbers, a, b, we use $a \sim b$, $a \prec b$, $a \preceq b$, and $a \approx b$ to represent "$a/b \to 1$", "$a/b \to 0$", "a/b is bounded" and "$a \preceq b$, $a \succeq b$" respectively.

Recall that a singular sequence satisfy the property $\|\mathcal{Z}^{(n)}\| = \|\mathcal{W}^{(n)}\| \to \infty$. Set $\|\mathcal{Z}^{(n)}\| = \|\mathcal{W}^{(n)}\| = 1/\epsilon^2$. Then $\epsilon \to 0$. Following Albouy-Kaloshin, [1], the *two-colored diagram* was introduced in [22] to classify the singular sequences. Given a singular sequence, the indices of the vertices will be written down. The first color, called the z-color (red), is used to mark the maximal order components of $\mathcal{Z}$. If $z_k \approx \epsilon^{-2}$, draw a z-circle around the vertex **k**; If $Z_{jk} \approx \epsilon^{-2}$, draw a z-stroke between vertices **k** and **j**. They constitute the z-diagram. The second color, called the w-color (blue and dashed), is used to mark the maximal order components of $\mathcal{W}$ in similar manner. Then we also have the w-diagram. The two-colored diagram is the combination of the z-diagram and the w-diagram, see Fig. 1.

Fig. 1 On the left, vertices **1, 2** are z-circled, and a z-edge is between them; In the middle, vertices **1, 2** are z- and w-circled, and a zw-edge is between them; On the right, vertices **1, 2** are w-circled, and a w-edge is between them

If there is either a z-stroke, or a w-stroke, or both between vertex **k** and vertex **l**, we say that there is an edge between them. There are three types of edges, z-edges, w-edges and zw-edges, see Fig. 1.

The following concepts were introduced to characterize some features of singular sequences. In the z-diagram, vertices **k** and **l** are called *z-close*, if $z_{kl} \prec \epsilon^{-2}$; a z-stroke between vertices **k** and **l** is called a *maximal z-stroke* if $z_{kl} \approx \epsilon^{-2}$; a subset of vertices are called *an isolated component of the z-diagram* if there is no z-stroke between a vertex of this subset and a vertex of its complement. These concepts also apply to the w-diagram.

Proposition 4.1 (Estimate) [22] *For any (k, l), $1 \leq k < l \leq N$, we have $\epsilon^2 \preceq z_{kl} \preceq \epsilon^{-2}$, $\epsilon^2 \preceq w_{kl} \preceq \epsilon^{-2}$ and $\epsilon^2 \preceq r_{kl} \preceq \epsilon^{-2}$.*

*There is a z-stroke between **k** and **l** if and only if $w_{kl} \approx \epsilon^2$. Then $r_{kl} \preceq 1$.*

*There is a maximal z-stroke between **k** and **l** if and only if $z_{kl} \approx \epsilon^{-2}$, $w_{kl} \approx \epsilon^2$. Then $r_{kl} \approx 1$.*

*There is a z-edge between **k** and **l** if and only if $z_{kl} \succ \epsilon^2$, $w_{kl} \approx \epsilon^2$. Then $\epsilon^2 \prec r_{kl} \preceq 1$.*

*There is a maximal z-edge between **k** and **l** if and only if $z_{kl} \approx \epsilon^{-2}$, $w_{kl} \approx \epsilon^2$. Then $r_{kl} \approx 1$.*

*There is a zw-edge between **k** and **l** if and only if $z_{kl}, w_{kl} \approx \epsilon^2$. This can be characterized as $r_{kl} \approx \epsilon^2$.*

Remark 2 By the estimates above, the strokes in a zw-edge are not maximal. A maximal z-stroke is exactly a maximal z-edge.

The following rules for the two-colored diagrams are valid if "z" and "w" were switched.

Rule I: There is something at each end of any z-stroke: another z-stroke or/and a z-circle drawn around the name of the vertex. A z-circle cannot be isolated; there must be a z-stroke emanating from it. There is at least one z-stroke in the z-diagram.

Rule II: If vertices **k** and **l** are z-close, they are both z-circled or both not z-circled.

Rule III: The moment of vorticity of a set of vertices forming an isolated component of the z-diagram is z-close to the origin.

Rule IV: Consider the z-diagram or an isolated component of it. If there is a z-circled vertex, there is another one. If the z-circled vertices are all z-close together, the total vorticity of these z-circled vertices is zero.

Rule V: There is at least one z-circle at certain end of any maximal z-stroke. As a result, if an isolated component of the z-diagram has no z-circled vertex, then it has no maximal z-stroke.

Rule VI: If there are two consecutive z-stroke, there is a third z-stroke closing the triangle.

Remark 3 We would like to compare our rules for the N-vortex problem with those of Albouy and Kaloshin for the N-body problem [1]. Our Rules I to V correspond to their Rules 1a to 1e but are weaker due to the possibility of negative vorticities and clusters with zero total vorticity. Our Rule VI is similar to their Rule 2b but is stronger in the sense that we only need two consecutive z-strokes (w-strokes) to form a triangle of z-strokes (w-strokes), whereas they require two consecutive zw-edges to form a triangle of zw-edges.

We lack the two-color rules due to the differences between the two potentials. The Newtonian potential leads to $Z_{ij} = z_{ij}^{-\frac{1}{2}} w_{ij}^{-\frac{3}{2}}$, so the existence of a z-stroke provides estimates for both z_{ij} and w_{ij}, which in turn yields the two-color rules. In contrast, the logarithmic potential leads to $Z_{ij} = w_{ij}^{-1}$, so the existence of a z-stroke provides an estimate only for w_{ij}, thus we lack the two-color rules.

Example: Robert's continuum: Roberts found the following continuum of relative equilibria for five vortices. The vorticities are $\Gamma_1 = -1, \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 = 2$. The first vortex is at the origin $z_1 = w_1 = 0$, while the other four vortices form a rhombus

$$(z_2, z_3, z_4, z_5) = (a, b, -a, -b), \ (w_2, w_3, w_4, w_5) = (a, -b, -a, b),$$

where $a \in \mathbb{R}$ and $\mathbf{i}b \in \mathbb{R}$ for the real configurations. One can check that the configuration defined by the coordinates above and the restriction $r_{23}^2 = r_{34}^2 = r_{45}^2 = r_{52}^2 = a^2 - b^2 = 1$ satisfies system (13).

For the real configurations, when $a \to 0$, vertices **1**, **2**, and **4** collide. The corresponding singular sequence is a triple contact with $r_{12}, r_{14}, r_{24} \to 0$, so the corresponding diagram is a copy of Diagram 1 of Fig. 15 (the vertices of the triangle should be **1, 2, 4**). When $b \to 0$, vertices **1**, **3**, and **5** collide. Similarly, the corresponding diagram is a copy of Diagram 1 of Fig. 15 (the vertices of the triangle should be **1, 3, 5**).

For the complex configurations, when $a \to \infty$, we have two choices: $b \sim a$ or $b \sim -a$. We get two other singular sequences with $\{z_2, z_3, z_4, z_5, w_2, w_3, w_4, w_5\}$ all going to infinity. For instance, if $a \to \infty$ and $b \sim a$, the corresponding diagram is exactly Diagram 18 of Fig. 16.

5 Constraints when some sub-diagrams appear

We collect some useful results in this section. We will use notations such as $\Gamma_J, \Gamma_{j_1,\ldots,j_n}, L_J$, and $L_{j_1,\ldots,j_n}$ below. Please refer to Definition 2.1 for their meanings.

Proposition 5.1 [22] *Suppose that a diagram has two z-circled vertices (say **1** and **2**) which are also z-close, if none of all the other vertices is z-close with them, then* $\Gamma_1 + \Gamma_2 \neq 0$ *and* $\overline{\Lambda} z_{12} w_{12} \sim \frac{1}{\Gamma_1 + \Gamma_2}$. *In particular, vertices **1** and **2** cannot form a z-stroke.*

Corollary 5.1 *Suppose that a diagram has two z-circled vertices (say **1** and **2**) which also form a z-stroke. If none of all the other vertices is z-close with them, then* $z_{12} \approx \epsilon^{-2}$, $\Gamma_1 + \Gamma_2 \neq 0$, *and* $w_1, w_2 \preceq \epsilon^2$.

Proof If they are z-close, by Proposition 5.1, they cannot form a z-stroke, which is a contradiction. Note that Rule III implies

$$\epsilon^{-2} \succ \Gamma_1 z_1 + \Gamma_2 z_2 = (\Gamma_1 + \Gamma_2) z_1 + \Gamma_2 (z_2 - z_1).$$

We obtain $\Gamma_1 + \Gamma_2 \neq 0$.

Note that $z_{1j}, z_{2j} \approx \epsilon^{-2}$, $j \geq 3$. Then

$$\bar{\Lambda} \sum_{j\geq 3} \Gamma_j w_j = \sum_{j\geq 3} \frac{\Gamma_1 \Gamma_j}{z_{1j}} + \sum_{j\geq 3} \frac{\Gamma_2 \Gamma_j}{z_{2j}} \preceq \epsilon^2.$$

By the equation $\sum_j \Gamma_j w_j = 0$, we have

$$\epsilon^2 \succeq \Gamma_1 w_1 + \Gamma_2 w_2 = (\Gamma_1 + \Gamma_2) w_1 + \Gamma_2 w_{21}.$$

Since $w_{21} \approx \epsilon^2$, we have $w_1, w_2 \preceq \epsilon^2$. □

Proposition 5.2 [22] *Suppose that a fully z-stroked sub-diagram with vertices* $\{1, \ldots, k\}$, ($k \geq 3$) *exists in isolation in a diagram, and none of its vertices is z-circled, then*

$$L_{1,\ldots,k} = \sum_{i,j\in\{1,\ldots,k\}, i\neq j} \Gamma_i \Gamma_j = 0.$$

Corollary 5.2 *Suppose a fully z-stroked sub-diagram with vertices* $K = \{1, \ldots, k\}$, $k \geq 3$, *exists in isolation in a diagram, and none of its vertices is z-circled.*

1. *If there is an isolated component* I *of the* w*-diagram such that* $K \subset I$, *then the* w*-circled vertices in* I *cannot be exactly* $\{1, \ldots, k\}$.
2. *Consider any subset of* K *with cardinality* $(k-1)$, *say* $K_1 = \{2, \ldots, k\}$. *If there is an isolated component* I *of the* w*-diagram with* $K_1 \subset I$, *then the* w*-circled vertices in* I *cannot be exactly* K_1.
3. *If there is a vertex outside of* K, *say* $k+1$, *such that* $\{k+1\} \cup K$ *forms an isolated component of the* w*-diagram and these* $k+1$ *vertices are fully* w*-stroked, then there is at least one* w*-circle among them.*
4. *If there are several isolated components* $\{I_j, j = 1, \ldots, s\}$ *of the* w*-diagram with* $K \subset \cup_{j=1}^{s} I_j$, *then the* w*-circled vertices in* $\cup_{j=1}^{s} I_j$ *cannot be exactly* $\{1, \ldots, k\}$.

Proof First, we have $L_K = 0$ by Proposition 5.2, and the vertices of K are all w-close by the estimate of Proposition 4.1.

For part (1), if the w-circled vertices in I are exactly $\{1, \ldots, k\}$, then by Rule IV, we have $\sum_{i\in K} \Gamma_i = 0$. This leads to a contradiction because:

$$\left(\sum_{i\in K} \Gamma_i \right)^2 = \sum_{i\in K} \Gamma_i^2 + 2L_K.$$

The proof of part (4) is similar.

For part (2), if the w-circled vertices in I are exactly $K_1 = \{2, \ldots, k\}$, then by Rule IV, we have $\sum_{i=2}^{k} \Gamma_i = 0$. Therefore:

$$L_{K_1} = L_K - \Gamma_1 \left(\sum_{i=2}^{k} \Gamma_i \right) = 0,$$

which again leads to a contradiction since $\sum_{i\in K_1}\Gamma_i=0$.

For part (3), if the component $\{k+1\}\cup K$ is fully w-stroked but has no w-circle, then Rule IV implies that $L_K=0$ and $L_K+\Gamma_{k+1}\sum_{i\in K}\Gamma_i=0$. This leads to $\sum_{i\in K}\Gamma_i=0$, which is a contradiction. □

Proposition 5.3 *Suppose that a diagram has an isolated z-stroke in the z-diagram, and its two ends are z-circled. Let **1** and **2** be the ends of this z-stroke. Suppose there is no other z-circle in the diagram. Then $\Gamma_1+\Gamma_2\neq 0$, and z_{12} is maximal. The diagram forces $\Lambda=\pm 1$ or $\pm\mathbf{i}$. Furthermore,*

- *If $\Lambda=\pm 1$, we have $\sum_{j=3}^N\Gamma_j=0$;*
- *If $\Lambda=\pm\mathbf{i}$, we have $L=0$ and $\Gamma_1\Gamma_2=L_{3\ldots N}$.*

Proof The facts that $\Gamma_1+\Gamma_2\neq 0$ and z_{12} is maximal follow from Corollary 5.1. Without loss of generality, assume $z_1\sim-\Gamma_2 a\epsilon^{-2}$ and $z_2\sim\Gamma_1 a\epsilon^{-2}$, then

$$z_{12}\sim(\Gamma_1+\Gamma_2)a\epsilon^{-2},\ \frac{1}{z_2}-\frac{1}{z_1}\sim\left(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}\right)\frac{\epsilon^2}{a}.$$

The System (13) yields

$$\begin{array}{c}\overline{\Lambda}w_{12}=(\Gamma_1+\Gamma_2)W_{12}+\sum_{j=3}^N\Gamma_j\left(\frac{1}{z_{j2}}-\frac{1}{z_{j1}}\right)\\ \Lambda z_2\sim\Gamma_1 Z_{12}.\end{array}\tag{14}$$

The second equation of (14) implies $w_{12}\sim\frac{\epsilon^2}{a\Lambda}$. Note that $\frac{1}{z_{j2}}-\frac{1}{z_{j1}}\sim\frac{1}{z_2}-\frac{1}{z_1}$ for all $j>2$ and that $W_{12}=\frac{1}{z_{12}}$. The first equation of (14) implies

$$\overline{\Lambda}/\Lambda=1+\sum_{j=3}^N\Gamma_j\left(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}\right).\tag{15}$$

It follows that $\Lambda=\pm 1$ or $\pm\mathbf{i}$.

If $\Lambda=\pm 1$, we have

$$0=\sum_{j=3}^N\Gamma_j\left(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}\right),\ \Rightarrow\ \sum_{j=3}^N\Gamma_j=0.$$

If $\Lambda=\pm\mathbf{i}$, we obtain

$$-2=\sum_{j=3}^N\Gamma_j\left(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}\right),\ L=0,\ \Rightarrow L=0,\ \Gamma_1\Gamma_2=L_{3\ldots N}.$$

□

Similarly, we have the following result.

Proposition 5.4 *Suppose that a diagram has an isolated triangle of z-strokes in the z-diagram, where two of the vertices of the triangle are z-circled. Let* **2** *and* **3** *be the two z-circled vertices and* **1** *the other vertex. Suppose there is no other z-circle in the diagram. Then $\Gamma_2+\Gamma_3\neq 0$, and z_{23} is maximal. The diagram forces $\Lambda=\pm 1$ or $\pm\boldsymbol{i}$. Furthermore,*

- *If $\Lambda=\pm 1$, we have $\sum_{j=4}^N \Gamma_j=0$;*
- *If $\Lambda=\pm\boldsymbol{i}$, we have $L=0$ and $L_{123}=L_{4\ldots N}+\Gamma_1(\sum_{j=4}^N\Gamma_j)$.*

Proof The facts that $\Gamma_2+\Gamma_3\neq 0$ and z_{23} is maximal follow from Corollary 5.1. Note that

$$\Gamma_2 z_2+\Gamma_3 z_3\prec\epsilon^{-2},\ \ \Lambda z_1\sim\Gamma_2 Z_{21}+\Gamma_3 Z_{31}\prec\epsilon^{-2}.$$

Without loss of generality, assume

$$z_2\sim-\Gamma_3 a\epsilon^{-2},\ z_3\sim\Gamma_2 a\epsilon^{-2},\ Z_{21}\sim-\Gamma_3 b\epsilon^{-2},\ Z_{31}\sim\Gamma_2 b\epsilon^{-2}.$$

Then

$$z_{23}\sim(\Gamma_2+\Gamma_3)a\epsilon^{-2},\ \frac{1}{z_3}-\frac{1}{z_2}\sim\left(\frac{1}{\Gamma_2}+\frac{1}{\Gamma_3}\right)\frac{\epsilon^2}{a},$$
$$Z_{23}=\frac{1}{w_{23}}=\frac{1}{1/Z_{21}+1/Z_{13}}\sim-b\frac{\Gamma_2\Gamma_3}{(\Gamma_2+\Gamma_3)}\epsilon^{-2}.$$

Then similar to the above case, we have

$$\overline{\Lambda}w_{23}\sim(\Gamma_2+\Gamma_3)W_{23}+\sum_{j\neq 2,3}^{N}\Gamma_j\left(\frac{1}{z_3}-\frac{1}{z_2}\right)$$
$$\Lambda z_{23}\sim(\Gamma_2+\Gamma_3)Z_{23}+\Gamma_1(Z_{13}-Z_{12}).$$

Short computation reduces the two equations to

$$-\overline{\Lambda}\frac{a}{b}=\frac{\Gamma_2\Gamma_3}{\Gamma_2+\Gamma_3}\left(1+\sum_{j\neq 2,3}\Gamma_j\frac{\Gamma_2+\Gamma_3}{\Gamma_2\Gamma_3}\right),$$
$$-\Lambda\frac{a}{b}=\frac{L_{123}}{\Gamma_2+\Gamma_3}.$$

Then we obtain

$$\frac{\overline{\Lambda}}{\Lambda}L_{123}=\Gamma_2\Gamma_3+(\Gamma_2+\Gamma_3)\sum_{j\neq 2,3}\Gamma_j.$$

It follows that $\Lambda=\pm 1$ or $\pm\mathbf{i}$.

If $\Lambda = \pm 1$, we have $\sum_{j=4}^{N} \Gamma_j = 0$. If $\Lambda = \pm \mathbf{i}$, we have $L = 0$ and

$$L = 0, \ -L_{123} = \Gamma_2\Gamma_3 + (\Gamma_2 + \Gamma_3) \sum_{j \neq 2,3} \Gamma_j,$$

which is equivalent to $L_{123} = L_{4\ldots N} + \Gamma_1(\sum_{j=4}^{N} \Gamma_j), \ L = 0.$ □

Proposition 5.5 *Assume there is a triangle with vertices* $\mathbf{1}, \mathbf{2}, \mathbf{3}$ *that is fully z- and w-stroked, and fully z- and w-circled. Moreover, assume that the triangle is isolated in the z-diagram. Then there must exist some* $k > 3$ *such that* $z_{k1} \preceq 1$.

Proof By Proposition 4.1 and Rule IV, we have

$$z_1 \sim z_2 \sim z_3, \ w_1 \sim w_2 \sim w_3, \ \Gamma_1 + \Gamma_2 + \Gamma_3 = 0.$$

Suppose that it holds $z_{k1} \succ 1$ for all $k > 3$. Then $\frac{1}{z_{kj}} - \frac{1}{z_{k1}} = \frac{z_{1j}}{z_{kj}z_{k1}} \prec \epsilon^2$ for all $k > 3$, $1 \le j \le 3$, and so

$$\bar{\Lambda} \sum_{j=1}^{3} \Gamma_j w_j = \sum_{k \ge 4} \sum_{j=1}^{3} \frac{\Gamma_k \Gamma_j}{z_{kj}} = \sum_{k \ge 4} \sum_{j=1}^{3} \Gamma_k \Gamma_j \left(\frac{1}{z_{k1}} + \frac{1}{z_{kj}} - \frac{1}{z_{k1}} \right) \prec \epsilon^2.$$

By the fact that $w_{12}, w_{13}, w_{23} \approx \epsilon^2$, the equations

$$\sum_{j=1}^{3} \Gamma_j w_j = \Gamma_2 w_{12} + \Gamma_3 w_{13} = \Gamma_1 w_{21} + \Gamma_3 w_{23} = \Gamma_1 w_{31} + \Gamma_2 w_{32} \prec \epsilon^2$$

imply that

$$\frac{w_{12}}{\Gamma_3} \sim \frac{w_{23}}{\Gamma_1} \sim \frac{w_{31}}{\Gamma_2} \approx \epsilon^2. \tag{16}$$

By the isolation of this triangle in the z-diagram, it holds that

$$\Lambda z_1 \sim \frac{\Gamma_2}{w_{21}} + \frac{\Gamma_3}{w_{31}}, \ \Lambda z_2 \sim \frac{\Gamma_1}{w_{12}} + \frac{\Gamma_3}{w_{32}}, \ \Lambda z_3 \sim \frac{\Gamma_1}{w_{13}} + \frac{\Gamma_2}{w_{23}}. \tag{17}$$

Since $z_1 \sim z_2 \sim z_3$, the equations (16) and (17) lead to

$$\frac{\Gamma_1}{\Gamma_2} - \frac{\Gamma_2}{\Gamma_1} = \frac{\Gamma_2}{\Gamma_3} - \frac{\Gamma_3}{\Gamma_2} = \frac{\Gamma_3}{\Gamma_1} - \frac{\Gamma_1}{\Gamma_3}.$$

This contradicts with $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. □

Similarly, we have the following result.

Proposition 5.6 *Suppose that a diagram has an isolated triangle of z-strokes in the z-diagram, where all three vertices, say* $\mathbf{1}, \mathbf{2}, \mathbf{3}$, *are z-circled. If* $z_1 \sim z_2 \sim z_3$, *then there exists some* $k > 3$ *such that* $z_{k1} \prec \epsilon^{-2}$.

Proof Suppose that $z_1 \sim z_2 \sim z_3 \approx \epsilon^{-2}$. By Proposition 4.1 and Rule IV, we have $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. Suppose that it holds $z_{k1} \approx \epsilon^{-2}$ for all $k > 3$. Then $\frac{1}{z_{kj}} - \frac{1}{z_{k1}} = \frac{z_{1j}}{z_{kj}z_{k1}} \prec \epsilon^2$ for all $k > 3$, $1 \le j \le 3$. Similar to the argument of the above result, we have $\bar{\Lambda} \sum_{j=1}^{3} \Gamma_j w_j \prec \epsilon^2$,

$$\frac{w_{12}}{\Gamma_3} \sim \frac{w_{23}}{\Gamma_1} \sim \frac{w_{31}}{\Gamma_2} \approx \epsilon^2,$$
$$\Lambda z_1 \sim \frac{\Gamma_2}{w_{21}} + \frac{\Gamma_3}{w_{31}}, \ \Lambda z_2 \sim \frac{\Gamma_1}{w_{12}} + \frac{\Gamma_3}{w_{32}}, \ \Lambda z_3 \sim \frac{\Gamma_1}{w_{13}} + \frac{\Gamma_2}{w_{23}},$$

and $\frac{\Gamma_1}{\Gamma_2} - \frac{\Gamma_2}{\Gamma_1} = \frac{\Gamma_2}{\Gamma_3} - \frac{\Gamma_3}{\Gamma_2} = \frac{\Gamma_3}{\Gamma_1} - \frac{\Gamma_1}{\Gamma_3}$. This contradicts with $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. □

Proposition 5.7 *Assume that vertices **1** and **2** are both z- and w-circled and connected by a zw-edge, and the sub-diagram formed by the two vertices is isolated in the z-diagram. Assume that vertices **3** and **4** are also both z- and w-circled and connected by a zw-edge, and is isolated in the z-diagram. Then, there must exist some $k > 4$ such that at least one among $z_{k1}, w_{k1}, z_{k3}, w_{k3}$ is bounded (i.e., $\preceq 1$).*

Proof By Proposition 4.1 and Rule IV, we have

$$z_1 \sim z_2, z_3 \sim z_4, \ w_1 \sim w_2, w_3 \sim w_4, \ \Gamma_1 + \Gamma_2 = 0, \Gamma_3 + \Gamma_4 = 0.$$

Suppose that it holds that $w_{k1} \succ 1$ for all $k > 4$. Then $\frac{1}{w_{k2}} - \frac{1}{w_{k1}} = \frac{w_{12}}{w_{k2}w_{k1}} \prec \epsilon^2$ for all $k > 4$. Note that $z_{12} \approx \epsilon^2$, and

$$\Lambda z_{12} = (\Gamma_1 + \Gamma_2) Z_{12} + \Gamma_3 \left(\frac{w_{21}}{w_{32}w_{31}} - \frac{w_{21}}{w_{42}w_{41}} \right) + \sum_{k>4} \Gamma_k \left(\frac{1}{w_{k2}} - \frac{1}{w_{k1}} \right).$$

We conclude that

$$\frac{w_{21}}{w_{31}w_{32}} \approx \frac{w_{21}}{w_{41}w_{42}} \succeq \epsilon^2 \Rightarrow w_{31} \preceq 1 \Rightarrow w_1 \sim w_2 \sim w_3 \sim w_4.$$

Similarly, we have

$$z_1 \sim z_2 \sim z_3 \sim z_4.$$

Note that

$$\bar{\Lambda} \sum_{j=1}^{4} \Gamma_j w_j = \sum_{k>4} \sum_{j=1}^{4} \frac{\Gamma_k \Gamma_j}{z_{kj}} = \sum_{k>4} \Gamma_k \Gamma_1 \left(\frac{1}{z_{k1}} - \frac{1}{z_{k2}} \right) + \sum_{k>4} \Gamma_k \Gamma_3 \left(\frac{1}{z_{k3}} - \frac{1}{z_{k4}} \right) \prec \epsilon^2.$$

Then the equation $\Gamma_2 w_{12} + \Gamma_4 w_{34} = \sum_{j=1}^{4} \Gamma_j w_j$ leads to

$$\Gamma_2 w_{12} \sim -\Gamma_4 w_{34}, \ \text{or } \Gamma_2 Z_{34} \sim -\Gamma_4 Z_{12}.$$

On the other hand, the isolation of the two segments implies

$$\Lambda z_2 \sim \Gamma_1 Z_{12},\ \ \Lambda z_4 \sim \Gamma_3 Z_{34}, \Rightarrow \Gamma_1 Z_{12} \sim \Gamma_3 Z_{34}.$$

As a result, we have

$$\Gamma_1\Gamma_2 = -\Gamma_3\Gamma_4,\ \text{ or } \Gamma_1^2 + \Gamma_3^2 = 0,$$

which is a contradiction. □

Proposition 5.8 *Assume that there is a quadrilateral with vertices* ***1, 2, 3, 4****, that is fully z- and w-stroked, and fully w-circled. Moreover, the quadrilateral is isolated in the w-diagram. Then, there must exist some $k > 4$ such that $w_{k1} \preceq 1$.*

Proof We establish the result by contradiction. Rule IV implies that $\sum_{j=1}^{4} \Gamma_j = 0$. Suppose that it holds that $w_{k1} \succ 1$ for all $k > 4$. Then $\frac{1}{w_{kj}} - \frac{1}{w_{k1}} \prec \epsilon^2$ for $k > 4,\ j \leq 4$, so

$$\Lambda \sum_{j=1}^{4} \Gamma_j z_j = \sum_{k>4} \Gamma_k \sum_{j=1}^{4} \frac{\Gamma_j}{w_{kj}} = \sum_{k>4} \Gamma_k \sum_{j=1}^{4} \Gamma_j \left(\frac{1}{w_{k1}} + \frac{1}{w_{kj}} - \frac{1}{w_{k1}} \right) \prec \epsilon^2.$$

Then

$$\epsilon^2 \succ \sum_{j=1}^{4} \Gamma_j z_j = \sum_{j=2,3,4} \Gamma_j z_{1j} \Rightarrow \Gamma_2 z_{12} \sim -\Gamma_3 z_{13} - \Gamma_4 z_{14} \approx \epsilon^2.$$

Set $z_{13} \sim a\epsilon^2$, $z_{14} \sim b\epsilon^2$, where $a \neq b$ are some nonzero constants. Then

$$z_{12} \sim -\frac{\Gamma_3 a + \Gamma_4 b}{\Gamma_2}\epsilon^2, \qquad z_{23} \sim \frac{a(\Gamma_2 + \Gamma_3) + \Gamma_4 b}{\Gamma_2}\epsilon^2,$$
$$z_{24} \sim \frac{\Gamma_3 a + b(\Gamma_2 + \Gamma_4)}{\Gamma_2}\epsilon^2, \qquad z_{34} \sim (b - a)\epsilon^2.$$

Since $w_1 \sim w_2 \sim w_3 \sim w_4$, we set $\bar{\Lambda} w_k \sim \frac{1}{c\epsilon^2}$, $k = 1, 2, 3, 4$. Substituting those into the system

$$\bar{\Lambda} w_k \sim \sum_{j \neq k, j=1}^{4} \frac{\Gamma_j}{z_{jk}},\ \ k = 1, 2, 3, 4,$$

which is from the isolation of the quadrilateral in w-diagram. We obtain four homogeneous polynomials of the three variables a, b, c. Thus, we set $c = 1$, and obtain the following four polynomials of the five variables $a, b, \Gamma_1, \Gamma_3, \Gamma_4$,

$$a^2(-\Gamma_3)(b + \Gamma_4) + ab\left(-\Gamma_4(b - 2\Gamma_3) + \Gamma_1^2 + 2\Gamma_1(\Gamma_3 + \Gamma_4)\right) - b^2\Gamma_3\Gamma_4 = 0,$$

$$
\begin{aligned}
& a^3\Gamma_3^2(\Gamma_1+\Gamma_4)+a^2\Gamma_3\left(-b\left(\Gamma_1^2+\Gamma_1\Gamma_3+\Gamma_4(2\Gamma_3-\Gamma_4)\right)-(\Gamma_1-\Gamma_3+\Gamma_4)(\Gamma_1+\Gamma_3+\Gamma_4)^2\right)\\
&\quad -ab\left(\Gamma_1^2\Gamma_4(b-4\Gamma_3)+\Gamma_1\left(\Gamma_4^2(b+2\Gamma_4)+2\Gamma_3^3-2\Gamma_3^2\Gamma_4-2\Gamma_3\Gamma_4^2\right)-\Gamma_3^2\Gamma_4(b+2\Gamma_4)\right)\\
&\quad -ab\left(2b\Gamma_3\Gamma_4^2-\Gamma_1^4-2\Gamma_1^3(\Gamma_3+\Gamma_4)+\Gamma_3^4+\Gamma_4^4\right)\\
&\quad +b^2\Gamma_4\left(\Gamma_1\left(\Gamma_4(b+\Gamma_4)-3\Gamma_3^2-2\Gamma_3\Gamma_4\right)+\Gamma_3\Gamma_4(b+\Gamma_4)\right.\\
&\quad \left.-\Gamma_1^3-\Gamma_1^2(3\Gamma_3+\Gamma_4)-\Gamma_3^3-\Gamma_3^2\Gamma_4+\Gamma_4^3\right)=0,\\
& a^3(\Gamma_1+\Gamma_4)+a^2(\Gamma_3(2\Gamma_1+\Gamma_3+2\Gamma_4)-b(\Gamma_1+2\Gamma_4))\\
&\quad +ab\left(b\Gamma_4-2\Gamma_1\Gamma_3-\Gamma_3^2-2\Gamma_3\Gamma_4\right)-b^2\Gamma_1\Gamma_4=0,\\
& a^2\Gamma_3(b-\Gamma_1)-ab(b(\Gamma_1+2\Gamma_3)+\Gamma_4(2\Gamma_1+2\Gamma_3+\Gamma_4))\\
&\quad +b^2(b(\Gamma_1+\Gamma_3)+\Gamma_4(2\Gamma_1+2\Gamma_3+\Gamma_4))=0.
\end{aligned}
$$

Tedious but standard computation, such as calculating the Gröbner basis, yields

$$
b^5(\Gamma_1+\Gamma_3+\Gamma_4)\left(\Gamma_1^2+\Gamma_1\Gamma_3+\Gamma_1\Gamma_4+\Gamma_3^2+\Gamma_3\Gamma_4+\Gamma_4^2\right)=0.
$$

It is a contradiction since $b\neq 0$, $\Gamma_1+\Gamma_3+\Gamma_4=-\Gamma_2\neq 0$ and

$$
\Gamma_1^2+\Gamma_1\Gamma_3+\Gamma_1\Gamma_4+\Gamma_3^2+\Gamma_3\Gamma_4+\Gamma_4^2=\frac{1}{2}(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2+\Gamma_4^2)\neq 0.
$$

□

6 Construction of the 5-vortex diagrams

From now on, we focus on the planar 5-vortex problem. In this section, we identify all problematic diagrams for the 5-vortex central configurations. We adopt the approach used by Albouy and Kaloshin in the N-body problem [1], which involves analyzing diagrams according to the maximal number of strokes emanating from a two-colored vertex. During the analysis of all the possibilities we rule some of them out immediately. The ones we cannot exclude without further consideration are collected into a list of 31 diagrams, shown in Figs. 15 and 16 in Sect. 7.

We call a *two-colored vertex* of the diagram a vertex which connects at least a stroke of z-color with at least a stroke of w-color. The number of strokes from a two-colored vertex is at least 2 and at most 8. Given a diagram, we define C as *the maximal number of strokes from a two-colored vertex*. We use this number to classify all possible diagrams.

Recall that the z-diagram indicates the maximal terms among a finite set of terms. It is nonempty. If there is a circle, there is an edge of the same color emanating from it. So there is at least a z-stroke, and similarly, at least a w-stroke.

Remark 4 We developed a symbolic computation algorithm to determine the diagrams for the N-vortex problem [23], inspired by the work of Chang and Chen, who designed symbolic computation algorithms to implement the singular sequence method for the N-body problem in celestial mechanics [4]. The core idea is to represent each two-colored diagram as an $N \times 2N$ binary matrix. The rules from Sect. 4.3 and the results from Sect. 5 are then translated into conditions on these binary matrices. Using this algorithm, we can efficiently filter out invalid diagrams. This approach is particularly useful when applying the singular sequence method to the finiteness problem for $N \geq 6$, as it significantly reduces the required manual work. For $N = 5$, the algorithm outputs 31 diagrams, matching the list shown in Figs. 15 and 16 in Sect. 7. However, we emphasize that all computations in this section were carried out by hand; the algorithm was not used to obtain any of these diagrams.

6.1 No two-colored vertex

There is at least one isolated edge, which is not a zw-edge. Let us say it is a z-edge. The complement has 3 bodies. There three can have one or three w-edges according to Rule VI.

For one w-edge, the attached bodies have to be w-circled by Rule I. This is the first diagram in Fig. 2.

For three w-edges, the three edges form a triangle. There are three possibilities for the number of w-circled vertices: it is either zero, or two or three (one is not possible by Rule III.) They constitute the last three diagrams in Fig. 2.

Hence, we have *four* possible diagrams, as shown in Fig. 2.

6.2 $C = 2$

There are two cases: a zw-edge exists or not.

If it is present, it is isolated. Let us say, vertex **1** and vertex **2** are connected by one zw-edge. Note that there must be both z and w-circle among vertices **3**, **4**, and **5**. If none of the three vertices is z-circled, we have $\Gamma_1 + \Gamma_2 = 0$ by Rule IV. On the other hand, since vertices **3**, **4**, and **5** are not z-circled, they are not z-close to vertex **1** and vertex **2** . Then Proposition 5.1 implies that $\Gamma_1 + \Gamma_2 \neq 0$. This is a contradiction.

Then Rule I implies that there is at least one z-stroke and one w-stroke among the cluster of vertices **3**, **4**, and **5**. There are two possibilities: whether there is another zw-edge or not.

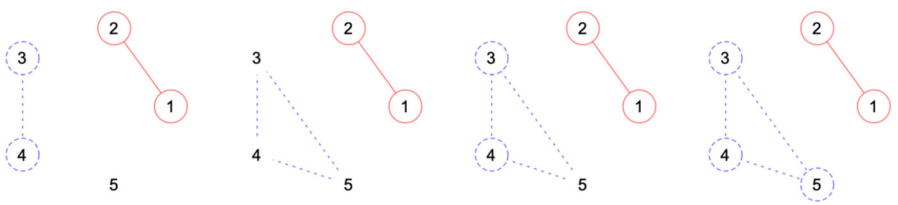


Fig. 2 Four possible diagram for no two-colored vertex. They correspond to Diagram 15 of Fig. 16, Diagram 1 of Fig. 15, Diagram 6 of Fig. 15, and Diagram 16 of Fig. 16 respectively

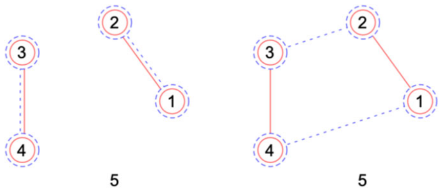


Fig. 3 Two diagrams for $C = 2$. The first one has been excluded. The second one corresponds to Diagram 18 of Fig. 16

If another zw-edge is present, then it is again isolated. This is the first diagram in Fig. 3. Note that $w_{15} \approx \epsilon^{-2}$, which contradicts with Proposition 5.7, thus impossible.

If another zw-edge is not present, there is at least one edge in both color. By the circling method, the adjacent vertex is z- and w-circled. By the Estimate, the z-edge implies the two attached vertices are w-close. Then the two ends are both w-circled by Rule II. Thus, all three vertices are z- and w-circled. Then there are more strokes in the diagram. This is a contradiction.

If there is no zw-edge, there are adjacent z-edges and w-edges. From any such adjacency there is no other edge. Suppose that vertex **1** connects with vertex **4** by w-edges and connects with **2** by z-edges. The circling method implies that **1** is z- and w-circled, 2 is w-circled and 4 is z-circled. The color of **2** and **4** forces the color of edges from the circle. If one of the two new edges completes the triangle with vertices **1, 2, 4**, then Rule VI implies that $C > 2$, a contradiction. If the two new edges go to the same vertex, we get the diagram corresponding to Roberts' continuum at infinity, shown as the second in Fig. 3.

If the two edges go to the different vertices, the circling method and Rule I demand a cycle with alternating colors, which is impossible since the cycle has five edges.

Hence, there is only *one* possible diagram, the second one in Fig. 3.

6.3 $C = 3$

Consider a two-colored vertex with three strokes. There are two cases: in the first, it is like vertex **1** in Fig. 4; in the second, it connects a single stroke to a zw-edge.

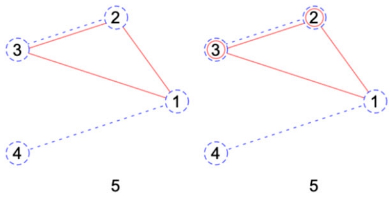
We start with the first case. Let us say vertex **1** connects with vertex **2** and vertex **3** by z-edges, and connects with vertex **4** by a w-edge. There is a z_{23}-stroke by Rule VI. The circling method implies that the vertices **1**, **2** and vertex **3** are all w-circled, see Fig. 4. Then there is w-stroke emanating from **2** and vertex **3**. The w-stroke may go to vertex **4**, vertex **5**, or it is a w_{23}-stroke.

If one w-stroke goes from vertex **2** to vertex **4**, then there is extra w_{12}-stroke by Rule VI, which contradict with $C = 3$. If all two w-strokes go to vertex **5**, then Rule VI implies the existence of w_{23}-stroke. This is again a contradiction with $C = 3$.

If the w-strokes emanating from **2** and vertex **3** are just the w_{23}-stroke, then we have a zw-edge between **2**, and **3**. Then it is not necessary to discuss the second case. Then, we consider the vertex **5**. It is connected with the previous four vertices or isolated.

If the diagram is connected, vertex **5** can only connects with vertex **4** by a z-edge (other cases is not possible by Rule VI). Then the circling method implies that vertex **5** is w-circled. Then there is w-stroke emanating from **5**. This is a contradiction.

Fig. 4 Two diagrams for $C = 3$. Both have been excluded



If vertex **5** is isolated. Then the circling method implies that all vertices except vertex **5** are w-circled. Only **2** and vertex **3** can be z-circled, and they are both z-circled or both not z-circled by Rule IV, see Fig. 4.

If both vertex **2** and vertex **3** are not z-circled, then we have $\Gamma_2 + \Gamma_3 = 0$ and $L_{123} = 0$ by Rule IV and Proposition 5.2. This is a contradiction since

$$L_{123} = \Gamma_1(\Gamma_2 + \Gamma_3) + \Gamma_2\Gamma_3.$$

If both vertex **2** and vertex **3** are z-circled, then Rule IV implies that $\Gamma_2 + \Gamma_3 = 0$. On the other hand, Corollary 5.1 implies that $\Gamma_2 + \Gamma_3 \neq 0$. This is a contradiction.

Thus, the two diagrams in Fig. 4 are both excluded. There is no possible diagram.

6.4 $C = 4$

There are five cases: two zw-edges, only one zw-edge and one edge of each color, only one zw-edge and two edge of the same color, one z-edge and three w-edges, or two edges of each color emanating from the same vertex.

6.4.1

Suppose that there are two zw-edges emanating from, e.g., vertex **1** as on the first diagram in Fig. 5.

We get a fully zw-edged triangle by Rule VI. This triangle is isolated since $C = 4$. Since vertices **1**, **2**, and **3** are z-close and w-close, if one of them is circled in some color, all of them will be circled in the same color. Thus, the first three vertices may be all z-circled, all z-and w-circled, or all not circled.

If vertices **1**, **2**, and **3** are z-circled but not w-circled, we have $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ and $L_{123} = 0$ by Rule IV and Proposition 5.2. This is a contradiction since $(\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - 2L_{123} \neq 0$. Then the first three vertices can only be all z-and w-circled, or all not circled.

The other two vertices can be disconnected, connected by one , e.g., z-edge, or by one zw-edge. Then there are six possibilities, according to whether the first three vertices are all z-and w-circled, or all not circled, and the connection between the other two vertices.

Suppose that vertex **4** and vertex **5** are connected by one zw-edge, and the first three vertices are all not circled. Then we have $\Gamma_4 + \Gamma_5 = 0$ by Rule IV. On the other hand, Corollary 5.1 implies that $\Gamma_4 + \Gamma_5 \neq 0$. This is a contradiction.

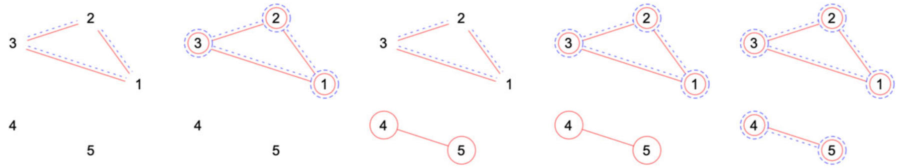


Fig. 5 Five diagrams for $C = 4$, two zw-edges. The second and fourth one has been excluded. The first, third and fifth one correspond to Diagram 1 of Fig. 16, Diagram 2 of Fig. 15, and Diagram 20 of Fig. 16 respectively

Then we have the five possibilities, see Fig. 5. Note that Proposition 5.5 can further exclude the second and fourth one.

Hence, there are *three* possible diagrams, the first, third and fifth one in Fig. 5.

6.4.2

Suppose that there is one zw-edge and one edge of each color emanating from vertex **1**, as in the first digram of Fig. 6. We complete the triangles by Rule VI. Note that no more strokes can emanating from vertex **1** and vertex **4** since $C = 4$. If there are more strokes from vertex **2**, it can not goes to vertex **3**, since it implies one more stroke emanating from vertex **1**. Similarly, there is no z_{35}-stroke or w_{25}-stroke. Then between vertex **5** and the first four vertices, there can have no edge, one z_{25}-stroke, or one z_{25}-stroke and one w_{35}-stroke.

For the disconnected diagram, vertex **1** and vertex **4** can not be circled. Otherwise, the circling method implies either vertex **2** is be z-circled or 3 is w-circled, there should be stroke emanating from vertex **2** or vertex **3** . This is a contradiction. Then vertex **2** can not be circled, otherwise, vertex **3** is of the same color by Rule IV. This is a contradiction. Hence, there is no circle in the diagram and this is the first diagram in Fig. 6.

If there is only z_{25}-stroke, then the circling method implies that vertices **1**, **2**, **4**, and **5** are z-circled. Note that vertex **3** and vertex **5** can not be w-circled, otherwise there are extra w-strokes. Then vertices **1**, **2**, and **4** are not w-circled by the circling method. Note that vertex **3** must be z-circled, otherwise, we have $L_{124} = 0$ and $\Gamma_1 + \Gamma_4 = 0$. This is a contradiction. Then we have the second diagram in Fig. 6.

If there are z_{25}-stroke and w_{35}-stroke. Then vertex **5** is z-and w-circled. By the circling method, all vertices are z-and w-circled. This the third diagram in Fig. 6.

Hence, there are *three* possible diagrams, the first three in Fig. 6.

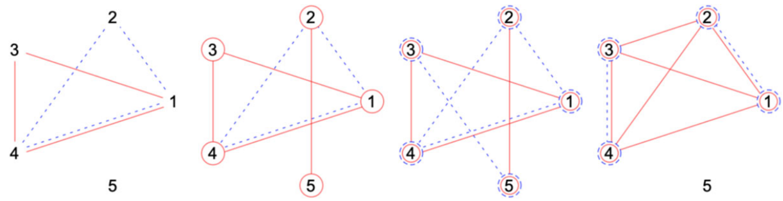


Fig. 6 Four diagrams for $C = 4$, one zw-edge. The fourth one has been excluded. The first three correspond to Diagram 2 of Fig. 16, Diagram 8 of Fig. 16, and Diagram 19 of Fig. 16 respectively

6.4.3

Suppose that there are one zw-edges and two z-edges emanating from vertex **1**, as in the fourth diagram in Fig. 6. Then there are z_{23}-stroke, z_{24}-stroke and z_{34}-stroke by Rule VI. Note that vertex **1** is w-circled, then the circling method implies all vertices except possibly vertex **5** are w-circled. Then there is w-stroke emanating from **3** and vertex **4**. The w-stroke may go to vertex **5**, or it is a w_{34}-stroke.

If all two w-strokes go to vertex **5**, then Rule VI implies the existence of w_{34}-stroke, which contradicts with $C = 4$. Then the w-strokes from **3** and vertex **4** is w_{34}-stroke, and vertex **5** is disconnected.

Consider the circling the the diagram. We have three different cases: all vertices are not z-circled, only two of the first four vertices, say, vertex **1** and vertex **2** are z-circled, or all the first four vertices are z-circled.

If none of the first four vertices is z-circled, then we have $\Gamma_{1234} = 0$ and $L_{1234} = 0$ by Rule IV and Proposition 5.2. This is a contradiction.

If only vertex **1** and vertex **2** are z-circled, we have $\Gamma_1 + \Gamma_2 = 0$ by Rule IV. On the other hand, Corollary 5.1 implies that $\Gamma_1 + \Gamma_2 \neq 0$. This is a contradiction.

Then all the first four vertices are z-circled. This gives the fourth diagram in Fig. 6. However, this contradicts with Proposition 5.7, so excluded.

Hence, there is no possible diagram, i.e., the fourth one in Fig. 6 is impossible.

6.4.4

Suppose that from vertex **1** there are three w-edges go to vertex **2**, vertex **3** and vertex **4** respectively and one z-edges goes to vertex **5** . There is a w-stroke between any pair of {**1, 2, 3, 4**} by Rule VI, and the four vertices are all z-circled.

Then there are z-strokes emanating from vertex **2**, vertex **3** and vertex **4** . None of them can go to vertex **5** by Rule VI. Then there are z_{23}-, z_{24}- and z_{34}-strokes. This contradict with $C = 4$. Hence, there is no possible diagram in this case.

6.4.5

Suppose that there are two w-edges and two z-edges emanating from vertex **1**, with numeration as in the first diagram of Fig. 7. By Rule VI, there is w_{23}- and z_{45}-stroke. Thus, we have two attached triangles. By Rule VI, if there is more stroke, it must be z_{23}- and/or w_{45}-stroke.

Case I: If there is no more stroke, then vertex **1** can not be circled, otherwise, the circling method would lead to contradiction. Rule IV implies that vertex **2** and vertex **3** can only be both or both not w-circled, and vertex **4** and vertex **5** can only be both or both not z-circled. Then we have the first three diagrams in Fig. 7.

Case II: If there is only z_{23}-stroke, the circling method implies vertex **1**, vertex **2** and vertex **3** are z-circled. Note that only vertex **2** and vertex **3** can be w-circled. There are two possibilities: whether vertex **2** and vertex **3** are both w-circled or both not w-circled.

If vertex **2** and vertex **3** are both w-circled, we have $\Gamma_2 + \Gamma_3 = 0$ by Rule IV. On the other hand, since vertices **1**, **4**, and **5** are not w-circled, they are not w-close to vertex **2** and vertex **3** . Then Proposition 5.1 implies that $\Gamma_2 + \Gamma_3 \neq 0$. This is a contradiction.

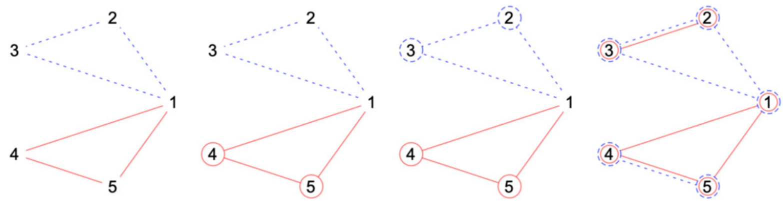


Fig. 7 Four diagrams for $C = 4$, no zw-edges. They correspond to Diagram 3 of Fig. 16, Diagram 3 of Fig. 15, Diagram 17 of Fig. 16, and Diagram 9 of Fig. 15 respectively

If vertex **2** and vertex **3** are both not w-circled, we have $\Gamma_2 + \Gamma_3 = 0$ and $L_{123} = 0$ by Rule IV and Proposition 5.2. This is a contradiction. Hence there is no possible diagram in this case.

Case III: If there are z_{23}- and w_{45}-strokes, the circling method implies vertex **2** and vertex **3** are z-circled, vertex **4** and vertex **5** are w-circled, and vertex **1** is z- and w-circled. By Rule I, vertex **2** and vertex **3** are both z-circled, and vertex **4** and vertex **5** are both w-circled. This is the last diagram in Fig. 7.

Hence, there are *four* possible diagrams in this case, as shown in Fig. 7.

In summary, among the thirteen diagrams in Figs. 5, 6 and 7, we have excluded the first and third one in Fig. 5 and the fourth one in Fig. 6. We have *10* possible diagrams.

6.5 $C = 5$

There are three cases: two zw-edges, one zw-edge with one z-edge and two w-edges, one zw-edge with three z-edges.

6.5.1

Suppose that there are two zw-edges and one z-edges emanating from vertex **1**, with numeration as in the first digram of Fig. 8. Rule VI implies the existence of z_{23}-, w_{23}-, z_{24}- and z_{34}-strokes. Then vertex **5** can be either disconnected or connects with vertex **4** by one w-edge, otherwise, it would contradict with $C = 5$.

For the disconnected diagram, note that any of the connected four vertices can not be w-circled. Otherwise, the circling method implies vertex **4** is w-circled, which is a contradiction. There are three cases: none of the vertices are circled, all four vertices except vertex **4** are z-circled, or all four vertices are z-circled.

If none of the vertices are circled, by Proposition 5.2 we have $L_{123} = 0$ since vertices **1**, **2**, and **3** form a triangle with no w-circled attached. Similarly, we have $L_{1234} = 0$ by Proposition 5.2. This is a contradiction since

$$L_{1234} = L_{123} + \Gamma_4(\Gamma_1 + \Gamma_2 + \Gamma_3), \ (\Gamma_1 + \Gamma_2 + \Gamma_3)^2 - 2L_{123} \neq 0.$$

If only vertices **1**, **2**, and **3** are z-circled, we also have $L_{123} = 0$. By Rule IV, we have $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. This is a contradiction.

Then we have only one possible diagram for the disconnected diagram, and it is the first in Fig. 8.

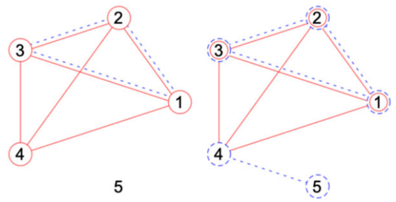


Fig. 8 Two diagrams for $C = 5$, two zw-edges. They correspond to Diagram 5 of Fig. 15, and Diagram 7 of Fig. 15 respectively

For the connected diagram with w_{45}-edge, the circling method implies that all vertices are w-circled. Note that vertex **4** and vertex **5** can not be z-circled. Then we have two cases: all the five vertices are not z-circled, or vertices **1**, **2**, and **3** are z-circled.

If all the five vertices are not z-circled, we have $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ by Rule IV and $L_{1234} = 0$ by Proposition 5.2. This is a contradiction.

Then, we have only one possible diagram for the connected diagram, and it is the second in Fig. 8.

Hence, we have *two* possible diagrams, shown in Fig. 8.

6.5.2

Suppose that there are one zw-edge, one z-edge, and two w-edges emanating from vertex **1**, with numeration as in the first diagram of Fig. 9. Rule VI implies that vertices **1**, **2**, **3**, and **4** are fully w-stroked, and that vertices **1**, **3**, and **5** are fully z-stroked. There is no more stroke emanating from vertices **1**, **3**, and **5**, since that would contradict with $C = 5$. There are two cases, z_{24}-stroke is present or not.

If it is not present, vertex **2** and vertex **4** can only be w-circled, and vertex **5** can only be z-circled. Then vertex **1** and vertex **3** can not be circled, otherwise, the circling method would lead to contradiction. Then we have the first two diagrams in Fig. 9 by Rule IV.

If z_{24}-stroke is present, then vertices **1**, **2**, **3**, and **4** are all z-circled. Vertex **1** and vertex **3** can not be w-circled, which would lead to vertex **5** also w-circled and a contradiction. Vertex **2** and vertex **4** are w-close, so they are both w-circled or both not w-circled. If they are both w-circled, then Rule IV implies that $\Gamma_2 + \Gamma_4 = 0$. On the other hand, Proposition 5.1 implies that $\Gamma_2 + \Gamma_4 \neq 0$. This is a contradiction.

Then according to whether vertex **5** is z-circled or not, we have two cases, which are the last two diagrams in Fig. 9.

The third diagram in Fig. 9 is impossible. We would have $L_{1234} = 0$ and $\sum_{j=1}^{4} \Gamma_j = 0$ by Proposition 5.2 and Rule IV. Hence, we have only one possible diagram if z_{24}-stroke is present, and it is the last diagram in Fig. 9.

Hence, we have *three* possible diagrams, the first two and the last one in Fig. 9.

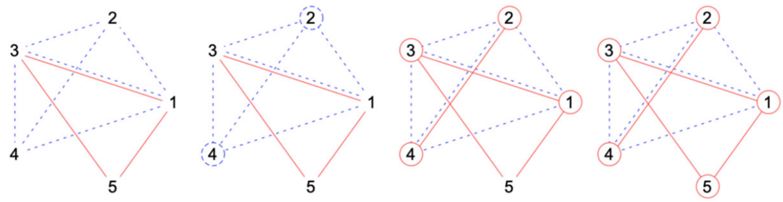


Fig. 9 Four diagrams for $C = 5$, one zw-edges. The third one has been excluded.The first, second and the fourth one correspond to Diagram 4 of Fig. 16, Diagram 4 of Fig. 15, and Diagram 12 of Fig. 16 respectively

6.5.3

Suppose that there is one zw-edge and three w-edges emanating from vertex **1** . Let us say, the zw-edge goes to vertex **2** and the other edges go to the other three vertices. Rule VI implies that vertices **1, 2, 3, 4, 5** are fully w-stroked. The circling method implies that all vertices are z-circled. Then there are z-strokes emanating from vertices **3**, **4**, and **5**. Since $C = 5$ at vertex **1** and vertex **2**, the z-strokes from vertices **3**, **4**, and **5** must go to vertices **3**, **4**, and **5**. By Rule VI, they form a triangle of z-strokes. This is a contradiction with $C = 5$. Hence, there is no possible diagram in this case.

In summary, among the six diagrams in Fig. 8 and 9, we have excluded the third one in Fig. 9. We have *five* possible diagrams.

6.6 $C = 6$

There are three cases: three zw-edges, two zw-edge with one edge in each color, two zw-edge with two z-edge.

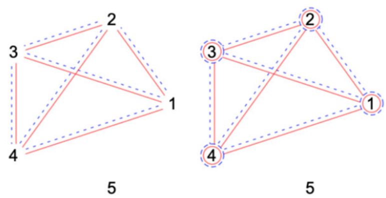
6.6.1

Suppose that there are three zw-edges emanating from vertex **1**, with numeration as in the first digram of Fig. 10. Rule VI implies that the vertices **1**, **2**, **3**, and **4** are fully zw-edged. Then vertex **5** must be disconnected since $C = 6$. The first four vertices can be circled in the same way by the circling method.

If all the first four vertices are z-circled but not w-circled, then we would have $L_{1234} = 0$ and $\sum_{j=1}^{4} \Gamma_j = 0$ by Proposition 5.2 and Rule IV. This is one contradiction.

Then we have two possible diagrams, as shown in Fig. 10. However, the second one is impossible by Proposition 5.8.

Fig. 10 Two diagrams for $C = 6$, three zw-edges. The second one has been excluded. The first one corresponds to Diagram 10 of Fig. 16



Hence, there is only *one* possible diagram if there are three zw-edges, the first one in Fig. 10.

6.6.2

Suppose that there are two zw-edges and two z-edges emanating from vertex **1**, with numeration as in the first diagram of Fig. 11. Rule VI implies the existence of z_{25}-, z_{35}-, z_{24}-, z_{45}- and z_{34}-strokes, and that the vertices **1**, **2**, and **3** are fully zw-edged. If there is more stroke, it must be w_{45}-stroke since $C = 6$.

If there is no more stroke, then the vertices can only be z-circled. There are two cases, either vertices **1**, **2**, and **3** are z-circled or not.

If vertices **1**, **2**, and **3** are z-circled, then either vertex **4** or vertex **5** must also be z-circled. Otherwise, we have $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ and $L_{123} = 0$ by Proposition 5.2 and Rule IV. This is one contradiction. Then we have the first two diagrams in Fig. 11.

If vertices **1**, **2**, and **3** are not z-circled, then vertex **4** and vertex **5** are both z-circled or both not z-circled. Then we have the second two diagrams in Fig. 11.

If w_{45}-stroke is present, then the circling method implies that all vertices are w-circled. Rule IV implies that $\Gamma_{123} = 0$. If at least one, hence all three vertices **1**, **2**, and **3** are z-circled. Then there are two cases, according to whether vertices **4** and **5** are z-circled. Hence, we have the last two diagrams in Fig. 11. If none of vertices **1**, **2**, and **3** are z-circled, there are also two cases, depending on whether vertices **4** and **5** are z-circled. If the two vertices are both z-circled, Rule IV implies that $\Gamma_{45} = 0$, which contradicts with $\Gamma_{45} \neq 0$ by Proposition 5.1. If the two vertices are both not z-circled, Proposition 5.2 yields $L_{12345} = 0$. But Rule IV implies that $\Gamma_{12345} = 0$, a contradiction.

Hence, there are *six* possible diagrams in this case, as shown in Fig. 11.

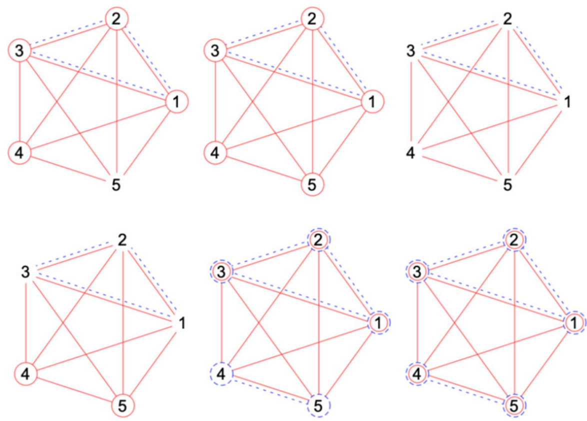
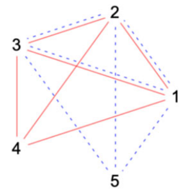


Fig. 11 Six diagrams for $C = 6$, two zw-edges, case 1. They correspond to Diagram 7 of Fig. 16, Diagram 9 of Fig. 16, Diagram 5 of Fig. 16, Diagram 6 of Fig. 16, Diagram 8 of Fig. 15, and Diagram 21 of Fig. 16 respectively

Fig. 12 One diagram for $C = 6$, two zw-edges, case 2. It corresponds to Diagram 11 of Fig. 16



6.6.3

Suppose that there are two zw-edges and one edge of each color emanating from vertex **1**, with numeration as in the first diagram of Fig. 12. Rule VI implies the existence of w_{25}-, w_{35}-, z_{24}-, and z_{34}-strokes, and that the vertices **1**, **2**, and **3** are fully zw-edged. If there is more stroke, it must be w_{45}-stroke, but it would lead to extra stroke emanating from vertex **1**, which contradicts with $C = 6$.

Vertex **5** can only be w-circled, and vertex **4** can only be z-circled. Then vertices **1**, 2 and 3 can not be circled, otherwise, the circling method would lead to contradiction. Then we have the diagram in Fig. 12 by Rule IV.

In summary, among the nine diagrams of Figs. 10, 11, and 12, we exclude the second one of Fig. 10. That is, we have *eight* possible diagrams.

6.7 $C = 7$

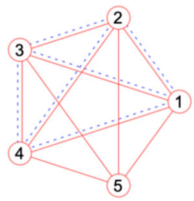
Suppose that there are three zw-edges and one z-edge emanating from vertex **1**, with numeration as in the diagram of Fig. 13. Rule VI implies that the vertices **1**, **2**, **3**, and **4** are fully zw-edged and that vertex **5** connects with the first four vertices by one z-edge. There is no more stroke since $C = 7$. If any vertex is w-circled, all are w-circled, which would lead to w-stroke emanating from vertex **5** . This is a contradiction. There are two cases, either vertex **1** is z-circled or not.

If vertex **1** is not z-circled, then none of the vertices is z-colored by the circling method. Then $L_{1234} = 0$ and $L = 0$ by Proposition 5.2, a contradiction.

If vertex **1** is z-circled, then vertices **1**, **2**, **3**, and **4** are all z-circled. In this case, vertex **5** must be z-circled. Otherwise, we have $L_{1234} = 0$ and $\sum_{j=1}^{4} \Gamma_j = 0$ by Proposition 5.2 and Rule IV, a contradiction.

Hence, we only have *one* diagram in the case of $C = 7$, as in Fig. 13.

Fig. 13 One diagram for $C = 7$. It corresponds to Diagram 13 of Fig. 16



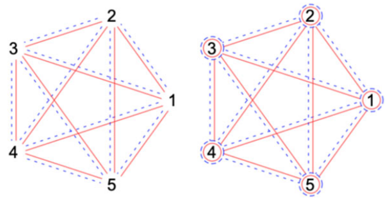


Fig. 14 Two diagrams for $C = 8$. They correspond to Diagram 14 and 22 of Fig. 16

6.8 $C = 8$

Suppose that there are four zw-edges emanating from vertex **1** . Rule VI implies that the vertices **1, 2, 3, 4, 5** are fully zw-edged. Since all vertices are both z-close and w-close, the vertices are all z-circled (respectively, w-circled) or all not z-circled (respectively, w-circled). If they are all just z-circled but not w-circled, then we have

$$\Gamma = 0, \quad L = 0,$$

by Rule IV and Proposition 5.2, a contradiction.

Hence, we have *two* diagrams in the case of $C = 8$, as in Fig. 14.

Summary: In the searching for all problematic 5-vortex two-colored diagrams, we have found 39 of them, as shown in Figs. 2-14. Among them, we have excluded eight of them, i.e., the first diagram in Fig. 3, the two diagrams in Fig. 4, the second and fourth diagram in Fig. 5, the fourth diagram in Fig. 6, the third diagram in Fig. 9, and the second diagram in Fig. 10. Hence, we conclude that any singular sequence should converge to one of the remaining 31 diagrams.

7 Further diagram exclusion and the vorticity constraints

The 31 diagrams found in Sect. 6 are collected into two classes, *9 diagrams that can be excluded* by further arguments, see Fig. 15 , and *22 diagrams that we can not exclude*, see Fig. 16.

We would like to point out that the diagrams in the two lists differ in appearance from those in Sect. 6. In Figs. 15 and 16, the diagrams are ordered by the number of circles, whereas in Sect. 6, they are ordered by the maximal number of strokes from a two-colored vertex. Additionally, there are differences in vertex labeling and in the switching of z- and w-diagrams. These differences are not mistakes. Each diagram in Sect. 6 represents an equivalence class under vertex permutations and color switching. It is therefore valid to present different representatives of the same equivalence classes, and we have intentionally done so to emphasize this fact, as it is often overlooked by readers.

For the 22 diagrams in Fig. 16, we list the vorticity constraints. Most of them are straightforward from the results in Sect. 5, while some requires additional work. We will use notations such as Γ_J, $\Gamma_{j_1,\ldots,j_n}$, L_J, and $L_{j_1,\ldots,j_n}$ below. Please refer to Definition 2.1 for their meanings.

7.1 The 9 impossible diagrams

The 9 impossible diagrams are presented in Fig. 15. We exclude them in the following.

1. Diagram 1: Consider the isolated component $\{3, 4, 5\}$of the z-digram, we see $L_{345} = 0$; Consider the isolated component $\{1, 2\}$ of the w-diagram. By Proposition 5.3, there are two cases:
   (a) $\Lambda = \pm 1$, then $\Gamma_3 + \Gamma_4 + \Gamma_5 = 0$, which contradicts with $L_{345} = 0$;
   (b) $\Lambda = \pm \mathbf{i}$, then $L = 0$,and $\Gamma_1\Gamma_2 = L_{345}$. Since $L_{345} = 0$, we obtain $\Gamma_1\Gamma_2 = 0$, which is impossible.
2. Diagram 2 can be excluded by the same argument as that of Diagram 1.
3. Diagram 3: Consider the isolated component $\{3, 4, 5\}$of the z-diagram, we see $L_{345} = 0$; Consider the isolated component $\{1, 2, 3\}$of the w-diagram. By Proposition 5.4, there are two cases:
   (a) $\Lambda = \pm 1$, then $\Gamma_3 + \Gamma_4 = 0$, which contradicts with $L_{345} = 0$ since $L_{345} = \Gamma_3\Gamma_4 + \Gamma_5(\Gamma_3 + \Gamma_4)$.
   (b) $\Lambda = \pm \mathbf{i}$, then $L = 0$,and $L_{125} = L_{345}$. Since $L_{345} = 0$, we obtain $\Gamma_3\Gamma_4 = -\Gamma_5(\Gamma_3 + \Gamma_4)$; similarly, it also holds $\Gamma_1\Gamma_2 = -\Gamma_5(\Gamma_1 + \Gamma_2)$. Note that

$$0 = L = L_{125} + (\Gamma_3 + \Gamma_4)(\Gamma_1 + \Gamma_2 + \Gamma_5) + \Gamma_3\Gamma_4 = (\Gamma_3 + \Gamma_4)(\Gamma_1 + \Gamma_2),$$

   which implies that $\Gamma_1\Gamma_2\Gamma_3\Gamma_4 = 0$, which is impossible.
4. Diagram 4: We will show that $z_1 \succ \epsilon^2$, then it contradicts with Corollary 5.1 and this diagram is thus excluded.

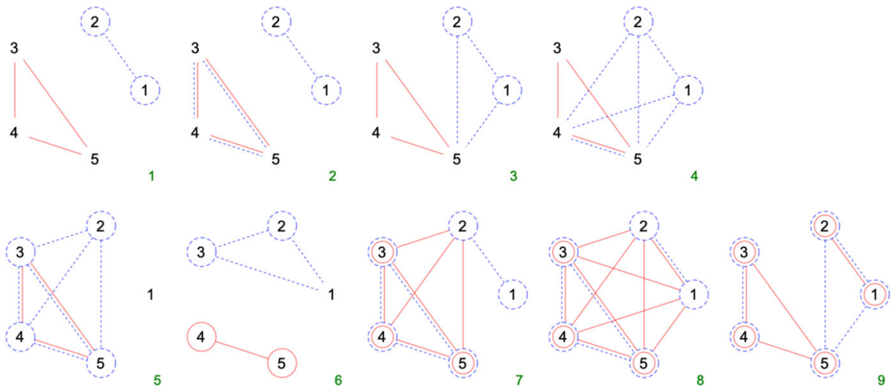


Fig. 15 The 9 diagrams that can be excluded. They correspond to the second diagram of Fig. 2, the third diagram of Fig. 5, the second diagram of Fig. 7, the second diagram of Fig. 9, the first diagram of Fig. 8, the third diagram of Fig. 2, the second diagram of Fig. 8, the fifth diagram of Fig. 11, and the fourth diagram of Fig. 7 respectively

Note that $w_{1j}, w_{2j} \approx \epsilon^{-2}$, $j \geq 3$. Then

$$\Lambda \sum_{j\geq 3} \Gamma_j z_j = \sum_{j\geq 3} \frac{\Gamma_1\Gamma_j}{w_{1j}} + \sum_{j\geq 3} \frac{\Gamma_2\Gamma_j}{w_{2j}} \preceq \epsilon^2.$$

Then the equation $\sum_{j=3}^5 \Gamma_j z_j = \Gamma_{345} z_3 + (\Gamma_4 + \Gamma_5) z_{34} + \Gamma_5 z_{45}$ and the fact that $z_{34} \succ \epsilon^2$, $z_{45} \approx \epsilon^2$ implies

$$\Gamma_{345} z_3 + (\Gamma_4 + \Gamma_5) z_{34} \preceq \epsilon^2.$$

By Proposition 5.2, it follows that $L_{345} = 0$ and then $\Gamma_{345} \neq 0,\ \Gamma_4 + \Gamma_5 \neq 0$. Then it holds that $z_3 \approx z_{34} \succ \epsilon^2$.
Finally, the equation $\sum_j \Gamma_j z_j = 0$ and the fact that $z_{jk} \approx \epsilon^2$, $j \neq k$, $j, k \in \{1, 2, 4, 5\}$ lead to

$$\Gamma_3 z_3 + \Gamma_{1245} z_1 \preceq \epsilon^2.$$

Hence, it holds that $z_1 \approx z_3 \succ \epsilon^2$.

5. Diagram 5: By Proposition 5.2, it follows that $L_{345} = 0$ and $\Gamma_{345} \neq 0,\ \Gamma_3 + \Gamma_4 \neq 0$.
We first claim that *the vertices* **2** *and* **3** *are not w-close, or,* $w_{23} \approx \epsilon^{-2}$, whose proof will be given later. Since $\frac{1}{w_{1j}} \approx \epsilon^2$, $j \geq 2$, we have $\Lambda z_1 = \sum_{j=2}^5 \frac{\Gamma_j}{w_{1j}} \preceq \epsilon^2$. Note that $z_{12} \succ \epsilon^2$, so $z_2 \succ \epsilon^2$. Then it is easy to see $z_2 \sim z_3 \sim z_4 \sim z_5 \succ \epsilon^2 \succeq z_1$. By the claim, $\frac{1}{w_{2j}} \approx \epsilon^2$, $j \geq 3$, so

$$\Lambda \sum_{j=3}^5 \Gamma_j z_j = \sum_{j=3}^5 \frac{\Gamma_1\Gamma_j}{w_{1j}} + \sum_{j=3}^5 \frac{\Gamma_2\Gamma_j}{w_{2j}} \preceq \epsilon^2.$$

By the equation $\sum_{j=3}^5 \Gamma_j z_j = \sum_{j=3}^5 \Gamma_j z_3 + \Gamma_4 z_{34} + \Gamma_5 z_{35}$, we conclude that $\sum_{j=3}^5 \Gamma_j = 0$, which is a contradiction.
We now prove the claim by contradiction. Suppose that **2** and **3** are w-close. Rule IV implies that $\sum_{j=2}^5 \Gamma_j = 0$. Note that $\frac{1}{w_{1j}} - \frac{1}{w_{12}} \prec \epsilon^2$, $j \geq 3$, so

$$\Lambda \sum_{j=2}^5 \Gamma_j z_j = \sum_{j=2}^5 \frac{\Gamma_1\Gamma_j}{w_{1j}} = \Gamma_1 \sum_{j=2}^5 \Gamma_j \left(\frac{1}{w_{12}} + \frac{1}{w_{1j}} - \frac{1}{w_{12}} \right) \prec \epsilon^2.$$

Then

$$\epsilon^2 \succ \sum_{j=2}^5 \Gamma_j z_j = \sum_{j=3,4,5} \Gamma_j z_{2j} \Rightarrow \Gamma_3 z_{23} \sim -\Gamma_4 z_{24} - \Gamma_5 z_{25} \approx \epsilon^2.$$

Note that the scenario is now the same as that described in Proposition 5.8, thus impossible.

6. Diagram 6: By Proposition 5.4, it holds that $\Gamma_4 + \Gamma_5 \neq 0$. There are two cases:
   (a) $\Lambda = \pm 1$. Consider the isolated component $\{2, 3\}$ of the w-diagram. Then $\Gamma_4 + \Gamma_5 = 0$, which is a contradiction.
   (b) $\Lambda = \pm \mathbf{i}$. Then $L = 0$. Consider the isolated component $\{4, 5\}$of the z-diagram. Then $\Gamma_4\Gamma_5 = L_{123}$. On the other hand, consider the isolated component $\{2, 3\}$ of the w-diagram. Then $L_{145} = L_{123}$. Thus, it holds

$$\Gamma_4\Gamma_5 = L_{145} = \Gamma_4\Gamma_5 + \Gamma_1(\Gamma_4 + \Gamma_5),$$

   which is a contradiction.

7. Diagram 7: It is easy to see that $w_2 \sim w_3 \sim w_4 \sim w_5 \approx \epsilon^{-2}$, $\Gamma_1 w_1 + \Gamma_2 w_2 \prec \epsilon^{-2}$, and

$$\Gamma_3 + \Gamma_4 + \Gamma_5 = 0.$$

Set $w_2 \sim b\epsilon^{-2}$. Then $w_1 \sim -\frac{\Gamma_2}{\Gamma_1} b\epsilon^{-2}$. We claim that $\Gamma_1 + \Gamma_2 \neq 0$. Otherwise, we have

$$\Gamma_2 w_{12} = \Gamma_1 w_1 + \Gamma_2 w_2 = \Lambda \left(\sum_{j=3}^{5} \frac{\Gamma_j \Gamma_1}{z_{j1}} + \sum_{j=3}^{5} \frac{\Gamma_j \Gamma_2}{z_{j2}} \right) \prec \epsilon^2,$$

which contradicts with Proposition 4.1. Hence, $\Gamma_1 + \Gamma_2 \neq 0$ and $w_{12} \approx \epsilon^{-2}$. By $\Lambda z_1 = \sum_{j=3}^{5} \frac{\Gamma_j}{w_{j1}} + \frac{\Gamma_2}{w_{21}}$ and $\sum_{j=3}^{5} \frac{\Gamma_j}{w_{j1}} \prec \epsilon^2$, it follows that

$$z_1 \sim -\frac{\Gamma_1\Gamma_2}{b\Lambda(\Gamma_1 + \Gamma_2)} \epsilon^2.$$

Similarly, by $w_1 = \Lambda \sum_{j=3}^{5} \frac{\Gamma_j}{z_{j1}} + \frac{\Gamma_2}{z_{21}}$ and $\sum_{j=3}^{5} \frac{\Gamma_j}{z_{j1}} \prec \epsilon^2$, it follows that

$$z_{12} \sim \frac{\Gamma_1 \Lambda}{b} \epsilon^2.$$

Since $\sum_{j=1}^{5} \Gamma_j z_j = 0$, we obtain

$$\sum_{j=3}^{5} \Gamma_j z_j = -(\Gamma_1 + \Gamma_2) z_1 - \Gamma_2 z_{12} = \begin{cases} \sim \left(\frac{1}{\Lambda} - \Lambda\right) \frac{\Gamma_1\Gamma_2}{b} \epsilon^2 & if\ \Lambda^2 \neq 1; \\ \prec \epsilon^2, & if\ \Lambda^2 = 1. \end{cases}$$

Note that $\sum_{j=3}^{5} \Gamma_j z_j = \Gamma_4 z_{34} + \Gamma_5 z_{35} = \Gamma_3 z_{43} + \Gamma_5 z_{45}$. Set $\frac{z_{34}}{\Gamma_5} \sim c\epsilon^2$.
Case 1. $\Lambda^2 = 1$: Then $\frac{z_{34}}{\Gamma_5} \sim \frac{z_{45}}{\Gamma_3} \sim \frac{z_{53}}{\Gamma_4} \sim c\epsilon^2$. Note that $\bar{\Lambda} w_3 \sim \frac{\Gamma_4}{z_{43}} + \frac{\Gamma_5}{z_{53}}$ and

$\bar{\Lambda} w_4 \sim \frac{\Gamma_3}{z_{34}} + \frac{\Gamma_5}{z_{54}}$. Then

$$b c \Gamma_4 \Gamma_5 + \Gamma_4^2 \Lambda - \Gamma_5^2 \Lambda = 0, b c \Gamma_3 \Gamma_5 + \Gamma_3^2 (-\Lambda) + \Gamma_5^2 \Lambda = 0.$$

Eliminating bc, we obtain $\frac{(\Gamma_3+\Gamma_4)(\Gamma_3\Gamma_4-\Gamma_5^2)}{\Gamma_4}\Lambda = 0$, which is a contradiction since $\Gamma_3 + \Gamma_4 = -\Gamma_5 \neq 0$ and $\Gamma_3\Gamma_4 - \Gamma_5^2 = L_{345} \neq 0$.
Case 2. $\Lambda^2 \neq 1$: Then

$$\frac{z_{45}}{\Gamma_3} \sim \left[c + \left(\frac{1}{\Lambda} - \Lambda\right) \frac{\Gamma_1\Gamma_2}{b\Gamma_3\Gamma_5}\right] \epsilon^2 \approx \epsilon^2, \ \frac{z_{35}}{\Gamma_4} \sim \left[-c + \left(\frac{1}{\Lambda} - \Lambda\right) \frac{\Gamma_1\Gamma_2}{b\Gamma_4\Gamma_5}\right] \epsilon^2 \approx \epsilon^2.$$

Note that $\bar{\Lambda} w_3 \sim \frac{\Gamma_4}{z_{43}} + \frac{\Gamma_5}{z_{53}}$ and $\bar{\Lambda} w_4 \sim \frac{\Gamma_3}{z_{34}} + \frac{\Gamma_5}{z_{54}}$. Then

$$\begin{aligned} & b^2c^2\Gamma_4\Gamma_5^2\Lambda + bc\Gamma_1\Gamma_2\Gamma_5\Lambda^2 - bc\Gamma_1\Gamma_2\Gamma_5 + bc\Gamma_4^2\Gamma_5\Lambda^2 - bc\Gamma_5^3\Lambda^2 \\ & \quad + \Gamma_1\Gamma_2\Gamma_4\Lambda^3 - \Gamma_1\Gamma_2\Gamma_4\Lambda = 0, \\ & b^2c^2\Gamma_3\Gamma_5^2\Lambda - bc\Gamma_1\Gamma_2\Gamma_5\Lambda^2 + bc\Gamma_1\Gamma_2\Gamma_5 - bc\Gamma_3^2\Gamma_5\Lambda^2 + bc\Gamma_5^3\Lambda^2 \\ & \quad + \Gamma_1\Gamma_2\Gamma_3\Lambda^3 - \Gamma_1\Gamma_2\Gamma_3\Lambda = 0. \end{aligned}$$

Eliminating bc, we obtain $\Lambda^2 = \frac{\Gamma_1\Gamma_2}{\Gamma_1\Gamma_2+\Gamma_3\Gamma_4-\Gamma_5^2}$. Hence, $\Lambda^2 \in \mathbb{R}$. It must be -1, and then $\Lambda = \pm\sqrt{-1}$, $L = 0$, which is a contradiction since

$$L = \Gamma_1\Gamma_2 + (\Gamma_1 + \Gamma_2)(\Gamma_3 + \Gamma_4 + \Gamma_5) + L_{345} = \Gamma_1\Gamma_2 + \Gamma_3\Gamma_4 - \Gamma_5^2.$$

8. Diagram 8: It is easy to see that $\Gamma_3 + \Gamma_4 + \Gamma_5 = 0$, $\Gamma_1 + \Gamma_2 = 0$. Note that $\frac{1}{z_{31}} - \frac{1}{z_{j1}}, \frac{1}{z_{32}} - \frac{1}{z_{j2}} \prec \epsilon^2$, $j \geq 3$, so

$$\begin{aligned} \bar{\Lambda}(\Gamma_1 w_1 + \Gamma_2 w_2) = \Gamma_1 \sum_{j=3}^{5} \Gamma_j \left(\frac{1}{z_{31}} + \frac{1}{z_{j1}} - \frac{1}{z_{31}}\right) \\ + \Gamma_2 \sum_{j=3}^{5} \Gamma_j \left(\frac{1}{z_{32}} + \frac{1}{z_{j2}} - \frac{1}{z_{32}}\right) \prec \epsilon^2. \end{aligned}$$

Hence

$$\epsilon^2 \succ \Gamma_1 w_1 + \Gamma_2 w_2 = (\Gamma_1 + \Gamma_2) w_1 + \Gamma_2 w_{12} = \Gamma_2 w_{12},$$

which contradicts with Proposition 4.1.

9. Diagram 9: By Considering the z- and w-diagrams, we obtain

$$\Gamma_1 + \Gamma_2 = 0, \Gamma_3 + \Gamma_4 = 0,$$

and

$$\Gamma_1 z_1 + \Gamma_2 z_2 = \Gamma_2 z_{12} \approx \epsilon^2, \Gamma_3 z_3 + \Gamma_4 z_4 = \Gamma_4 z_{34} \approx \epsilon^2.$$

By $\sum_{j=1}^{5} \Gamma_j z_j = 0$, we see $\Gamma_5 z_5 \preceq \epsilon^2$, which is a contradiction.

7.2 The remaining 22 diagrams

1. Diagram 1: $L_{345} = 0$. *This is the case of Roberts' example.*
2. Diagram 2: $L_{345} = L_{245} = 0$, or equivalently, $L_{345} = 0, \Gamma_2 = \Gamma_3$.
3. Diagram 3: $\Lambda = \pm 1$, $L_{345} = L_{125} = 0$.
   It is easy to see that $L_{345} = L_{125} = 0$. We claim that $\Lambda = \pm 1$. Otherwise, we have $L = 0$. Then

$$\begin{aligned} L &= L_{125} + \Gamma_3\Gamma_4 + (\Gamma_1 + \Gamma_2 + \Gamma_5)(\Gamma_3 + \Gamma_4) \\ &= L_{125} + L_{345} + (\Gamma_1 + \Gamma_2)(\Gamma_3 + \Gamma_4) = (\Gamma_1 + \Gamma_2)(\Gamma_3 + \Gamma_4) = 0. \end{aligned}$$

   Without loss of generality, assume that $\Gamma_1 + \Gamma_2 = 0$. Then $L_{125} = 0$ implies that $\Gamma_1\Gamma_2 = 0$, which is a contradiction. Hence, it holds that $\Lambda = \pm 1$.
4. Diagram 4: $L_{345} = L_{1245} = 0$.
5. Diagram 5: $L_{345} = L = 0$.
6. Diagram 6: $L_{345} = 0$.
7. Diagram 7: $L_{345} = 0$.
8. Diagram 8: $L_{345} = 0$, $[\Gamma_1(\Gamma_3 + \Gamma_4) - \Gamma_2\Gamma_5](\Gamma_1 + \Gamma_5)(\Gamma_2 + \Gamma_3 + \Gamma_4) = 0$.
   It is easy to see $L_{345} = 0$. Set $w_3 \sim w_4 \sim w_5 \sim a\epsilon^{-2}$. Then $w_1 \sim -\frac{\Gamma_5}{\Gamma_1} a\epsilon^{-2}$, and $w_2 \sim -\frac{\Gamma_3+\Gamma_4}{\Gamma_2} a\epsilon^{-2}$. If $w_{12} \prec \epsilon^{-2}$, then

$$\Gamma_1(\Gamma_3 + \Gamma_4) = \Gamma_2\Gamma_5.$$

   We now assume that $w_{12} \approx \epsilon^{-2}$. Since $z_{12} \succ \epsilon^2$, it holds that $z_1 \succ \epsilon^2$, or $z_2 \succ \epsilon^2$.
   Case I: $z_1 \succ \epsilon^2$. Then $\epsilon^2 \prec \Lambda z_1 = \sum_{j=3}^{5} \frac{\Gamma_j}{w_{j1}} + \frac{\Gamma_2}{w_{21}}$, and $\sum_{j=3}^{5} \frac{\Gamma_j}{w_{j1}} \succ \epsilon^2$. If $\Gamma_1 + \Gamma_5 \neq 0$, then $\frac{1}{w_{31}}, \frac{1}{w_{41}}, \frac{1}{w_{51}} \approx \epsilon^2$, which is a contradiction. Thus, we obtain

$$\Gamma_1 + \Gamma_5 = 0.$$

   Case II: $z_2 \succ \epsilon^2$. Then $\epsilon^2 \prec \Lambda z_2 = \sum_{j=3}^{5} \frac{\Gamma_j}{w_{j2}} + \frac{\Gamma_1}{w_{12}}$, and $\sum_{j=3}^{5} \frac{\Gamma_j}{w_{j2}} \succ \epsilon^2$. If $\Gamma_2 + \Gamma_3 + \Gamma_4 \neq 0$, then $\frac{1}{w_{32}}, \frac{1}{w_{42}}, \frac{1}{w_{52}} \approx \epsilon^2$, which is a contradiction. Thus, we obtain

$$\Gamma_2 + \Gamma_3 + \Gamma_4 = 0.$$

9. Diagram 9: $L_{345} = 0$.
10. Diagram 10: $L_{2345} = 0$.

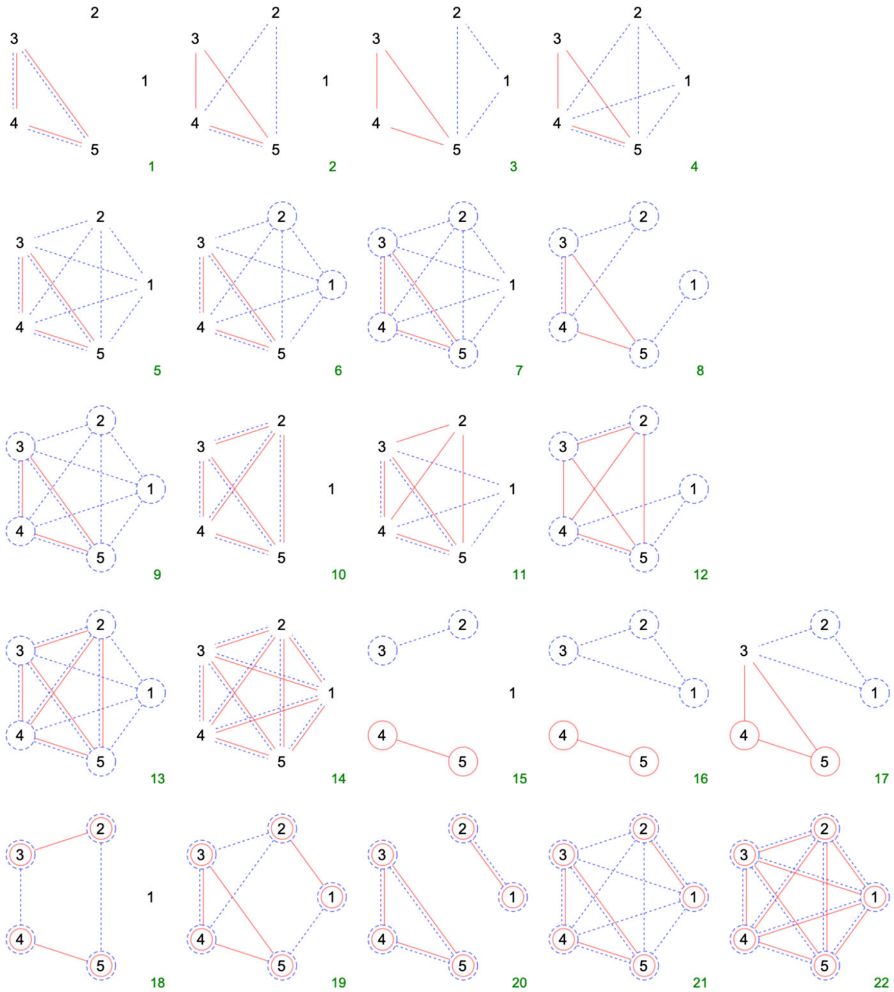


Fig. 16 The remaining 22 diagrams. They correspond to the first diagram of Fig. 5, the first diagram of Fig. 6, the first diagram of Fig. 7, the first diagram of Fig. 9, the third diagram of Fig. 11, the fourth diagram of Fig. 11, the first diagram of Fig. 11, the second diagram of Fig. 6, the second diagram of Fig. 11, the first diagram of Fig. 10, the diagram of Fig. 12, the fourth diagram of Fig. 9, the diagram of Fig. 13, the first diagram of Fig. 14, the first diagram of Fig. 2, the fourth diagram of Fig. 2, the third diagram of Fig. 7, the second diagram of Fig. 3, the third diagram of Fig. 6, the fifth diagram of Fig. 5, the sixth diagram of Fig. 11, and the second diagram of Fig. 14 respectively

11. Diagram 11: $L_{1345} = L_{2345} = 0$, or equivalently, $L_{1345} = 0$, $\Gamma_1 = \Gamma_2$.
12. Diagram 12: $L_{2345} = 0 = \Gamma_2 + \Gamma_3$.
13. Diagram 13: $L_{2345} = 0$.
14. Diagram 14: $L = 0$. It is easy to see that $r_{kl} \approx \epsilon^2$ for any $k \neq l, k, l \in \{1, 2, 3, 4, 5\}$.
15. Diagram 15. $\Lambda = \pm 1$, $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$, $\Gamma_1 + \Gamma_4 + \Gamma_5 = 0$.
    By Proposition 5.3, there are two cases:

(a) $\Lambda = \pm 1$, then $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$, and $\Gamma_1 + \Gamma_4 + \Gamma_5 = 0$.
(b) $\Lambda = \pm \mathbf{i}$, then $L = 0$, and $\Gamma_2\Gamma_3 = L_{145}$, $\Gamma_4\Gamma_5 = L_{123}$. Then it holds

$$\begin{aligned} L &= L_{145} + \Gamma_2\Gamma_3 + (\Gamma_1 + \Gamma_4 + \Gamma_5)(\Gamma_2 + \Gamma_3) \\ &= L_{145} + L_{123} + (\Gamma_4 + \Gamma_5)(\Gamma_2 + \Gamma_3) \\ &= \Gamma_2\Gamma_3 + \Gamma_4\Gamma_5 + (\Gamma_4 + \Gamma_5)(\Gamma_2 + \Gamma_3) = L_{2345} = 0. \end{aligned}$$

On the other hand, $L = \Gamma_1\Gamma_{2345} + L_{2345} = 0$ implies that $\Gamma_{2345} = 0$, which is a contradiction. Hence, this case is impossible.

16 Diagram 16: Consider the isolated component $\{4, 5\}$ of the z-digram. By Proposition 5.3, it holds that $\Gamma_4 + \Gamma_5 \neq 0$. There are two cases:

(a) $\Lambda = \pm 1$, then $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$.
(b) $\Lambda = \pm \mathbf{i}$, then $L = 0$ and $\Gamma_4\Gamma_5 = L_{123}$.

It is easy to see that

$$r_{14}, r_{15}, r_{24}, r_{25}, r_{34}, r_{35} \approx \epsilon^{-2}.$$

According to Proposition 5.3, z_{45} is a maximal stroke. *We claim that* w_{12}, w_{13}, w_{23} *are all maximal strokes*. Therefore, by Proposition 4.1,

$$r_{12}, r_{13}, r_{23}, r_{45} \approx 1.$$

Now, we prove the above claim, i.e., $w_1 \nsim w_2 \nsim w_3 \nsim w_1$. Since, $w_{41}, w_{51} \approx \epsilon^{-2}$, by Proposition 5.6, it can not happen that $w_1 \sim w_2 \sim w_3$. If exactly two of w_1, w_2, w_3 are w-close, without loss of generality, assume that $w_{12} \prec \epsilon^{-2}$, $w_{13} \approx \epsilon^{-2}$. Then

$$\Lambda z_1 = \frac{\Gamma_2}{w_{21}} + \sum_{j=3}^{5} \frac{\Gamma_j}{w_{j1}} \succ \epsilon^2, \ \Lambda z_3 = \sum_{j=1, j\neq 3}^{5} \frac{\Gamma_j}{w_{j3}} \preceq \epsilon^2.$$

Hence, $z_{13} \sim -z_1 \succ \epsilon^2$, which contradicts with Proposition 4.1. Hence, the claim is proved.

17. Diagram 17: $\Lambda = \pm \mathbf{i}$, $L = 0$ and $L_{345} = L_{123}$.
Consider the isolated component $\{3, 4, 5\}$ of the z-diagram and the component $\{1, 2, 3\}$ of the w-diagram. By Proposition 5.4, it holds that $\Gamma_4 + \Gamma_5 \neq 0$ and $\Gamma_1 + \Gamma_2 \neq 0$. There are two cases:

(a) $\Lambda = \pm 1$, then $\Gamma_1 + \Gamma_2 = 0$, which is a contradiction. Hence, this case is impossible.
(b) $\Lambda = \pm \mathbf{i}$, $L = 0$ and $L_{345} = L_{123}$.

It is easy to see that

$$r_{14}, r_{15}, r_{24}, r_{25} \approx \epsilon^{-2}.$$

According to Proposition 5.4, z_{45}, w_{12} are maximal strokes. It is easy to see that $z_{34}, z_{35}, w_{23}, w_{13}$ are all maximal stokes. By Proposition 4.1, it holds that

$$r_{12}, r_{13}, r_{23}, r_{34}, r_{35}, r_{45} \approx 1.$$

18. Diagram 18: $\Gamma_3\Gamma_5 = \Gamma_2\Gamma_4$. *This is the case of Roberts' example.*
Consider the isolated component z_{23}-edge. By Rule III, $\Gamma_2 z_2 + \Gamma_3 z_3 \prec \epsilon^{-2}$. Without loss of generality, assume $z_2 \sim -\Gamma_3 a\epsilon^{-2}$ and $z_3 \sim \Gamma_2 a\epsilon^{-2}$.Similarly, we assume $z_4 \sim -\Gamma_5 c\epsilon^{-2}$ and $z_5 \sim \Gamma_4 c\epsilon^{-2}$, where a and c are two non-zero complex constants. Since $z_{34} \approx z_{25} \approx \epsilon^2$ by the estimate of Proposition 4.1, we obtain $\Gamma_2 a = -\Gamma_5 b$, $-\Gamma_3 a = \Gamma_4 b$. Hence, we obtain

$$\Gamma_2\Gamma_4 = \Gamma_3\Gamma_5.$$

Apply similar argument to the isolated component w_{25}-edge. Then we have

$$z_2 \sim z_5 \sim -\Gamma_3 a\epsilon^{-2}, \quad z_3 \sim z_4 \sim \Gamma_2 a\epsilon^{-2},$$
$$w_2 \sim w_3 \sim -\Gamma_5 b\epsilon^{-2}, \quad w_4 \sim w_5 \sim \Gamma_2 b\epsilon^{-2},$$

where b is a non-zero complex constants. It follows that

$$\Gamma_2\Gamma_4 = \Gamma_3\Gamma_5.$$

If $\Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 \neq 0$, then $(\Gamma_5 + \Gamma_2)(\Gamma_2 + \Gamma_3)(\Gamma_3 + \Gamma_4)(\Gamma_4 + \Gamma_5) \neq 0$. Otherwise, without loss of generality, assume $\Gamma_2 + \Gamma_3 = 0$, it follows that $\Gamma_4 + \Gamma_5 = 0$, which contradicts with $\Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 \neq 0$.
In this case, it is easy to see that

$$r_{23}^2 \sim \frac{\Gamma_2 + \Gamma_3}{-\Lambda}, r_{34}^2 \sim \frac{\Gamma_3 + \Gamma_4}{-\overline{\Lambda}}, r_{45}^2 \sim \frac{\Gamma_4 + \Gamma_5}{-\Lambda}, r_{25}^2 \sim \frac{\Gamma_2 + \Gamma_5}{-\overline{\Lambda}};$$
$$r_{24}^2 \sim (\Gamma_2 + \Gamma_3)(\Gamma_2 + \Gamma_5)ab\epsilon^{-4}, r_{35}^2 \sim -(\Gamma_2 + \Gamma_3)(\Gamma_2 + \Gamma_5)ab\epsilon^{-4}.;$$
$$r_{12}^2 \sim \Gamma_3\Gamma_5 ab\epsilon^{-4}, r_{13}^2 \sim -\Gamma_2\Gamma_5 ab\epsilon^{-4}, r_{14}^2 \sim \Gamma_2\Gamma_2 ab\epsilon^{-4}, r_{15}^2 \sim -\Gamma_2\Gamma_3 ab\epsilon^{-4}.$$

19. Diagram 19: $\Gamma_1(\Gamma_3 + \Gamma_4) = \Gamma_2\Gamma_5$. Moreover, it holds that either $|\Gamma_1 + \Gamma_2| + |\Gamma_1 + \Gamma_5| = 0$ or $(\Gamma_1 + \Gamma_2)(\Gamma_1 + \Gamma_5) \neq 0$.
Consider the isolated component z_{12}-edge. By Rule III, $\Gamma_1 z_1 + \Gamma_2 z_2 \prec \epsilon^{-2}$. Without loss of generality, assume $z_1 \sim -\Gamma_2 a\epsilon^{-2}$ and $z_2 \sim \Gamma_1 a\epsilon^{-2}$. Similarly, we assume $w_1 \sim -\Gamma_5 b\epsilon^{-2}$ and $w_5 \sim \Gamma_1 b\epsilon^{-2}$, where a and b are two non-zero complex constants. Then

$$z_1 \sim z_5 \sim -\Gamma_2 a\epsilon^{-2}, \quad z_2 \sim z_3 \sim z_4 \sim \Gamma_1 a\epsilon^{-2},$$
$$w_1 \sim w_2 \sim -\Gamma_5 b\epsilon^{-2}, \; w_3 \sim w_4 \sim w_5 \sim \Gamma_1 b\epsilon^{-2}.$$

Since $\sum_{j=1}^{5} \Gamma_j z_j = 0$, we obtain $\Gamma_1(\Gamma_3 + \Gamma_4) = \Gamma_2\Gamma_5$. Note that if $\Gamma_1 + \Gamma_2 = 0$, then $\Gamma_3 + \Gamma_4 + \Gamma_5 = 0$; if $\Gamma_1 + \Gamma_5 = 0$, then $\Gamma_2 + \Gamma_3 + \Gamma_4 = 0$.

(a) If it holds that $\Gamma_1+\Gamma_2=0$ and $\Gamma_1+\Gamma_5=0$, i.e., $\Gamma_1=\Gamma_3+\Gamma_4=-\Gamma_2=-\Gamma_5$. It holds that $z_j\sim\Gamma_1 a\epsilon^{-2}$, $w_j\sim\Gamma_1 b\epsilon^{-2}$, $z_{jk}\prec\epsilon^{-2}$, $w_{jk}\prec\epsilon^{-2}$ for any $j,k\in\{1,2,3,4,5\}$. Thus,

$$\epsilon^4\prec r_{12},r_{15},r_{23},r_{24},r_{35},r_{45}\prec 1,\quad r_{34}\approx\epsilon^2,$$
$$\epsilon^4\prec r_{14}^2=z_{14}w_{14}\approx z_{45}w_{45}\prec\epsilon^{-4},\quad \epsilon^4\prec r_{13},r_{25}\prec\epsilon^{-4}.$$

(b) If it holds that $\Gamma_1+\Gamma_2\neq 0$ and $\Gamma_1+\Gamma_5\neq 0$. It holds that $z_1\sim z_5\sim -\Gamma_2 a\epsilon^{-2}$, $z_2\sim z_3\sim z_4\sim\Gamma_1 a\epsilon^{-2}$, $w_1\sim w_2\sim-\Gamma_5 b\epsilon^{-2}$, $w_3\sim w_4\sim w_5\sim\Gamma_1 b\epsilon^{-2}$. Thus,

$$r_{34}^2\approx\epsilon^4, r_{12},r_{35},r_{45}\approx 1,$$
$$r_{15},r_{23},r_{24}\approx 1, r_{14}^2,r_{13}^2,r_{25}^2\approx\epsilon^{-4}.$$

(c) If it holds that $\Gamma_1+\Gamma_2=0$ and $\Gamma_1+\Gamma_5\neq 0$. It holds that $z_j\sim\Gamma_1 a\epsilon^{-2}$, $z_{jk}\prec\epsilon^{-2}$ for any $j,k\in\{1,2,3,4,5\}$, $w_1\sim w_2\sim-\Gamma_5 b\epsilon^{-2}$, $w_3\sim w_4\sim w_5\sim\Gamma_1 b\epsilon^{-2}$. Note that

$$\Lambda\sum_{j=3}^5\Gamma_j z_j=\sum_{j=3}^5\frac{\Gamma_1\Gamma_j}{w_{1j}}+\sum_{j=3}^5\frac{\Gamma_2\Gamma_j}{w_{2j}}=\sum_{j=3}^5\frac{\Gamma_2\Gamma_j w_{12}}{w_{1j}w_{2j}}\prec\epsilon^2$$

and $\sum_{j=3}^5\Gamma_j z_j=\sum_{j=3}^5\Gamma_j z_3+\Gamma_4 z_{34}+\Gamma_5 z_{35},\ z_{34}\approx\epsilon^2$. It holds that

$$z_{35}\approx\epsilon^2,$$

which is a contradiction. Hence, this case is impossible.

20. Diagram 20: $\Gamma_1+\Gamma_2=0$, $\Gamma_3+\Gamma_4+\Gamma_5=0$.
21. Diagram 21: $\Gamma_1+\Gamma_2=0$, $\Gamma_3+\Gamma_4+\Gamma_5=0$.
22. Diagram 22: $\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5=0$. It is easy to see that $r_{kl}\approx\epsilon^2$ for any $k\neq l, k,l\in\{1,2,3,4,5\}$.

8 Proofs of the main results

Proof of Theorem 1.1 Suppose that there are infinitely many solutions of system (13) in the complex domain. At least one of the squared distances r_{kl}^2, say r_{12}^2, must take infinitely many values. By Lemma 4.1, $r_{12}^2=z_{12}w_{12}$ is dominating. There exists a sequence of complex normalized central configurations such that $r_{12}^2\to 0$. Then $z_{12}^{(n)}w_{12}^{(n)}\to 0$, and either $\mathcal{Z}$ or $\mathcal{W}$ is unbounded along this sequence. We extract a singular sequence, which must correspond to one of the 22 diagrams in Fig. 16. Consequently, some explicit linear or quadratic relations on the five vorticities must be satisfied.

Consider first the case of relative equilibria. We can further exclude Diagram 17. Recall that r_{12}^2 is dominating. By pushing it to ∞, we cannot be in Diagram 14 or 22. Thus, there are finitely many relative equilibria unless at least one of the following fourteen polynomial systems holds. (Recall that $L_{j_1,\ldots,j_n} = \sum_{1\le k<l\le n} \Gamma_{j_k}\Gamma_{j_l}$),

$$\begin{aligned}
&L_{345} = 0;\\
&L_{345} = 0, \Gamma_2 = \Gamma_3;\\
&L_{345} = L_{125} = 0;\\
&L_{345} = L_{1245} = 0;\\
&L_{345} = L = 0;\\
&L_{345} = 0, [\Gamma_1(\Gamma_3+\Gamma_4) - \Gamma_2\Gamma_5](\Gamma_1+\Gamma_5)(\Gamma_2+\Gamma_3+\Gamma_4) = 0;\\
&L_{2345} = 0;\\
&L_{1345} = 0, \Gamma_1 - \Gamma_2 = 0;\\
&L_{2345} = 0, \Gamma_2 + \Gamma_3 = 0;\\
&\Gamma_1+\Gamma_2+\Gamma_3 = 0, \Gamma_1+\Gamma_4+\Gamma_5 = 0;\\
&\Gamma_1+\Gamma_2+\Gamma_3 = 0;\\
&\Gamma_3\Gamma_5 = \Gamma_2\Gamma_4;\\
&\Gamma_1(\Gamma_3+\Gamma_4) = \Gamma_2\Gamma_5;\\
&\Gamma_1+\Gamma_2 = 0, \Gamma_3+\Gamma_4+\Gamma_5 = 0.
\end{aligned} \tag{18}$$

Now consider the case of collapse configurations. In this case, we have an additional constraint $L = 0$. It is straightforward to see that we can further exclude 10 diagrams, namely, Diagram 3, 4, 10, 11, 12, 13, 15, 20, 21, and 22. Hence, for collapse configurations, we have only 12 remaining diagrams: Diagram 1, 2, 5, 6, 7, 8, 9, 14, 16, 17, 18, and 19. Recall that r_{12}^2 is dominating. By pushing it to ∞, we cannot be in Diagram 14. Therefore, there are finitely many collapse configurations unless at least one of the following seven polynomial systems holds,

$$\begin{aligned}
&L = 0, L_{345} = 0;\\
&L = 0, L_{345} = 0, \Gamma_2 = \Gamma_3;\\
&L = 0, L_{345} = 0, [\Gamma_1(\Gamma_3+\Gamma_4) - \Gamma_2\Gamma_5](\Gamma_1+\Gamma_5)(\Gamma_2+\Gamma_3+\Gamma_4) = 0;\\
&L = 0, L_{123} = \Gamma_4\Gamma_5;\\
&L = 0, L_{345} = L_{123};\\
&L = 0, \Gamma_3\Gamma_5 = \Gamma_2\Gamma_4;\\
&L = 0, \Gamma_1(\Gamma_3+\Gamma_4) = \Gamma_2\Gamma_5.
\end{aligned} \tag{19}$$

□

To further improve the above result, we first recall some simple tricks to estimate the distances between vorticities when a singular sequence approaches one of the diagrams in Fig. 16.

By Proposition 4.1, it is easy to see the following: A zw-edge corresponds to a distance $r_{kl} \approx \epsilon^2$. The other cases, no edge or a simple edge, i.e., a z-edge or a w-edge, correspond to a distance $r_{kl} \succ \epsilon^2$. A maximal simple edge corresponds to a distance $r_{kl} \approx 1$. A simple edge is maximal if there is exactly one of its ends is circled in one of z- and w-diagram. The non-maximal simple edges correspond to distances r_{kl} such that $\epsilon^2 \prec r_{kl} \prec 1$. For distances r_{kl} without edge, it holds $\epsilon^2 \prec r_{kl} \preceq \epsilon^{-2}$. For a distance r_{kl} without edge that the two ends are not equally circled, it holds that $1 \prec r_{kl}$, i.e., unbounded.

We will avoid further case-by-case analysis and use only these simple estimates, along with those stated in Sect. 7. We call a 4-product a quantity $p_{ij} = r_{ij}^2 r_{kl}^2 r_{lm}^2 r_{mk}^2$, where i, j, k, l, m are all the indices from 1 to 5 .

Proposition 8.1 *In Diagram 5, 6, 7, 9, 10, 13 and 14, any 4-product is bounded.*

Proof Note that all distances in Diagram 5, 6, 7, 9, 13 and 14 are bounded. For Diagram 10, we arrive the estimate by Proposition 4.1. □

In the following proof, we assume that $\sum_{j=2}^{5} \Gamma_j \neq 0$ in Diagram 18, otherwise its constraint is a codimension 2 set. Similarly, we assume that it holds in Diagram 19 that $\Gamma_1 + \Gamma_2 \neq 0$ and $\Gamma_1 + \Gamma_5 \neq 0$. Hence, in Diagram 18 and 19, a simple edge corresponds to a distance $r_{kl} \approx 1$, and no edge corresponds to a distance $r_{kl} \approx \epsilon^{-2}$.

Proof of Theorem 1.2 *Part 1:* For relative equilibria, repeating the first paragraph of the proof of Theorem 1.1 shows that there is a singular sequence of complex normalized central configurations approaching one of the 21 diagrams (the 22 diagrams minus Diagram 17). Since each of Diagram 2, 3, 4, 5, 8, 11, 12, 15, 20, and 21 have two independent constraints, which defines a codimension 2 set, we put them, and all the similar sets obtained by renumbering the five vorticities, in the exceptional set $\mathcal{A}$. We assume that the singular sequence approaches one of Diagram 1, 6, 7, 9, 10, 13, 14, 16, 18, 19 and 22.

Suppose that it is Diagram 1. We number the vertices as in Fig. 16. Then $r_{34}^2 \approx \epsilon^4$, and is a dominating function. Let $r_{34}^2 \to \infty$. Since all distances in Diagram 6, 7, 9, 13, 14 and 22 are bounded, we are in either Diagram 1 or in Diagram 10, 16, 18, and 19. If we are in Diagram 1, it must be renumbered such that r_{34} is not in the fully edged triangle, so there is a new independent constraint on the five vorticities. In the remaining cases, there is always a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{A}$, and Diagram 1 is now excluded.

Suppose that it is one of Diagram 6, 7, 9, 10 and 13. We number the vertices as in Fig. 16. Then the 4-product $p_{12} \to 0$, and is a dominating function. Let $p_{12} \to \infty$. By Proposition 8.1, we are in one of Diagram 16, 18, and 19. In either cases, there is always a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{A}$, and Diagram 6, 7, 9, 10 and 13 are now excluded.

Suppose that it is one of Diagram 14 and 22. Then $r_{34}^2 \approx \epsilon^4$, and is a dominating function. Let $r_{34}^2 \to \infty$. By the estimates from Sect. 7, we are in one of Diagram 16, 18, and 19. In either cases, there is also a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{A}$, and Diagram 14 and 22 are now excluded.

Suppose that it is one of Diagram 16 and 18. We number the vertices as in Fig. 16. Then $r_{14}^2 \approx \epsilon^{-4}$, and is a dominating function. Let $r_{14}^2 \to 0$. We are in Diagram 19. There is a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{A}$, and Diagram 16 and 18 are now excluded.

Suppose that it is Diagram 19. We number the vertices as in Fig. 16. Then $r_{34}^2 \to 0$, and is a dominating function. Let $r_{34}^2 \to \infty$. We are in Diagram 19, but it must be renumbered such that r_{34} is not a zw-edge, so there is a new independent constraint on the five vorticities. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{A}$. This concludes the construction of $\mathcal{A}$. The last possibility for a singular sequence is now forbidden. There is no continuum of relative equilibria if the vorticities do not belong to $\mathcal{A}$.

Part 2: For collapse configurations, the vorticity space is now $\{(\Gamma_1, \ldots, \Gamma_5) : \Gamma_i \in \mathbb{R}^*, i = 1, \ldots, 5, \sum_{1 \leq i < j \leq 5} \Gamma_i \Gamma_j = 0\}$. Repeating the first paragraph of the proof of Theorem 1.1 shows that there is a singular sequence of complex normalized central configurations approaching one of the 12 diagrams (Diagram 1, 2, 5–9, 14, 16–19). Since each of Diagram 2 and 8 have two independent constraints, which defines a codimension 2 set, we put them, and all the similar sets obtained by renumbering the five vorticities, in the exceptional set $\mathcal{B}$. We assume that the singular sequence approaches one of Diagram 1, 5, 6, 7, 9, 14, 16, 17, 18 and 19.

Suppose that it is Diagram 1. We number the vertices as in Fig. 16. Then $r_{34}^2 \approx \epsilon^4$, and is a dominating function. Let $r_{34}^2 \to \infty$. Since all distances in Diagram 5, 6, 7, 9 and 14 are bounded, we are in either Diagram 1 or in Diagram 16, 17, 18, and 19. If we are in Diagram 1, it must be renumbered such that r_{34} is not in the fully edged triangle, so there is a new independent constraint on the five vorticities. In the remaining cases, there is always a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{B}$, and Diagram 1 is now excluded.

Suppose that it is one of Diagram 5, 6, 7, 9. We number the vertices as in Fig. 16. Then the 4-product $p_{12} \to 0$, and is a dominating function. Let $p_{12} \to \infty$. By Proposition 8.1, we are in one of Diagram 16, 17, 18, and 19. In either cases, there is always a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{B}$, and Diagram 5, 6, 7, and 9 are now excluded.

Suppose that it is Diagram 19. We number the vertices as in Fig. 16. Then $r_{34}^2 \approx \epsilon^4$, and is a dominating function. Let $r_{34}^2 \to \infty$. By the estimates from Sect. 7, we are in one of Diagram 16, 17, 18, 19. If we are in Diagram 19, it must be renumbered such that r_{34} is not a zw-edge, so there is a new independent constraint on the five vorticities. In the remaining cases, there is always a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{B}$, and Diagram 19 is now excluded.

Suppose that it is Diagram 16. We number the vertices as in Fig. 16. Then $r_{14}^2 \approx \epsilon^{-4}$, and is a dominating function. Let $r_{14}^2 \to 0$. We are in Diagram 14, where $r_{12}^2 r_{13}^2 r_{23}^2 \approx \epsilon^{12}$, and is a dominating function. Let $r_{12}^2 r_{13}^2 r_{23}^2 \to \infty$, then we are in one of Diagram 16, 17, 18. If we are in Diagram 16, it must be renumbered such that $\{1, 2, 3\}$ is no longer one isolated component of the w-diagram, so there is a new independent constraint on the five vorticities. In the remaining cases, there is always a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{B}$, and Diagram 16 is now excluded.

Suppose that it is Diagram 17. We number the vertices as in Fig. 16. Then $r_{14}^2 \approx \epsilon^{-4}$, and is a dominating function. Let $r_{14}^2 \to 0$. We are in Diagram 14, where $r_{12}^2 r_{13}^2 r_{23}^2 \approx \epsilon^{12}$, and is a dominating function. Let $r_{12}^2 r_{13}^2 r_{23}^2 \to \infty$, then we are in one of Diagram 17, 18. If we are in Diagram 17, it must be renumbered such that $\{1, 2, 3\}$ is no longer one isolated component of the w-diagram, so there is a new independent constraint on the five vorticities. If it is Diagram 18, there is a new independent constraint. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{B}$, and Diagram 17 is now excluded.

Suppose that it is Diagram 18. We number the vertices as in Fig. 16. Then $r_{14}^2 \approx \epsilon^{-4}$, and is a dominating function. Let $r_{14}^2 \to 0$. We are in Diagram 14, where $r_{23}^2 r_{34}^2 r_{45}^2 r_{25}^2 \approx \epsilon^{16}$, and is a dominating function. Let $r_{23}^2 r_{34}^2 r_{45}^2 r_{25}^2 \to \infty$, then we are in one copy of Diagram 18. It must be renumbered such that one of the vertices $\{2, 3, 4, 5\}$ is no longer in the quadrilateral formed by the four simple edges, so there is a new independent constraint on the five vorticities. We add the corresponding codimension 2 set to the exceptional set $\mathcal{B}$, and Diagram 18 is now excluded.

Suppose that it is Diagram 14. We number the vertices as in Fig. 16. Then $r_{14}^2 \approx \epsilon^4$, and is a dominating function. Let $r_{14}^2 \to \infty$, which is impossible. This concludes the construction of $\mathcal{B}$. The last possibility for a singular sequence is now forbidden. There is no continuum of collapse configurations if the vorticities do not belong to $\mathcal{B}$. □

We also confirm the finiteness under the following restrictions of Theorem 1.3, namely,

$$\sum_{j\in J} \Gamma_j \neq 0, \quad \sum_{j,k\in J, j\neq k} \Gamma_j \Gamma_k \neq 0, \text{ for any nonempty subset } J \text{ of } \{1, 2, 3, 4, 5\}.$$

Under the above restriction, a singular sequence can only approach Diagram 18 and 19.

Proof of Theorem 1.3 Suppose that there are infinitely many solutions of system (13) in the complex domain. The argument in the proof of Theorem 1.1 implies that there is a singular sequence corresponding to Diagram 18 or 19. In either case, the polynomial $\prod_{j\neq k} r_{jk}^2$ approach ∞, which follows easily from the estimation of the distances of Sect. 7. Then it is a dominating function. Push the polynomial to zero and extract a singular sequence. However, the singular sequence would correspond to none of the two diagrams. This is a contradiction. □

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Declarations

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