MEMOIRS
of the
American Mathematical Society
ISSN 0065-9266 (print)
ISSN 1947-6221 (online)
Number 1532

On the Problem of Infinite Spin in Total Collisions of the Planar N-Body Problem

Xiang Yu

December 2024 • Volume 304 • Number 1532 (fifth of 7 numbers)

AMERICAN MATHEMATICAL SOCIETY

MEMOIRS
of the
American Mathematical Society

ISSN 0065-9266 (print)
ISSN 1947-6221 (online)

Number 1532

On the Problem of Infinite Spin in Total Collisions of the Planar N-Body Problem

Xiang Yu

December 2024 • Volume 304 • Number 1532 (fifth of 7 numbers)

AMS
American Mathematical Society

Library of Congress Cataloging-in-Publication Data

Cataloging-in-Publication Data has been applied for by the AMS.
See http://www.loc.gov/publish/cip/.
DOI: https://doi.org/10.1090/memo/1532

Memoirs of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics.

Subscription information. Beginning in 2024, Memoirs will be published monthly through 2026. Memoirs is also accessible from www.ams.org/journals. The 2024 subscription begins with volume 293 and consists of twelve mailings, each containing one or more numbers. Individual subscription prices for 2024 are as follows. For electronic only: US$1013. For paper delivery: US$1,163. Add US$22 for delivery within the United States; US$150 for surface delivery outside the United States. Upon request, subscribers to paper delivery of this journal are also entitled to receive electronic delivery. For information on institutional pricing, please visit `https://www.ams.org/publications/journals/subscriberinfo`. Subscription renewals are subject to late fees. See `www.ams.org/journal-faq` for more journal subscription information. Each number may be ordered separately; please specify number when ordering an individual number.

Back number information. For back issues see `www.ams.org/backvols`.

Subscriptions and orders should be addressed to the American Mathematical Society, P. O. Box 845904, Boston, MA 02284-5904 USA. *All orders must be accompanied by payment.* Other correspondence should be addressed to 201 Charles Street, Providence, RI 02904-2213 USA.

Memoirs of the American Mathematical Society (ISSN 0065-9266 (print); 1947-6221 (online)) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213 USA. Periodicals postage paid at Providence, RI. Postmaster: Send address changes to Memoirs, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2213 USA.

This publication is indexed in *Mathematical Reviews*®, *Zentralblatt MATH*, *Science Citation Index*®, *Science Citation Index*™*-Expanded*, *ISI Alerting Services*℠, *SciSearch*®, *Research Alert*®, *CompuMath Citation Index*®, *Current Contents*®*/Physical, Chemical & Earth Sciences.*
This publication is archived in *Portico* and *CLOCKSS.*
Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.
Visit the AMS home page at `https://www.ams.org/`

10 9 8 7 6 5 4 3 2 1   29 28 27 26 25 24

Contents

Abstract

For the planar N-body problem, we introduce a class of moving coordinates suitable for orbits near central configurations, especially for total collision orbits, which is the main new ingredient of this paper. The moving coordinates allow us to reduce the degeneracy of the N-Body problem from its intrinsic symmetrical characteristic. First, we give a full answer to the infinite spin or *Painlevé-Wintner* problem in the case corresponding to nondegenerate central configurations. Then following some original ideas of C.L. Siegel, especially the idea of normal forms, and applying the theory of central manifolds, we give a partial answer to the problem in the case corresponding to degenerate central configurations. We completely answer the problem in the case corresponding to central configurations with degree of degeneracy one. Combining some results on the planar nonhyperbolic equilibrium point, we give a criterion in the case corresponding to central configurations with degree of degeneracy two. We further answer the problem in the case corresponding to all known central configurations of four bodies. Therefore, we solve the problem for almost every choice of the masses of the four-body problem. Finally, we give a measure of the set of initial conditions leading to total collisions.

Received by the editor June 16, 2021, and, in revised form, July 14, 2021, October 28, 2021, February 18, 2022, June 20, 2022, and July 26, 2022.

Article electronically published on December 10, 2024.

DOI: https://doi.org/10.1090/memo/1532

2020 *Mathematics Subject Classification.* Primary 70F10, 70F16, 34A34, 70F15.

Key words and phrases. N-body problem, collisions, infinite spin, central configurations, *Painlevé-Wintner* problem, moving frame, normal forms, central manifolds.

I wish to express my gratitude to Shanzhen Chen, Rick Moeckel, Shiqing Zhang and Shuqiang Zhu for their very helpful discussions. I am also deeply indebted to the reviewer for carefully reading the original manuscript and suggesting very helpful improvements with paying enormous energy.

I would like to acknowledge the support from NSFC (No.11701464).

CHAPTER 1

Introduction

We consider N particles with positive masses moving in an Euclidean plane $\mathbb{R}^2$ interacting under the law of universal gravitation. Let the k-th particle have mass m_k and position $\mathbf{r}_k \in \mathbb{R}^2$ $(k = 1, 2, \cdots, N)$, then the equations of motion of the N-body problem are written as

$$m_k \ddot{\mathbf{r}}_k = \sum_{1 \leq j \leq N, j \neq k} \frac{m_k m_j (\mathbf{r}_j - \mathbf{r}_k)}{|\mathbf{r}_j - \mathbf{r}_k|^3}, \qquad k = 1, 2, \cdots, N. \tag{1.1}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$.

In the study of the N-body problem, collision singularities have shown to be the main difficulties, and therefore they are among the center of interest.

It is relatively simple to understand binary collisions of the N-body problem. Indeed, for the Newtonian two-body problem, one can change the variables and rescale time so that a binary collision solution transforms to a regular solution of equation of motions. Such a transformation is called a regularization of the binary collision. The solution can then be extended through the singularity. Sundman [**25**] showed that binary collisions can also be regularized in the three-body problem. That is, one can transform the variables in such a way that the solution can be continued through the binary collision as an analytic function of a new time variable. This is also true for several binary collisions occurring simultaneously in the N-body problem [**24**].

Collisions involving more than two particles are more complicated, only some partial results are known. Consider the *normalized configuration* of the particles to be the configuration divided by a norm which corresponds physically to the square root of the moment of inertia. Sundman [**25**] showed that, for triple collision in the three-body problem, the normalized configuration approaches the set of central configurations (cf. [**27**] and Section 2 and Section 3 below). Wintner [**27**] observed that Sundman's techniques can be used to show that the normalized configurations of solutions ending in total collision in the N-body problem also approach the set of central configurations. This is also true for general collision singularity of the N-body problem in which several clusters of particles collapse simultaneously [**24**].

It is natural to ask whether this implies that the normalized configuration of the particles must approach a certain central configuration, or, may the normalized configuration of the particles make an infinite number of revolutions before arriving at a collision. This is a long standing open problem on the collision singularity of the Newtonian N-body problem. This problem was posed by Painlevé and discussed by Wintner [**27**, p.283] in the total collision of the N-body problem. So this problem is usually called the *Painlevé-Wintner* problem or the problem of infinite spin. For simplicity, the abbreviation "*PISPW*" will be used to mean "the problem of infinite spin or *Painlevé-Wintner*" in this paper.

Although there has been tremendous interest in the problem, so far, only few progress has been made. Indeed, ones knew that $PISPW$ could be solved in the case corresponding to nondegenerate central configurations for a long time [**7**, **19**, **22**]; for example, one of the ideas is to apply the theory of normally hyperbolic invariant manifolds, and there are several papers mentioned this [**19**, etc]. On the other hand, because little is known about central configurations for $N > 3$ [**2**; **8**; **18**; **23**, etc], especially on the degeneracy of central configurations, $PISPW$ is completely solved only for the three-body problem.

All in all, though several claims of even stronger results were published [**19**, etc], however, as pointed out in [**1**], the proofs in the corresponding papers could not be found up till now. Thus recently Chenciner and Venturelli [**1**] asked for a solution to $PISPW$ even in the basic case: *in the total collision of the planar N-body problem.*

The main goal of this article is to study $PISPW$ in the total collision of the planar N-body problem. To this end, we introduce a class of moving coordinates suitable for describing orbits near central configurations, especially for total collision orbits.

In the moving coordinates, the degeneracy of the equations of motion from intrinsic symmetrical characteristic of N-body problem can be easily reduced. As a result, $PISPW$ can be well described. In fact, once the moving coordinates are successfully set, one can describe the motion of collision orbit effectively, and give the equations of motion in a form suitable for $PISPW$.

As by-products, the moving coordinates are found useful in investigating other questions of the planar N-body problem. Indeed, in addition to $PISPW$, we have found that the moving coordinates are also useful in investigating the stability of relative equilibrium solutions, degenerate central configurations and periodic orbits of the planar N-body problem so far.

It is shown that orbits starting at total collision belong to unstable manifolds of the origin with regard to a subsystem of equations. Unfortunately, results on stable manifolds and unstable manifolds cannot be applied to $PISPW$ directly. However we find that some original ideas of Siegel [**22**] are applicable. The ideas are related to normal forms, which is especially important for us. Since the original results on normal forms in [**22**] can only be applied to the case corresponding to nondegenerate central configurations, it is necessary to generalize the results of normal forms in [**22**] to the case corresponding to degenerate central configurations. Thus the theory of central manifolds is also introduced to explore the case corresponding to degenerate central configurations.

First we give a full answer to $PISPW$ in the case corresponding to nondegenerate central configurations: the normalized configuration of the particles must approach a certain central configuration without undergoing infinite spin for total collision orbits. This result is an immediate application of the theory of hyperbolic dynamics, or equivalently, the theory of normal forms. Therefore, as a separate method, we give a new rigorous and simple proof of the above result in this paper.

However, $PISPW$ in the case corresponding to degenerate central configurations, is unexpectedly difficult.

Therefore, in the paper, we mainly study $PISPW$ corresponding to central configurations with degree of degeneracy two or less. We completely solve the problem in the case corresponding to central configurations with degree of degeneracy zero or one: the configuration of the particles must approach a certain central configuration without undergoing infinite spin. Combining some results on the planar nonhyperbolic equilibrium point, we give a criterion for the case corresponding to central configurations with degree of degeneracy two.

Because it has been shown that for the four-body problem the exceptional masses corresponding to degenerate central configurations form a proper algebraic subset of the mass space [**16**], we conclude that, for almost every choice of the masses of the four-body problem, the configuration of the particles must approach a certain central configuration without undergoing infinite spin for collision orbits. Furthermore, based upon our investigation of a kind of symmetrical degenerate central configurations of four bodies, we answer $PISPW$ in the case corresponding to all known central configurations of four bodies.

After $PISPW$ is investigated, we naturally study the manifold of all the collision orbits (i.e., the set of initial conditions leading to total collisions). We show that this set is a finite union of real submanifold in the neighbourhood of the collision instant, and the dimensions of the submanifolds depend upon the index of the limiting central configuration (i.e., the number of positive eigenvalues of the limiting central configuration).

Finally, we examine the question of whether orbits can be extended through total collision from the viewpoint of Sundman and Siegel, that is, whether a single solution can be extended as an analytic function of time. We only consider the case corresponding to nondegenerate central configurations.

The paper is structured as follows. In Chapter 2, we introduce some notations, and some preliminary results of central configurations; in particular, we introduce the moving coordinates. In Chapter 3, it will be seen that collision orbits are well described in the moving coordinates. In particular, $PISPW$ can be expressed in a form suitable for deeper investigation. In Chapter 4, we investigate $PISPW$. In Chapter 5, we investigate the set of initial conditions leading to total collisions locally and examines the question whether a single solution can be extended as an analytic function of time. In Chapter 6, we summarize the main results and give some interesting open questions. Finally, in Appendix A, we give a criterion for the degeneracy of central configurations by using the cartesian coordinates; in Appendix B, we investigate degenerate central configurations of the planar four-body problem with an axis of symmetry; in Appendix C, we show how to diagonalize the linear part of the equations of motion in detail; in Appendix D, we give the theory of normal forms (or reduction theorems) to simplify equations of the problem in this paper; in Appendix E, we discuss some aspects of planar equilibrium points.

CHAPTER 2

Preliminaries

In this chapter we fix notations and give some definitions, in particular, we will introduce moving coordinates to describe motions near central configurations.

2.1. Central Configurations

Let $(\mathbb{R}^2)^N$ denote the space of configurations for N point particles in the Euclidean plane $\mathbb{R}^2$: $(\mathbb{R}^2)^N = \{\mathbf{r} = (\mathbf{r}_1, \cdots, \mathbf{r}_N) | \mathbf{r}_j \in \mathbb{R}^2, j = 1, \cdots, N\}$. In this paper, unless otherwise specified, the cartesian space $(\mathbb{R}^2)^N$ is considered as a column space. In particular, when necessary, one may identify $\mathbb{R}^2$ with $\mathbb{C}$ and $(\mathbb{R}^2)^N$ with $\mathbb{C}^N$ and so on.

For each pair of indices $j, k \in \{1, \ldots, N\}$, let $\Delta_{(j,k)} = \{\mathbf{r} \in (\mathbb{R}^2)^N | \mathbf{r}_j = \mathbf{r}_k\}$ denote the collision set of the j-th and k-th particles. Let $\Delta = \bigcup_{j,k} \Delta_{(j,k)}$ be the collision set in $(\mathbb{R}^2)^N$. Then $(\mathbb{R}^2)^N \backslash \Delta$ is the space of collision-free configurations.

The *mass scalar product* in the space $(\mathbb{R}^2)^N$ is defined as:

$$\langle \mathbf{r}, \mathbf{s} \rangle = \sum_{j=1}^{N} m_j (\mathbf{r}_j, \mathbf{s}_j),$$

where $\mathbf{r} = (\mathbf{r}_1, \cdots, \mathbf{r}_N)$ and $\mathbf{s} = (\mathbf{s}_1, \cdots, \mathbf{s}_N)$ are two configurations in $(\mathbb{R}^2)^N$, and $(\cdot, \cdot)$ denotes the standard scalar product in $\mathbb{R}^2$. We denote $\|\cdot\|$ the Euclidean norm associated to the mass scalar product, that is

$$\|\mathbf{r}\| = \sqrt{\langle \mathbf{r}, \mathbf{r} \rangle}.$$

Given a configuration $\mathbf{r}$, let $\hat{\mathbf{r}} := \frac{\mathbf{r}}{\|\mathbf{r}\|}$ be the unit vector corresponding to $\mathbf{r}$ henceforth. In particular, the unit vector $\hat{\mathbf{r}}$ is called the **normalized configuration** of the configuration $\mathbf{r}$.

Let $\mathbf{r}_c = \frac{\sum_{k=1}^{N} m_k \mathbf{r}_k}{\mathfrak{m}}$ be the center of mass, where $\mathfrak{m} = \sum_{k=1}^{N} m_k$ is the total mass. Observe that the equations (1.1) of motion are invariant by translation, so without loss of generality, we can assume that the center of mass is fixed at the origin. Let $\mathcal{X}$ denote the space of configurations whose center of mass is at the origin; that is $\mathbf{r} \in \mathcal{X}$, where we define:

$$\mathcal{X} = \{\mathbf{r} = (\mathbf{r}_1, \cdots, \mathbf{r}_N) \in (\mathbb{R}^2)^N | \sum_{k=1}^{N} m_k \mathbf{r}_k = 0\}.$$

Then $\mathcal{X}$ is a $2(N-1)$-dimensional subspace of the Euclidean space $(\mathbb{R}^2)^N$. The subset $\mathcal{X} \backslash \Delta$ is open in $\mathcal{X}$, and it is called the space of collision-free configurations with center of mass at the origin.

Let us recall the important concept of central configurations [**27**]:

DEFINITION 2.1. A configuration $\mathbf{r} \in \mathcal{X}\backslash\Delta$ is called a central configuration if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\sum_{j=1,j\neq k}^{N} \frac{m_j m_k}{|\mathbf{r}_j - \mathbf{r}_k|^3}(\mathbf{r}_j - \mathbf{r}_k) = -\lambda m_k \mathbf{r}_k, 1 \leq k \leq N. \tag{2.1}$$

The coefficient λ in (2.1) is uniquely determined, and is given by

$$\lambda = \frac{\mathcal{U}(\mathbf{r})}{I(\mathbf{r})},$$

where $\mathcal{U}$ is the opposite of the potential energy (or force function) and I is the moment of inertia with respect to the origin. These two functions are defined as:

$$\mathcal{U}(\mathbf{r}) = \sum_{1\leq k<j\leq N} \frac{m_k m_j}{|\mathbf{r}_k - \mathbf{r}_j|},$$

$$I(\mathbf{r}) = \sum_{j=1}^{N} m_j |\mathbf{r}_j|^2.$$

Hereafter, for given $m_j (j = 1, 2, \cdots, N)$ and a fixed λ, let $\mathbf{CC}_\lambda$ be the set of central configurations satisfying (2.1).

Although there were a lot of works on central configurations, many significant problems of central configurations are open up to now. Among these open problems, the most famous one is the conjecture on the Finiteness of Central Configurations [**23**]: for any given masses $m_1, \cdots, m_N$, is the number of central configurations in the associated N-body Problem finite? Note that if $\mathbf{r} \in \mathcal{X}\backslash\Delta$ is a central configuration, so is $\mathsf{z}\mathbf{r}$ for any $\mathsf{z} \in \mathbb{C}^*(:= \mathbb{C}\setminus\{0\})$, where we set $\mathsf{z}\mathbf{r} = (\mathsf{z}\mathbf{r}_1, \mathsf{z}\mathbf{r}_2, \cdots, \mathsf{z}\mathbf{r}_N)$. The transformation z defines an equivalence relation on $\mathcal{X}$. Thus when counting central configurations, we actually count the equivalence classes of them.

We remark that $\mathbf{ir}$ is just $\mathbf{r}$ rotated anticlockwise by an angle $\frac{\pi}{2}$, where $\mathbf{i}$ is the imaginary unit; and $\mathbb{U} = \{e^{\mathbf{i}\theta} | \theta \in \mathbb{R}\}$, the unit circle in $\mathbb{C}$, is identified with the special orthogonal group $S\mathbb{O}(2)$ of the plane.

There are several equivalent definitions of central configurations. One of the equivalent definitions considers a central configuration as a critical point of the *normalized potential* $\widetilde{\mathcal{U}} := I^{\frac{1}{2}}\mathcal{U}$. In fact, it is easy to see that the equations (2.1) are equivalent to

$$\nabla\mathcal{U}(\mathbf{r}) = -\lambda\mathbf{r},$$

where ∇f is the gradient of a differentiable function f on $\mathcal{X}$ (or $\mathcal{X}\backslash\Delta$) with respect to the mass scalar product, i.e., given $\mathbf{r} \in \mathcal{X}$,

$$df(\mathbf{r})(\mathbf{v}) = \langle\nabla f(\mathbf{r}), \mathbf{v}\rangle, \qquad \forall\mathbf{v} \in \mathrm{T}_{\mathbf{r}}\mathcal{X},$$

where $\mathrm{T}_{\mathbf{r}}\mathcal{X}$ is the tangent space of $\mathcal{X}$ at the point $\mathbf{r}$; note that $\mathrm{T}_{\mathbf{r}}\mathcal{X}$ is naturally identified with $\mathcal{X}$, since $\mathcal{X}$ is a vector space. Then it follows from

$$\nabla\widetilde{\mathcal{U}}(\mathbf{r}) = I^{-\frac{1}{2}}(\mathbf{r})\mathcal{U}(\mathbf{r})\mathbf{r} + I^{\frac{1}{2}}(\mathbf{r})\nabla\mathcal{U}(\mathbf{r}),$$

that a central configuration is exactly a critical point of $\widetilde{\mathcal{U}}$.

In the following, let $D^2\widetilde{\mathcal{U}}(\mathbf{r})$ be the Hessian of $\widetilde{\mathcal{U}}$ at the point $\mathbf{r}$, i.e., the linear operator $D^2\widetilde{\mathcal{U}}(\mathbf{r}) : \mathrm{T}_{\mathbf{r}}\mathcal{X} \to \mathrm{T}_{\mathbf{r}}\mathcal{X}$ characterized by

$$d^2\widetilde{\mathcal{U}}(\mathbf{r})(\mathbf{u}, \mathbf{v}) = \langle D^2\widetilde{\mathcal{U}}(\mathbf{r})\mathbf{u}, \mathbf{v}\rangle, \qquad \forall\mathbf{u}, \mathbf{v} \in \mathrm{T}_{\mathbf{r}}\mathcal{X}.$$

It is well known that the critical points of $\widetilde{\mathcal{U}}$ are not isolated but rather occur as manifolds of critical points. Thus these critical points are always degenerate in the ordinary sense. More specifically, let $\mathbf{r} \in \mathcal{X}\backslash\Delta$ be a critical point of $\widetilde{\mathcal{U}}$. By $\widetilde{\mathcal{U}}(\mathsf{z}\mathbf{r}) = \widetilde{\mathcal{U}}(\mathbf{r})$ for any $\mathsf{z} \in \mathbb{C}^*$, it follows that the set $\{\mathsf{z}\mathbf{r}|\mathsf{z} \in \mathbb{C}^*\}$ is a critical manifold points of $\widetilde{\mathcal{U}}$; furthermore, the Hessian of $\widetilde{\mathcal{U}}$ evaluated at a central configuration $\mathbf{r}$ must contain the real plane

$$\mathcal{P}_{\mathbf{r}} := \{\mathsf{z}\mathbf{r}|\mathsf{z} \in \mathbb{C}\}$$

in its kernel. Taking into account these facts, a central configuration $\mathbf{r}$ will be called **nondegenerate**, if the kernel of the Hessian of $\widetilde{\mathcal{U}}$ evaluated at $\mathbf{r}$ is exactly $\mathcal{P}_{\mathbf{r}}$. Obviously, this definition of nondegeneracy is equivalent to the one used by Palmore in his study of planar central configurations [**18**]. Furthermore, in this paper, we call a central configuration is **isolated**, if it is an isolated critical point of the function $\widetilde{\mathcal{U}}$ in the sense of equivalence classes of central configurations. It is well known that, due to a result of Shub on compactness of the set of normalized central configurations [**21**], the famous conjecture on the Finiteness of Central Configurations is equivalent to the following problem: for any given masses $m_1, \cdots, m_N$, is every central configuration in the associated N-body Problem isolated?

If all the central configurations are nondegenerate for any choice of positive masses, then the conjecture on the Finiteness of Central Configurations is correct. However, it has been shown that degenerate central configurations exist in the N-body problem for any $N > 3$. Indeed, in the N-body problem, the set of masses for which a degenerate central configuration exists has a positive $(N-1)$-dimensional (Haussdorff) measure, provided $N > 3$ [**18, 28**]. Despite all this, it is conjectured that all the central configurations are nondegenerate for almost every choice of positive masses [**15, 18**]. Unfortunately, no practical progress has been made for this open problem so far, except for $N = 4$. For the four-body problem it has been shown by Moeckel [**16**] that the exceptional masses for which the degenerate central configuration exists form a proper algebraic subset of the mass space.

In this paper, the following concept of **degree of degeneracy** for central configurations is important.

Definition 2.2. Given a central configuration $\mathbf{r} \in \mathcal{X}\backslash\Delta$, consider the dimension of the kernel of $D^2\widetilde{\mathcal{U}}(\mathbf{r})$. The number

$$n_0 := dimKerD^2\widetilde{\mathcal{U}}(\mathbf{r}) - 2$$

is called the degree of degeneracy of $\mathbf{r}$. In particular, a central configuration with degree of degeneracy zero is nondegenerate.

Given a central configuration $\mathbf{r}_0 \in \mathcal{X}\backslash\Delta$, let $\mathcal{P}^{\perp}_{\mathbf{r}_0}$ be the orthogonal complement of $\mathcal{P}_{\mathbf{r}_0}$ in $\mathcal{X}$, i.e.,

$$\mathcal{X} = \mathcal{P}_{\mathbf{r}_0} \oplus \mathcal{P}^{\perp}_{\mathbf{r}_0}.$$

It is noteworthy that $\mathcal{P}_{\mathbf{r}_0}$ and $\mathcal{P}^{\perp}_{\mathbf{r}_0}$ are both complex vector subspaces of $\mathcal{X}(\simeq \mathbb{C}^{N-1})$.

Note that $\mathcal{P}_{\mathbf{r}_0}$ is an invariant subspace of $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$, because $\widetilde{\mathcal{U}}$ is invariant under the action of the transformation z, i.e., $\widetilde{\mathcal{U}}(\mathsf{z}\mathbf{r}) = \widetilde{\mathcal{U}}(\mathbf{r})$ for any $\mathsf{z} \in \mathbb{C}^*$ and for any $\mathbf{r} \in \mathcal{X}\backslash\Delta$. By the fact that the Hessian $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$ is a symmetric linear operator, it follows that $\mathcal{P}^{\perp}_{\mathbf{r}_0}$ is also an invariant subspace of $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$. Therefore, $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$ can be diagonalized in an orthogonal basis of $\mathcal{X}$ consisting of eigenvectors of $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$, where the first two vectors of the basis can be chosen as $\mathbf{r}_0$ and $\mathbf{i}\mathbf{r}_0$, in $\mathcal{P}_{\mathbf{r}_0}$; and the remaining $2N-4$ vectors

$$\{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_{2N-5}, \mathcal{E}_{2N-4}\}$$

of the basis are in $\mathcal{P}^{\perp}_{\mathbf{r}_0}$.

Assume

$$D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)\mathcal{E}_j = \mu_j \mathcal{E}_j, \qquad j \in \{1, 2, \cdots, 2N-4\}. \tag{2.2}$$

It is noteworthy that the values of μ_j ($j \in \{1, \cdots, 2N-4\}$) depend on the central configuration $\mathbf{r}_0$. Indeed, the Hessian $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$ is homogeneous of degree -2, i.e.,

$$D^2\widetilde{\mathcal{U}}(\rho\mathbf{r}_0) = \frac{1}{\rho^2} D^2\widetilde{\mathcal{U}}(\mathbf{r}_0), \qquad \forall \rho > 0.$$

Furthermore, by the invariance of $\widetilde{\mathcal{U}}$ under the action of the transformation z, it follows that

$$D^2\widetilde{\mathcal{U}}(\rho e^{\mathbf{i}\theta}\mathbf{r}_0) = \frac{1}{\rho^2} e^{\mathbf{i}\theta} D^2\widetilde{\mathcal{U}} e^{-\mathbf{i}\theta}, \quad \forall \rho > 0, \forall \theta \in \mathbb{R}. \tag{2.3}$$

Similarly, it is easy to see that

$$\nabla\widetilde{\mathcal{U}}(\rho e^{\mathbf{i}\theta}\mathbf{r}) = \frac{e^{\mathbf{i}\theta}}{\rho} \nabla\widetilde{\mathcal{U}}(\mathbf{r}), \quad \forall \rho > 0, \forall \theta \in \mathbb{R}, \forall \mathbf{r} \in \mathcal{X} \backslash \Delta. \tag{2.4}$$

By definition of n_0, the cardinality of the set of $j \in \{1, \cdots, 2N-4\}$ such that $\mu_j = 0$ is equal to n_0. Let n_p be the cardinality of the set of $j \in \{1, \cdots, 2N-4\}$ such that $\mu_j > 0$. By (2.3), it follows that n_0 and n_p are invariant under the action of the transformation z, i.e.,

$$n_0(\mathsf{z}\mathbf{r}_0) = n_0(\mathbf{r}_0), \qquad n_p(\mathsf{z}\mathbf{r}_0) = n_p(\mathbf{r}_0).$$

Note that, by a classic result (see [**14, 18**]), it follows that $n_p \geq N-2$.

The degeneracy of central configurations can also be described in a specific coordinate system of $(\mathbb{R}^2)^N$. In particular, this method offers some convenience at practical calculations of degeneracy of central configurations, for more detail please refer to Appendix A and Appendix B.

Obviously, $\{\hat{\mathbf{r}}_0, \mathbf{i}\hat{\mathbf{r}}_0, \hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N-4}\}$ is an orthonormal basis of $\mathcal{X}$. Every configuration $\mathbf{r} \in \mathcal{X}$ can be written uniquely as $\mathbf{r} = y_{\mathbf{r}_0}\hat{\mathbf{r}}_0 + y_{\mathbf{i}\mathbf{r}_0}\mathbf{i}\hat{\mathbf{r}}_0 + \sum_{j=1}^{2N-4} y_j \hat{\mathcal{E}}_j$. The equations (1.1) of motion can be expressed clearly in these coordinates, which would be especially useful in studying relative equilibrium solutions of the Newtonian N-body problem. However, we will adopt another coordinate system originating from a kind of moving frame, which is more suitable for collision orbits and relative equilibrium solutions.

2.2. Moving Coordinates

In this subsection, we shall introduce moving coordinates to describe the orbits near central configurations. Here we say a configuration $\mathbf{r}$ is near the central configuration $\mathbf{r}_0$, if $\mathbf{r}$ is near the $S\mathbb{O}(2)$-orbit $\mathbf{S}$ of $\mathbf{r}_0$, where $\mathbf{S} = \{e^{\mathbf{i}\theta}\mathbf{r}_0 | \theta \in \mathbb{R}\}$.

The notations in this subsection are inherited from the previous subsection.

For any configuration $\mathbf{r} \in \mathcal{X} \backslash \mathcal{P}^{\perp}_{\mathbf{r}_0}$, set $r = \|\mathbf{r}\|$, then $\mathbf{r} = r\hat{\mathbf{r}}$. It is easy to see that there exist $\theta(\mathbf{r}) \in \mathbb{R}$, unique up to integer multiple of 2π, such that

$$\langle e^{-\mathbf{i}\theta(\mathbf{r})}\hat{\mathbf{r}}, \mathbf{i}\hat{\mathbf{r}}_0 \rangle = 0$$

and

$$\langle e^{-\mathbf{i}\theta(\mathbf{r})}\hat{\mathbf{r}}, \hat{\mathbf{r}}_0\rangle > 0.$$

Moreover, by decomposing $e^{-\mathbf{i}\theta(\mathbf{r})}\hat{\mathbf{r}}$ with respect to the basis $\{\hat{\mathbf{r}}_0, \mathbf{i}\hat{\mathbf{r}}_0, \hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N-4}\}$, and denoting $z = (z_1, \cdots, z_{2N-4})^\top$ the coordinates with respect to the vectors $\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N-4}$, it holds

$$\mathbf{r} = r\hat{\mathbf{r}} = re^{\mathbf{i}\theta(\mathbf{r})}(\sqrt{1-|z|^2}\hat{\mathbf{r}}_0 + \sum_{k=1}^{2N-4} z_k\hat{\mathcal{E}}_k); \tag{2.5}$$

where "⊤" denotes transposition of matrices, and

$$|z|^2 = z^\top z = \sum_{j=1}^{2N-4} z_j^2.$$

Indeed, we have defined a real analytic diffeomorphism:

$$(0, +\infty) \times \mathbb{U} \times \mathbf{B}^{2N-4} \to \mathcal{X}\backslash\mathcal{P}_{\mathbf{r}_0}^{\perp};$$

$$(r, e^{\mathbf{i}\theta}, z) \mapsto \mathbf{r} = re^{\mathbf{i}\theta}(\sqrt{1-|z|^2}\hat{\mathbf{r}}_0 + \sum_{k=1}^{2N-4} z_k\hat{\mathcal{E}}_k),$$

where $\mathbf{B}^{2N-4}$ is the $(2N-4)$-dimensional unit ball in $\mathbb{R}^{2N-4}$. Therefore, the total set of the variables r, θ, z can be thought as the coordinates of $\mathbf{r} \in \mathcal{X}\backslash\mathcal{P}_{\mathbf{r}_0}^{\perp}$, and r, θ, z are referred as the **moving coordinates**.

2.3. Invariant Set

Let us finish the section by recalling some well known notions of differential equations [**4**, **9**, **20**].

Given a differential system

$$\dot{q} = v(q), \tag{2.6}$$

where $v : \Omega \to \mathbb{R}^n$ is a continuously differentiable vector field and Ω is an open set in $\mathbb{R}^n$. For any $p \in \Omega$, let $\phi(t, p)$ be the solution of (2.6) passing through p at $t = 0$, i.e., if $q(t) = \phi(t, p)$, then $\dot{q}(t) = v(q(t))$ and $q(0) = p$. We also call ϕ the flow of (2.6) if $\phi(t, p)$ is defined for all $t \in \mathbb{R}$ and all $p \in \Omega$. The orbit $\mathcal{O}(p)$ of (2.6) through p is defined by $\mathcal{O}(p) = \{q = \phi(t, p) \mid t \in \mathbb{R}\}$, the positive semiorbit through p is $\mathcal{O}^+(p) = \{q = \phi(t, p) \mid t \geq 0\}$ and the negative semiorbit through p is $\mathcal{O}^-(p) = \{q = \phi(t, p) \mid t \leq 0\}$.

An **equilibrium point** of (2.6) is a point p such that $v(p) = 0$. A set Σ in Ω is called an **invariant set** of (2.6) if $\mathcal{O}(p) \subset \Sigma$ for any $p \in \Sigma$. Any orbit $\mathcal{O}$ of (2.6) is obviously an invariant set of (2.6). A set Σ in Ω is called positively (resp. negatively) invariant if $\mathcal{O}^+(p) \subset \Sigma$ (resp. $\mathcal{O}^-(p) \subset \Sigma$) for any $p \in \Sigma$.

Definition 2.3. The positive or ω-limit set of an orbit $\mathcal{O}$ is the set

$$\omega(\mathcal{O}) = \bigcap_{p\in\mathcal{O}} \overline{\mathcal{O}^+(p)}.$$

where the bar denotes closure. Similarly, The negative or α-limit set of a point p is the set

$$\alpha(\mathcal{O}) = \bigcap_{p\in\mathcal{O}} \overline{\mathcal{O}^-(p)}.$$

Recall that $\omega(\mathcal{O})$ and $\alpha(\mathcal{O})$ are invariant and closed, and if $\phi(t,p)$ for $t \geq 0$ (resp. $t \leq 0$) is bounded, then $\omega(\mathcal{O})$ (resp. $\alpha(\mathcal{O})$) is nonempty compact and connected, furthermore, $\phi(t,p) \to \omega(\mathcal{O})$ (resp. $\alpha(\mathcal{O})$) as $t \to +\infty$ (resp. $t \to -\infty$), that is, $dist(\phi(t,p), \omega(\mathcal{O})) \to 0$ (resp. $dist(\phi(t,p), \alpha(\mathcal{O})) \to 0$) as $t \to +\infty$ (resp. $t \to -\infty$), here $dist(p,q)$ denotes the distance of $p, q \in \mathbb{R}^n$.

Definition 2.4. The stable set of a positively invariant set Σ is the set

$$\mathcal{W}^s(\Sigma) = \{p \in \Omega \mid dist(\phi(t,p), \Sigma) \to 0 \text{ as } t \to +\infty\};$$

The unstable set of a negatively invariant set Σ is the set

$$\mathcal{W}^u(\Sigma) = \{p \in \Omega \mid dist(\phi(t,p), \Sigma) \to 0 \text{ as } t \to -\infty\};$$

In particular, in case of Σ consisting of one equilibrium point p_0, we have

Definition 2.5. The stable set of an equilibrium point p_0 is the set

$$\mathcal{W}^s(p_0) = \{p \mid \phi(t,p) \to p_0 \text{ as } t \to +\infty\};$$

The unstable set of an equilibrium point p_0 is the set

$$\mathcal{W}^u(p_0) = \{p \mid \phi(t,p) \to p_0 \text{ as } t \to -\infty\}.$$

In a small neighbourhood of an equilibrium point p_0, we can expand v in a Taylor series

$$v(p_0 + q) = \frac{\partial v(p_0)}{\partial q} q + \cdots .$$

Let us consider the following linearized system of the system (2.6)

$$\dot{q} = \frac{\partial v(p_0)}{\partial q} q.$$

Definition 2.6. The equilibrium point p_0 of (2.6) is hyperbolic if all of the eigenvalues of $\frac{\partial v(p_0)}{\partial q}$ have nonzero real parts. Otherwise, i.e., if at least one of the eigenvalues of $\frac{\partial v(p_0)}{\partial q}$ is on the imaginary axis, the equilibrium point p_0 is nonhyperbolic. Furthermore, if at least one of the eigenvalues of $\frac{\partial v(p_0)}{\partial q}$ is zero, the equilibrium point p_0 is degenerate.

It is well known that if the equilibrium point p_0 of (2.6) is hyperbolic, then the system (2.6) is topologically equivalent to its linearized system in a small neighbourhood of p_0. This is the famous Hartman-Grobman Theorem. Moreover, by Hartman-Grobman Theorem it follows that the stable (resp. unstable) set $\mathcal{W}^s(p_0)$ (resp. $\mathcal{W}^u(p_0)$) is an immersed submanifold and it is called stable (resp. unstable) manifold of the hyperbolic equilibrium point p_0.

However, the behavior near a nonhyperbolic equilibrium point is more complicated. In particular, if the equilibrium point p_0 is degenerate, it is generally very difficult to understand the system (2.6) near p_0. For instance, we could not find any literature on system (2.6) with three vanishing eigenvalues of $\frac{\partial v(p_0)}{\partial q}$.

CHAPTER 3

Equations of Motion for Collision Orbits and $PISPW$

Let us recall the concepts of the kinetic energy, the total energy, and the angular momentum, respectively, defined by

$$\mathcal{K}(\dot{\mathbf{r}}) = \sum_{j=1}^{N} \frac{1}{2} m_j |\dot{\mathbf{r}}_j|^2,$$

$$\mathcal{H}(\mathbf{r}, \dot{\mathbf{r}}) = \mathcal{K}(\dot{\mathbf{r}}) - \mathcal{U}(\mathbf{r}),$$

$$\mathcal{J}(\mathbf{r}) = \sum_{j=1}^{N} m_j \mathbf{r}_j \times \dot{\mathbf{r}}_j,$$

where $\times$ denotes the standard cross product in $\mathbb{R}^2$. Note that

$$\mathcal{K}(\dot{\mathbf{r}}) = \frac{1}{2}\langle \dot{\mathbf{r}}, \dot{\mathbf{r}} \rangle,$$

$$\mathcal{J}(\mathbf{r}) = \langle \mathbf{i}\mathbf{r}, \dot{\mathbf{r}} \rangle,$$

and the total energy and the angular momentum are first integrals for the N-body problem.

Recall that, an orbit $\mathbf{r}(t)$ of the N-body problem starts (resp. arrives) at a **total collision** at some instant t_0 if and only if $\mathbf{r}(t) \to 0$ as $t \to t_0+$ (resp. $t \to t_0-$), that is to say, $r(t) = \|\mathbf{r}(t)\| \to 0$ as $t \to t_0+$ (resp. $t \to t_0-$). Without loss of generality, assume the instant $t_0 = 0$ and we consider only that $r(t) \to 0$ as $t \to 0+$ henceforth.

Some classical results concerning the total collision orbits can be found in [**27**]. We summarize the results as follows.

THEOREM 3.1. *Suppose a solution $\mathbf{r}(t)$ of the N-body problem arrives at a total collision at the instant 0, then there exists a constant $\kappa > 0$, such that*

- $I(\mathbf{r}(t)) \sim (\frac{3}{2})^{\frac{4}{3}} \kappa^{\frac{2}{3}} t^{\frac{4}{3}}, \dot{I}(\mathbf{r}(t)) \sim (12)^{\frac{1}{3}} \kappa^{\frac{2}{3}} t^{\frac{1}{3}}, \ddot{I}(\mathbf{r}(t)) \sim (\frac{2}{3})^{\frac{2}{3}} \kappa^{\frac{2}{3}} t^{-\frac{2}{3}}$ *as* $t \to 0+$.
- $\mathcal{U}(\mathbf{r}(t)) \sim (\frac{1}{18})^{\frac{1}{3}} \kappa^{\frac{2}{3}} t^{-\frac{2}{3}}, \mathcal{K}(\dot{\mathbf{r}}(t)) \sim (\frac{1}{18})^{\frac{1}{3}} \kappa^{\frac{2}{3}} t^{-\frac{2}{3}}$ *as* $t \to 0+$.
- $\hat{\mathbf{r}}(t) \to \mathbf{CC}_\lambda$ *(i.e.,* $dist(\hat{\mathbf{r}}(t), \mathbf{CC}_\lambda) \to 0$*)* *as* $t \to 0+$*, where* $\lambda = \frac{\kappa}{2}$.
- $\mathcal{J}(\mathbf{r}(t)) \equiv 0$.

Therefore, it is natural to ask whether there exists a certain central configuration $\mathbf{s}_0 \in \mathbf{CC}_\lambda$ such that

$$\hat{\mathbf{r}}(t) \to \mathbf{s}_0, \qquad \text{as} \qquad t \to 0+.$$

The point is that:

(1) if the number of central configurations is infinite for given masses $m_1, \cdots, m_N$, then it would be possible that the normalized configuration $\hat{\mathbf{r}}(t)$ comes closer and closer to more than one central configuration, which are not in the same equivalence classes, in such a way as to oscillate between these central configurations;
(2) if the number of central configurations is finite for given masses $m_1, \cdots, m_N$, then it would be possible that the normalized configuration $\hat{\mathbf{r}}(t)$ moves in spirals without asymptotes (or in other words, make an infinite number of revolutions) as $t \to 0+$. This is the so-called *problem of infinite spin*.

For more detail please refer to [**27**, p.282–p.283].

To investigate the problem of infinite spin or *Painlevé-Wintner* (abbreviated "*PISPW*"), we will focus on a collisions solution $t \mapsto \mathbf{r}(t)$ such that $\hat{\mathbf{r}}(t)$ converges to the $S\mathbb{O}(2)$-orbit $\mathbf{S}$ of an isolated central configuration $\mathbf{r}_0 = \hat{\mathbf{r}}_0 \in \mathbf{CC}_\lambda$. As a result, it is easy to see that *PISPW* explores *whether there exists a fixed central configuration* $e^{\mathbf{i}\theta_0}\mathbf{r}_0 \in \mathbf{S}$ *such that* $\hat{\mathbf{r}}(t) \to e^{\mathbf{i}\theta_0}\mathbf{r}_0$ *as* $t \to 0+$.

3.1. Equations of Motion

In this subsection we will write equations of motion for total collision orbits by using McGehee's coordinates and the moving coordinates.

3.1.1. McGehee Equations. In the subsection let us write equations of motion in McGehee's coordinates.

First the equations (1.1) of motion can be written:

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{v}, \\ \dot{\mathbf{v}} = \nabla \mathcal{U}(\mathbf{r}), \end{cases} \tag{3.1}$$

where $(\mathbf{r}, \mathbf{v}) \in \mathrm{T}(\mathcal{X}\backslash\Delta) = (\mathcal{X}\backslash\Delta) \times \mathcal{X}$. Equations (3.1) determine a vector field on $\mathrm{T}(\mathcal{X}\backslash\Delta)$.

It is well known that the total energy and the angular momentum are conserved: along solutions of the equation (3.1),

$$\mathcal{H} = \frac{1}{2}\langle \mathbf{v}, \mathbf{v} \rangle - \mathcal{U}(\mathbf{r}) \equiv constant,$$

$$\mathcal{J} = \langle \mathbf{i}\mathbf{r}, \mathbf{v} \rangle \equiv constant.$$

We now introduce the following McGehee's variables:

$$\begin{cases} r = \|\mathbf{r}\|, \\ \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}, \\ \Upsilon = r^{\frac{1}{2}} \langle \mathbf{v}, \hat{\mathbf{r}} \rangle, \\ \mathbf{R} = r^{\frac{1}{2}} \mathbf{v} - \Upsilon \hat{\mathbf{r}}, \end{cases}$$

note that $\mathbf{R}$ is orthogonal to $\hat{\mathbf{r}}$ with respect to the mass scalar product. Certainly, the old variables can be written in terms of the new variables:

$$\begin{cases} \mathbf{r} = r\hat{\mathbf{r}}, \\ \mathbf{v} = r^{-\frac{1}{2}}(\Upsilon \hat{\mathbf{r}} + \mathbf{R}). \end{cases}$$

Hence the velocity $\mathbf{v}$ has been decomposed into a radial component Υ and a tangential component $\mathbf{R}$. Furthermore, the relations of the total energy and the angular momentum become:

$$\frac{1}{2}(\Upsilon^2 + \|\mathbf{R}\|^2) - \widetilde{\mathcal{U}}(\hat{\mathbf{r}}) = r\mathcal{H},$$

$$r^{\frac{1}{2}}\langle \mathbf{i}\hat{\mathbf{r}}, \mathbf{R}\rangle = \mathcal{J}.$$

Let us introduce the time transformation:

$$dt = r^{\frac{3}{2}} d\tau. \tag{3.2}$$

Differentiation with respect to time t is denoted by $\dot{}$: $\frac{df}{dt} = \dot{f}$ in the previous pages. Similarly, differentiation with respect to the new variable τ will be denoted by $'$: $\frac{df}{d\tau} = f'$ henceforth. Note that

$$\dot{f} = r^{-\frac{3}{2}} f', \quad \ddot{f} = r^{-3} f'' - \frac{3}{2} r^{-4} r' f'.$$

Now a straightforward computation shows that the equations (3.1) of motion become

$$\begin{cases} r' = r\Upsilon, \\ \hat{\mathbf{r}}' = \mathbf{R}, \\ \Upsilon' = \frac{1}{2}\Upsilon^2 + \|\mathbf{R}\|^2 - \widetilde{\mathcal{U}}(\hat{\mathbf{r}}), \\ \mathbf{R}' = \nabla\widetilde{\mathcal{U}}(\hat{\mathbf{r}}) - \|\mathbf{R}\|^2\hat{\mathbf{r}} - \frac{1}{2}\Upsilon\mathbf{R}. \end{cases} \tag{3.3}$$

In the calculation, please note that

$$\mathcal{U}(\mathbf{r}), \widetilde{\mathcal{U}}(\mathbf{r}), \nabla\mathcal{U}(\mathbf{r}), \nabla\widetilde{\mathcal{U}}(\mathbf{r})$$

are homogeneous and use Euler's formula.

3.1.2. Equations of Motion for Collision Orbits. Given a central configuration $\mathbf{r}_0$, let us describe the motion of a total collision orbit near $\mathbf{r}_0$ by using the system of coordinates r, θ, z introduced in subsection 2.2. Without loss of generality, assume $\hat{\mathbf{r}}_0 = \mathbf{r}_0$ below, i.e., $\|\mathbf{r}_0\| = 1$.

Following the notations introduced in subsection 2.2, consider the decomposition of $\hat{\mathbf{r}}$:

$$\hat{\mathbf{r}} = e^{\mathbf{i}\theta}(\sqrt{1-|z|^2}\mathbf{r}_0 + \sum_{k=1}^{2N-4} z_k \hat{\mathcal{E}}_k).$$

For the sake of compact notations, we temporarily set

$$z_0 = \sqrt{1-|z|^2}, \qquad \mathrm{z} = \sum_{k=1}^{2N-4} z_k \hat{\mathcal{E}}_k,$$

By differentiating

$$\hat{\mathbf{r}} = e^{\mathbf{i}\theta}(z_0\mathbf{r}_0 + \mathrm{z})$$

with respect to τ, it follows that

$$\mathbf{R} = e^{\mathbf{i}\theta}(z_0'\mathbf{r}_0 + \Theta z_0 \mathbf{i}\mathbf{r}_0 + \Theta\mathbf{i}\mathrm{z} + \mathrm{z}),$$

where $\Theta = \theta'$ and $\mathrm{z} = \mathrm{z}'$.

By Theorem 3.1, the relation of the angular momentum reduces to

$$\langle \mathbf{i}\hat{\mathbf{r}}, \mathbf{R}\rangle = \Theta + \langle \mathbf{i}\mathrm{z}, \mathrm{z}\rangle = 0.$$

Consequently,

$$\Theta = -\langle \mathbf{i}\mathrm{z}, \mathrm{z}\rangle = \langle \mathrm{z}, \mathbf{i}\mathrm{z}\rangle,$$
$$\Theta' = -\langle \mathbf{i}\mathrm{z}, \mathrm{z}'\rangle.$$

Therefore, by (2.4) and $z_0^2 + \|\mathrm{z}\|^2 = 1$, a straightforward computation shows that the equations (3.3) become:

(3.4)

$$\begin{cases} r' = r\Upsilon, \\ \Upsilon' = \frac{1}{2}\Upsilon^2 + z_0'^2 - \Theta^2 + \|\mathrm{z}\|^2 - \widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}), \\ \mathrm{z}' = \mathrm{z}, \\ \mathrm{z}' = [\Theta^2 - z_0'^2 - \|\mathrm{z}\|^2](z_0\mathbf{r}_0 + \mathrm{z}) - \frac{1}{2}\Upsilon(z_0'\mathbf{r}_0 + \Theta z_0\mathbf{i}\mathbf{r}_0 + \Theta\mathbf{i}\mathrm{z} + \mathrm{z}) \\ -[(z_0'' - \Theta^2 z_0)\mathbf{r}_0 + (\Theta' z_0 + 2\Theta z_0')\mathbf{i}\mathbf{r}_0 - \Theta^2\mathrm{z} + \Theta'\mathbf{i}\mathrm{z} + 2\Theta\mathbf{i}\mathrm{z}] + \nabla\widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}); \end{cases}$$

and

$$\begin{cases} \theta' = \Theta, \\ \Theta = \langle \mathrm{z}, \mathbf{i}\mathrm{z}\rangle, \\ \Theta' = -\langle \mathbf{i}\mathrm{z}, \mathrm{z}'\rangle. \end{cases} \tag{3.5}$$

Note that $\mathrm{z}, \mathbf{i}\mathrm{z}, \mathrm{z}, \mathbf{i}\mathrm{z}, \mathrm{z}' \in \mathcal{P}_{\mathbf{r}_0}^{\perp}$. By considering the projection of the last equation in (3.4) along the directions $\mathbb{R}\mathbf{r}_0$, $\mathbb{R}\mathbf{i}\mathbf{r}_0$ and $\mathcal{P}_{\mathbf{r}_0}^{\perp}$, it follows that the equations (3.4) can be reduced to

$$\begin{cases} r' = r\Upsilon, \\ \Upsilon' = \frac{1}{2}\Upsilon^2 + z_0'^2 - \Theta^2 + \|\mathrm{z}\|^2 - \widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}), \\ \mathrm{z}' = \mathrm{z}, \\ \mathrm{z}' = \nabla\widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}) - \langle \nabla\widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}), \mathbf{r}_0\rangle\mathbf{r}_0 - \langle \nabla\widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}), \mathbf{i}\mathbf{r}_0\rangle\mathbf{i}\mathbf{r}_0 \\ +[2\Theta^2 - z_0'^2 - \|\mathrm{z}\|^2]\mathrm{z} - (\frac{1}{2}\Upsilon\Theta + \Theta')\mathbf{i}\mathrm{z} - \frac{1}{2}\Upsilon\mathrm{z} - 2\Theta\mathbf{i}\mathrm{z}, \end{cases} \tag{3.6}$$

where

$$\langle \nabla\widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}), \mathbf{r}_0\rangle = (\|\mathrm{z}\|^2 + z_0'^2 - 2\Theta^2)z_0 + \frac{1}{2}\Upsilon z_0' + z_0'',$$
$$\langle \nabla\widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}), \mathbf{i}\mathbf{r}_0\rangle = \frac{1}{2}\Upsilon\Theta z_0 + \Theta' z_0 + 2\Theta z_0'.$$

It is noteworthy that, the last three equations in (3.6) is a closed subsystem in $\Upsilon, \mathrm{z}, \mathrm{z}$. Once the subsystem of $\Upsilon, \mathrm{z}, \mathrm{z}$ is solved, the variables r, θ can be solved by using $r' = r\Upsilon, \theta' = \langle \mathrm{z}, \mathbf{i}\mathrm{z}\rangle$.

Now let us use the variables $z, Z = z'$ to write the equations above in a more concise form. Set $q_{jk} = \langle \hat{\mathcal{E}}_j, \mathbf{i}\hat{\mathcal{E}}_k\rangle$, then the square matrix

$$Q := (q_{jk})_{(2N-4)\times(2N-4)}$$

is an anti-symmetric orthogonal matrix. Since

$$\widetilde{\mathcal{U}}(z_0\mathbf{r}_0 + \mathrm{z}) = \widetilde{\mathcal{U}}(\sqrt{1 - |z|^2}\mathbf{r}_0 + \sum_{k=1}^{2N-4} z_k\hat{\mathcal{E}}_k)$$

only contains the variables z_j $(j = 1, \cdots, 2N-4)$, we will simply write it as $U(z)$ henceforth. As a result, it is easy to see that the equations of motion above become:
(3.7)
$$\begin{cases} z' = Z, \\ Z' = \frac{\partial U(z)}{\partial z} - \frac{\Upsilon}{2} Z + (\frac{z_0''}{z_0} + \frac{\Upsilon}{2}\frac{z_0'}{z_0})z - (\frac{\Upsilon}{2} z^\top QZ + z^\top QZ')Qz - 2(z^\top QZ)QZ, \\ r' = r\Upsilon, \\ \Upsilon' = \frac{1}{2}\Upsilon^2 + \frac{(z^\top Z)^2}{1-|z|^2} + |Z|^2 - (z^\top QZ)^2 - U(z), \\ \theta' = \Theta, \end{cases}$$

where
$$\frac{z_0'}{z_0} = -\frac{z^\top Z}{1-|z|^2},$$
$$\frac{z_0''}{z_0} = -\frac{z^\top Z' + |Z|^2}{1-|z|^2} - \frac{(z^\top Z)^2}{(1-|z|^2)^2},$$

and
$$\Theta = z^\top QZ.$$

In fact, it suffices to note that
$$\frac{\partial U(z)}{\partial z_i} = \langle \nabla\widetilde{\mathfrak{U}}(z_0\mathbf{r}_0+\mathrm{z}), \frac{\partial}{\partial z_i}(z_0\mathbf{r}_0+\mathrm{z})\rangle = \frac{\partial z_0}{\partial z_i}\langle \nabla\widetilde{\mathfrak{U}}(z_0\mathbf{r}_0+\mathrm{z}), \mathbf{r}_0\rangle + \langle \nabla\widetilde{\mathfrak{U}}(z_0\mathbf{r}_0+\mathrm{z}), \hat{\mathcal{E}}_i\rangle.$$

Remark 3.2. Equations (3.7) look more complex than the Newtonian equations (1.1), however, an advantage of using equations (3.7) is that there is no degeneracy according to intrinsic symmetrical characteristic of the N-body problem. More specifically, the subsystem of (3.7) below

$$\begin{cases} z' = Z, \\ Z' = \frac{\partial U(z)}{\partial z} - \frac{\Upsilon}{2} Z + (\frac{z_0''}{z_0} + \frac{\Upsilon}{2}\frac{z_0'}{z_0})z - (\frac{\Upsilon}{2} z^\top QZ + z^\top QZ')Qz - 2(z^\top QZ)QZ, \\ \Upsilon' = \frac{1}{2}\Upsilon^2 + \frac{(z^\top Z)^2}{1-|z|^2} + |Z|^2 - (z^\top QZ)^2 - U(z), \end{cases}$$

is a reduction of (1.1) with the aid of the 6 classical integrals of the planar N-body problem (i.e., the four center of mass integrals, the angular momentum integral and the total energy integral) and the "scale invariance" of the N-body problem (i.e., the Newton equations (1.1) are invariant under the transformation $(\mathbf{r}, t) \mapsto (\rho\mathbf{r}, \rho^{\frac{3}{2}}t)$).

Remark 3.3. To solve Z' in (3.7), note that the equation below
$$Z' = \frac{\partial U(z)}{\partial z} - \frac{\Upsilon}{2} Z + (\frac{z_0''}{z_0} + \frac{\Upsilon}{2}\frac{z_0'}{z_0})z - (\frac{\Upsilon}{2} z^\top QZ + z^\top QZ')Qz - 2(z^\top QZ)QZ$$
is equivalent to
$$(\mathbb{I} + \frac{zz^\top}{1-|z|^2} + Qzz^\top Q)Z' = v(z, Z, \Upsilon),$$
where
$$v(z, Z, \Upsilon) = \frac{\partial U(z)}{\partial z} - \frac{\Upsilon}{2} Z - (\frac{|Z|^2}{1-|z|^2} + \frac{(z^\top Z)^2}{(1-|z|^2)^2} - \frac{\Upsilon}{2}\frac{z_0'}{z_0})z$$
$$- (\frac{\Upsilon}{2} z^\top QZ)Qz - 2(z^\top QZ)QZ.$$

It is easy to see that the symmetric matrix $\mathbb{I} + \frac{zz^\top}{1-|z|^2} + Qzz^\top Q$ is positive definite and thus is invertible when $|z| < 1$. Moreover, by

$$\|\frac{zz^\top}{1-|z|^2} + Qzz^\top Q\| \leq \frac{|z|^2}{1-|z|^2} + |z|^2,$$

it is easy to see that, if

$$|z| < \frac{\sqrt{5}-1}{2},$$

then

$$(\mathbb{I} + \frac{zz^\top}{1-|z|^2} + Qzz^\top Q)^{-1} = \sum_{k=0}^{\infty}(-1)^k(\frac{zz^\top}{1-|z|^2} + Qzz^\top Q)^k;$$

consequently, when $|z| < \frac{\sqrt{5}-1}{2}$,

(3.8)

$$Z' = v(z,Z,\Upsilon) - (\frac{zz^\top}{1-|z|^2} + Qzz^\top Q)v(z,Z,\Upsilon) - (\frac{zz^\top}{1-|z|^2} + Qzz^\top Q)^2 v(z,Z,\Upsilon) + \cdots.$$

As a result, when considering Taylor Expansion, if $v(z,Z,\Upsilon)$ has no constant term, then it is obvious that the quadratic terms in $v(z,Z,\Upsilon)$ are exactly the quadratic terms of the right side of (3.8).

Incidentally, the relation of total energy becomes

$$2r\mathcal{H} = \Upsilon^2 + [\frac{(z^\top Z)^2}{1-|z|^2} + |Z|^2 - (z^\top QZ)^2] - 2U(z), \tag{3.9}$$

or

$$2r\mathcal{H} = \Upsilon' + \frac{1}{2}\Upsilon^2 - U(z).$$

REMARK 3.4. Consider the system of the first four equations of (3.7) in $(z, Z, r, \Upsilon) \in \mathbf{B}^{2N-4} \times \mathbb{R}^{2N-4} \times [0, +\infty) \times \mathbb{R}$. It is easy to see that the system does not have singularities at $r = 0$ and has an invariant manifold $\{r = 0\}$. Set

$$\mathcal{N}_{\mathcal{H}} = \{(z,Z,r,\Upsilon) \in \mathbf{B}^{2N-4} \times \mathbb{R}^{2N-4} \times [0,+\infty) \times \mathbb{R} \mid (3.9) \quad \textit{holds}\}$$

for a fixed real number $\mathcal{H}$. Then $\mathcal{N}_{\mathcal{H}}$ is an invariant manifold of the system. Similar to results in [**13**], one can show that the flow of the system restricted to the invariant manifold

$$\begin{aligned}&\{r=0\}\bigcap \mathcal{N}_{\mathcal{H}}\\ &= \{(z,Z,0,\Upsilon) \in \mathcal{N}_{\mathcal{H}} \mid \Upsilon^2 + [\frac{(z^\top Z)^2}{1-|z|^2} + |Z|^2 - (z^\top QZ)^2] - 2U(z) = 0\}\end{aligned}$$

is gradient-like with respect to

$$\pi_\Upsilon : \quad \{r=0\}\bigcap \mathcal{N}_{\mathcal{H}} \to \mathbb{R}; \quad (z,Z,0,\Upsilon) \mapsto -\Upsilon.$$

For more detail please refer to [**13**].

We conclude this subsection with some discussions of $U(z)$ near the point $z = 0$. First, we can expand $U(z)$ as

(3.10)
$$\begin{aligned} U(z) &= \mathcal{U}(z_0\hat{\mathbf{r}}_0 + \textstyle\sum_{j=1}^{2N-4} z_j\hat{\mathcal{E}}_j) = \widetilde{\mathcal{U}}(z_0\hat{\mathbf{r}}_0 + \textstyle\sum_{j=1}^{2N-4} z_j\hat{\mathcal{E}}_j) \\ &= \widetilde{\mathcal{U}}(\hat{\mathbf{r}}_0) + \textstyle\sum_{k=1}^{2N-4} d\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\hat{\mathcal{E}}_k)z_k + d\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\hat{\mathbf{r}}_0)(z_0 - 1) \\ &\quad + \tfrac{1}{2}[\textstyle\sum_{j,k=1}^{2N-4} d^2\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\hat{\mathcal{E}}_j, \hat{\mathcal{E}}_k)z_j z_k + 2\textstyle\sum_{k=1}^{2N-4} d^2\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\hat{\mathbf{r}}_0, \hat{\mathcal{E}}_k)(z_0 - 1)z_k] \\ &\quad + \tfrac{1}{3!}\textstyle\sum_{i,j,k=1}^{2N-4} d^3\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\hat{\mathcal{E}}_i, \hat{\mathcal{E}}_j, \hat{\mathcal{E}}_k)z_i z_j z_k + \cdots, \end{aligned}$$

where "$\cdots$" denotes power-series in z_j ($j = 1, \cdots, 2N-4$) starting with quartic terms, and $d\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}$, $d^2\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}$, $d^3\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}$ denote the differential, second order differential, third order differential of $\widetilde{\mathcal{U}}$ at $\hat{\mathbf{r}}_0$ respectively. The computation to expand $U(z)$ is straightforward if one notes

$$z_0 = \sqrt{1 - |z|^2} = 1 - \frac{|z|^2}{2} + O(|z|^4).$$

Then, by $d\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\cdot) = 0$, $d^2\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\hat{\mathbf{r}}_0, \cdot) = 0$ and (2.2), it follows that

$$U(z) = \lambda + \frac{1}{2}\sum_{k=1}^{2N-4} \mu_k z_k^2 + \frac{1}{6}\sum_{i,j,k=1}^{2N-4} a_{ijk} z_i z_j z_k + \cdots, \tag{3.11}$$

where $a_{ijk} = d^3\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\hat{\mathcal{E}}_i, \hat{\mathcal{E}}_j, \hat{\mathcal{E}}_k)$, thus a_{ijk} is symmetric with respect to the subscripts i, j, k. We remark that

$$a_{ijk} = d^3\widetilde{\mathcal{U}}|_{\hat{\mathbf{r}}_0}(\hat{\mathcal{E}}_i, \hat{\mathcal{E}}_j, \hat{\mathcal{E}}_k) = d^3\mathcal{U}|_{\hat{\mathbf{r}}_0}(\hat{\mathcal{E}}_i, \hat{\mathcal{E}}_j, \hat{\mathcal{E}}_k) = \frac{\partial^3 U(0)}{\partial z_i \partial z_j \partial z_k}.$$

By (3.11), if the Hessian of $U(z)$ is nondegenerate, that is, the central configuration $\hat{\mathbf{r}}_0$ is nondegenerate or $n_0 = 0$, then the function $U(z)$ has exactly one critical point $z = 0$ in some small neighbourhood of the point $z = 0$. However, even if the central configuration $\hat{\mathbf{r}}_0$ is degenerate, the function $U(z)$ still has exactly one critical point $z = 0$ in a small neighbourhood of $z = 0$ provided $\hat{\mathbf{r}}_0$ is an isolated central configuration.

PROPOSITION 3.5. *The function $U(z)$ has exactly one critical point $z = 0$ in a small neighbourhood of $z = 0$, provided that the central configuration $\mathbf{r}_0$ is isolated.*

PROOF. Note that $U(z) = \mathcal{U}(z_0\hat{\mathbf{r}}_0 + \sum_{j=1}^{2N-4} z_j\hat{\mathcal{E}}_j) = \widetilde{\mathcal{U}}(z_0\hat{\mathbf{r}}_0 + \sum_{j=1}^{2N-4} z_j\hat{\mathcal{E}}_j)$, thus U is the composition of the two analytic functions $\widetilde{\mathcal{U}}$ and π, where the function

$$\pi: \quad \mathbf{B}^{2N-4} \to \mathbf{S}_+^{2N-4}; \quad z \mapsto z_0\hat{\mathbf{r}}_0 + \sum_{j=1}^{2N-4} z_j\hat{\mathcal{E}}_j$$

is a diffeomorphism between $\mathbf{B}^{2N-4}$ and upper hemisphere $\mathbf{S}_+^{2N-4}$ in $span\{\mathbf{r}_0, \mathcal{P}_{\mathbf{r}_0}^{\perp}\}$. Therefore, a critical point of U in a small neighbourhood of $z = 0$ corresponds precisely to a central configuration in a small neighbourhood of $\hat{\mathbf{r}}_0$ confined to the upper hemisphere $\mathbf{S}_+^{2N-4}$. It is easy to see that in such a small neighbourhood $\hat{\mathbf{r}}_0$ is the only central configuration.

So the proposition is proved. □

An argument similar to the above proof, if we note that the space $span\{\mathbf{r}_0, \mathcal{P}^{\perp}_{\mathbf{r}_0}\}$ has removed the rotation freedom of $e^{\mathbf{i}\theta}$, shows an extension of Proposition 3.5 below.

PROPOSITION 3.6. *The function $U(z)$ has finitely many isolated critical points in $\mathbf{B}^{2N-4}$, provided that the number of central configurations in the N-body Problem is finite for given masses $m_1, \cdots, m_N$.*

3.2. The Problem of Infinite Spin or Painlevé-Wintner ($PISPW$)

In order to make preparations for resolving $PISPW$, we need to write the equations of motion for collision orbits in a form as simple as possible. Let us first describe some features of total collision orbits in the variables z, Z, r, Υ.

3.2.1. Notes on Collision Orbits. When considering a total collision orbit $\mathbf{r}(t)$ such that $\hat{\mathbf{r}}(t)$ converges to the $S\mathbb{O}(2)$-orbit of an isolated central configuration $\mathbf{r}_0$, we have legitimate rights to use the moving coordinates. By Sundman's theorem, the coordinates $r(t), \theta(t), z(t)$ of the total collision orbit $\mathbf{r}(t)$ are real analytic functions for $t > 0$; and by Theorem 3.1, it is easy to see that $r(t), \theta(t), z(t)$ satisfy the following relations:

$$
\begin{cases}
r(t) \sim (\frac{3}{2})^{\frac{2}{3}}\kappa^{\frac{1}{3}}t^{\frac{2}{3}}, & as \quad t \to 0+ \\
\dot{r}(t) \sim (\frac{2}{3})^{\frac{1}{3}}\kappa^{\frac{1}{3}}t^{-\frac{1}{3}}, & as \quad t \to 0+ \\
\ddot{r}(t) \sim -(\frac{2}{81})^{\frac{1}{3}}\kappa^{\frac{1}{3}}t^{-\frac{4}{3}}, & as \quad t \to 0+ \\
z \to 0, & as \quad t \to 0+
\end{cases}
\tag{3.12}
$$

As a result, the total collision solution $\mathbf{r}(t)$ of equations (1.1) corresponds to a solution $(z, Z, r, \Upsilon, \theta)$ of the equations (3.7) such that $r \to 0, z \to 0$. Furthermore, we have

PROPOSITION 3.7. *The total collision orbit $\mathbf{r}(t)$ of equations (1.1) corresponds exactly to a solution $(z(\tau), Z(\tau), r(\tau), \Upsilon(\tau), \theta(\tau))$ of equations (3.7) such that*

$$z \to 0, \quad Z \to 0, \quad r \to 0, \quad \Upsilon \to \kappa^{\frac{1}{2}}, \quad \theta' = \Theta \to 0, \quad as \tau \to -\infty.$$

PROOF. We only need to prove that a solution $(z(\tau), Z(\tau), r(\tau), \Upsilon(\tau), \theta(\tau))$ of equations (3.7) corresponding to the total collision orbit $\mathbf{r}(t)$ satisfies

$$Z \to 0, \quad \Upsilon \to \kappa^{\frac{1}{2}}.$$

First, by (3.12), it is easy to show that $\Upsilon \to \kappa^{\frac{1}{2}}$.

Denote by $\alpha(\mathbf{r})$ the α-limit set of the solution $(z(\tau), Z(\tau), r(\tau), \Upsilon(\tau), \theta(\tau))$. Then

$$\alpha(\mathbf{r}) \subset \{(z, Z, r, \Upsilon, \theta) | r = 0, \Upsilon = \kappa^{\frac{1}{2}}, z = 0\}.$$

Let us investigate the maximum invariant set included in

$$\{(z, Z, r, \Upsilon, \theta) | r = 0, \Upsilon = \kappa^{\frac{1}{2}}, z = 0\}.$$

It is easy to show that this set is precisely

$$\{(z, Z, r, \Upsilon, \theta) | r = 0, \Upsilon = \kappa^{\frac{1}{2}}, z = 0, Z = 0\}.$$

The relation $Z \to 0$ therefore follows from the following fact

$$\alpha(\mathbf{r}) \subset \{(z, Z, r, \Upsilon, \theta) | r = 0, \Upsilon = \kappa^{\frac{1}{2}}, z = 0, Z = 0\}. \qquad \square$$

Remark 3.8. For an isolated central configuration $\mathbf{r}_0$, it follows from Proposition 3.5 that all the equilibrium points of equations (3.7), which geometrically constitute a circle, satisfy $z = 0, Z = 0, r = 0, \Upsilon = \pm\kappa^{\frac{1}{2}}, \theta = const$ so long as z is small.

Let us investigate all the solutions of equations (3.7) satisfying the above asymptotic conditions in Proposition 3.7.

Now define a new variable $\gamma = \Upsilon - \kappa^{\frac{1}{2}}$ and substitute γ into equations (3.7). By (3.8), it follows that

$$\begin{pmatrix} z' \\ Z' \\ \gamma' \end{pmatrix} = \mathfrak{A} \begin{pmatrix} z \\ Z \\ \gamma \end{pmatrix} + \begin{pmatrix} 0 \\ \chi_Z(z, Z, \gamma) \\ \chi_0(z, Z, \gamma) \end{pmatrix}; \tag{3.13}$$

$$r' = r(\kappa^{\frac{1}{2}} + \gamma); \tag{3.14}$$

$$\theta' = z^\top QZ; \tag{3.15}$$

where

$$\mathfrak{A} = \begin{pmatrix} 0 & \mathbb{I} & \\ \Lambda & -\frac{\kappa^{\frac{1}{2}}}{2}\mathbb{I} & \\ & & \kappa^{\frac{1}{2}} \end{pmatrix}$$

denotes the square matrix of the coefficients of the linear terms, $\Lambda = diag(\mu_1, \cdots, \mu_{2N-4})$; $\chi_Z = (\chi_1, \cdots, \chi_{2N-4})^\top$ and the functions χ_0, χ_k $(k = 1, \cdots, 2N-4)$ are power-series in the $4N-7$ real variables z, Z, γ starting with quadratic terms and all converge for sufficiently small z, Z, γ:

$$\begin{aligned} \chi_k(z, Z, \gamma) &= \tfrac{1}{2}[a_{kkk}z_k^2 + 2\textstyle\sum_{j=1, j\neq k}^{2N-4} a_{jkk}z_kz_j \\ &\quad + \textstyle\sum_{i,j=1, i,j\neq k}^{2N-4} a_{ijk}z_iz_j - \gamma Z_k] + \cdots \\ &= \tfrac{1}{2}[\textstyle\sum_{i,j=1,}^{2N-4} a_{ijk}z_iz_j - \gamma Z_k] + \cdots, \end{aligned} \tag{3.16}$$

$$\chi_0(z, Z, \gamma) = \frac{1}{2}\gamma^2 + |Z|^2 - \frac{1}{2}z^\top\Lambda z + \cdots. \tag{3.17}$$

As a result, the total collision orbit $\mathbf{r}(t)$ corresponds exactly to a solution of equations (3.13), (3.14) and (3.15), $(z(\tau), Z(\tau), r(\tau), \gamma(\tau), \theta(\tau))$, such that

$$z \to 0, Z \to 0, r \to 0, \gamma \to 0, \theta' = \Theta \to 0, \quad \text{as } \tau \to -\infty.$$

In particular, *PISPW* explores exactly whether $\theta(\tau)$ approaches a fixed limit as $\tau \to -\infty$ if $(z(\tau), Z(\tau), \gamma(\tau))$ is a solution of equations (3.13) such that

$$z \to 0, Z \to 0, \gamma \to 0, \quad \text{as } \tau \to -\infty.$$

We conclude this subsection with an interesting characterization for total collision orbits of the planar N-body problem.

Given masses $m_1, \ldots, m_N$, suppose that there are n central configurations. Then it is easy to see that

Theorem 3.9. *A total collision orbit* $\mathbf{r}(t)$ *of equations* (1.1) *reduces exactly to a solution* $(z(\tau), Z(\tau), \gamma(\tau))$ *of equations* (3.13) *corresponding to some central configuration* $\mathbf{r}_0$ *such that*

$$z \to 0, Z \to 0, \gamma \to 0 \quad \textit{as } \tau \to -\infty.$$

Therefore, the set of all the total collision orbits corresponds exactly to the union of the unstable set of the origin of the n systems similar to (3.13).

3.2.2. $PISPW$**.** Now $PISPW$ can be formulated as:

PROBLEM 3.10 ($PISPW$). for all the solutions of equations (3.13) satisfying

$$z \to 0, Z \to 0, \gamma \to 0 \tag{3.18}$$

does $\theta(\tau)$ in (3.15) approach a fixed limit as $\tau \to -\infty$?

Let $\alpha(z, Z, \gamma)$ denote the α-limit set of the solution (z, Z, γ) of equations (3.13) and $\alpha(z, Z, \gamma, \theta)$ denote the α-limit set of the solution (z, Z, γ, θ) of equations (3.13) (3.15). Then $PISPW$ can also be stated as following:

PROBLEM 3.11. given a solution (z, Z, γ) of equations (3.13), does the implication

$$\alpha(z, Z, \gamma) = \{0\} \quad \Rightarrow \quad \alpha(z, Z, \gamma, \theta) \text{ is a single point},$$

hold?

We remark that there is exactly one equilibrium point $(z, Z, \gamma) = 0$ of equations (3.13) in some small neighbourhood of the original point $(z, Z, \gamma) = 0$ according to isolation of the central configuration $\mathbf{r}_0$. A complete answer to $PISPW$ is given if we can prove that

$$\mathcal{W}^u(\Sigma) = \bigcup_{p \in \Sigma} \mathcal{W}^u(p)$$

where $\Sigma = \{(z, Z, \gamma, \theta) | z = 0, Z = 0, \gamma = 0, \theta \in \mathbb{R}\}$ is the set of all the equilibrium points of equations (3.13) (3.15).

Before discussing formally the above problem, we wish to give some examples to illustrate some ideas proper or not for $PISPW$.

EXAMPLE 1. Consider the system of differential equations

$$\begin{cases} u' = -u^2(u^2 + 1)v \\ v' = v \end{cases} \tag{3.19}$$

The set of all the equilibrium points of the above equations is the u-axis $\Sigma = \{(u, v) | v = 0\}$ and forms an invariant manifold, furthermore, Σ is a central manifold (see [**6**, **20**]). We cannot simply utilize the theory of central manifolds to prove that $\mathcal{W}^u(\Sigma) = \bigcup_{p \in \Sigma} \mathcal{W}^u(p)$, that is, it does not simply follow from the theory of central manifolds that $v(\tau) \to 0$ implies that $u(\tau)$ approaches a fixed limit as $\tau \to -\infty$. Indeed, the solution

$$\begin{cases} \frac{1}{u} + \arctan u = \exp \tau + \frac{\pi}{2} \\ v = \exp \tau \end{cases}$$

or phase portrait of equations (3.19) in Figure 3.1 shows that

$$\mathcal{W}^u(\Sigma) \neq \bigcup_{p \in \Sigma} \mathcal{W}^u(p).$$

REMARK 3.12. The example also reveals one of difficulties of the application of the theory of normally hyperbolic invariant manifold to $\mathbf{v}$-flow, the flow generated by a vector field $\mathbf{v}$. Indeed, for an invariant manifold Σ of $\mathbf{v}$ to be normally hyperbolic, except the case of $\Sigma = onepoint$, even if Σ is a circle, the condition that the linearized vector field of $\mathbf{v}$ at Σ have "its normal (to Σ) eigenvalues off the

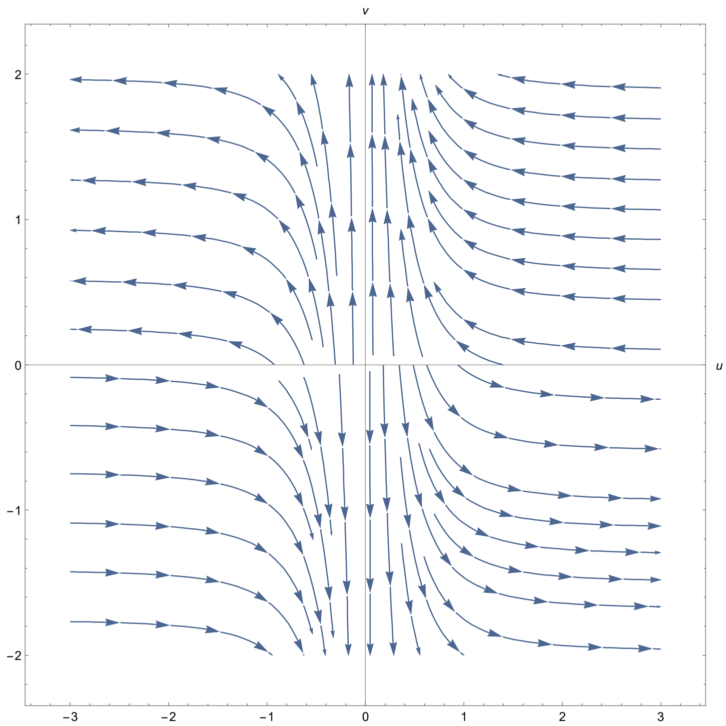

FIGURE 3.1. Phase portrait of equations (3.19) in Example 1

imaginary axis is neither necessary nor sufficient for the **v**-flow. It thus remains an open, fuzzy question to formulate an integrated conditions on **v** at Σ that guarantee normally hyperbolicity of the **v**-flow" [**11**, p.8].

EXAMPLE 2. Consider the following functions

$$\begin{cases} u(\tau) = \frac{1}{\sqrt{\tau}\ln\tau}\sin\tau, \\ v(\tau) = \frac{1}{\sqrt{\tau}\ln\tau}\cos\tau. \end{cases}$$

Then $u(\tau), v(\tau), u'(\tau), v'(\tau)$ approach zero as $\tau \to +\infty$, however, it is easy to see that all the following improper integrals

$$\int^{+\infty} u(\tau)v'(\tau)d\tau, \int^{+\infty} v(\tau)u'(\tau)d\tau, \int^{+\infty} u(\tau)v'(\tau) - v(\tau)u'(\tau)d\tau$$

are not convergent.

Similarly, we cannot simply claim that $\theta(\tau)$ approaches a fixed limit as $\tau \to -\infty$, although

$$z \to 0, Z \to 0,$$

and $\int_{-\infty} Z(\tau)d\tau$ is convergent.

Let us finish the section with further simplifying equations (3.13) to make preparations for resolving $PISPW$.

The aim is to diagonalize the linear part of equations (3.13). More specifically, it is well known that the $(4N-7)\times(4N-7)$ square matrix $\mathfrak{A}$ in equations (3.13)

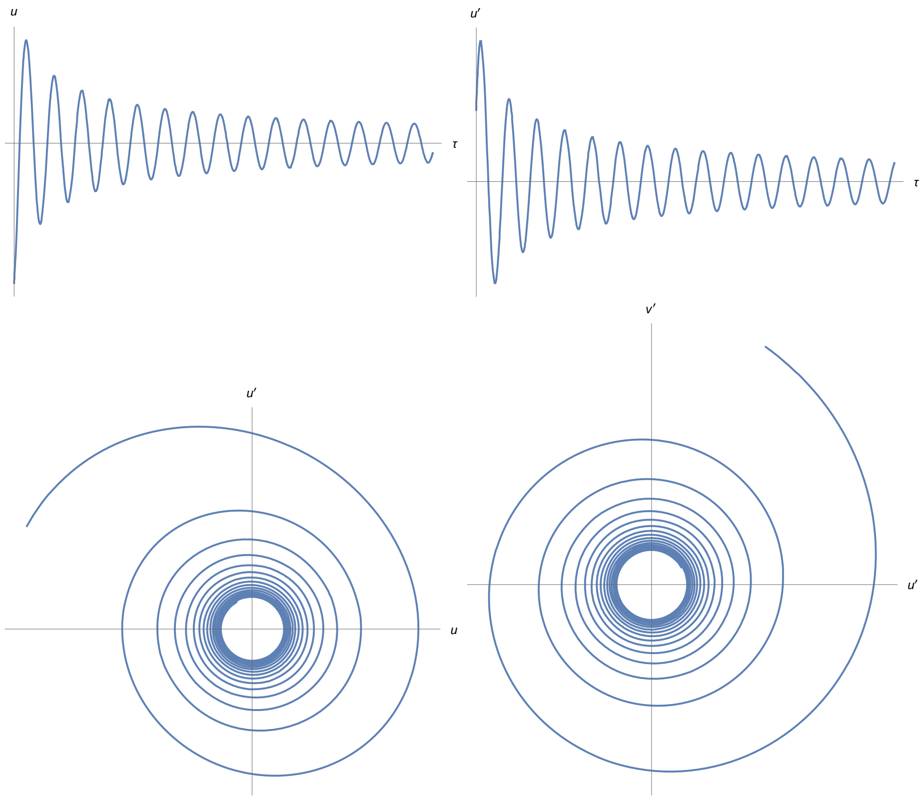

FIGURE 3.2. plots in Example 2

is generically diagonalizable, that is, $\mathfrak{A}$ is similar to a diagonal matrix, provided that $\mathfrak{A}$ has $4N-7$ distinct eigenvalues; on the other hand, $\mathfrak{A}$ is always "almost" diagonalizable, that is, for any $\epsilon > 0$, there exists an upper triangular matrix, such that all the elements above the diagonal are less than ϵ in absolute value, is similar to $\mathfrak{A}$. However, to avoid complexifying the system (3.13), it is enough to find a block diagonal matrix $\mathfrak{C}$ similar to $\mathfrak{A}$. Indeed, in Appendix C we show that for any $\epsilon > 0$, we find that a $(4N-8)\times(4N-8)$ invertible square matrix $\mathfrak{P}$ such that if we set

$$\mathfrak{C} = \begin{pmatrix} \mathfrak{P}^{-1} & \\ & 1 \end{pmatrix} \mathfrak{A} \begin{pmatrix} \mathfrak{P} & \\ & 1 \end{pmatrix},$$

then $\mathfrak{C}$ is in block-diagonal form.

Consequently, after applying the linear substitution

$$\begin{pmatrix} z \\ Z \end{pmatrix} = \mathfrak{P} q \tag{3.20}$$

to the equations (3.13), we arrive at the following equations:

$$\begin{pmatrix} q' \\ \gamma' \end{pmatrix} = \mathfrak{C} \begin{pmatrix} q \\ \gamma \end{pmatrix} + \varphi(q,\gamma), \tag{3.21}$$

see again Appendix C.

After applying the above linear substitution to equations (3.15), it follows that

$$\theta' = z^\top QZ = \sum_{1\le k\le n_0}\sum_{j=n_0+1}^{n_0+n_p} q_{kj}\tilde{\mu}_j q_k q_j + \sum_{j,k=n_0+1}^{n_0+n_p} q_{kj}\tilde{\mu}_j q_k q_j + \cdots, \tag{3.22}$$

the right hand side is a quadratic form of q, where "$\cdots$" denotes all the quadratic terms which contain at least one of q_k $(k > n_0 + n_p)$ as a factor.

To simplify notations, by applying a permutation, $\mathcal{C}$ is reduced to the following block-diagonal form:

$$\mathfrak{C} = \begin{pmatrix} \mathfrak{C}^0 & & \\ & \mathfrak{C}^+ & \\ & & \mathfrak{C}^- \end{pmatrix},$$

where $\mathfrak{C}^0$ is the $n_0 \times n_0$ null matrix, $\mathfrak{C}^+$ is a $(n_p+1)\times(n_p+1)$ diagonal matrix with positive diagonal elements, $\mathfrak{C}^-$ is a matrix such that all eigenvalues have negative real part; correspondingly, the components of (q, γ) are reordered as (q^0, q^+, q^-), and the components of φ are reordered as $(\varphi^0, \varphi^+, \varphi^-)$. More specifically,

$$\begin{aligned} q^0 &= (q_1, \cdots, q_{n_0})^\top, \\ q^+ &= (\gamma, q_{n_0+1}, \cdots, q_{n_0+n_p})^\top, \\ q^- &= (q_{n_0+n_p+1}, \cdots, q_{2N-4}, q_{2N-3}, \cdots, q_{4N-8})^\top, \\ \varphi^0 &= (\varphi_1, \cdots, \varphi_{n_0})^\top, \\ \varphi^+ &= (\varphi_0, \varphi_{n_0+1}, \cdots, \varphi_{n_0+n_p})^\top, \\ \varphi^- &= (\varphi_{n_0+n_p+1}, \cdots, \varphi_{2N-4}, \varphi_{2N-3}, \cdots, \varphi_{2N-8})^\top. \end{aligned}$$

Consequently, the system (3.21) can be rewritten as the following system of equations:

$$\begin{cases} q'^0 = \varphi^0(q^0, q^+, q^-), \\ q'^+ = \mathfrak{C}^+ q^+ + \varphi^+(q^0, q^+, q^-), \\ q'^- = \mathfrak{C}^- q^- + \varphi^-(q^0, q^+, q^-). \end{cases} \tag{3.23}$$

Please refer to Appendix C for more detail.

CHAPTER 4

Resolving $PISPW$

Let us turn to discuss $PISPW$ in this section. As the work of Siegel [**22**] indicates, the theory of normal forms is useful for discussing $PISPW$. Furthermore, in the discussion of $PISPW$ for the case corresponding to central configurations with degree of degeneracy two, it is natural to investigate equilibrium points in two-dimensional systems and degenerate central configurations of the planar four-body problem. To avoid disrupting the flow of the discussion, these materials are deferred to Appendix B, D and E.

The main result on $PISPW$ is the following theorem.

THEOREM 4.1. *For given masses of the N-body problem, if all the central configurations are isolated and have degree of degeneracy less than or equal to two, and central configurations with degree of degeneracy two satisfy the condition* (4.16)*, then the normalized configuration of any total collision orbit of the given N-body problem approaches a certain central configuration as time t approaches the collision instant.*

REMARK 4.2. For collision orbits of the four-body problem, $PISPW$ is now unanswered in very few cases corresponding to central configurations with degree of degeneracy two. Recall that the masses for which the degenerate central configuration exists form a proper algebraic subset of the mass space for the four-body problem [**16**]. Furthermore, following a crude dimension count, almost all degenerate central configurations have degree of degeneracy one, and the masses that admit central configurations with degree of degeneracy two should form a subset of the mass space consisting of finite points.

Indeed in Appendix B.4 we venturesomely conjectured that:

CONJECTURE. *All the four-body central configurations except the degenerate equilateral central configuration have degree of degeneracy equal or less than one.*

Here, the degenerate equilateral central configuration is a configuration with three particles m_1, m_2, m_3 lying at three vertices of a regular triangle and the fourth particle m_4 lying at the center, here $m_1 = m_2 = m_3 = \frac{m_4}{(64\sqrt{3}+81)/249}$, see Appendix B.2 for more detail.

As a result, we have the following corollary.

COROLLARY 4.3. *If the above conjecture is correct, the normalized configuration of any total collision orbit of the four-body problem approaches a certain central configuration as time t approaches the collision instant.*

In the following we divide the proof of Theorem 4.1 into several cases according to the degree of degeneracy of central configurations. One of the key points is to estimate the speed of convergence to the origin for z, Z or q, γ; if $n_0 = 0$, the system

(3.23) is hyperbolic, then it is easy to estimate the speed; if $n_0 = 1$, the system (3.23) is nonhyperbolic, however, the center manifold of (3.23) is one-dimensional, then it is not difficult to estimate the speed; if $n_0 = 2$, the system (3.23) is nonhyperbolic and its center manifold is two-dimensional, thanks to the fact that the origin is an isolated equilibrium, one can successfully estimate the speed under fairly general conditions. Unfortunately, if $n_0 \geq 3$, the dimension of center manifold of the system (3.23) exceeds two, then it is too difficult to estimate the speed; indeed, it is even too difficult to understand orbits on the center manifold qualitatively.

4.1. $n_0 = 0$

First, let us discuss what happen if $n_0 = 0$, i.e., the central configuration $\mathbf{r}_0$ is nondegenerate.

Now the problem reduces to the following form:

PROBLEM 4.4. given a solution $(q(\tau), \gamma(\tau))$ of the system (3.23) such that

$$(q(\tau), \gamma(\tau)) \to 0 \text{ as } \tau \to -\infty,$$

if $\theta(\tau)$ satisfies (3.22) then does $\theta(\tau)$ approach a fixed limit as $\tau \to -\infty$?

By solving the above problem, we have the following theorem.

THEOREM 4.5. *Given a total collision orbit of the N-body problem, if the orbit converges to the $S\mathbb{O}(2)$-orbit of a nondegenerate central configuration $\mathbf{r}_0$, then the normalized configuration of the orbit approaches a certain central configuration as time t approaches the collision instant.*

PROOF. It is well known that an orbit on the unstable manifold of a hyperbolic equilibrium point approaches the hyperbolic equilibrium point exponentially fast as $\tau \to -\infty$ (for example, see Corollary D.9). Since the origin is hyperbolic for the system (3.23), it follows that, for a fixed τ_0, there are two positive constants ϖ_1 and σ such that

$$\|u\| \leq \varpi_1 e^{\sigma\tau} \qquad \forall \tau \leq \tau_0,$$

where

$$u = (\gamma, q_1, \cdots, q_{4N-8})^\top$$

and

$$\|u\| = \sqrt{|\gamma|^2 + |q_1|^2 + \cdots + |q_{4N-8}|^2}.$$

Thanks to the equation (3.22), θ' is a quadratic form in the variable u. It follows that there exists a positive constants ϖ_2 such that

$$|\theta'| \leq \varpi_2 \|u\|^2.$$

Then

$$|\theta'| \leq \varpi_2 \varpi_1^2 e^{2\sigma\tau} \qquad \forall \tau \leq \tau_0.$$

It follows from Cauchy's test for convergence that $\theta(\tau)$ approaches a fixed limit as $\tau \to -\infty$.

In conclusion, the proof of Theorem 4.5 is completed. That is, we have solved *PISPW* for nondegenerate central configurations. □

4.2. $n_0 > 0$

When $n_0 > 0$, i.e., the central configuration $\mathbf{r}_0$ is degenerate, the problem reduces to the following form:

PROBLEM 4.6. given a solution $(q(\tau), \gamma(\tau))$ of the system (3.23) such that

$$(q(\tau), \gamma(\tau)) \to 0 \quad \text{as} \quad \tau \to -\infty,$$

if $\theta(\tau)$ satisfies (3.22) then does $\theta(\tau)$ approach a fixed limit as $\tau \to -\infty$?

4.2.1. Preliminaries of Proof.

(A) The first step to solve the above problem is to simplify equations (3.23) by using the reduction theorems (see Appendix D). The idea is as follows: for any given orbit on the center-unstable manifold of the system (3.23), we can find an orbit on the center manifold of the system (3.23) approaching it exponentially fast.

It follows from Corollary D.5 that we can introduce a nonlinear substitution in the form of

$$\begin{cases} u_0 = \gamma, & \\ u_k = q_k, & k \in \{1, \cdots, n_0 + n_p\} \\ u_k = q_k - F_k^{cu}(q^0, q^+), & k \in \{n_0 + n_p + 1, \cdots, 4N - 8\} \end{cases} \tag{4.1}$$

so that the system (3.23) can be written as the simpler form below

$$\begin{cases} u'^0 = \varphi^0(u^0, u^+, F^{cu}(u^0, u^+)) + \psi^0(u)u^-, \\ u'^+ = \mathfrak{C}^+ u^+ + \varphi^+(u^0, u^+, F^{cu}(u^0, u^+)) + \psi^+(u)u^-, \\ u'^- = \mathfrak{C}^- u^- + \psi^-(u)u^-, \end{cases} \tag{4.2}$$

where

$$u^0 = (u_1, \cdots, u_{n_0})^\top$$
$$u^+ = (u_0, u_{n_0+1}, \cdots, u_{n_0+n_p})^\top$$
$$u^- = (u_{n_0+n_p+1}, \cdots, u_{4N-8})^\top,$$

the equation

$$q^- = F^{cu}(q^0, q^+)$$

defines a center-unstable manifold of class C^l, moreover F^{cu} and all the partial derivative of F^{cu} are vanishing at $(q^0, q^+) = 0$. The functions ψ^0, ψ^+ are C^l-smooth and ψ^- is C^{l-1}-smooth; in addition, all the functions ψ^0, ψ^+, ψ^- are vanishing at the origin, i.e., $\psi^*(0) = 0$, where $* \in \{0, +, -\}$.

By Theorem D.3, an orbit such that $(q(\tau), \gamma(\tau)) \to 0$ as $\tau \to -\infty$ is necessarily contained on any center-unstable manifold, so that $u^-(\tau)$ vanishes identically.

Therefore, for a solution of Problem 4.6 the system (4.2) reduces to

$$\begin{cases} u'^0 = \varphi^0(u^0, u^+, F^{cu}(u^0, u^+)), \\ u'^+ = \mathfrak{C}^+ u^+ + \varphi^+(u^0, u^+, F^{cu}(u^0, u^+)). \end{cases} \tag{4.3}$$

Then it follows from the Theorems D.7 and D.8 that there exists a solution $\upsilon(\tau) = (\upsilon_1(\tau), \cdots, \upsilon_{n_0}(\tau))^\top$ of the following system

$$u'^0 = \varphi^0\left(u^0, F^c(u^0), F^{cu}\left(u^0, F^c(u^0)\right)\right), \tag{4.4}$$

such that

$$\begin{cases} u^0(\tau) = \upsilon(\tau) + O(e^{\sigma\tau}) \\ u^+(\tau) = F^c(\upsilon(\tau)) + O(e^{\sigma\tau}) \end{cases} \quad \text{as } \tau \to -\infty, \tag{4.5}$$

where the equation $u^+ = F^c(u^0)$ defines a center manifold of class C^l, moreover F^c and all the partial derivative of F^c are vanishing at $u^0 = 0$; the term $\sigma > 0$ is a constant depending only on $\mathfrak{C}^+$.

(B) Let us first simplify the Equation (3.22) by using the variables υ.

Due to the equation (3.22) and the nonlinear substitution (4.1), now we have

$$\begin{aligned} \theta' &= \left(u^0, u^+, F^{cu}(u^0, u^+)\right)^\top \mathfrak{P}^\top \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix} Q \begin{pmatrix} 0 & \mathbb{I} \end{pmatrix} \mathfrak{P} \left(u^0, u^+, F^{cu}(u^0, u^+)\right) \\ &= \textstyle\sum_{j=1}^{n_0} \sum_{k=n_0+1}^{n_0+n_p} q_{jk}\tilde{\mu}_k u_k u_j + \sum_{j,k=n_0+1}^{n_0+n_p} q_{jk}\tilde{\mu}_k u_k u_j + \cdots \end{aligned} \tag{4.6}$$

where "$\cdots$" denotes all the terms which contain at least one of F_k^{cu} $(k > n_0 + n_p)$ as a factor.

By Taylor's formula, the equation (4.6) can be rewritten as

$$\theta' = \sum_{k=1}^{n_0} \sum_{j=n_0+1}^{n_0+n_p} q_{kj}\tilde{\mu}_j u_k u_j + \sum_{j,k=n_0+1}^{n_0+n_p} q_{kj}\tilde{\mu}_j u_k u_j + \sum_{|\alpha|=3}^{l} b_\alpha (u^{0+})^\alpha + o_l(u^{0+}), \tag{4.7}$$

where o_l denotes the reminder term which vanishes at the origin along with the first l derivatives,

$$\begin{aligned} u^{0+} &= (u^0, u^+) = (u_1, \cdots, u_{n_0}, u_0, u_{n_0+1}, \cdots, u_{n_0+n_p})^\top, \\ \alpha &= (\alpha^0, \alpha^+) = (\alpha_1, \cdots, \alpha_{n_0}, \alpha_0, \alpha_{n_0+1}, \cdots, \alpha_{n_0+n_p}) \end{aligned}$$

is a multiindex and

$$|\alpha| = \alpha_1 + \cdots + \alpha_{n_0} + \alpha_0 + \alpha_{n_0+1} + \cdots + \alpha_{n_0+n_p}.$$

Obviously, to determine the coefficients b_α, by Theorem D.3, it suffices to find the coefficients of $F^{cu}(q^0, q^+)$ according to the following relationship:

$$\begin{aligned} &\mathfrak{C}^- F^{cu}(q^0, q^+) + \varphi^-(q^0, q^+, F^{cu}(q^0, q^+)) \\ &\quad = \tfrac{\partial F^{cu}(q^0,q^+)}{\partial q^0} \varphi^0(q^0, q^+, F^{cu}(q^0, q^+)) \\ &\quad + \tfrac{\partial F^{cu}(q^0,q^+)}{\partial q^+} \left[\mathfrak{C}^+ q^+ + \varphi^+(q^0, q^+, F^{cu}(q^0, q^+))\right]. \end{aligned} \tag{4.8}$$

As a matter of fact, we can use the method of undetermined coefficients to find $\mathbf{c}_\alpha$, the coefficient of $(q^0)^{\alpha^0}(q^+)^{\alpha^+}$ in $F^{cu}(q^0, q^+)$. Suppose that we have already determined all coefficients in $F^{cu}(q^0, q^+)$ of total degrees $2, \cdots, |\alpha| - 1$. Then, by equating the coefficients of $(q^0)^{\alpha^0}(q^+)^{\alpha^+}$ in the above equation, we determine $\mathbf{c}_\alpha$ from the following recurrence formula:

$$\begin{aligned} &\mathfrak{C}^- \mathbf{c}_\alpha + \{\varphi^-(q^0, q^+, F^{cu}(q^0, q^+))\}_\alpha \\ &\quad = \{\tfrac{\partial F^{cu}(q^0,q^+)}{\partial q^0} \varphi^0(q^0, q^+, F^{cu}(q^0, q^+))\}_\alpha \\ &\quad + (\alpha^+, \mathfrak{C}^+)\mathbf{c}_\alpha + \{\tfrac{\partial F^{cu}(q^0,q^+)}{\partial q^+} \varphi^+(q^0, q^+, F^{cu}(q^0, q^+))\}_\alpha, \end{aligned}$$

this is because $\mathfrak{C}^- - (\alpha^+, \mathfrak{C}^+)\mathbb{I}$ is an invertible matrix; where

$$(\alpha^+, \mathfrak{C}^+) = \alpha_0 \kappa^{\frac{1}{2}} + \alpha_{n_0+1}\tilde{\mu}_{n_0+1} + \cdots + \alpha_{n_0+n_p}\tilde{\mu}_{n_0+n_p} > 0,$$

$\{f\}_\alpha$ stands for the coefficient of the term $(q^0)^{\alpha^0}(q^+)^{\alpha^+}$ in the Taylor expansion of f. Note that three terms $\{f\}_\alpha$ in the above recurrence formula only involve coefficients in F^{cu} of total degrees $2, \cdots, |\alpha|-1$. If we consider $|\alpha| = 2$, then it is easy to see that

$$(\mathfrak{C}^- - (\alpha^+, \mathfrak{C}^+)\mathbb{I})\mathbf{c}_\alpha = -\{\varphi^-(q^0, q^+, F^{cu}(q^0, q^+))\}_\alpha,$$

and

$$\{\varphi^-(q^0, q^+, F^{cu}(q^0, q^+))\}_\alpha = \{\varphi^-(q^0, q^+, 0)\}_\alpha.$$

Consequently, we can explicitly compute the coefficients b_α for $|\alpha| = |\alpha^0| = 3$, where $|\alpha^0| = \alpha_1 + \cdots + \alpha_{n_0}$. To this end, it suffices to pay attention to the terms in "$\cdots$" in (4.6) containing $u_j F_k^{cu}(u^0, u^+)$ such $n_0 \geq j > 0$ and $k > n_0 + n_p$. More precisely, a straightforward computation shows that

$$\sum_{|\alpha|=|\alpha^0|=3} b_\alpha (u^0)^{\alpha^0} = \sum_{|\alpha|=|\alpha^0|=2} (u^0)^\top \begin{pmatrix}\mathbb{I}_{n_0} & 0\end{pmatrix} Q\mathfrak{P}\left(0, 0, \mathbf{c}_\alpha (u^0)^{\alpha^0}\right), \tag{4.9}$$

where $\mathbb{I}_{n_0}$ is the identity matrix of order n_0.

Similarly, the relation

$$\begin{aligned}&\mathfrak{C}^+ F^c(u^0) + \varphi^+\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right)\\ &= \tfrac{\partial F^c(u^0)}{\partial u^0}\varphi^0\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right)\end{aligned} \tag{4.10}$$

gives an algorithm for computing the Taylor's coefficients of $F^c(u^0)$. Here only the quadratic terms are computed explicitly.

PROPOSITION 4.7. *The quadratic form in Taylor's formula of $F_k^c(u^0)$ is*

$$\begin{cases} 0, & k = 0, \\ \frac{\sum_{i,j=1}^{n_0} a_{ijk} u_i u_j}{4\tilde{\mu}_k \sqrt{\mu_k + \frac{\kappa}{16}}}, & k = n_0 + 1, \cdots, n_0 + n_p; \end{cases} \tag{4.11}$$

the quadratic form of $\varphi_k^0\left(u^0, F^c(u^0), F^{cu}\left(u^0, F^c(u^0)\right)\right)$ for $k = 1, \cdots, n_0$ is

$$\frac{\sum_{i,j=1}^{n_0} a_{ijk} u_i u_j}{\sqrt{\kappa}}.$$

PROOF. Suppose that $\mathbf{a}_{\alpha^0}$ is the coefficient of $(u^0)^{\alpha^0}$ in $F^c(u^0)$, and $\mathbf{b}_{\alpha^0}$ is the coefficient of $(u^0)^{\alpha^0}$ in $\varphi^0\left(u^0, F^c(u^0), F^{cu}\left(u^0, F^c(u^0)\right)\right)$. By comparing the coefficients of $(u^0)^{\alpha^0}$ in (4.10), we obtain the following recurrence formula:

$$\begin{aligned}&\mathfrak{C}^+ \mathbf{a}_{\alpha^0} + \{\varphi^+\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right)\}_{\alpha^0}\\ &= \{\tfrac{\partial F^c(u^0)}{\partial u^0}\varphi^0\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right)\}_{\alpha^0}.\end{aligned}$$

If we consider $|\alpha^0| = \alpha_1 + \cdots + \alpha_{n_0} = 2$, then it is easy to see that

$$\mathfrak{C}^+ \mathbf{a}_{\alpha^0} = -\{\varphi^+\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right)\}_{\alpha^0},$$

$$\mathbf{b}_{\alpha^0} = \{\varphi^0\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right)\}_{\alpha^0} = \{\varphi^0\left(u^0, 0, 0\right)\}_{\alpha^0},$$

and

$$\{\varphi^+\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right)\}_{\alpha^0} = \{\varphi^+\left(u^0, 0, 0\right)\}_{\alpha^0}.$$

By the substitutions (3.20) and (4.1), recall that,

$$\begin{cases} z_k = q_k + \cdots = u_k + \cdots, & k \in \{1, \cdots, n_0\} \\ z_k = \cdots, & k \in \{n_0 + 1, \cdots, 2N - 4\} \\ Z_k = \cdots, & k \in \{1, \cdots, 2N - 4\} \\ \gamma = u_0 = \cdots, & \end{cases}$$

where "$\cdots$" denotes all the terms which contain at least one of u_k ($k = 0, n_0 + 1, \cdots, 4N - 8$) as a factor. As a result, if we set

$$u_k = 0, \qquad \text{for } k = 0, n_0 + 1, \cdots, 4N - 8,$$

it follows that

$$\begin{cases} z_k = u_k, & k \in \{1, \cdots, n_0\} \\ z_k = 0, & k \in \{n_0 + 1, \cdots, 2N - 4\} \\ Z_k = 0, & k \in \{1, \cdots, 2N - 4\} \\ \gamma = 0. & \end{cases}$$

Then, by (3.16), (3.17) and (C.6), we have

$$\begin{cases} \varphi_k\left(u^0, 0, 0\right) = \frac{\frac{1}{2}[\sum_{i,j=1,}^{n_0} a_{ijk} z_i z_j]}{2\sqrt{\mu_k + \frac{\kappa}{16}}} + \cdots = \frac{\sum_{i,j=1,}^{n_0} a_{ijk} u_i u_j}{4\sqrt{\mu_k + \frac{\kappa}{16}}} + \cdots, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in \{1, \cdots, n_0 + n_p\} \\ \\ \varphi_0\left(u^0, 0, 0\right) = -\frac{1}{2} u_0^\top \Lambda_0 u_0 + \cdots = 0 + \cdots, \end{cases}$$

where "$\cdots$" denotes all the terms of degree higher than or equal to 3.

Thus we have proved the proposition. □

Consequently, based on the Taylor expansion, the system (4.4) can be written as

$$v_k' = \frac{\sum_{i,j=1}^{n_0} a_{ijk} v_i v_j}{\sqrt{\kappa}} + \cdots + o_l(v), \quad k \in \{1, \cdots, n_0\}. \tag{4.12}$$

where "$\cdots$" denotes all the terms of degrees from 3 to l.

Taking into consideration of (4.5), by (4.9) and (4.11), it follows that the equation (4.7) can be further rewritten as

$$\theta' = \sum_{k=n_0+1}^{n_0+n_p} \sum_{j,m,n=1}^{n_0} \frac{q_{jk} a_{mnk} v_j v_m v_n}{4\sqrt{\mu_k + \frac{\kappa}{16}}} + \sum_{|\alpha|=|\alpha^0|=3} b_\alpha v^{\alpha^0} + \cdots + o_l(v) + O(e^{\sigma\tau}).$$

where "$\cdots$" denotes all the terms of degree from 4 to l. Obviously, we can discard the term $O(e^{\sigma\tau})$ without affecting the convergence of the integral $\int_{-\infty}^{\tau_0} \theta'(\tau) d\tau$. Therefore it suffices to consider

$$\theta' = \sum_{k=n_0+1}^{n_0+n_p} \sum_{j,m,n=1}^{n_0} \frac{q_{jk} a_{mnk} v_j v_m v_n}{4\sqrt{\mu_k + \frac{\kappa}{16}}} + \sum_{|\alpha|=|\alpha^0|=3} b_\alpha v^{\alpha^0} + \cdots + o_l(v) \tag{4.13}$$

to establish the convergence of $\theta(\tau)$ as $\tau \to -\infty$.

As a result, by (4.5) it is easy to see that Problem 4.6 reduces to

PROBLEM 4.8. given a solution $\upsilon(\tau)$ of the system (4.4) or (4.12) such that

$$\upsilon(\tau) \to 0 \quad \text{as} \quad \tau \to -\infty,$$

if $\theta(\tau)$ satisfies (3.22) or (4.13) then does $\theta(\tau)$ approach a fixed limit as $\tau \to -\infty$?

(C) In general, the quadratic forms in (4.12) may be zero. However, we claim that the Taylor's coefficients of the right side of (4.12) are not all zero, that is,

PROPOSITION 4.9. *For sufficiently large l, there exists some nonzero Taylor's coefficient in the Taylor expansion of the right side of the system* (4.4) *or* (4.12).

PROOF. The proposition is an inference of the fact that the origin is an isolated equilibrium. However, since it is probable that $\varphi^0\left(u^0, F^c(u^0), F^{cu}\left(u^0, F^c(u^0)\right)\right)$ is not an analytic function, the proof is not immediate.

It suffices to prove the proposition in the case of formal power series.

If the statement is false, then

$$\varphi^0\left(u^0, F^c(u^0), F^{cu}\left(u^0, F^c(u^0)\right)\right) \equiv 0$$

in the sense of formal power series.

It follows from (4.10) that

$$\mathfrak{C}^+ F^c(u^0) + \varphi^+\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right) \equiv 0$$

By (4.8), it turns out that

$$\mathfrak{C}^- F^{cu}(u^0, F^c(u^0)) + \varphi^-\left(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))\right) \equiv 0$$

As a result, u^0 and two formal power series

$$f_1(u^0) = F^c(u^0), f_2(u^0) = F^{cu}(u^0, F^c(u^0))$$

satisfy

$$\begin{cases} \varphi^0\left(u^0, f_1, f_2\right) = 0, \\ \mathfrak{C}^+ f_1 + \varphi^+\left(u^0, f_1, f_2\right) = 0, \\ \mathfrak{C}^- f_2 + \varphi^-\left(u^0, f_1, f_2\right) = 0. \end{cases}$$

It is noteworthy that the two equations

$$\begin{cases} \mathfrak{C}^+ f_1 + \varphi^+\left(u^0, f_1, f_2\right) = 0, \\ \mathfrak{C}^- f_2 + \varphi^-\left(u^0, f_1, f_2\right) = 0, \end{cases}$$

are enough to determine $f_1(u^0), f_2(u^0)$; furthermore, by the fact that $\varphi^+(u^0, u^+, u^-)$ and $\varphi^-\left(u^0, u^+, u^-\right)$ are two analytic functions, it follows from the analytic version of implicit function theorem that $f_1(u^0), f_2(u^0)$ are more than just formal power series, they are really analytic functions of u^0.

Therefore, we have infinitely many critical points of the system (3.23) in a small neighbourhood of the origin. However, we know that the origin is an isolated equilibrium point of the system (3.23). This leads to a contradiction. □

4.2.2. $n_0 = 1$**.** By solving the Problem 4.8 for $n_0 = 1$, we completely solve *PISPW* for $n_0 \le 1$ in this subsection. That is

THEOREM 4.10. *Given a total collision orbit of the N-body problem, if the orbit converges to the* $S\mathbb{O}(2)$*-orbit of an isolated central configuration* $\mathbf{r}_0$ *and* $\mathbf{r}_0$ *has degree of degeneracy one, then the normalized configuration of the orbit approaches a certain central configuration as time t approaches the collision instant.*

PROOF. Consider Problem 4.8 for $n_0 = 1$.

As a matter of notational convenience, set $v_1 = w$. Then the system (4.12) becomes

$$w' = c_2 w^2 + \cdots + c_l w^l + o_l(w).$$

By Proposition 4.9, suppose

$$c_2 = \cdots = c_{m-1} = 0, c_m \neq 0, 2 \le m < l,$$

then

$$w' = c_m w^m + \cdots + c_l w^l + o_l(w). \tag{4.14}$$

If $w(\tau) = 0$ for some τ, then $w(\tau) \equiv 0$, and (4.13) becomes $\theta'(\tau) = 0$, therefore $\theta(\tau)$ is obviously convergent as $\tau \to -\infty$.

So we consider $w(\tau) \neq 0$ as $\tau \to -\infty$. Without loss of generality, we assume $w(\tau) > 0$, i.e., $w(\tau) \to 0+$ as $\tau \to -\infty$.

According to L'Hópital's rule, it follows from (4.14) that

$$\lim_{\tau \to -\infty} \frac{1}{\tau w^{m-1}} = (1-m)c_m,$$

or

$$w = \left(\frac{1}{(1-m)c_m \tau}\right)^{\frac{1}{m-1}} + o\left(\left(\frac{1}{-\tau}\right)^{\frac{1}{m-1}}\right) \qquad \text{as } \tau \to -\infty.$$

We cannot prove the convergence of $\theta(\tau)$ by (4.13), but a proof based on the original equation (3.15) will be given right now.

By substitutions (3.20), (4.1) and (4.5), it is simple to show that for every $j = 1, \cdots, 2N-4$,

$$Z_j = c_{j,2} w^2 + \cdots + c_{j,l} w^l + o_l(w).$$

If $c_{j,2}, \cdots, c_{j,l}$ are all zero, then it is easy to show that

$$|Z_j| = O\left(\left(\frac{1}{-\tau}\right)^{1+\frac{2}{l-2}}\right),$$

it follows that the improper integral $\int_{-\infty}^{\tau_0} |Z_j| d\tau$ converges. If at least one of $c_{j,2}, \cdots, c_{j,l}$ is not zero, then Z_j takes the following form

$$Z_j = \tilde{c}_j \left(\frac{1}{-\tau}\right)^{\frac{m_j}{m-1}} + o\left(\left(\frac{1}{-\tau}\right)^{\frac{m_j}{m-1}}\right),$$

where $m_j \in \{2, \cdots, l\}$. So Z_j converges to zero with constant sign. Then it follows from $z_j' = Z_j$ that the improper integral $\int_{-\infty}^{\tau_0} |Z_j| d\tau$ also converges.

As a result, the improper integral $\int_{-\infty}^{\tau_0} |Z_j| d\tau$ converges for every $j = 1, \cdots, 2N-4$. Consequently, the improper integral

$$\left| \int_{-\infty}^{\tau_0} \theta'(\tau) d\tau \right| \le \int_{-\infty}^{\tau_0} \sum_{j,k=1}^{2N-4} | q_{kj} || Z_j || z_k | \, d\tau$$

also converges.

Therefore $\theta(\tau)$ is obviously convergent as $\tau \to -\infty$.

In conclusion, the proof of Theorem 4.10 is completed. □

4.2.3. $n_0 = 2$. Due to the intrinsic difficulty of degenerate or nonhyperbolic differential equations, the difficulty of the problem increases rapidly for $n_0 \geq 2$. Indeed, we cannot completely resolve $PISPW$ even in the case of $n_0 = 2$. The main difficulty comes from estimating the speed of tending to the equilibrium point for an orbit on a center-unstable manifold.

In particular, we could not apply a similar method as that of $n_0 = 1$, because generally we can not prove that every Z_j $(j = 1, \cdots, 2N-4)$ converges with constant sign to zero. Indeed, in general, Z_j can converges to zero with alternating signs as shown in Example 2.

Naturally, one hopes that researches on central configurations would help for resolving $PISPW$. Unfortunately, the problem of central configurations is also very difficult, as indicated by researches on this topic in the past decades. Therefore, we can only utilize results of central configurations in some special cases.

We consider Problem 4.8 for $n_0 = 2$ in this subsection. First, we give a criterion for the case of $n_0 = 2$; then we give an answer to $PISPW$ for all known degenerate central configurations of four bodies. Therefore, for almost every choice of the masses of the four-body problem, $PISPW$ is answered.

Theorem 4.11. *Given a total collision orbit of the N-body problem, if the orbit converges to the $S\mathbb{O}(2)$-orbit of an isolated central configuration $\mathbf{r}_0$ and $\mathbf{r}_0$ has degree of degeneracy two and satisfies the condition* (4.16)*, then the normalized configuration of the orbit approaches a certain central configuration as time t approaches the collision instant.*

As an example, we have the following result:

Corollary 4.12. *Given a total collision orbit of the four-body problem, if the orbit converges to the $S\mathbb{O}(2)$-orbit of $\mathbf{r}_0$, where $\mathbf{r}_0$ is the degenerate equilateral central configuration discovered by J. Palmore in* [**18**]*, then the normalized configuration of the orbit approaches a certain central configuration as time t approaches the collision instant.*

Proof of Theorem 4.11. Consider Problem 4.8 for $n_0 = 2$.

As a matter of notational convenience, set $(\upsilon_1, \upsilon_2) = (\zeta, \eta)$. Then the system (4.12) becomes

$$\begin{cases} \zeta' = c_1\zeta^2 + 2c_2\zeta\eta + c_3\eta^2 + o(\zeta^2 + \eta^2) \\ \eta' = c_2\zeta^2 + 2c_3\zeta\eta + c_4\eta^2 + o(\zeta^2 + \eta^2) \end{cases} \tag{4.15}$$

where

$$c_1 = \frac{a_{111}}{\sqrt{\kappa}}, c_2 = \frac{a_{112}}{\sqrt{\kappa}}, c_3 = \frac{a_{122}}{\sqrt{\kappa}}, c_4 = \frac{a_{222}}{\sqrt{\kappa}}.$$

Let us introduce polar coordinates as in Appendix E

$$\zeta = \rho\cos\vartheta, \eta = \rho\sin\vartheta,$$

then the system (4.15) becomes

$$\begin{cases} \rho' = \rho^2\Phi(\vartheta) + o(\rho^2), \\ \vartheta' = \rho\Psi(\vartheta) + o(\rho), \end{cases}$$

where

$$\begin{cases} \Phi(\vartheta) = c_1 \cos^3\vartheta + 3c_2 \cos^2\vartheta \sin\vartheta + 3c_3 \cos\vartheta \sin^2\vartheta + c_4 \sin^3\vartheta, \\ \Psi(\vartheta) = c_2 \cos^3\vartheta + (2c_3 - c_1) \cos^2\vartheta \sin\vartheta + (c_4 - 2c_2) \cos\vartheta \sin^2\vartheta - c_3 \sin^3\vartheta; \end{cases}$$

thus Φ, Ψ are homogeneous polynomials of degree 3 in $\cos\vartheta, \sin\vartheta$.

To prove the theorem, we need two lemmas.

LEMMA 4.13. *Assume c_1, c_2, c_3, c_4 are not all zero. Then*

$$\vartheta_0 = \lim_{\tau\to-\infty} \vartheta(\tau) \quad \textit{exists (and is finite)}$$

and $\Psi(\vartheta_0) = 0$.

LEMMA 4.14. *Assume*

$$c_1^2 c_4^2 - 6c_1c_2c_3c_4 + 4c_1c_3^3 + 4c_2^3c_4 - 3c_2^2c_3^2 \neq 0.$$

Then $\Phi(\vartheta_0) \neq 0$ and moreover

$$\rho = -\frac{1}{\tau\Phi(\vartheta_0)} + o(\frac{1}{\tau}),$$

where ϑ_0 is defined in Lemma 4.13.

Note that (4.13) becomes

$$\theta' = P_3(\zeta, \eta) + o(\rho^3),$$

where $P_3(\zeta, \eta)$ is a homogeneous polynomial of degree 3. It follows that $|\theta'| \leq \sigma\rho^3$ for sufficiently small ρ, where σ is some positive number. Then it follows from Lemma 4.14 and Cauchy's test for convergence that $\theta(\tau)$ approaches a fixed limit as $\tau \to -\infty$, provided that

$$c_1^2 c_4^2 - 6c_1c_2c_3c_4 + 4c_1c_3^3 + 4c_2^3c_4 - 3c_2^2c_3^2 \neq 0,$$

i.e.,

$$a_{111}^2 a_{222}^2 - 6a_{111}a_{112}a_{122}a_{222} + 4a_{111}a_{122}^3 + 4a_{112}^3a_{222} - 3a_{112}^2a_{122}^2 \neq 0. \tag{4.16}$$

In conclusion, the proof of Theorem 4.11 is completed. □

PROOF OF LEMMA 4.13. Since the origin is an isolated equilibrium point of the system (3.23), and the system (4.12) is just the restriction of the system (3.23) to the center manifold $u^+ = F^c(u^0)$, it follows that the origin is also an isolated equilibrium point of the system (4.12).

A routine computation gives rise to

$$\Psi(\vartheta) = \frac{c_2 + c_4}{4}\cos\vartheta + \frac{-c_1 - c_3}{4}\sin\vartheta + \frac{3c_2 - c_4}{4}\cos 3\vartheta + \frac{3c_3 - c_1}{4}\sin 3\vartheta.$$

As a result, $\Psi(\vartheta) \equiv 0$ if and only if

$$\begin{cases} c_2 + c_4 = 0, \\ -c_1 - c_3 = 0, \\ 3c_2 - c_4 = 0, \\ 3c_3 - c_1 = 0, \end{cases}$$

or

$$c_1 = c_2 = c_3 = c_4 = 0.$$

The lemma is now a direct consequence of the Theorem E.3. □

Proof of Lemma 4.14. Obviously, c_1, c_2, c_3, c_4 are not all zero. Therefore

$$\vartheta_0 = \lim_{\tau\to-\infty} \vartheta(\tau) \quad \text{exists (and is finite)}$$

and

$$\Psi(\vartheta_0) = 0.$$

Furthermore, we claim that $\Phi(\vartheta_0) \neq 0$. Otherwise, it is easy to see that

$$\begin{cases} c_1\zeta^2 + 2c_2\zeta\eta + c_3\eta^2 = 0 \\ c_2\zeta^2 + 2c_3\zeta\eta + c_4\eta^2 = 0 \end{cases}$$

for $\zeta = \cos\vartheta_0, \eta = \sin\vartheta_0$.

If $\sin\vartheta_0 \neq 0$, we conclude that

$$\begin{cases} c_1 w^2 + 2c_2 w + c_3 = 0 \\ c_2 w^2 + 2c_3 w + c_4 = 0 \end{cases}$$

for $w = \frac{\cos\vartheta_0}{\sin\vartheta_0}$. By the fact that the resultant of two polynomials having a common root vanishes (see [**26**, p.104]), it follows that the resultant

$$\begin{vmatrix} c_1 & 2c_2 & c_3 & 0 \\ 0 & c_1 & 2c_2 & c_3 \\ c_2 & 2c_3 & c_4 & 0 \\ 0 & c_2 & 2c_3 & c_4 \end{vmatrix}$$

is zero, i.e.,

$$c_1^2c_4^2 - 6c_1c_2c_3c_4 + 4c_1c_3^3 + 4c_2^3c_4 - 3c_2^2c_3^2 = 0.$$

This is contrary to the assumption of the lemma.

If $\sin\vartheta_0 = 0$, then $\cos\vartheta_0 \neq 0$. Using the same argument as above, we can always obtain $\Phi(\vartheta_0) \neq 0$.

The lemma is now a direct consequence of Theorem E.5. □

Proof of Corollary 4.12. We only need to verify the condition (4.16) for the degenerate equilateral central configurations.

As in Appendix B.2, we consider

$$m_1 = m_2 = m_3 = 1, m_4 = \frac{81 + 64\sqrt{3}}{249}.$$

Recall that

$$a_{ijk} = d^3\mathcal{U}|_{\mathbf{r}_0}(\hat{\mathcal{E}}_i, \hat{\mathcal{E}}_j, \hat{\mathcal{E}}_k), \qquad i, j, k \in \{1, 2\},$$

and

$$\mathcal{E}_5 = \mathbf{r}_0 = (-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}, 0, 1, 0, 0)^\top,$$

$$\mathcal{E}_6 = \mathbf{i}\mathbf{r}_0 = (\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, -1, 0, 0, 0)^\top,$$

$$\mathcal{E}_1 = \left(\tfrac{64\sqrt{3}+81}{498}, -\tfrac{741\sqrt{3}+908}{1494}, \tfrac{64\sqrt{3}+81}{498}, \tfrac{741\sqrt{3}+908}{1494}, 0, 0, -1, 0\right)^\top,$$

$$\mathcal{E}_2 = \left(\tfrac{165\sqrt{3}+179}{747}, -\tfrac{371\sqrt{3}+738}{2241}, -\tfrac{165\sqrt{3}+179}{747}, -\tfrac{371\sqrt{3}+738}{2241}, 0, \tfrac{2\sqrt{3}+9}{27}, 0, 1\right)^\top.$$

Some tedious computation yields

$$a_{111} = 0,$$

$$a_{112} = -6630331032\sqrt{\frac{2}{13129701006956661\sqrt{3} + 22740709543896356}},$$

$$a_{122} = 0,$$

$$a_{222} = 3269394\sqrt{\frac{2}{6312834009\sqrt{3} + 10926270656}}.$$

Obviously,

$$a_{111}^2 a_{222}^2 - 6a_{111}a_{112}a_{122}a_{222} + 4a_{111}a_{122}^3 + 4a_{112}^3 a_{222} - 3a_{112}^2 a_{122}^2 \neq 0.$$

The proof is completed. □

4.3. Summary

We conclude this section with simply proving Theorem 4.1 and Corollary 4.3.

Obviously, by Theorem 4.5, Theorem 4.10 and Theorem 4.11, it follows that Theorem 4.1 is correct. By Corollary 4.12, it follows that Corollary 4.3 is correct.

CHAPTER 5

Manifold of Collision Orbits

Based upon the work on $PISPW$, we can consider now the manifold of all the collision orbits or the set of initial conditions leading to total collisions locally. We also have to divide the discussion into several cases according to the value of n_0.

5.1. $n_0 = 0$

First, let us consider the case that all the central configurations of the given N-body problem are nondegenerate.

THEOREM 5.1. *For the planar N-body problem, the manifold of all collision orbits corresponding to a fixed nondegenerate central configuration is a real analytic manifold of dimension $n_p + 3$ in a neighbourhood of the collision instant. Therefore the set of initial conditions leading to total collisions is locally a finite union of real analytic submanifolds in a neighbourhood of the collision instant, here the dimension of each submanifold depends upon the index of the limiting central configuration and is at most $2N-1$.*

PROOF. Recall that, given a total collision orbit $\mathbf{r}(t)$, there is a central configuration $\mathbf{r}_0$ such that $\hat{\mathbf{r}}(t) \to \mathbf{S} = \{e^{\mathbf{i}\theta}\mathbf{r}_0 | \theta \in \mathbb{R}\}$ as $t \to 0+$. The orbit $\mathbf{r}(t)$ now reduces to a solution, such that $(q(\tau), \gamma(\tau)) \to 0$ as $\tau \to -\infty$, of the following equations of motion:

$$\begin{cases} q'^{+} = \mathfrak{C}^{+} q^{+} + \varphi^{+}(q^{+}, q^{-}), \\ q'^{-} = \mathfrak{C}^{-} q^{-} + \varphi^{-}(q^{+}, q^{-}); \end{cases} \tag{5.1}$$

and

$$\begin{cases} r' = r(\kappa^{\frac{1}{2}} + \gamma), \\ \theta' = z^{\top} Q Z = q^{\top} \mathfrak{P}^{\top} \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix} Q \begin{pmatrix} 0 & \mathbb{I} \end{pmatrix} \mathfrak{P} q. \end{cases} \tag{5.2}$$

Since the origin is a hyperbolic equilibrium point of the subsystem (5.1), it follows that $(q(\tau), \gamma(\tau))$ is exactly a solution in the (real analytic) unstable manifold of the origin, for the system (5.1). This unstable manifold has dimension $n_p + 1$.

Since the map $\mathbf{r} \mapsto (z, Z, \gamma, r, e^{\mathbf{i}\theta})$ is an analytic diffeomorphism, and the map $(z, Z, \gamma, r, \theta) \mapsto (q, \gamma)$ is a submersion, we conclude that the set of initial conditions leading to a total collision with normalized configuration approaching $\mathbf{S}$ is a real analytic manifold of dimension $n_p + 3$ in a neighbourhood of the collision instant.

Here, for the fixed nondegenerate central configuration $\mathbf{r}_0$, n_p satisfies $N - 2 \leq n_p \leq 2N - 4$. Hence

$$n_p + 3 \leq 2N - 1 < 4N - 4 \quad \text{for} N \geq 2,$$

i.e., the dimension of each above collision manifold is strictly less than that of the phase space $\mathrm{T}\mathfrak{X}$.

As a result, the set of initial conditions leading to total collisions is locally a finite union of real analytic submanifolds in a neighbourhood of the collision instant, and the dimension of each submanifolds depends upon the index n_p of the limiting central configuration and is at most $2N-1$ for the N-body problem.

The proof is completed. □

5.2. $n_0 > 0$

Let us consider the case that central configurations of the given N-body problem are assumed isolated but possibly degenerate.

THEOREM 5.2. *The set of initial conditions leading to total collisions is included in a finite union of real submanifolds in a neighbourhood of the collision instant, and the dimension of each submanifolds depends upon the index of the limiting central configuration and is at most $2N-1$ for the N-body problem.*

REMARK 5.3. Since $2N-1 < 4N-4$ for $N \geq 2$, we get the result that the set of initial conditions leading to total collisions has zero measure in a neighbourhood of the collision instant. Since the invariant set $\mathfrak{J} \equiv 0$ is of dimension $4N-5$, it follows that the set of initial conditions leading to total collisions has zero measure even when restricted to the invariant set $\mathfrak{J} \equiv 0$ for $N \geq 3$. However, let us quote a remark by Siegel in [**22**]:"We remark that since our solutions are described only near $t=0$, the above description of the collision orbits is purely local. It is not possible to describe the manifold of collision orbits in the large, that is, for all t, by our method."

PROOF. Given a total collision orbit $\mathbf{r}(t)$, there is a central configuration $\mathbf{r}_0$ such that $\hat{\mathbf{r}}(t) \to \mathbf{S}$ as $t \to 0+$. The orbit $\mathbf{r}(t)$ reduces to a solution, such that $(q(\tau), \gamma(\tau)) \to 0$ as $\tau \to -\infty$, of the following equations of motion:

$$
\begin{cases}
q'^0 = \varphi^0(q^0, q^+, q^-), \\
q'^+ = \mathfrak{C}^+ q^+ + \varphi^+(q^0, q^+, q^-), \\
q'^- = \mathfrak{C}^- q^- + \varphi^-(q^0, q^+, q^-),
\end{cases}
\tag{5.3}
$$

and

$$
\begin{cases}
r' = r(\kappa^{\frac{1}{2}} + \gamma), \\
\theta' = q^\top \mathfrak{P}^\top \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix} Q \begin{pmatrix} 0 & \mathbb{I} \end{pmatrix} \mathfrak{P} q.
\end{cases}
\tag{5.4}
$$

Although we cannot completely resolve $PISPW$ in this case, we could give a measure of the set of initial conditions leading to total collisions.

Obviously, $(q(\tau), \gamma(\tau))$ is a solution in a (smooth) center-unstable manifold of the origin, for the system (5.3). This center-unstable manifold has dimension $n_0 + n_p + 1$. Similar to the argument used in Theorem 5.1, we conclude that the set of initial conditions leading to total collisions with normalized configuration approaching $\mathbf{S}$ is a subset of a real smooth manifold of dimension no more than $n_0 + n_p + 3$ in a neighbourhood of the collision instant.

Here, for the fixed nondegenerate central configuration $\mathbf{r}_0$, we have

$$n_0 + n_p + 3 \leq 2N - 1 < 4N - 4 \quad \text{for} N \geq 2,$$

i.e., the dimension of each above collision set is strictly less than that of the phase space $T\mathfrak{X}$.

As a result, the set of initial conditions leading to total collisions is included in a finite union of real smooth submanifolds in a neighbourhood of the collision instant, and the dimension of each submanifolds depends upon the index $n_0 + n_p$ of the limiting central configuration and is at most $2N - 1$ for the N-body problem.

The proof is completed. □

REMARK 5.4. In Theorem 5.1 and 5.2, if we remove the assumption that the center of mass is fixed at the origin, then each present dimension need to be increased by 4.

5.3. Analytic Extension

Finally, let us examine the question of whether orbits can be extended through total collision from the viewpoint of Sundman and Siegel, that is, whether a single solution can be extended as an analytic function of time. The problem has been studied by Sundman and Siegel in the case of the three-body problem.

It is easy to show that the nature of the singularity corresponding to a total collision depends on the arithmetical nature of the eigenvalues $\kappa^{\frac{1}{2}}, \tilde{\mu}_1, \cdots, \tilde{\mu}_{n_p}$. Here we discuss only the case corresponding to rhombic central configurations for the four-body problem as a demonstration.

Recall the facts (B.17) and (B.16). For simplicity, we consider only the case $\sqrt{3} < \zeta < \zeta_1 \approx 1.7889580612081344$, then

$$0 < \kappa^{\frac{1}{2}} < \mu_1 < \mu_2 < \mu_4 < \mu_3.$$

It is clear that $\kappa^{\frac{1}{2}}, \mu_1, \mu_2, \mu_3, \mu_4$ are nonresonant for almost all $\zeta \in (\sqrt{3}, \zeta_1)$.

On the other hand, one can show that the coordinate functions q, γ, r, θ of a collision orbit are regular analytic functions of the following variables

$$\tilde{u}_0 = c_0 e^{\kappa^{\frac{1}{2}}\tau}, \tilde{u}_1 = c_1 e^{\tilde{\mu}_1 \tau}, \tilde{u}_2 = c_2 e^{\tilde{\mu}_2 \tau}, \tilde{u}_3 = c_3 e^{\tilde{\mu}_3 \tau}, \tilde{u}_4 = c_4 e^{\tilde{\mu}_4 \tau}.$$

As a matter of fact, since rhombic central configurations are nondegenerate, the equations of motion reduce to an analytic system with a hyperbolic equilibrium, then the proof is straightforward by the celebrated Poincaré- Lyapunov Theorem on analytic invariant manifolds, for more detail please refer to [**22**].

Furthermore, it follows from (2.5), (3.2) and (3.20) that $\mathbf{r}, \dot{\mathbf{r}}$ are regular analytic functions of the following variables

$$\breve{u}_0 = c_0 t^{\frac{2}{3}}, \quad \breve{u}_1 = c_1 t^{(2\tilde{\mu}_1)/(3\kappa^{\frac{1}{2}})}, \quad \breve{u}_2 = c_2 t^{(2\tilde{\mu}_2)/(3\kappa^{\frac{1}{2}})},$$
$$\breve{u}_3 = c_3 t^{(2\tilde{\mu}_3)/(3\kappa^{\frac{1}{2}})}, \quad \breve{u}_4 = c_4 t^{(2\tilde{\mu}_4)/(3\kappa^{\frac{1}{2}})}.$$

Since all the numbers $\kappa^{\frac{1}{2}}, \mu_1, \mu_2, \mu_3, \mu_4$ and their ratios are irrational for almost all $\zeta \in (\sqrt{3}, \zeta_1)$, thus generically we have an essential singularity at collision instant $t = 0$. In this case it is not possible to continue the solutions analytically beyond the collision.

REMARK 5.5. Although one can conclude that the eigenvalues μ_j $(j = 1, \cdots, 2N-4)$ depend continuously upon the value of the masses m_k $(k = 1, \cdots, N)$, it is obvious that one can not simply claim that there must exist values of m_k yielding some μ_j being irrational, since we can not simply exclude the case in which μ_j are constant as m_k vary.

CHAPTER 6

Conclusion and Questions

For the planar N-body problem, based on the moving coordinates, which allows us to describe the motion of collision orbit effectively, we discussed $PISPW$: whether the normalized configuration of the particles must approach a certain central configuration without undergoing infinite spin for a total collision orbit. In the cases corresponding to central configurations with degree of degeneracy less than or equal to one, we completely solve the problem. We also give a criterion of the problem for the case corresponding to central configurations with degree of degeneracy two; we further give an answer to the problem in the case corresponding to all known degenerate central configurations of four bodies. Therefore, for almost every choice of the masses of the four-body problem, $PISPW$ is solved. For all the solved cases, we conclude that the normalized configuration of the particles must approach a certain central configuration without undergoing infinite spin for a total collision orbit. Finally, we give a measure of the set of initial conditions leading to total collisions: the set has zero measure in the neighbourhood of collision instant.

This work indicates the fact that $PISPW$ is the link of many interesting subjects in dynamical system. It is our hope that this work may spark interest to the problems on degenerate central configurations and/or degenerate equilibrium points etc.

Many questions remain to be answered. For example, some concrete questions are the following:

(i) In the cases corresponding to central configurations with three or higher degrees of degeneracy, how should one study $PISPW$?

(ii) For the spatial N-body problem, how should one study $PISPW$?

(iii) If a solution of a strongly degenerate system (there are many zeros in the eigenvalues of linear part of the system at an equilibrium point O) approaches O, how should one estimate the speed of tending to the equilibrium point O?

(iv) Is it true that all the four-body central configurations except the degenerate equilateral central configuration have degree of degeneracy equal or less than one?

We hope to explore some of these questions in future work.

By the way, this work hints the fact that the moving coordinates will be useful when we study the Newtonian N-body problem near relative equilibrium solutions. Inspired by this, we will investigate the stability of relative equilibrium solutions in future work by making use of the moving coordinates. Indeed, it is shown that the theory of KAM is successfully applied to study the N-body problem near relative equilibrium solutions.

APPENDIX A

Degeneracy of Central Configurations

A.1. Degeneracy of a Constrained Critical Point

Let f and g_k be smooth functions defined on Ω, $1 \le k \le m < n$, where Ω is an open set in $\mathbb{R}^n$. Assume $\mathcal{M}$ is a smooth submanifold given as the common zero-set of the m smooth functions g_k, $1 \le k \le m$.

The classical approach to find the critical points of $\tilde{f} := f|_{\mathcal{M}}$, which is the restriction of f on $\mathcal{M}$, involves the well-known *method of Lagrange multipliers*. The method avoids looking for local coordinates on the submanifold $\mathcal{M}$, but consists of introducing m "undetermined multipliers" λ_k, defining the following auxiliary function

$$L = f + \sum_{k=1}^{m} \lambda_k g_k$$

and finding its critical points; i.e., by regarding L as a function of

$$(x, \bar{\lambda}) = (x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_m) \in \Omega \times \mathbb{R}^m.$$

The critical points are obtained by solving the equations

$$\nabla L = 0.$$

Similarly, one can investigate the degeneracy of a critical point in a straightforward manner: assume $\underline{a}$ is a critical point of $\tilde{f}$, let $(\underline{a}, \underline{\lambda})$ be the corresponding auxiliary critical point of L; by regarding L as a function of $(x, \bar{\lambda})$, the degeneracy of $\tilde{f}$ at $\underline{a}$ is the same as that of L at $(\underline{a}, \underline{\lambda})$, i.e., we have the following result (please refer to [**10**] for more detail)

Theorem A.1 ([**10**]). *The nullity (i.e., the dimension of the kernel) of the Hessian $D^2\tilde{f}(\underline{a})$ equals the nullity of the Hessian $D^2L(\underline{a}, \underline{\lambda})$.*

Note that the Hessian $D^2\tilde{f}(\underline{a})$ is an $(n-m)\times(n-m)$ symmetric matrix, and the Hessian $D^2L(\underline{a}, \underline{\lambda})$, called the bordered Hessian, is an $(n+m)\times(n+m)$ symmetric matrix.

A.2. Degeneracy of Central Configurations

When investigating the degeneracy of central configurations, the method in A.1 is certainly useful. For example, $\mathcal{X}$ is the submanifold of $(\mathbb{R}^2)^N$ such that the center of mass is at the origin; then the nullity of the Hessian $D^2\widetilde{U}$ in $\mathcal{X}$ is naturally reduced to that in $(\mathbb{R}^2)^N \times \mathbb{R}^2$. The benefit of this approach is that we do not need to look for local coordinates of $\mathcal{X}$ for various masses. As it happens often, it is difficult to find appropriate local coordinates for concrete problems.

In the following, we use the cartesian coordinates of $(\mathbb{R}^2)^N$ to give a slightly better criterion than the bordered Hessian in $(\mathbb{R}^2)^N \times \mathbb{R}^2$ for investigating the degeneracy of central configurations.

First of all, recall that $\hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$ for any configuration $\mathbf{r}$, and given a central configuration $\mathbf{r}_0$, $\{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_{2N-5}, \mathcal{E}_{2N-4}, \mathbf{r}_0, \mathbf{i}\mathbf{r}_0\}$ is an orthogonal basis of $\mathcal{X}$ composed by eigenvectors of $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$.

Let

$$\{\mathbf{e}_1, \cdots, \mathbf{e}_{2N}\}$$

be the standard basis of $(\mathbb{R}^2)^N$, where $\mathbf{e}_j \in (\mathbb{R}^2)^N$ has unity at the j-th component and zero at all others. Then every N-body configuration $\mathbf{r} \in (\mathbb{R}^2)^N$ can be written as

$$\mathbf{r} = \sum_{j=1}^{2N} x^j \mathbf{e}_j,$$

and

$$(x^1, x^2, \cdots, x^{2N})^\top$$

are coordinates of $\mathbf{r}$ in the standard basis. It is also true that $\mathbf{r}_j = (x^{2j-1}, x^{2j})^\top$ for $j = 1, 2, \cdots, N$. Then

$$\mathcal{X} = \{\mathbf{r} \in (\mathbb{R}^2)^N | \langle \mathbf{r}, \mathcal{E}_{2N-1}\rangle = 0, \langle \mathbf{r}, \mathcal{E}_{2N}\rangle = 0\},$$

where

$$\mathcal{E}_{2N-1} = \sum_{j=1}^{N} \mathbf{e}_{2j-1} = (1, 0, \cdots, 1, 0)^\top, \mathcal{E}_{2N} = \sum_{j=1}^{N} \mathbf{e}_{2j} = (0, 1, \cdots, 0, 1)^\top.$$

Let $\mathfrak{M}$ be the matrix

$$diag(m_1, m_1, m_2, m_2, \cdots, m_N, m_N),$$

where "diag" means diagonal matrix. Then

$$\langle \mathbf{r}, \mathbf{r}\rangle = \mathbf{r}^\top \mathfrak{M}\mathbf{r}.$$

Given a central configuration $\mathbf{r}_0 = (\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathcal{X}\backslash\Delta$, a straightforward computation shows that the Hessian $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$ with respect to the mass scalar product in $(\mathbb{R}^2)^N$, that is the linear operator $D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)$ satisfying

$$\langle D^2\widetilde{\mathcal{U}}(\mathbf{r}_0)\mathbf{u}, \mathbf{v}\rangle = d^2\widetilde{\mathcal{U}}(\mathbf{r}_0)(\mathbf{u}, \mathbf{v}), \qquad \text{for any } \mathbf{u}, \mathbf{v} \in (\mathbb{R}^2)^N,$$

is

$$I^{\frac{1}{2}}(\lambda\mathbb{I} + \mathfrak{M}^{-1}\mathfrak{B}) - 3I^{-\frac{1}{2}}\lambda\mathbf{r}_0\mathbf{r}_0^\top\mathfrak{M},$$

where $\lambda = \frac{\mathcal{U}(\mathbf{r}_0)}{I(\mathbf{r}_0)}$, $\mathfrak{B}$ is the Hessian of $\mathcal{U}$ evaluated at $\mathbf{r}_0$ with respect to the standard scalar product of $(\mathbb{R}^2)^N$, that is the linear operator $\mathfrak{B}$ satisfying

$$\mathbf{u}^\top\mathfrak{B}\mathbf{v} = d^2\mathcal{U}(\mathbf{r}_0)(\mathbf{u}, \mathbf{v}), \qquad \text{for any } \mathbf{u}, \mathbf{v} \in (\mathbb{R}^2)^N.$$

$\mathfrak{B}$ can be viewed as an $N \times N$ array of 2×2 blocks:

$$\mathfrak{B} = \begin{pmatrix} B_{11} & \cdots & B_{1N} \\ \vdots & \ddots & \vdots \\ B_{N1} & \cdots & B_{NN} \end{pmatrix}$$

The off-diagonal blocks are given by:

$$B_{jk} = \frac{m_j m_k}{r_{jk}^3}[\mathbb{I} - \frac{3(\mathbf{r}_k - \mathbf{r}_j)(\mathbf{r}_k - \mathbf{r}_j)^\top}{r_{jk}^2}],$$

where $r_{jk} = |\mathbf{r}_k - \mathbf{r}_j|$, and $\mathbb{I}$ is the identity matrix of order 2. However, as a matter of notational convenience, the **identity matrix** of any order will always be denoted by $\mathbb{I}$, and the order of $\mathbb{I}$ can be determined according to the context. The diagonal blocks are given by:

$$B_{kk} = -\sum_{1 \le j \le N, j \ne k} B_{jk}.$$

Set

$$\mathcal{E}_{2N-3} = \mathbf{r}_0, \qquad \mathcal{E}_{2N-2} = \mathbf{i}\mathbf{r}_0.$$

Then

$$\mathcal{P}_{\mathbf{r}_0} = span\{\mathcal{E}_{2N-3}, \mathcal{E}_{2N-2}\}, \qquad \mathcal{P}_{\mathbf{r}_0}^{\perp} = span\{\mathcal{E}_1, \cdots, \mathcal{E}_{2N-4}\},$$

and $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, \cdots, \mathcal{E}_{2N}\}$ is just an orthogonal basis of $(\mathbb{R}^2)^N$ such that

$$D^2\widetilde{\mathfrak{U}}(\mathbf{r}_0)\mathcal{E}_j = \mu_j \mathcal{E}_j, \qquad j = 1, \cdots, 2N,$$

$$\mu_{2N-1} = \mu_{2N} = I^{\frac{1}{2}}\lambda = I^{-\frac{1}{2}}U, \qquad \mu_{2N-3} = \mu_{2N-2} = 0.$$

Thus

$$\{\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2, \hat{\mathcal{E}}_3, \hat{\mathcal{E}}_4, \cdots, \hat{\mathcal{E}}_{2N}\}$$

is an orthonormal basis of the space $(\mathbb{R}^2)^N$, for the scalar product $\langle , \rangle$, consisting of eigenvectors of $D^2\widetilde{\mathfrak{U}}(\mathbf{r}_0)$, that is,

$$(\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2, \hat{\mathcal{E}}_3, \hat{\mathcal{E}}_4, \cdots, \hat{\mathcal{E}}_{2N})^\top \mathfrak{M} (\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2, \hat{\mathcal{E}}_3, \hat{\mathcal{E}}_4, \cdots, \hat{\mathcal{E}}_{2N}) = \mathbb{I}.$$

By

$$D^2\widetilde{\mathfrak{U}}(\mathbf{r}_0)(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) = diag(\mu_1, \mu_2, \cdots, \mu_{2N})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}),$$

it follows that

(A.1)
$$(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N})^\top (I^{\frac{1}{2}}(\lambda\mathfrak{M} + \mathfrak{B}) - 3I^{-\frac{1}{2}}\lambda\mathfrak{M}\mathbf{r}_0\mathbf{r}_0^\top\mathfrak{M})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) = diag(\mu_1, \cdots, \mu_{2N}).$$

Thanks to

$$(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N})^\top (\mathfrak{M}\mathcal{E}_{2N-3}\mathcal{E}_{2N-3}^\top\mathfrak{M})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) = diag(0, 0, \cdots, I, 0, 0, 0),$$

$$(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N})^\top (\mathfrak{M}\mathcal{E}_{2N-2}\mathcal{E}_{2N-2}^\top\mathfrak{M})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) = diag(0, 0, \cdots, 0, I, 0, 0),$$

$$(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N})^\top (\mathfrak{M}\mathcal{E}_{2N-1}\mathcal{E}_{2N-1}^\top\mathfrak{M})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) = diag(0, 0, \cdots, 0, 0, \mathfrak{m}, 0),$$

and

$$(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N})^\top (\mathfrak{M}\mathcal{E}_{2N}\mathcal{E}_{2N}^\top\mathfrak{M})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) = diag(0, 0, \cdots, 0, 0, 0, \mathfrak{m}),$$

where recall that $\mathfrak{m} = \sum_{k=1}^N m_k$ is the total mass, we have

(A.2)
$$(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N})^\top (\lambda\mathfrak{M} + \mathfrak{B})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) = diag(\frac{\mu_1}{\|\mathbf{r}_0\|}, \cdots, \frac{\mu_{2N-4}}{\|\mathbf{r}_0\|}, 3\lambda, 0, \lambda, \lambda),$$

(A.3)
$$\begin{aligned}(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N})^\top (\lambda\mathfrak{M} + \mathfrak{B} + \mathfrak{M}\mathcal{E}_{2N-2}\mathcal{E}_{2N-2}^\top\mathfrak{M})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) \\ = diag(\tfrac{\mu_1}{\|\mathbf{r}_0\|}, \cdots, \tfrac{\mu_{2N-4}}{\|\mathbf{r}_0\|}, 3\lambda, I, \lambda, \lambda),\end{aligned}$$

$$(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N})^\top (\lambda\mathfrak{M} + \mathfrak{B} - \tfrac{3\lambda}{I}\mathfrak{M}\mathcal{E}_{2N-3}\mathcal{E}_{2N-3}^\top\mathfrak{M} - \tfrac{\lambda}{\mathfrak{m}}\mathfrak{M}\mathcal{E}_{2N-1}\mathcal{E}_{2N-1}^\top\mathfrak{M} - \tfrac{\lambda}{\mathfrak{m}}\mathfrak{M}\mathcal{E}_{2N}\mathcal{E}_{2N}^\top\mathfrak{M})(\hat{\mathcal{E}}_1, \cdots, \hat{\mathcal{E}}_{2N}) = diag(\tfrac{\mu_1}{\|\mathbf{r}_0\|}, \cdots, \tfrac{\mu_{2N-4}}{\|\mathbf{r}_0\|}, 0, 0, 0, 0). \tag{A.4}$$

These results will be useful in investigating the degeneracy of central configurations. For example, the nullity of $\lambda\mathfrak{M} + \mathfrak{B} + \mathfrak{M}\mathcal{E}_{2N-2}\mathcal{E}_{2N-2}^\top\mathfrak{M}$ in (A.3) is just the degree of degeneracy of $\mathbf{r}_0$; when investigating central configurations of four bodies with degree of degeneracy two, the problem reduces to show that the matrix

$$\lambda\mathfrak{M} + \mathfrak{B} - \frac{3\lambda}{I}\mathfrak{M}\mathcal{E}_{2N-3}\mathcal{E}_{2N-3}^\top\mathfrak{M} - \frac{\lambda}{\mathfrak{m}}\mathfrak{M}\mathcal{E}_{2N-1}\mathcal{E}_{2N-1}^\top\mathfrak{M} - \frac{\lambda}{\mathfrak{m}}\mathfrak{M}\mathcal{E}_{2N}\mathcal{E}_{2N}^\top\mathfrak{M}$$

is a positive semi-definite matrix with rank 2 by (A.4).

Of course, if one finds a basis of $\mathcal{P}_{\mathbf{r}_0}^\perp$, say $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_{2N-4}$, then it is simpler to consider the nullity of

$$(\mathcal{F}_1, \cdots, \mathcal{F}_{2N-4})^\top (\lambda\mathfrak{M} + \mathfrak{B})(\mathcal{F}_1, \cdots, \mathcal{F}_{2N-4}).$$

Starting from these considerations, in the next section we will investigate the degeneracy of central configurations with an axis of symmetry and in the four body problem.

APPENDIX B

Central Configurations of Four Bodies

Viewing as a preliminary study on the problem of degeneracy, in the section let us investigate the degeneracy of central configurations with an axis of symmetry in the four body problem. They consist of systems of point particles in $\mathbb{R}^2$ whose configurations have the following geometric properties:

(1) There are 2 particles m_3, m_4 lying on a fixed line which is the axis of symmetry of problem, which is supposed to coincide with the y-axis.
(2) Two other particles m_1, m_2 are symmetric with respect to the y-axis.

This kind of configurations are usually called **kite configurations**. It is easy to see that $m_1 = m_2$ is a necessary condition to have a kite central configuration. Geometry of kite configurations may be seen in Figure B.1.

Without loss of generality, suppose

$$\begin{aligned} m_1 = m_2 &= 1, \\ \mathbf{r}_1 &= (-s, -t)^\top, \\ \mathbf{r}_2 &= (s, -t)^\top, \\ \mathbf{r}_3 &= (0, u)^\top, \\ \mathbf{r}_4 &= (0, u-1)^\top, \end{aligned}$$

where $s > 0$ and $u = \frac{2t+m_4}{m_3+m_4}$.

Then

$$\begin{aligned} \mathcal{E}_5 = \mathbf{r}_0 &= (-s, -t, s, -t, 0, u, 0, u-1)^\top, \\ \mathcal{E}_6 = \mathbf{ir}_0 &= (t, -s, t, s, -u, 0, 1-u, 0)^\top, \end{aligned}$$

and the central configurations equation (2.1) becomes:

$$\text{(B.1)} \qquad \begin{cases} \frac{1}{4s^3} + \frac{m_3}{[s^2+(u+t)^2]^{\frac{3}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} = \lambda \\ \frac{m_3(u+t)}{[s^2+(u+t)^2]^{\frac{3}{2}}} + \frac{m_4(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} = \lambda t \\ \frac{2(u+t)}{[s^2+(u+t)^2]^{\frac{3}{2}}} + m_4 = \lambda u \\ \frac{2(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} - m_3 = \lambda(u-1). \end{cases}$$

It is easy to see that it follows from the system (B.1) that

$$\text{(B.2)}\qquad \begin{cases} \frac{1}{4s^3} + \frac{m_3}{[s^2+(u+t)^2]^{\frac{3}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} = \lambda, \\ \frac{m_3(u+t)}{[s^2+(u+t)^2]^{\frac{3}{2}}} + \frac{m_4(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} + \frac{2(u+t)}{[s^2+(u+t)^2]^{\frac{3}{2}}} + m_4 = \lambda(u+t), \\ \frac{m_3(u+t)}{[s^2+(u+t)^2]^{\frac{3}{2}}} + \frac{m_4(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} + \frac{2(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} - m_3 = \lambda(u+t-1). \end{cases}$$

We remark that, in fact, by $u = \frac{2t+m_4}{m_3+m_4}$ the systems (B.1) and (B.2) are equivalent. The proof of the equivalence is not immediate, please refer to the following Remark B.3 for more detail.

By considering (B.2) as a linear system on m_3 and m_4, it follows that

$$\text{(B.3)}\qquad \begin{cases} \frac{1}{4s^3} + \frac{m_3}{[s^2+(u+t)^2]^{\frac{3}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} = \lambda, \\ \left(\frac{1}{[s^2+(u+t)^2]^{\frac{3}{2}}} - 1\right) m_3 = 2\,(u+t-1)\left(\frac{1}{8s^3} - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}\right), \\ \left(1 - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}\right) m_4 = 2\,(u+t)\left(\frac{1}{8s^3} - \frac{1}{[s^2+(u+t)^2]^{\frac{3}{2}}}\right). \end{cases}$$

An easy argument shows that, except the cases $s = \frac{\sqrt{3}}{2}, t = \frac{1}{2}, u = 1$ and $s = \frac{\sqrt{3}}{2}, t = -\frac{1}{2}, u = 0$, the masses can be presented by geometric elements of the central configuration:

$$\begin{cases} m_3 = \dfrac{2(u+t-1)\left(\frac{1}{8s^3} - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}\right)}{\frac{1}{[s^2+(u+t)^2]^{\frac{3}{2}}} - 1}, \\ m_4 = \dfrac{2(u+t)\left(\frac{1}{8s^3} - \frac{1}{[s^2+(u+t)^2]^{\frac{3}{2}}}\right)}{1 - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}}. \end{cases}$$

For instance, if

$$\text{(B.4)}\qquad \frac{1}{[s^2+(u+t)^2]^{\frac{3}{2}}} - 1 = 0,$$

then we have $s = \frac{\sqrt{3}}{2}, t = -\frac{1}{2}, u = 0$. As a matter of fact, by (B.3) it follows that

$$(u+t-1)\left(\frac{1}{8s^3} - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}\right) = 0.$$

Note that, the relation $u+t-1 = 0$ is conflict with (B.4), since they imply that $s = 0$. Thus we have

$$\text{(B.5)}\qquad \frac{1}{8s^3} - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} = 0.$$

By (B.4) and (B.5), it follows that

$$\begin{cases} s^2 + (u+t)^2 = 1, \\ (u+t-1)^2 = 3s^2. \end{cases}$$

It is easy to see that it holds that

$$\text{(B.6)} \qquad \begin{cases} s = \frac{\sqrt{3}}{2}, \\ u + t = -\frac{1}{2}. \end{cases}$$

By substituting (B.6) into the equation

$$\left(1 - \frac{1}{[s^2 + (u+t-1)^2]^{\frac{3}{2}}}\right) m_4 = 2\,(u+t)\left(\frac{1}{8s^3} - \frac{1}{[s^2+(u+t)^2]^{\frac{3}{2}}}\right)$$

in (B.3), it follows that

$$m_4 = 1.$$

Then, by considering the equation

$$\frac{2(u+t)}{[s^2+(u+t)^2]^{\frac{3}{2}}} + m_4 = \lambda u$$

in (B.1), we arrive at the conclusion that $s = \frac{\sqrt{3}}{2}, t = -\frac{1}{2}, u = 0$. A similar argument shows that, if

$$1 - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} = 0,$$

then we have $s = \frac{\sqrt{3}}{2}, t = \frac{1}{2}, u = 1$.

Central configurations in the cases $s = \frac{\sqrt{3}}{2}, t = \frac{1}{2}, u = 1$ and $s = \frac{\sqrt{3}}{2}, t = -\frac{1}{2}, u = 0$ are formed by the three particles at the vertices of an equilateral triangle and a fourth particle at the centroid. We shall call them **equilateral central configurations**.

Therefore, provided that equilateral central configurations are ruled out of our discussion, the system (B.3) can be written as

$$\text{(B.7)} \qquad \begin{cases} m_3 = \dfrac{2(u+t-1)\left(\frac{1}{8s^3} - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}\right)}{\frac{1}{[s^2+(u+t)^2]^{\frac{3}{2}}} - 1}, \\[2ex] m_4 = \dfrac{2(u+t)\left(\frac{1}{8s^3} - \frac{1}{[s^2+(u+t)^2]^{\frac{3}{2}}}\right)}{1 - \frac{1}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}}, \\[2ex] \lambda = \frac{1}{4s^3} + \frac{m_3}{[s^2+(u+t)^2]^{\frac{3}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}. \end{cases}$$

The main results obtained on kite central configurations are as follows:

THEOREM B.1. *A degenerate equilateral central configuration has degree of degeneracy two. Except equilateral central configurations, if a kite central configuration is degenerate, then the degree of degeneracy is one.*

THEOREM B.2. *All the rhombic central configurations are nondegenerate. Moreover, for almost every rhombic central configuration, all the corresponding values* $\kappa^{\frac{1}{2}}, \mu_1, \mu_2, \mu_3, \mu_4$ *and their ratios are irrational.*

Here, we remark that a rhombic central configuration is a kite central configuration with four particles located at vertices of a rhombus.

B.1. Preliminaries

First, recall that

$$\mathfrak{M} = diag(m_1, m_1, m_2, m_2, m_3, m_3, m_4, m_4) = diag(1, 1, 1, 1, m_3, m_3, m_4, m_4),$$

and

$$\mathfrak{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix},$$

where

$$B_{11} = \begin{pmatrix} \frac{m_3\left(2s^2-(t+u)^2\right)}{(s^2+(t+u)^2)^{5/2}} + \frac{m_4\left(2s^2-(t+u-1)^2\right)}{(s^2+(t+u-1)^2)^{5/2}} + \frac{1}{4s^3} & 3s\left(\frac{m_3(t+u)}{(s^2+(t+u)^2)^{5/2}} + \frac{m_4(t+u-1)}{(s^2+(t+u-1)^2)^{5/2}}\right) \\ 3s\left(\frac{m_3(t+u)}{(s^2+(t+u)^2)^{5/2}} + \frac{m_4(t+u-1)}{(s^2+(t+u-1)^2)^{5/2}}\right) & -\frac{m_3\left(s^2-2(t+u)^2\right)}{(s^2+(t+u)^2)^{5/2}} - \frac{m_4\left(s^2-2(t+u-1)^2\right)}{(s^2+(t+u-1)^2)^{5/2}} - \frac{1}{8s^3} \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} \frac{m_3\left(2s^2-(t+u)^2\right)}{(s^2+(t+u)^2)^{5/2}} + \frac{m_4\left(2s^2-(t+u-1)^2\right)}{(s^2+(t+u-1)^2)^{5/2}} + \frac{1}{4s^3} & -\frac{3m_3s(t+u)}{(s^2+(t+u)^2)^{5/2}} - \frac{3m_4s(t+u-1)}{(s^2+(t+u-1)^2)^{5/2}} \\ -\frac{3m_3s(t+u)}{(s^2+(t+u)^2)^{5/2}} - \frac{3m_4s(t+u-1)}{(s^2+(t+u-1)^2)^{5/2}} & -\frac{m_3\left(s^2-2(t+u)^2\right)}{(s^2+(t+u)^2)^{5/2}} - \frac{m_4\left(s^2-2(t+u-1)^2\right)}{(s^2+(t+u-1)^2)^{5/2}} - \frac{1}{8s^3} \end{pmatrix},$$

$$B_{33} = \begin{pmatrix} m_3\left(\frac{4s^2-2(t+u)^2}{(s^2+(t+u)^2)^{5/2}} - m_4\right) & 0 \\ 0 & 2m_3\left(m_4 + \frac{2(t+u)^2-s^2}{(s^2+(t+u)^2)^{5/2}}\right) \end{pmatrix},$$

$$B_{44} = \begin{pmatrix} m_4\left(\frac{4s^2-2(t+u-1)^2}{(s^2+(t+u-1)^2)^{5/2}} - m_3\right) & 0 \\ 0 & 2m_4\left(m_3 + \frac{2(t+u-1)^2-s^2}{(s^2+(t+u-1)^2)^{5/2}}\right) \end{pmatrix},$$

$$B_{12} = B_{21} = \begin{pmatrix} -\frac{1}{4s^3} & 0 \\ 0 & \frac{1}{8s^3} \end{pmatrix},$$

$$B_{13} = B_{31} = \begin{pmatrix} -\frac{m_3\left(2s^2-(t+u)^2\right)}{(s^2+(t+u)^2)^{5/2}} & -\frac{3m_3s(t+u)}{(s^2+(t+u)^2)^{5/2}} \\ -\frac{3m_3s(t+u)}{(s^2+(t+u)^2)^{5/2}} & \frac{m_3\left(s^2-2(t+u)^2\right)}{(s^2+(t+u)^2)^{5/2}} \end{pmatrix},$$

$$B_{14} = B_{41} = \begin{pmatrix} -\frac{m_4\left(2s^2-(t+u-1)^2\right)}{(s^2+(t+u-1)^2)^{5/2}} & -\frac{3m_4s(t+u-1)}{(s^2+(t+u-1)^2)^{5/2}} \\ -\frac{3m_4s(t+u-1)}{(s^2+(t+u-1)^2)^{5/2}} & \frac{m_4\left(s^2-2(t+u-1)^2\right)}{(s^2+(t+u-1)^2)^{5/2}} \end{pmatrix},$$

$$B_{23} = B_{32} = \begin{pmatrix} -\frac{m_3\left(2s^2-(t+u)^2\right)}{(s^2+(t+u)^2)^{5/2}} & \frac{3m_3s(t+u)}{(s^2+(t+u)^2)^{5/2}} \\ \frac{3m_3s(t+u)}{(s^2+(t+u)^2)^{5/2}} & \frac{m_3\left(s^2-2(t+u)^2\right)}{(s^2+(t+u)^2)^{5/2}} \end{pmatrix},$$

$$B_{24} = B_{42} = \begin{pmatrix} -\frac{m_4\left(2s^2-(t+u-1)^2\right)}{(s^2+(t+u-1)^2)^{5/2}} & \frac{3m_4s(t+u-1)}{(s^2+(t+u-1)^2)^{5/2}} \\ \frac{3m_4s(t+u-1)}{(s^2+(t+u-1)^2)^{5/2}} & \frac{m_4\left(s^2-2(t+u-1)^2\right)}{(s^2+(t+u-1)^2)^{5/2}} \end{pmatrix},$$

$$B_{34} = B_{43} = \begin{pmatrix} m_3m_4 & 0 \\ 0 & -2m_3m_4 \end{pmatrix}.$$

Set

$$\mathcal{V} = (-1, 0, 1, 0, 0, 0, 0, 0)^\top,$$
$$\mathcal{P} = (0, -1, 0, -1, 0, \tfrac{2}{m_3+m_4}, 0, \tfrac{2}{m_3+m_4})^\top,$$

$$\tilde{\mathcal{V}} = \mathcal{V} - \frac{\langle \mathcal{V}, \mathcal{E}_5\rangle}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle}\mathcal{E}_5, \quad \tilde{\mathcal{P}} = \mathcal{P} - \frac{\langle \mathcal{P}, \mathcal{E}_5\rangle}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle}\mathcal{E}_5.$$

Then

$$\mathbf{i}\mathcal{V} = (0, -1, 0, 1, 0, 0, 0, 0)^\top,$$
$$\mathbf{i}\mathcal{P} = (1, 0, 1, 0, -\tfrac{2}{m_3+m_4}, 0, -\tfrac{2}{m_3+m_4}, 0)^\top,$$

$$\mathbf{i}\tilde{\mathcal{V}} = \mathbf{i}\mathcal{V} - \frac{\langle \mathbf{i}\mathcal{V}, \mathcal{E}_6\rangle}{\langle \mathcal{E}_6, \mathcal{E}_6\rangle}\mathcal{E}_6, \quad \mathbf{i}\tilde{\mathcal{P}} = \mathbf{i}\mathcal{P} - \frac{\langle \mathbf{i}\mathcal{P}, \mathcal{E}_6\rangle}{\langle \mathcal{E}_6, \mathcal{E}_6\rangle}\mathcal{E}_6.$$

By the fact that $\{\mathbf{r}_0, \mathcal{V}, \mathcal{P}\}$ is a $\mathbb{C}$-linearly independent family, it is easy to see that $\{\tilde{\mathcal{V}}, \tilde{\mathcal{P}}, \mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}}\}$ is a basis of $\mathcal{P}^\perp_{\mathbf{r}_0}$. And a straightforward computation shows that

$$\tilde{\mathcal{V}}^\top(\lambda\mathfrak{M} + \mathfrak{B})\mathbf{i}\tilde{\mathcal{V}} = 0,$$
$$\tilde{\mathcal{V}}^\top(\lambda\mathfrak{M} + \mathfrak{B})\mathbf{i}\tilde{\mathcal{P}} = 0,$$
$$\tilde{\mathcal{P}}^\top(\lambda\mathfrak{M} + \mathfrak{B})\mathbf{i}\tilde{\mathcal{V}} = 0,$$
$$\tilde{\mathcal{P}}^\top(\lambda\mathfrak{M} + \mathfrak{B})\mathbf{i}\tilde{\mathcal{P}} = 0,$$

to verify the above relations, we only point out that,

$$\langle \mathbf{i}\mathcal{V}, \mathcal{E}_5\rangle = \langle \mathbf{i}\mathcal{P}, \mathcal{E}_5\rangle = 0,$$
$$\langle \mathcal{V}, \mathcal{E}_6\rangle = \langle \mathcal{P}, \mathcal{E}_6\rangle = 0,$$
$$\langle \mathcal{V}, \mathcal{P}\rangle = \langle \mathbf{i}\mathcal{V}, \mathbf{i}\mathcal{P}\rangle = 0,$$
$$\langle \mathcal{V}, \mathbf{i}\mathcal{V}\rangle = \langle \mathcal{V}, \mathbf{i}\mathcal{P}\rangle = 0,$$
$$\langle \mathcal{P}, \mathbf{i}\mathcal{V}\rangle = \langle \mathcal{P}, \mathbf{i}\mathcal{P}\rangle = 0,$$
$$\mathcal{V}^\top\mathfrak{B}(\mathbf{i}\mathcal{V}) = 0,$$
$$\mathcal{V}^\top\mathfrak{B}\mathbf{i}\mathcal{P} = 0,$$
$$\mathcal{P}^\top\mathfrak{B}(\mathbf{i}\mathcal{V}) = 0,$$
$$\mathcal{P}^\top\mathfrak{B}(\mathbf{i}\mathcal{P}) = 0,$$

and by (A.2) it follows that

$$\mathfrak{B}\mathcal{E}_5 = 2\lambda\mathfrak{M}\mathcal{E}_5,$$
$$\mathfrak{B}\mathcal{E}_6 = -\lambda\mathfrak{M}\mathcal{E}_6.$$

Consequently,

$$(\tilde{\mathcal{V}}, \tilde{\mathcal{P}}, \mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}})^\top(\lambda\mathfrak{M} + \mathfrak{B})(\tilde{\mathcal{V}}, \tilde{\mathcal{P}}, \mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}}) = \begin{pmatrix} \mathfrak{H}_1 & \\ & \mathfrak{H}_2 \end{pmatrix}$$

where

$$\begin{aligned}
\mathfrak{H}_1 &= \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \\
&:= (\tilde{\mathcal{V}}, \tilde{\mathcal{P}})^\top (\lambda \mathfrak{M} + \mathfrak{B})(\tilde{\mathcal{V}}, \tilde{\mathcal{P}}) \\
&= \begin{pmatrix} \mathcal{V}^\top (\lambda\mathfrak{M}+\mathfrak{B})\mathcal{V} - \frac{3\lambda|\langle \mathcal{V}, \mathcal{E}_5\rangle|^2}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle} & \mathcal{V}^\top (\lambda\mathfrak{M}+\mathfrak{B})\mathcal{P} - \frac{3\lambda\langle \mathcal{V}, \mathcal{E}_5\rangle\langle \mathcal{P}, \mathcal{E}_5\rangle}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle} \\ \mathcal{P}^\top (\lambda\mathfrak{M}+\mathfrak{B})\mathcal{V} - \frac{3\lambda\langle \mathcal{V}, \mathcal{E}_5\rangle\langle \mathcal{P}, \mathcal{E}_5\rangle}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle} & \mathcal{P}^\top (\lambda\mathfrak{M}+\mathfrak{B})\mathcal{P} - \frac{3\lambda|\langle \mathcal{P}, \mathcal{E}_5\rangle|^2}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{V}^\top \mathfrak{B}\mathcal{V} + \lambda\langle \mathcal{V}, \mathcal{V}\rangle - \frac{3\lambda|\langle \mathcal{V}, \mathcal{E}_5\rangle|^2}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle} & \mathcal{V}^\top \mathfrak{B}\mathcal{P} - \frac{3\lambda\langle \mathcal{V}, \mathcal{E}_5\rangle\langle \mathcal{P}, \mathcal{E}_5\rangle}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle} \\ \mathcal{P}^\top \mathfrak{B}\mathcal{V} - \frac{3\lambda\langle \mathcal{V}, \mathcal{E}_5\rangle\langle \mathcal{P}, \mathcal{E}_5\rangle}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle} & \mathcal{P}^\top \mathfrak{B}\mathcal{P} + \lambda\langle \mathcal{P}, \mathcal{P}\rangle - \frac{3\lambda|\langle \mathcal{P}, \mathcal{E}_5\rangle|^2}{\langle \mathcal{E}_5, \mathcal{E}_5\rangle} \end{pmatrix}, \\
\mathfrak{H}_2 &= \begin{pmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{pmatrix} \\
&:= (\mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}})^\top (\lambda \mathfrak{M} + \mathfrak{B})(\mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}}) \\
&= (\mathbf{i}\mathcal{V}, \mathbf{i}\mathcal{P})^\top (\lambda \mathfrak{M} + \mathfrak{B})(\mathbf{i}\mathcal{V}, \mathbf{i}\mathcal{P}) \\
&= \begin{pmatrix} (\mathbf{i}\mathcal{V})^\top \mathfrak{B}(\mathbf{i}\mathcal{V}) + \lambda\langle \mathcal{V}, \mathcal{V}\rangle & (\mathbf{i}\mathcal{V})^\top \mathfrak{B}(\mathbf{i}\mathcal{P}) \\ (\mathbf{i}\mathcal{P})^\top \mathfrak{B}(\mathbf{i}\mathcal{V}) & (\mathbf{i}\mathcal{P})^\top \mathfrak{B}(\mathbf{i}\mathcal{P}) + \lambda\langle \mathcal{P}, \mathcal{P}\rangle \end{pmatrix}.
\end{aligned}$$

A routine computation shows that

$$\langle \mathcal{E}_5, \mathcal{E}_5\rangle = I = 2(s^2+t^2) + \tfrac{4t^2+m_3m_4}{m_3+m_4},$$

$$\langle \mathcal{V}, \mathcal{V}\rangle = 2,$$

$$\langle \mathcal{P}, \mathcal{P}\rangle = 2 + \tfrac{4}{m_3+m_4},$$

$$\langle \mathcal{V}, \mathcal{E}_5\rangle = 2s,$$

$$\langle \mathcal{P}, \mathcal{E}_5\rangle = 2t + 2u - \tfrac{2m_4}{m_3+m_4} = 2t + \tfrac{4t}{m_3+m_4},$$

$$\mathcal{V}^\top \mathfrak{B}\mathcal{V} = \tfrac{2m_3}{[s^2+(u+t)^2]^{\frac{3}{2}}}\left(\tfrac{3s^2}{s^2+(u+t)^2} - 1\right) + \tfrac{2m_4}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}\left(\tfrac{3s^2}{s^2+(u+t-1)^2} - 1\right) + \tfrac{1}{s^3},$$

$$\mathcal{V}^\top \mathfrak{B}\mathcal{P} = \mathcal{P}^\top \mathfrak{B}\mathcal{V} = \tfrac{6m_3 s(u+t)}{[s^2+(u+t)^2]^{\frac{5}{2}}}\left(\tfrac{2}{m_3+m_4} + 1\right) + \tfrac{6m_4 s(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}}\left(\tfrac{2}{m_3+m_4} + 1\right),$$

$$\begin{aligned}\mathcal{P}^\top \mathfrak{B}\mathcal{P} = {}& \left[\tfrac{2m_3}{[s^2+(u+t)^2]^{\frac{3}{2}}}\left(\tfrac{3(u+t)^2}{s^2+(u+t)^2} - 1\right) + \tfrac{2m_4}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}\left(\tfrac{3(u+t-1)^2}{s^2+(u+t-1)^2} - 1\right)\right] \\ & \cdot \left(\tfrac{2}{m_3+m_4} + 1\right)^2,\end{aligned}$$

$$\begin{aligned}(\mathbf{i}\mathcal{V})^\top \mathfrak{B}(\mathbf{i}\mathcal{V}) = {}& \tfrac{2m_3}{[s^2+(u+t)^2]^{\frac{3}{2}}}\left(\tfrac{3(u+t)^2}{s^2+(u+t)^2} - 1\right) + \tfrac{2m_4}{[s^2+(u+t-1)^2]^{\frac{3}{2}}} \\ & \cdot \left(\tfrac{3(u+t-1)^2}{s^2+(u+t-1)^2} - 1\right) - \tfrac{1}{2s^3},\end{aligned}$$

$$\begin{aligned}(\mathbf{i}\mathcal{V})^\top \mathfrak{B}(\mathbf{i}\mathcal{P}) &= (\mathbf{i}\mathcal{P})^\top \mathfrak{B}(\mathbf{i}\mathcal{V}) \\ &= \tfrac{-6m_3 s(u+t)}{[s^2+(u+t)^2]^{\frac{5}{2}}}\left(\tfrac{2}{m_3+m_4} + 1\right) + \tfrac{-6m_4 s(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}}\left(\tfrac{2}{m_3+m_4} + 1\right),\end{aligned}$$

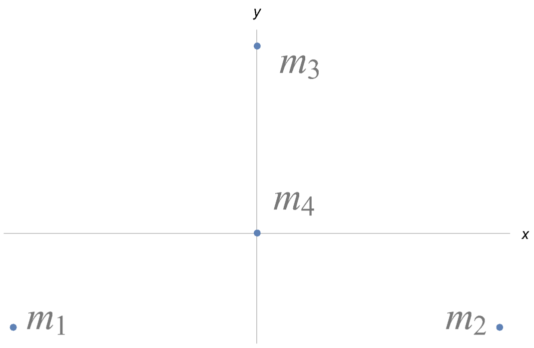

FIGURE B.1. equilateral central configuration

$$
\begin{aligned}
(\mathbf{i}\mathcal{P})^{\top}\mathfrak{B}(\mathbf{i}\mathcal{P}) = {} & [\frac{2m_3}{[s^2+(u+t)^2]^{\frac{3}{2}}}(\frac{3s^2}{s^2+(u+t)^2}-1) + \frac{2m_4}{[s^2+(u+t-1)^2]^{\frac{3}{2}}}(\frac{3s^2}{s^2+(u+t-1)^2}-1)] \\
& \cdot (\frac{2}{m_3+m_4}+1)^2.
\end{aligned}
$$

By (B.1), it follows that
(B.8)

$$
\begin{aligned}
h_{11} &= 6s^2\left(\frac{m_3}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda}{I} + \frac{1}{4s^5}\right), \\
h_{12} = h_{21} &= 6s(\frac{2}{m_3+m_4}+1)\left(\frac{m_3(u+t)}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda t}{I}\right), \\
h_{22} &= 6(\frac{2}{m_3+m_4}+1)^2\left(\frac{m_3(u+t)^2}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)^2}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda t^2}{I} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)}\right),
\end{aligned}
$$

(B.9)

$$
\begin{aligned}
h_{33} &= 6\left(\frac{m_3(u+t)^2}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)^2}{[s^2+(u+t-1)^2]^{\frac{5}{2}}}\right), \\
h_{34} = h_{43} &= -6s(\frac{2}{m_3+m_4}+1)\left(\frac{m_3(u+t)}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}}\right), \\
h_{44} &= 6s^2(\frac{2}{m_3+m_4}+1)^2\left(\frac{m_3}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} + \frac{1}{12s^5} - \frac{2\lambda}{3s^2(2+m_3+m_4)}\right).
\end{aligned}
$$

In conclusion, by (A.2), the central configuration $\mathcal{E}_5$ is degenerate if and only if one of the 2×2 matrixes $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is degenerate.

B.2. Equilateral Central Configurations

In this subsection, let us investigate the degeneracy of equilateral central configurations, i.e., $s=\frac{\sqrt{3}}{2}, t=\frac{1}{2}, u=1$ or $s=\frac{\sqrt{3}}{2}, t=-\frac{1}{2}, u=0$. Without loss of generality, we only investigate the case $s=\frac{\sqrt{3}}{2}, t=\frac{1}{2}, u=1$ as illustrated in Figure B.1.

Then $m_3=1$, and the central configurations equation (B.1) becomes:

$$
\lambda = \frac{1}{\sqrt{3}} + m_4.
$$

It is a classical result that $m_4 = (64\sqrt{3}+81)/249$ is the unique value of the mass parameter m_4 corresponding to degenerate central configurations, and the degree of degeneracy is two by Palmore [**17**, **18**]. In the following we shall reproduce this result and further find the corresponding vectors $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$.

A straightforward computation shows that the $\mathfrak{H}_1$ and $\mathfrak{H}_2$ become

$$\begin{pmatrix} \frac{1}{2}\left(3m_4+\sqrt{3}\right) & \frac{(\sqrt{3}m_4-1)(3+m_4)}{2+2m_4} \\ \frac{(\sqrt{3}m_4-1)(3+m_4)}{2+2m_4} & \frac{(m_4+3)\left(9m_4^2+\left(11\sqrt{3}-45\right)m_4+9\sqrt{3}\right)}{18(m_4+1)^2} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{2}\left(3m_4+\sqrt{3}\right) & -\frac{(m_4+3)\left(3\sqrt{3}m_4+1\right)}{2(m_4+1)} \\ -\frac{(m_4+3)\left(3\sqrt{3}m_4+1\right)}{2(m_4+1)} & \frac{(m_4+3)\left(81m_4^2+\left(11\sqrt{3}+171\right)m_4+9\sqrt{3}\right)}{18(m_4+1)^2} \end{pmatrix}$$

respectively. Since both of the determinants of the above two matrices are

$$\frac{m_4\,(m_4+3)\left(\left(5\sqrt{3}-18\right)m_4+3\sqrt{3}+2\right)}{3\,(m_4+1)^{\,2}},$$

the central configuration $\mathcal{E}_5$ is degenerate if and only if

$$(5\sqrt{3}-18)m_4+3\sqrt{3}+2=0.$$

Obviously, $m_4 = \frac{3\sqrt{3}+2}{18-5\sqrt{3}} = \frac{81+64\sqrt{3}}{249}$ is the unique solution of above equation. And in the meantime, it is clear that the corresponding degree of degeneracy is two.

Furthermore, it is easy to see that the 4×4 matrix

$$(\tilde{\mathcal{V}}, \tilde{\mathcal{P}}, \mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}})^{\top}(\lambda\mathfrak{M}+\mathfrak{B})(\tilde{\mathcal{V}}, \tilde{\mathcal{P}}, \mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}}) = \begin{pmatrix} \mathfrak{H}_1 & \\ & \mathfrak{H}_2 \end{pmatrix}$$

is positive definite for $0 < m_4 < \frac{81+64\sqrt{3}}{249}$ and indefinite for $m_4 > \frac{81+64\sqrt{3}}{249}$. So the equilateral central configuration is a local minimum of the function $I^{\frac{1}{2}}\mathcal{U}$ for $0 < m_4 < \frac{81+64\sqrt{3}}{249}$, and the equilateral central configuration is a saddle point of the function $I^{\frac{1}{2}}\mathcal{U}$ for $m_4 > \frac{81+64\sqrt{3}}{249}$.

As a result of (A.2), $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ can be obtained by calculating eigenvectors of the matrix $\lambda\mathbb{I}+\mathfrak{M}^{-1}\mathfrak{B}$.

A straightforward computation shows that, for the case $m_4 = \frac{81+64\sqrt{3}}{249}$,

(B.10)

$$\begin{aligned}
\mathcal{E}_1 &= \left(\tfrac{64\sqrt{3}+81}{498}, -\tfrac{741\sqrt{3}+908}{1494}, \tfrac{64\sqrt{3}+81}{498}, \tfrac{741\sqrt{3}+908}{1494}, 0, 0, -1, 0\right)^{\top},\\
\mathcal{E}_2 &= \left(\tfrac{165\sqrt{3}+179}{747}, -\tfrac{371\sqrt{3}+738}{2241}, -\tfrac{165\sqrt{3}+179}{747}, -\tfrac{371\sqrt{3}+738}{2241}, 0, \tfrac{2\sqrt{3}+9}{27}, 0, 1\right)^{\top},\\
\mathcal{E}_3 &= \left(\tfrac{275\sqrt{3}+243}{1494}, \tfrac{9\sqrt{3}+49}{166}, \tfrac{275\sqrt{3}+243}{1494}, -\tfrac{9\sqrt{3}+49}{166}, -\tfrac{1}{3\sqrt{3}}, 0, -1, 0\right)^{\top},\\
\mathcal{E}_4 &= \left(\tfrac{9\sqrt{3}+49}{166}, \tfrac{81-19\sqrt{3}}{1494}, -\tfrac{9\sqrt{3}+49}{166}, \tfrac{81-19\sqrt{3}}{1494}, 0, \tfrac{211\sqrt{3}+162}{747}, 0, -1\right)^{\top}.
\end{aligned}$$

The corresponding eigenvalues of the matrix $\lambda\mathbb{I}+\mathfrak{M}^{-1}\mathfrak{B}$ are

$$\frac{\mu_1}{\sqrt{3}}=0, \quad \frac{\mu_2}{\sqrt{3}}=0, \quad \frac{\mu_3}{\sqrt{3}}=\frac{799\sqrt{3}+1233}{498}, \quad \frac{\mu_4}{\sqrt{3}}=\frac{799\sqrt{3}+1233}{498}.$$

The computation above was performed with the aid of the software Mathematica. For example, by inputting $\mathfrak{B}$, $\mathfrak{M}$, $s=\frac{\sqrt{3}}{2}, t=\frac{1}{2}, u=1$, $\lambda=\frac{1}{\sqrt{3}}+m_4$, $m_3=1$

and $m_4 = (64\sqrt{3} + 81)/249$, Mathematica outputs $\lambda\mathbb{I} + \mathfrak{M}^{-1}\mathfrak{B}$ as following

$$\begin{pmatrix}
\frac{3305\sqrt{3}+2187}{2988} & \frac{81\sqrt{3}+275}{332} & -\frac{2}{3\sqrt{3}} & 0 & \frac{1}{12\sqrt{3}} & -\frac{1}{4} & \frac{5(64\sqrt{3}+81)}{-996} & \frac{3(27\sqrt{3}+64)}{-332} \\
\frac{81\sqrt{3}+275}{332} & \frac{1655\sqrt{3}+729}{2988} & 0 & \frac{1}{3\sqrt{3}} & -\frac{1}{4} & -\frac{5}{12\sqrt{3}} & \frac{3(27\sqrt{3}+64)}{-332} & \frac{64\sqrt{3}+81}{996} \\
-\frac{2}{3\sqrt{3}} & 0 & \frac{3305\sqrt{3}+2187}{2988} & -\frac{81\sqrt{3}+275}{332} & \frac{1}{12\sqrt{3}} & \frac{1}{4} & \frac{5(64\sqrt{3}+81)}{-996} & \frac{3(27\sqrt{3}+64)}{332} \\
0 & \frac{1}{3\sqrt{3}} & -\frac{81\sqrt{3}+275}{332} & \frac{1655\sqrt{3}+729}{2988} & \frac{1}{4} & -\frac{5}{12\sqrt{3}} & \frac{3(27\sqrt{3}+64)}{332} & \frac{64\sqrt{3}+81}{996} \\
\frac{1}{12\sqrt{3}} & -\frac{1}{4} & \frac{1}{12\sqrt{3}} & \frac{1}{4} & \frac{5}{6\sqrt{3}} & 0 & \frac{64\sqrt{3}+81}{249} & 0 \\
-\frac{1}{4} & -\frac{5}{12\sqrt{3}} & \frac{1}{4} & -\frac{5}{12\sqrt{3}} & 0 & \frac{81}{83}+\frac{2065}{498\sqrt{3}} & 0 & \frac{2(64\sqrt{3}+81)}{-249} \\
-\frac{5}{4} & -\frac{3\sqrt{3}}{4} & -\frac{5}{4} & \frac{3\sqrt{3}}{4} & 1 & 0 & \frac{98\sqrt{3}+303}{166} & 0 \\
-\frac{3\sqrt{3}}{4} & \frac{1}{4} & \frac{3\sqrt{3}}{4} & \frac{1}{4} & 0 & -2 & 0 & \frac{98\sqrt{3}+303}{166}
\end{pmatrix}$$

and by calling the command *Eigensystem*, Mathematica outputs the $\mathcal{E}_j$ and μ_j as above.

B.3. Rhombic Central Configurations

Let us investigate the degeneracy of rhombic central configurations in this subsection, that is, $t = 0, u = \frac{1}{2}$ for (B.1). Then $m_3 = m_4$, and the central configurations equation (B.1) becomes:

$$\begin{cases} \frac{1}{4s^3} + \frac{2\tilde{m}}{\tilde{r}^3} = \lambda \\ \frac{2}{\tilde{r}^3} + 2\tilde{m} = \lambda \end{cases} \tag{B.11}$$

where $\tilde{m} = m_3 = m_4, \tilde{r} = \sqrt{s^2 + \frac{1}{4}}$. Note that, $(\tilde{r}, s)$ forms a part of a hyperbola, and the two variables $\tilde{r}, s$ can be rationally parameterized simultaneously by the following transformations

$$\begin{cases} \tilde{r} = \frac{\zeta^2+1}{4\zeta}, \\ s = \frac{\zeta^2-1}{4\zeta}. \end{cases} \tag{B.12}$$

Since $s > 0$, we assume that $\zeta > 1$.

Proof of Theorem B.2. It is easy to see that the 4×4 matrix $(\tilde{\mathcal{V}}, \tilde{\mathcal{P}}, \mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}})^\top \cdot (\lambda \mathfrak{M} + \mathfrak{B})(\tilde{\mathcal{V}}, \tilde{\mathcal{P}}, \mathbf{i}\tilde{\mathcal{V}}, \mathbf{i}\tilde{\mathcal{P}})$ becomes

$$\begin{pmatrix} \frac{12\tilde{m}s^2}{\tilde{r}^5} + \frac{3}{2s^3} - \frac{12\lambda s^2}{2s^2+\frac{\tilde{m}}{2}} & 0 & & \\ 0 & (\frac{3\tilde{m}}{\tilde{r}^5} + \frac{1}{2s^3} - \frac{2\lambda}{\tilde{m}+1})(\frac{1}{\tilde{m}}+1)^2 & & \\ & & \frac{3\tilde{m}}{\tilde{r}^5} & 0 \\ & & 0 & (\frac{12\tilde{m}s^2}{\tilde{r}^5} + \frac{1}{2s^3} - \frac{2\lambda}{\tilde{m}+1})(\frac{1}{\tilde{m}}+1)^2 \end{pmatrix},$$

so the central configuration $\mathcal{E}_5$ is degenerate if and only if

$$(\frac{12\tilde{m}s^2}{\tilde{r}^5} + \frac{3}{2s^3} - \frac{12\lambda s^2}{2s^2 + \frac{\tilde{m}}{2}})(\frac{3\tilde{m}}{\tilde{r}^5} + \frac{1}{2s^3} - \frac{2\lambda}{\tilde{m}+1})(\frac{12\tilde{m}s^2}{\tilde{r}^5} + \frac{1}{2s^3} - \frac{2\lambda}{\tilde{m}+1}) = 0. \tag{B.13}$$

Note that $\tilde{r} = 1$ is impossible for the equation (B.11). Then, by the equation (B.11), we have

$$\begin{cases} \tilde{m} = \frac{\frac{\tilde{r}^3}{8s^3} - 1}{\tilde{r}^3 - 1}, \\ \lambda = \frac{2}{\tilde{r}^3} + 2\tilde{m} = \frac{\frac{2\tilde{r}^3}{8s^3} - \frac{2}{\tilde{r}^3}}{\tilde{r}^3-1}. \end{cases} \tag{B.14}$$

By the rational transformations (B.12), it follows that (B.14) becomes

$$\begin{cases} \tilde{m} = \frac{-8\zeta^3\left(\zeta^2-3\right)\left(7\zeta^4-6\zeta^2+3\right)}{(\zeta^2-1)^3(\zeta^2-4\zeta+1)(\zeta^4+4\zeta^3+18\zeta^2+4\zeta+1)}, \\ \lambda = \frac{16\zeta^3\left(\zeta^{12}+6\zeta^{10}-512\zeta^9+15\zeta^8+1536\zeta^7+20\zeta^6-1536\zeta^5+15\zeta^4+512\zeta^3+6\zeta^2+1\right)}{(\zeta^4-1)^3(\zeta^6+3\zeta^4-64\zeta^3+3\zeta^2+1)}. \end{cases} \tag{B.15}$$

Note that

$$\tilde{m} > 0 \Leftrightarrow \sqrt{3} < \zeta < \sqrt{3} + 2.$$

Therefore, all rhombic central configurations are

$$\left(-\frac{\zeta^2-1}{4\zeta}, 0, \frac{\zeta^2-1}{4\zeta}, 0, 0, \frac{1}{2}, 0, -\frac{1}{2}\right), \qquad \text{for } \sqrt{3} < \zeta < \sqrt{3} + 2.$$

By (B.12) and (B.15), a routine computation shows that the equation (B.13) becomes

$$\begin{aligned} &[(\zeta^2-3)(7\zeta^{10} - 45\zeta^8 + 70\zeta^6 + 256\zeta^5 - 90\zeta^4 + 35\zeta^2 - 9) \\ &(7\zeta^{16} - 88\zeta^{14} - 448\zeta^{13} - 44\zeta^{12} + 12352\zeta^{11} + 184\zeta^{10} - 37504\zeta^9 - 70\zeta^8 \\ &+ 34176\zeta^7 - 296\zeta^6 - 13248\zeta^5 - 12\zeta^4 + 576\zeta^3 + 72\zeta^2 - 9) \\ &(17\zeta^{16} - 56\zeta^{14} - 2432\zeta^{13} - 4\zeta^{12} + 14720\zeta^{11} + 248\zeta^{10} - 32768\zeta^9 + 70\zeta^8 \\ &+ 30720\zeta^7 - 136\zeta^6 - 14976\zeta^5 + 60\zeta^4 + 2688\zeta^3 + 72\zeta^2 - 15)]/[\left(\zeta^2 - 4\zeta + 1\right) \\ &\left(\zeta^{12} - 120\zeta^9 - 3\zeta^8 + 408\zeta^7 - 360\zeta^5 + 3\zeta^4 + 136\zeta^3 - 1\right) \\ &\left(\zeta^{12} - 4\zeta^{10} - 64\zeta^9 + 5\zeta^8 + 224\zeta^7 - 160\zeta^5 - 5\zeta^4 + 64\zeta^3 + 4\zeta^2 - 1\right)] = 0 \end{aligned}$$

By the software Mathematica, there is no solution in the interval $(\sqrt{3}, \sqrt{3}+2)$ for the above equation. Therefore, we claim that all the rhombic central configurations are nondegenerate.

Moreover, by (A.2) and some tedious computation (with the aid of Mathematica), it follows that the eight eigenvalues of the matrix $\lambda\mathbb{I} + \mathfrak{M}^{-1}\mathfrak{B}$ of a rhombic central configuration $\left(-\frac{\zeta^2-1}{4\zeta}, 0, \frac{\zeta^2-1}{4\zeta}, 0, 0, \frac{1}{2}, 0, -\frac{1}{2}\right)$ are

$$3\lambda, \quad 0, \quad \frac{\mu_7}{\sqrt{I}} = \lambda, \quad \frac{\mu_8}{\sqrt{I}} = \lambda;$$

and

(B.16)

$$\frac{\mu_1}{\sqrt{I}} = -\frac{48\zeta^3\left(7\zeta^{10}-45\zeta^8+70\zeta^6+256\zeta^5-90\zeta^4+35\zeta^2-9\right)}{(\zeta^2-1)^3(\zeta^2+1)^2(\zeta^6+3\zeta^4-64\zeta^3+3\zeta^2+1)},$$

$$\frac{\mu_2}{\sqrt{I}} = \frac{384\zeta^3\left(\zeta^{12}-4\zeta^{10}-64\zeta^9+5\zeta^8+224\zeta^7-160\zeta^5-5\zeta^4+64\zeta^3+4\zeta^2-1\right)}{(\zeta^2-1)^3(\zeta^2+1)^3(\zeta^6+3\zeta^4-64\zeta^3+3\zeta^2+1)},$$

$$\frac{\mu_3}{\sqrt{I}} = \begin{cases} 16\zeta^3(7\zeta^{16} - 88\zeta^{14} - 448\zeta^{13} - 44\zeta^{12} + 12352\zeta^{11} + 184\zeta^{10} - 37504\zeta^9 \\ -70\zeta^8 + 34176\zeta^7 - 296\zeta^6 - 13248\zeta^5 - 12\zeta^4 + 576\zeta^3 + 72\zeta^2 - 9) \\ / \left(1-\zeta^2\right)^3\left(\zeta^2+1\right)^5\left(\zeta^6+3\zeta^4-64\zeta^3+3\zeta^2+1\right), \end{cases}$$

$$\frac{\mu_4}{\sqrt{I}} = \begin{cases} 16\zeta^3(17\zeta^{16} - 56\zeta^{14} - 2432\zeta^{13} - 4\zeta^{12} + 14720\zeta^{11} + 248\zeta^{10} - 32768\zeta^9 \\ +70\zeta^8 + 30720\zeta^7 - 136\zeta^6 - 14976\zeta^5 + 60\zeta^4 + 2688\zeta^3 + 72\zeta^2 - 15) \\ / \left(\zeta^2-1\right)^3\left(\zeta^2+1\right)^5\left(\zeta^6+3\zeta^4-64\zeta^3+3\zeta^2+1\right). \end{cases}$$

where

$$I = \frac{\left(\zeta^2+1\right)^2\left(\zeta^{12}-4\zeta^{10}-64\zeta^9+5\zeta^8+224\zeta^7-160\zeta^5-5\zeta^4+64\zeta^3+4\zeta^2-1\right)}{8\zeta^2(\zeta^2-1)^3(\zeta^6+3\zeta^4-64\zeta^3+3\zeta^2+1)}.$$

A straightforward computation shows that

$$\frac{\kappa^{\frac{1}{2}}}{\sqrt{I}} = \frac{\sqrt{2\lambda}}{\sqrt{I}} = 16\sqrt{\frac{\zeta^5(\zeta^{12}+6\zeta^{10}-512\zeta^9+15\zeta^8+1536\zeta^7+20\zeta^6-1536\zeta^5+15\zeta^4+512\zeta^3+6\zeta^2+1)}{(\zeta^2+1)^5(\zeta^{12}-4\zeta^{10}-64\zeta^9+5\zeta^8+224\zeta^7-160\zeta^5-5\zeta^4+64\zeta^3+4\zeta^2-1)}},$$

and

$$\begin{aligned}
&0 < \kappa^{\frac{1}{2}}, \mu_1 < \mu_2 < \mu_3, \mu_4, && \text{for } \sqrt{3} < \zeta < \sqrt{3}+2 \\
&\mu_1 < \sqrt{\kappa}, && \text{for } \zeta_1 < \zeta < \zeta_2 \\
&\mu_1 = \sqrt{\kappa}, && \text{for } \zeta = \zeta_1, \zeta_2 \\
&\mu_1 > \sqrt{\kappa}, && \text{for } \sqrt{3} < \zeta < \zeta_1 \text{ or } < \zeta_2 < \zeta < \sqrt{3}+2 \qquad \text{(B.17)} \\
&\mu_4 < \mu_3, && \text{for } \sqrt{3} < \zeta < \sqrt{2}+1 \\
&\mu_3 < \mu_4, && \text{for } \sqrt{2}+1 < \zeta < \sqrt{3}+2 \\
&\mu_3 = \mu_4, && \text{for } \zeta = \sqrt{2}+1
\end{aligned}$$

where$\zeta_1 \approx 1.7889580612081344, \zeta_2 \approx 3.705602221466667$.

It is clear that all the numbers $\kappa^{\frac{1}{2}}, \mu_1, \mu_2, \mu_3, \mu_4$ and their ratios are irrational for almost all $\zeta \in (\sqrt{3}, \sqrt{3}+2)$, since each of them is rational only for countably many ζ.

In conclusion, Theorem B.2 holds. □

B.4. Central Configurations With Degree of Degeneracy Two

Let us investigate central configurations with degree of degeneracy two in this subsection. Note that equilateral and rhombic central configurations are ruled out of our discussion in the following.

PROOF OF THEOREM B.1. First, let us introduce the following rational transformation:

$$\begin{cases} u+t = s\frac{\xi^2-1}{2\xi}, \\ u+t-1 = s\frac{\eta^2-1}{2\eta}. \end{cases}$$

Without losing generality, we may assume that $\xi > \eta > 0$. Then

$$\begin{cases} \sqrt{s^2+(u+t)^2} = s\frac{\xi^2+1}{2\xi}, \\ \sqrt{s^2+(u+t-1)^2} = s\frac{\eta^2+1}{2\eta}, \\ s = \frac{1}{\frac{\xi^2-1}{2\xi}-\frac{\eta^2-1}{2\eta}}. \end{cases}$$

By (B.7), it follows that

$$\begin{cases} m_3 = \frac{-(\eta-1)(\eta+1)\left(\eta^2-4\eta+1\right)\left(\eta^4+4\eta^3+18\eta^2+4\eta+1\right)\left(\xi^2+1\right)^3(\eta-\xi)^2(\eta\xi+1)^2}{32(\eta^2+1)^3\xi^2(\eta^2\xi+2\eta-\xi)(\eta^4\xi^2-3\eta^3\xi^3+\eta^3\xi+3\eta^2\xi^4-2\eta^2\xi^2+\eta^2+3\eta\xi^3-\eta\xi+\xi^2)}, \\ m_4 = \frac{\left(\eta^2+1\right)^3(\xi-1)(\xi+1)\left(\xi^2-4\xi+1\right)\left(\xi^4+4\xi^3+18\xi^2+4\xi+1\right)(\eta-\xi)^2(\eta\xi+1)^2}{32\eta^3(\xi^2+1)^3(2\eta\xi-\xi^2+1)(\eta^4\xi^2-\eta^3\xi^3+\eta^3\xi+\eta^2\xi^4-2\eta^2\xi^2+\eta^2+3\eta\xi^3-3\eta\xi+3\xi^2)}, \\ \lambda = \frac{1}{4s^3} + \frac{8\xi^3 m_3}{(\xi^2+1)^3 s^3} + \frac{8\eta^3 m_4}{(\eta^2+1)^3 s^3}. \end{cases}$$

Note that

$$\begin{cases} m_3 > 0 \Leftrightarrow (1-\eta)\left(\eta^2-4\eta+1\right)\left(\eta^2\xi+2\eta-\xi\right) > 0, \\ m_4 > 0 \Leftrightarrow (\xi-1)\left(\xi^2-4\xi+1\right)\left(2\eta\xi-\xi^2+1\right) > 0. \end{cases}$$

By $u = \frac{2t+m_4}{m_3+m_4}$, it follows that

$$t = \frac{(m_3+m_4)(u+t)-m_4}{2+m_3+m_4} = \frac{(m_3+m_4)(s\frac{\xi^2-1}{2\xi})-m_4}{2+m_3+m_4}.$$

Thus the variables s, u, t, m_3, m_4 and λ are all explicit rational functions of the variables ξ and η.

REMARK B.3. We claim that the system (B.1) is equivalent to the system (B.7). To prove the claim, it suffices to show the fact that the system (B.1) holds when the six variables s, u, t, m_3, m_4 and λ are substituted by the obtained functions in terms of ξ and η. Indeed, this fact is proved by a routine computation with the aid of the software Mathematica.

It is clear now that the entries of the matrixes $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are rational functions of two variables ξ, η, which can be well handled by the symbolic computation of the software Mathematica. This is one reason of introducing the above rational transformation.

Below we investigate only the degeneracy in the cases corresponding to $\xi, \eta \neq 1, 2 \pm \sqrt{3}$ and $\xi\eta \neq 1$. Since it is easy to see that

- If $\xi = 1$ or $\eta = 1$, then the corresponding configurations are formed by the three particles on a common straight line but the fourth particle not on the straight line. By the well-known *perpendicular bisector theorem* [**14**], they cannot be central configurations.
- If $\xi\eta = 1$, then the corresponding configurations are rhombus.
- If $\xi = 2 \pm \sqrt{3}$, then $m_4 = 0$ or the corresponding configurations are equilateral configurations; similarly, if $\eta = 2 \pm \sqrt{3}$, then $m_3 = 0$ or the corresponding configurations are equilateral configurations.

By (B.9), it holds

$$h_{33} = \frac{6m_3(u+t)^2}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{6m_4(u+t-1)^2}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} > 0.$$

Therefore, the 2×2 matrix $\mathfrak{H}_2$ is not zero. As a result, central configurations with degree of degeneracy two satisfy

$$\mathfrak{H}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{B.18}$$

or

$$\begin{cases} Det\mathfrak{H}_1 = 0, \\ Det\mathfrak{H}_2 = 0. \end{cases} \tag{B.19}$$

We claim that there is no solution for the equation (B.18) (resp. (B.19)) such that

$$(\xi - 1)(\xi^2 - 4\xi + 1)(\eta - 1)(\eta^2 - 4\eta + 1)(\xi\eta - 1) \neq 0 \tag{B.20}$$

and $\xi > \eta > 0$.

We remark that it is necessary to include computer-aided proofs in this part, due to the size of the polynomials we are working with. We will not write all the explicit expressions of the polynomials. Instead, we shall provide the steps followed to calculate all the important polynomials.

A. The Case for (B.18)**.**

By (B.8), the equation (B.18) becomes

$$\begin{cases} \frac{m_3}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda}{I} + \frac{1}{4s^5} = 0, \\ \frac{m_3(u+t)}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda t}{I} = 0, \\ \frac{m_3(u+t)^2}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)^2}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda t^2}{I} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)} = 0. \end{cases}$$

It follows that

$$\begin{cases} (1-u)\left[\frac{m_3 u}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{(u+t-1)t}{4s^5} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)}\right] = 0, \\ u\left[\frac{m_4(1-u)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} + \frac{(u+t)t}{4s^5} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)}\right] = 0, \\ \frac{2\lambda u(1-u)}{I} + \frac{(u+t-1)(u+t)}{4s^5} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)} = 0. \end{cases} \tag{B.21}$$

By $u = \frac{2t+m_4}{m_3+m_4}$, (B.1) and (B.21), it is easy to shows that,

$$u(1-u) = 0$$

yields

$$s = \frac{\sqrt{3}}{2}, t = \frac{1}{2}, u = 1 \text{ or } s = \frac{\sqrt{3}}{2}, t = -\frac{1}{2}, u = 0.$$

They are equilateral central configurations ruled out by us.

Therefore, the equation (B.18) becomes

$$\begin{cases} \frac{m_3 u}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{(u+t-1)t}{4s^5} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)} = 0, \\ \frac{m_4(1-u)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} + \frac{(u+t)t}{4s^5} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)} = 0, \\ \frac{2\lambda u(1-u)}{I} + \frac{(u+t-1)(u+t)}{4s^5} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)} = 0, \end{cases}$$

or

$$\begin{cases} Q_{11}(\xi,\eta) := \frac{m_3 u}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{(u+t-1)t}{4s^5} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)} = 0, \\ Q_{12}(\xi,\eta) := -\frac{m_3 u}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(1-u)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} + \frac{t}{4s^5} = 0, \\ Q_{22}(\xi,\eta) := \frac{2\lambda(1-u)}{I} + \frac{(u+t-1)}{4s^5} - \frac{m_3}{[s^2+(u+t)^2]^{\frac{5}{2}}} = 0. \end{cases}$$

It is clear that Q_{11}, Q_{12}, Q_{22} are rational functions of ξ, η.

Let P_{11}, P_{12}, P_{22} be the numerators of the following rational functions

$$\frac{Q_{11}(\xi,\eta)}{(\eta-\xi)^6(\eta\xi+1)^5},$$

$$\frac{Q_{12}(\xi,\eta)}{(\eta-1)(\eta+1)(\xi-1)(\xi+1)(\eta-\xi)^7(\eta\xi-1)(\eta\xi+1)^6},$$

$$\frac{Q_{22}(\xi,\eta)}{(\eta-1)(\eta+1)(\xi-1)(\xi+1)(\eta-\xi)^7(\eta\xi+1)^6}$$

respectively. By calling the command *Solve* of Mathematica for seeking solutions of

(B.22) $$P_{11}(\xi,\eta) = P_{12}(\xi,\eta) = P_{22}(\xi,\eta) = 0$$

such that

$$\xi > \eta > 0,$$

we arrive at the conclusion that there is no solution for the equation (B.22), this computation takes less than half a minute on a desktop computer. Therefore, there is no solution for the equation (B.18).

It is noteworthy that we can clearly demonstrate that there is no solution for (B.18) by using a Gröbner basis of polynomials. As a matter of fact, it is well known that the set of polynomials in a Gröbner basis have the same collection of roots as the original polynomials. Therefore, by calling the command *GroebnerBasis* of Mathematica for P_{11}, P_{12}, P_{22}, one finds that it also holds

(B.23) $$G_{(\eta,\xi)}(\xi) = G_{(\xi,\eta)}(\eta) = 0,$$

where

$$G_{(\eta,\xi)}(\xi) = (\xi-1)^3\xi^4(\xi+1)^3\left(\xi^2+1\right)^7\left(\xi^2-4\xi+1\right)\left(\xi^4+4\xi^3+18\xi^2+4\xi+1\right)$$

is the first element of the list in the *GroebnerBasis* of P_{11}, P_{12}, P_{22} for the ordered $\{\eta, \xi\}$, and

$$G_{(\xi,\eta)}(\eta) = (\eta-1)^2\eta^5(\eta+1)^2\left(\eta^2+1\right)^7\left(\eta^2-4\eta+1\right)\left(\eta^4+4\eta^3+18\eta^2+4\eta+1\right)$$

is that for the ordered $\{\xi, \eta\}$. Note that it is not easy to find a Gröbner basis in general, for more detail please refer to `https://reference.wolfram.com/language/ref/GroebnerBasis.html`. It is now clear that there is no solution for the equation (B.23), thus there is no solution for the equation (B.18), under the condition that $\xi, \eta \neq 1, 2 \pm \sqrt{3}$ and $\xi > \eta > 0$.

B. The Case for (B.19)**.**

By (B.8) and (B.9), the equation (B.19) becomes

$$\begin{cases} Q_1(\xi,\eta) := \left(\frac{m_3(u+t)^2}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)^2}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda t^2}{I} + \frac{1}{12s^3} - \frac{2\lambda}{3(2+m_3+m_4)} \right) \\ \qquad \cdot \left(\frac{m_3}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda}{I} + \frac{1}{4s^5} \right) \\ \qquad - \left(\frac{m_3(u+t)}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} - \frac{2\lambda t}{I} \right)^2 \\ \quad = 0, \\ Q_2(\xi,\eta) := \left(\frac{m_3}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} + \frac{1}{12s^5} - \frac{2\lambda}{3s^2(2+m_3+m_4)} \right) \\ \qquad \left(\frac{m_3(u+t)^2}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)^2}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} \right) - \left(\frac{m_3(u+t)}{[s^2+(u+t)^2]^{\frac{5}{2}}} + \frac{m_4(u+t-1)}{[s^2+(u+t-1)^2]^{\frac{5}{2}}} \right)^2 \\ \quad = 0. \end{cases}$$

Similarly, let P_1, P_2 be the numerators of the following rational functions

$$\frac{Q_1(\xi,\eta)}{(\eta^2-1)(\eta^2-4\eta+1)(\eta^4+4\eta^3+18\eta^2+4\eta+1)(\xi^2-1)(\xi^2-4\xi+1)(\xi^4+4\xi^3+18\xi^2+4\xi+1)(\eta-\xi)^{14}(\eta\xi+1)^{13}},$$

$$\frac{Q_2(\xi,\eta)}{(\eta^2-1)(\eta^2-4\eta+1)(\eta^4+4\eta^3+18\eta^2+4\eta+1)(\xi^2-1)(\xi^2-4\xi+1)(\xi^4+4\xi^3+18\xi^2+4\xi+1)(\eta-\xi)^{14}(\eta\xi+1)^{13}},$$

respectively.

It is noteworthy that using the resultant of two polynomials can evidently reduce calculating time. Let R_ξ (resp. R_η) be the resultant of P_1 and P_2 for the variable ξ (resp. η). Then the equation (B.19) reduces to

$$P_1(\xi,\eta) = P_2(\xi,\eta) = R_\xi(\eta) = R_\eta(\xi) = 0. \tag{B.24}$$

By calling the command *Solve* of Mathematica for seeking solutions of (B.24) such that

$$\xi > \eta > 0,$$

we obtain two solutions

$$\xi = \frac{1}{\sqrt{3}}, \eta = 2 - \sqrt{3}; \qquad \xi = \sqrt{3} + 2, \eta = \sqrt{3}.$$

this computation takes less than ten seconds on a desktop computer. However, the two solutions should be excluded, since, for example, (B.20) does not hold for them. Thus we arrive at the conclusion that there is no solution for the equation (B.19).

In conclusion, Theorem B.1 holds. □

Following the result obtained and a crude dimension count, we can venturesomely conjecture that:

All the four-body central configurations except equilateral central configurations have degree of degeneracy equal or less than one.

That is to say, we believe that the equilateral central configurations founded by Palmore [**17**, **18**] are the only degenerate central configurations with degree of degeneracy two for the four-body problem, although we cannot prove it now. This conjecture is important for resolving $PISPW$ of the four-body problem. Indeed, if this conjecture is true, one can completely solve $PISPW$ of the four-body problem, please see Corollary 4.3.

Remark B.4. The results and method in this section are novel according to what I know, even though for kite central configurations, Leandro [**12**] has investigated a more general case, including works on degenerate central configurations. Indeed the above rational transformations are inspired by his work. However, the definition of degeneracy in his work is different from ours, thus the results in [**12**] cannot be directly applied. The point is that the degenerate central configuration by his definition is still degenerate by our definition; the opposite, however, isn't necessarily true. The problem can be explained in this way:

Given a function f defined on a manifold Ω, $x \in \Omega$ is a critical point of the function f. Let Ω_1 be a submanifold of Ω and $x \in \Omega_1$, although the point x may still be a critical point of the function f restricted to Ω_1, the degeneracy of the critical point x may change.

APPENDIX C

Diagonalization of the Linear Part

The aim of the section is to discuss the diagonalization of the linear part of equations (3.13) in detail.

Without loss of generality, suppose that the eigenvalues $\mu_1, \cdots, \mu_{2N-4}$ of Λ satisfy

$$\begin{aligned} \mu_j = 0 \quad &\text{for} \quad j \in \{1, \cdots, n_0\}, \\ \mu_j > 0 \quad &\text{for} \quad j \in \{n_0+1, \cdots, n_0+n_p\}, \\ 0 > \mu_j > -\frac{\kappa}{16} \quad &\text{for} \quad j \in \{n_0+n_p+1, \cdots, n_0+n_p+n_1\}, \\ \mu_j < -\frac{\kappa}{16} \quad &\text{for} \quad j \in \{n_0+n_p+n_1+1, \cdots, n_0+n_p+n_1+n_2\}, \\ \mu_j = -\frac{\kappa}{16} \quad &\text{for} \quad j \in \{n_0+n_p+n_1+n_2+1, \cdots, 2N-4\}. \end{aligned}$$

Thus n_p is the number of positive eigenvalues μ_j (counted with their multiplicity), n_1 is the number of eigenvalues contained in the interval $(-\frac{\kappa}{16}, 0)$, n_2 is the number of eigenvalues strictly smaller than $-\frac{\kappa}{16}$, and therefore the multiplicity of $-\frac{\kappa}{16}$ is equal to $2N-4-(n_0+n_p+n_1+n_2+1)$.

To simplify the notation, let

$$d_0 = n_0, \quad d_p = n_0+n_p, \quad d_1 = n_0+n_p+n_1, \quad d_2 = n_0+n_p+n_1+n_2, \quad d = 2N-4.$$

Set

$$\tilde{\mu}_j^{\pm} = -\frac{\kappa^{\frac{1}{2}}}{4} \pm \sqrt{\mu_j + \frac{\kappa}{16}}$$

for any $j \in \{1, \cdots, d\}$; note that, if $\mu_j + \frac{\kappa}{16} < 0$, we agree on that

$$\sqrt{\mu_j + \frac{\kappa}{16}} = \mathbf{i}\sqrt{|\mu_j + \frac{\kappa}{16}|},$$

hence

$$\begin{cases} \tilde{\mu}_j^{+} = 0, \quad \tilde{\mu}_j^{-} = -\frac{\kappa^{\frac{1}{2}}}{2}, & \text{if } j \in \{1, \cdots, d_0\}; \\ \tilde{\mu}_j^{+} > 0 > \tilde{\mu}_j^{-}, & \text{if } j \in \{d_0+1, \cdots, d_p\}; \\ Re\tilde{\mu}_j^{\pm} < 0, & \text{if } j \in \{d_p+1, \cdots, d\}; \\ 0 > \tilde{\mu}_j^{+} > \tilde{\mu}_j^{-}, & \text{if } j \in \{d_p+1, \cdots, d_1\}; \\ \tilde{\mu}_j^{+} = \overline{\tilde{\mu}_j^{-}}, & \text{if } j \in \{d_1+1, \cdots, d_2\}; \\ \tilde{\mu}_j^{+} = \tilde{\mu}_j^{-} = -\frac{\kappa^{\frac{1}{2}}}{4}, & \text{if } j \in \{d_2+1, \cdots, d\}. \end{cases}$$

Here and below "$\bar{\cdot}$" denotes the notation of complex conjugate.

Let

$$\{\mathbf{e}_1, \cdots, \mathbf{e}_d\}$$

be the standard basis of $\mathbb{R}^d$, it is obvious that

$$\Lambda \mathbf{e}_j = \mu_j \mathbf{e}_j, \qquad \forall j \in \{1, \cdots, d\}.$$

Then a straightforward computation shows that

$$\tilde{\Lambda} \mathbf{v}_j^{\pm} = \tilde{\mu}_j^{\pm} \mathbf{v}_j^{\pm}, \qquad \forall j \in \{1, \cdots, d\},$$

where

$$\tilde{\Lambda} = \begin{pmatrix} 0 & \mathbb{I} \\ \Lambda & -\frac{\kappa^{\frac{1}{2}}}{2}\mathbb{I} \end{pmatrix}, \qquad \mathbf{v}_j^{\pm} = \begin{pmatrix} \mathbf{e}_j \\ \tilde{\mu}_j^{\pm} \mathbf{e}_j \end{pmatrix}.$$

To obtain a basis of $\mathbb{R}^{2d}$ starting from the eigenvectors $\mathbf{v}_j^{\pm}$ of $\tilde{\Lambda}$, let us introduce the vectors

$$\begin{cases} \mathbf{p}_j^{\pm} = \mathbf{v}_j^{\pm}, & \text{if } j \in \{1, \cdots, d_1\}; \\ \mathbf{p}_j^{+} = Im \mathbf{v}_j^{+} = \begin{pmatrix} \mathbf{0} \\ \sqrt{|\mu_j + \frac{\kappa}{16}|}\mathbf{e}_j \end{pmatrix}, & \text{if } j \in \{d_1+1, \cdots, d_2\}; \\ \mathbf{p}_j^{-} = Re \mathbf{v}_j^{+} = \begin{pmatrix} \mathbf{e}_j \\ -\frac{\kappa^{\frac{1}{2}}}{4}\mathbf{e}_j \end{pmatrix}, & \text{if } j \in \{d_1+1, \cdots, d_2\}; \\ \mathbf{p}_j^{+} = \mathbf{v}_j^{+} = \begin{pmatrix} \mathbf{e}_j \\ -\frac{\kappa^{\frac{1}{2}}}{4}\mathbf{e}_j \end{pmatrix}, & \text{if } j \in \{d_2+1, \cdots, d\}; \\ \mathbf{p}_j^{-} = \mathbf{v}_j^{-} + \begin{pmatrix} \mathbf{0} \\ \epsilon \mathbf{e}_j \end{pmatrix} = \begin{pmatrix} \mathbf{e}_j \\ (\epsilon - \frac{\kappa^{\frac{1}{2}}}{4})\mathbf{e}_j \end{pmatrix}, & \text{if } j \in \{d_2+1, \cdots, d\}. \end{cases}$$

It is easy to see that the family $\{\mathbf{p}_j^{\pm}\}_{j \in \{1, \cdots, d\}}$ is a basis of $\mathbb{R}^{2d}$ and moreover, it holds

$$\begin{cases} \tilde{\Lambda}\mathbf{p}_j^{\pm} = \tilde{\mu}_j^{\pm} \mathbf{p}_j^{\pm}, & \text{if } j \in \{1, \cdots, d_1\}; \\ \tilde{\Lambda}\mathbf{p}_j^{+} = -\frac{\kappa^{\frac{1}{2}}}{4}\mathbf{p}_j^{+} + \sqrt{|\mu_j + \frac{\kappa}{16}|}\mathbf{p}_j^{-}, & \text{if } j \in \{d_1+1, \cdots, d_2\}; \\ \tilde{\Lambda}\mathbf{p}_j^{-} = -\sqrt{|\mu_j + \frac{\kappa}{16}|}\mathbf{p}_j^{+} - \frac{\kappa^{\frac{1}{2}}}{4}\mathbf{p}_j^{-}, & \text{if } j \in \{d_1+1, \cdots, d_2\}; \\ \tilde{\Lambda}\mathbf{p}_j^{+} = -\frac{\kappa^{\frac{1}{2}}}{4}\mathbf{p}_j^{+}, & \text{if } j \in \{d_2+1, \cdots, d\}; \\ \tilde{\Lambda}\mathbf{p}_j^{-} = \epsilon \mathbf{p}_j^{+} - \frac{\kappa^{\frac{1}{2}}}{4}\mathbf{p}_j^{-}, & \text{if } j \in \{d_2+1, \cdots, d\}. \end{cases}$$

Now set

$$\begin{aligned} \mathfrak{P} &= (\mathbf{p}_1^{+}, \cdots, \mathbf{p}_d^{+}, \mathbf{p}_1^{-}, \cdots, \mathbf{p}_d^{-}), \\ D_1 &= diag(\tilde{\mu}_1^{+}, \cdots, \tilde{\mu}_{d_1}^{+}), \\ D_2 &= diag\left(\frac{1}{\sqrt{\mu_1 + \frac{\kappa}{16}}}, \cdots, \frac{1}{\sqrt{\mu_{d_1} + \frac{\kappa}{16}}}\right), \\ D_3 &= diag\left(\frac{1}{\sqrt{|\mu_{d_1+1} + \frac{\kappa}{16}|}}, \cdots, \frac{1}{\sqrt{|\mu_{d_2} + \frac{\kappa}{16}|}}\right), \\ D_4 = D_3^{-1} &= diag\left(\sqrt{|\mu_{d_1+1} + \frac{\kappa}{16}|}, \cdots, \sqrt{|\mu_{d_2} + \frac{\kappa}{16}|}\right), \end{aligned}$$

then

$$\mathfrak{P} = \begin{pmatrix} \mathbb{I} & & & \mathbb{I} & & \\ & 0 & & & \mathbb{I} & \\ & & \mathbb{I} & & & \mathbb{I} \\ D_1 & & & -\frac{\kappa^{\frac{1}{2}}}{2}\mathbb{I} - D_1 & & \\ & D_4 & & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I} & \\ & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I} & & & (\epsilon - \frac{\kappa^{\frac{1}{2}}}{4})\mathbb{I} \end{pmatrix},$$

and $\mathfrak{P}$ is invertible for any $\epsilon > 0$. It holds that

$$\mathfrak{P}^{-1}\tilde{\Lambda}\mathfrak{P} = \begin{pmatrix} D_1 & & & & & \\ & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I}_{n_2} & & & -D_4 & \\ & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I} & & & \epsilon\mathbb{I} \\ & & & -\frac{\kappa^{\frac{1}{2}}}{2}\mathbb{I} - D_1 & & \\ & D_4 & & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I}_{n_2} & \\ & & & & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I} \end{pmatrix},$$

where

$$\mathfrak{P}^{-1} = \begin{pmatrix} \frac{1}{2}\mathbb{I} + \frac{\kappa^{\frac{1}{2}}}{8}D_2 & & & \frac{1}{2}D_2 & & \\ & \frac{\kappa^{\frac{1}{2}}}{4}D_3 & & & D_3 & \\ & & (1 - \frac{\kappa^{\frac{1}{2}}}{4\epsilon})\mathbb{I} & & & -\frac{\mathbb{I}}{\epsilon} \\ \frac{1}{2}\mathbb{I} - \frac{\kappa^{\frac{1}{2}}}{8}D_2 & & & -\frac{1}{2}D_2 & & \\ & \mathbb{I} & & & 0 & \\ & & \frac{\kappa^{\frac{1}{2}}}{4\epsilon}\mathbb{I} & & & \frac{\mathbb{I}}{\epsilon} \end{pmatrix}.$$

As a result, set

$$q = (q_1, \cdots, q_{2N-4}, q_{2N-3}, \cdots, q_{4N-8})^\top,$$

and by applying the linear substitution

$$\begin{pmatrix} z \\ Z \end{pmatrix} = \mathfrak{P}q,$$

the equations (3.13) are transformed to the following equations

$$\text{(C.1)} \qquad \begin{cases} q_k' = \phi_k(q,\gamma), & k \in \{1,\cdots,d_0\} \\ q_{2N-4+k}' = -\frac{\kappa^{\frac{1}{2}}}{2} q_{2N-4+k} - \phi_k(q,\gamma), & k \in \{1,\cdots,d_0\} \end{cases}$$

$$\text{(C.2)} \qquad \begin{cases} q_k' = \tilde{\mu}_k q_k + \phi_k(q,\gamma), & k \in \{d_0+1,\cdots,d_1\} \\ q_{2N-4+k}' = (-\frac{\kappa^{\frac{1}{2}}}{2} - \tilde{\mu}_k) q_{2N-4+k} - \phi_k(q,\gamma), & k \in \{d_0+1,\cdots,d_1\} \end{cases}$$

$$\text{(C.3)} \qquad \begin{cases} q_k' = -\frac{\kappa^{\frac{1}{2}}}{4} q_k - \sqrt{|\mu_k + \frac{\kappa}{16}|}\, q_{2N-4+k} + \phi_k(q,\gamma), & k \in \{d_1+1,\cdots,d_2\} \\ q_{2N-4+k}' = \sqrt{|\mu_k + \frac{\kappa}{16}|}\, q_k - \frac{\kappa^{\frac{1}{2}}}{4} q_{2N-4+k}, & k \in \{d_1+1,\cdots,d_2\} \end{cases}$$

$$\text{(C.4)} \qquad \begin{cases} q_k' = -\frac{\kappa^{\frac{1}{2}}}{4} q_k + \epsilon q_{2N-4+k} + \phi_k(q,\gamma), & k \in \{d_2+1,\cdots,d\} \\ q_{2N-4+k}' = -\frac{\kappa^{\frac{1}{2}}}{4} q_{2N-4+k} - \phi_k(q,\gamma), & k \in \{d_2+1,\cdots,d\} \end{cases}$$

$$\text{(C.5)} \qquad \gamma' = \kappa^{\frac{1}{2}} \gamma + \phi_0(q,\gamma),$$

where the functions φ_k, φ_0 are power-series in the $4N-7$ variables q, γ starting with quadratic terms:

$$\text{(C.6)} \qquad \begin{cases} \phi_k(q,\gamma) = \frac{1}{2\sqrt{\mu_k + \frac{\kappa}{16}}} \chi_k(z,Z,\gamma), & k \in \{1,\cdots,d_1\} \\ \phi_k(q,\gamma) = \frac{1}{\sqrt{|\mu_k + \frac{\kappa}{16}|}} \chi_k(z,Z,\gamma), & k \in \{d_1+1,\cdots,d_2\} \\ \phi_k(q,\gamma) = -\frac{1}{\epsilon} \chi_k(z,Z,\gamma), & k \in \{d_2+1,\cdots,2N-4\} \\ \phi_0(q,\gamma) = \chi_0(z,Z,\gamma); \end{cases}$$

and the equation (3.15) is transformed to (3.22):

$$\begin{aligned} \theta' &= z^\top Q Z = q^\top \mathfrak{P}^\top \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix} Q \begin{pmatrix} 0 & \mathbb{I} \end{pmatrix} \mathfrak{P} q \\ &= \sum_{1 \le k \le n_0} \sum_{j=n_0+1}^{n_0+n_p} q_{kj} \tilde{\mu}_j q_k q_j + \sum_{j,k=n_0+1}^{n_0+n_p} q_{kj} \tilde{\mu}_j q_k q_j + \cdots, \end{aligned}$$

where "$\cdots$" denotes all the quadratic terms which contain at least one of q_k ($k > n_0 + n_p$) as a factor.

To simplify the notations, the equations (C.1), (C.2), (C.3), (C.4) and (C.5) can be rewritten in the compact form (3.21):

$$\begin{pmatrix} q' \\ \gamma' \end{pmatrix} = \mathcal{C} \begin{pmatrix} q \\ \gamma \end{pmatrix} + \varphi(q,\gamma),$$

where

$$\varphi = (\varphi_1, \cdots, \varphi_{2N-4}, \varphi_{2N-3}, \cdots, \varphi_{4N-8}, \varphi_0)^\top,$$

$$\begin{cases} \varphi_k = \phi_k(q,\gamma), & k \in \{0\} \bigcup \{1, \cdots, d\}; \\ \varphi_{2N-4+k} = 0, & k \in \{d_1+1, \cdots, d_2\}; \\ \varphi_{2N-4+k} = -\phi_k(q,\gamma), & k \in \{1, \cdots, d_1\} \bigcup \{d_2+1, \cdots, d\}; \end{cases}$$

$$\mathcal{C} = \begin{pmatrix} \mathfrak{P}^{-1} & \\ & 1 \end{pmatrix} \mathfrak{A} \begin{pmatrix} \mathfrak{P} & \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} D_1 & & & & & & & \\ & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I}_{n_2} & & & -D_4 & & & \\ & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I} & & & & \epsilon\mathbb{I} & \\ & & & -\frac{\kappa^{\frac{1}{2}}}{2}\mathbb{I} - D_1 & & & & \\ & D_4 & & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I}_{n_2} & & & \\ & & & & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I} & & \\ & & & & & & & \kappa^{\frac{1}{2}} \end{pmatrix}.$$

By applying the permutation

$$\begin{cases} q_k \mapsto q_k, & k \in \{1, \cdots, n_0\}; \\ q_k \mapsto q_{k+1}, & k \in \{n_0+1, \cdots, 4N-9\}; \\ q_{4N-8} \mapsto \gamma, & \\ \gamma \mapsto q_{n_0+1}, & \end{cases}$$

it is easy to see that $\mathcal{C}$ is reduced to the following block-diagonal form:

$$\mathfrak{C} = \begin{pmatrix} \mathfrak{C}^0 & & \\ & \mathfrak{C}^+ & \\ & & \mathfrak{C}^- \end{pmatrix},$$

and the system (3.21) can be rewritten as the system (3.23):

$$\begin{cases} q'^0 = \varphi^0(q^0, q^+, q^-), \\ q'^+ = \mathfrak{C}^+ q^+ + \varphi^+(q^0, q^+, q^-), \\ q'^- = \mathfrak{C}^- q^- + \varphi^-(q^0, q^+, q^-); \end{cases}$$

where

$$\mathfrak{C}^0 = \begin{pmatrix} \tilde{\mu}_1^+ & & \\ & \ddots & \\ & & \tilde{\mu}_{n_0}^+ \end{pmatrix} = 0,$$

$$\mathfrak{C}^+ = \begin{pmatrix} \kappa^{\frac{1}{2}} & & & \\ & \tilde{\mu}_{n_0+1}^+ & & \\ & & \ddots & \\ & & & \tilde{\mu}_{n_0+n_p}^+ \end{pmatrix},$$

$$
\mathfrak{C}^{-} = \begin{pmatrix}
\tilde{\Lambda}_1 & & & & & \\
& -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I}_{n_2} & & & -D_4 & \\
& & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I} & & & \epsilon\mathbb{I} \\
& & & -\frac{\kappa^{\frac{1}{2}}}{2}\mathbb{I} - D_1 & & \\
& D_4 & & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I}_{n_2} & \\
& & & & & -\frac{\kappa^{\frac{1}{2}}}{4}\mathbb{I}
\end{pmatrix}, \tag{C.7}
$$

$$
\begin{aligned}
q^0 &= (q_1, \cdots, q_{n_0})^\top, \\
q^+ &= (\gamma, q_{n_0+1}, \cdots, q_{n_0+n_p})^\top, \\
q^- &= (q_{n_0+n_p+1}, \cdots, q_{2N-4}, q_{2N-3}, \cdots, q_{4N-8})^\top, \\
\varphi^0 &= (\varphi_1, \cdots, \varphi_{n_0})^\top, \\
\varphi^+ &= (\varphi_0, \varphi_{n_0+1}, \cdots, \varphi_{n_0+n_p})^\top, \\
\varphi^- &= (\varphi_{n_0+n_p+1}, \cdots, \varphi_{2N-4}, \varphi_{2N-3}, \cdots, \varphi_{2N-8})^\top.
\end{aligned}
$$

Note that the eigenvalues of the matrix $\mathfrak{C}^0$ are all on the imaginary axis, the eigenvalues of the matrix $\mathfrak{C}^+$ lie to the right of the imaginary axis, and the eigenvalues of the matrix $\mathfrak{C}^-$ lie to the left of the imaginary axis.

APPENDIX D

Normal Forms

One of the key ideas of resolving $PISPW$ is estimating the speed of convergence to zero in (3.18) to ensure that $\theta(\tau)$ approaches a fixed limit as $\tau \to -\infty$. For this purpose, we need to further simplify equations (3.13) by the theory of normal forms (or reduction theorems).

Note that equations in (C.1) are degenerate when $n_0 > 0$ as a result of the degeneracy of central configuration. In general, degenerate equations are very difficult to handle.

Based on some original ideas of Siegel [**22**], we develop some results of normal forms, more general than that of [**22**], to explore $PISPW$ involving with degenerate equations.

Here is one reminder on the notations of this section. To simplify the notation, we shall utilize some notations used in previous sections, but with meanings slightly different.

Suppose the origin 0 is an equilibrium point of (2.6), then it follows from Taylor expansion near the origin that

$$v(q) = \frac{\partial v(0)}{\partial q} q + o(q),$$

the system (2.6) becomes

$$\dot{q} = \mathfrak{C} q + \varphi(q),$$

where $\mathfrak{C} = \frac{\partial v(0)}{\partial q}$, and $\varphi(q) = o(q)$, i.e., $\varphi(0) = 0$, $\frac{\partial \varphi(0)}{\partial q} = 0$.

If the function v is C^m-smooth, then φ can be expanded as:

$$\varphi(q) = \varphi_2(q) + \cdots + \varphi_m(q) + o_m(q),$$

where each $\varphi_k(q)$ is a homogeneous polynomial of degree k; hereafter $o_m(q)$ stands for terms which vanish at the origin along with the first m derivatives. Furthermore, if v is analytic, then φ can be expanded as:

$$\varphi(q) = \varphi_2(q) + \cdots + \varphi_m(q) + \cdots ,$$

which is a power-series starting with quadratic terms.

Consider the system

$$q' = \mathfrak{C} q + \varphi(q). \tag{D.1}$$

When one of the eigenvalues $\mu_1, \cdots, \mu_n$ of the matrix $\mathfrak{C}$ is on the imaginary axis, we have to consider some smooth change of variables to simplify the system (D.1), even though the right side of the system (D.1) is an analytic function in the n independent variables $q_1, \cdots, q_n$. Suppose $\varphi(q)$ is C^l-smooth ($1 \leq l \leq \infty$) in $q_1, \cdots, q_n$ in the following.

Suppose the matrix $\mathfrak{C}$ in the system (D.1) is in block-diagonal form

$$\mathfrak{C} = \begin{pmatrix} \mathfrak{C}^0 & & \\ & \mathfrak{C}^+ & \\ & & \mathfrak{C}^- \end{pmatrix}.$$

Set

$$\begin{aligned} q^0 &= (q_1, \cdots, q_{n_0})^\top, \\ q^+ &= (q_{n_0+1}, \cdots, q_{n_0+m})^\top, \\ q^- &= (q_{n_0+m+1}, \cdots, q_n)^\top; \\ \varphi^0 &= (\varphi_1, \cdots, \varphi_{n_0})^\top, \\ \varphi^+ &= (\varphi_{n_0+1}, \cdots, \varphi_{n_0+m})^\top, \\ \varphi^- &= (\varphi_{n_0+m+1}, \cdots, \varphi_n)^\top. \end{aligned}$$

Then the system (D.1) has the form

$$\begin{cases} q'^0 = \mathfrak{C}^0 q^0 + \varphi^0(q^0, q^+, q^-), \\ q'^+ = \mathfrak{C}^+ q^+ + \varphi^+(q^0, q^+, q^-), \\ q'^- = \mathfrak{C}^- q^- + \varphi^-(q^0, q^+, q^-). \end{cases} \tag{D.2}$$

Suppose the eigenvalues of the matrix $\mathfrak{C}^0$ are all on the imaginary axis, the eigenvalues of the matrix $\mathfrak{C}^+$ lie to the right of the imaginary axis, and the eigenvalues of the matrix $\mathfrak{C}^-$ lie to the left of the imaginary axis.

It is well known that the following results hold (our main reference on this issue is [**20**]).

THEOREM D.1 (**Center-Stable Manifold**,[**20**, p.281]). *In a small neighborhood of the origin there exists an $(n-m)$-dimensional invariant center-stable manifold $\mathcal{W}^{cs}_{loc} : q^+ = F^{cs}(q^0, q^-)$ of class C^l $(l < \infty)$, which contains the origin and which is tangent to the subspace $q^+ = 0$ at the origin. The manifold $\mathcal{W}^{cs}_{loc}$ contains all orbits which stay in a small neighborhood of the origin for all positive times. Though the center-stable manifold is not defined uniquely, for any two center-stable manifolds $\mathcal{W}^{cs}_1$ and $\mathcal{W}^{cs}_2$ the functions F^{cs}_1 and F^{cs}_2 have the same Taylor expansion at the origin (and at each point whose positive semiorbit stays in a small neighborhood of the origin).*

REMARK D.2. Note that even if the system is C^∞-smooth, the center-stable manifold has, in general, only finite smoothness. Of course, if the original system is C^∞-smooth, it is C^l-smooth for any finite l. Therefore, in this case one may apply the center-stable manifold theorem with any given l which implies that: for any finite l there exists a neighborhood $\mathcal{N}_l$ of 0 where $\mathcal{W}^{cs}_{loc}$ is C^l-smooth. In principle, however, these neighborhoods may shrink to zero as $l \to \infty$.

THEOREM D.3 (**Center-Unstable Manifold**, [**20**, p.281]). *In a small neighborhood of the origin there exists an (n_0+m)-dimensional invariant center-unstable manifold $\mathcal{W}^{cu}_{loc} : q^- = F^{cu}(q^0, q^+)$ of class C^l $(l < \infty)$, which contains the origin and which is tangent to the subspace $q^- = 0$ at the origin. The manifold $\mathcal{W}^{cu}_{loc}$ contains all orbits which stay in a small neighborhood of the origin for all negative times. Though the center-unstable manifold is not defined uniquely, for any two center-unstable manifolds $\mathcal{W}^{cu}_1$ and $\mathcal{W}^{cu}_2$ the functions F^{cu}_1 and F^{cu}_2 have the same Taylor expansion at the origin (and at each point whose negative semiorbit stays in* *a small neighborhood of the origin). In the case where the system is C^∞-smooth, the center-unstable manifold has, in general, only finite smoothness.*

Note that the condition of invariance of the manifolds $\mathcal{W}^{cs}_{loc}$ and $\mathcal{W}^{cu}_{loc}$ may be expressed as

$$\begin{aligned} q'^{+} &= \tfrac{\partial F^{cs}(q^0,q^-)}{\partial q^0} q'^0 + \tfrac{\partial F^{cs}(q^0,q^-)}{\partial q^-} q'^- \quad \text{when } q^+ = F^{cs}(q^0,q^-),\\ q'^{-} &= \tfrac{\partial F^{cu}(q^0,q^+)}{\partial q^0} q'^0 + \tfrac{\partial F^{cu}(q^0,q^+)}{\partial q^+} q'^+ \quad \text{when } q^- = F^{cu}(q^0,q^+), \end{aligned}$$

or

$$\begin{aligned} &\mathfrak{C}^+ F^{cs}(q^0,q^-) + \varphi^+(q^0, F^{cs}(q^0,q^-), q^-)\\ &\quad = \tfrac{\partial F^{cs}(q^0,q^-)}{\partial q^0}[\mathfrak{C}^0 q^0 + \varphi^0(q^0, F^{cs}(q^0,q^-), q^-)]\\ &\qquad + \tfrac{\partial F^{cs}(q^0,q^-)}{\partial q^-}[\mathfrak{C}^- q^- + \varphi^-(q^0, F^{cs}(q^0,q^-), q^-)],\\ &\mathfrak{C}^- F^{cu}(q^0,q^+) + \varphi^-(q^0, q^+, F^{cu}(q^0,q^+))\\ &\quad = \tfrac{\partial F^{cu}(q^0,q^+)}{\partial q^0}[\mathfrak{C}^0 q^0 + \varphi^0(q^0, q^+, F^{cu}(q^0,q^+))]\\ &\qquad + \tfrac{\partial F^{cu}(q^0,q^+)}{\partial q^+}[\mathfrak{C}^+ q^+ + \varphi^+(q^0, q^+, F^{cu}(q^0,q^+))]. \end{aligned} \tag{D.3}$$

The above relations yield an algorithm for computing the invariant manifolds.

COROLLARY D.4 (**Reduction Theorem**). *One can introduce new variables:*

$$\begin{cases} u^0 = q^0\\ u^+ = q^+ - F^{cs}(q^0,q^-)\\ u^- = q^- \end{cases} \tag{D.4}$$

so that we can write the system (D.2) *in the simpler form*

$$\begin{cases} u'^0 = \mathfrak{C}^0 u^0 + \varphi^0(u^0, F^{cs}(u^0,u^-), u^-) + \psi^0(u)u^+,\\ u'^+ = \mathfrak{C}^+ u^+ + \psi^+(u)u^+,\\ u'^- = \mathfrak{C}^- u^- + \varphi^-(u^0, F^{cs}(u^0,u^-), u^-) + \psi^-(u)u^+ \end{cases} \tag{D.5}$$

where the functions ψ^0, ψ^- are C^l-smooth and ψ^+ is C^{l-1}-smooth; in addition, all the functions ψ^0, ψ^+, ψ^- are vanishing at the origin, i.e., $\psi(0) = 0$.

PROOF. As a matter of fact, for the new variables, it is easy to see that

$$\begin{cases} u'^0 = \mathfrak{C}^0 u^0 + \varphi^0(u^0, u^+ + F^{cs}(u^0,u^-), u^-),\\ u'^+ = q'^+ - \tfrac{\partial F^{cs}(q^0,q^-)}{\partial q^0} q'^0 - \tfrac{\partial F^{cs}(q^0,q^-)}{\partial q^-} q'^-,\\ u'^- = \mathfrak{C}^- u^- + \varphi^-(u^0, u^+ + F^{cs}(u^0,u^-), u^-), \end{cases} \tag{D.6}$$

By (D.3), the second equation in (D.6) may be rewritten as

$$\begin{aligned}
u'^{+} &= \mathfrak{C}^{+}u^{+} + \mathfrak{C}^{+}F^{cs}(u^0,u^-) + \varphi^{+}(u^0,u^{+}+F^{cs}(u^0,u^-),u^-) \\
&\quad - \frac{\partial F^{cs}(u^0,u^-)}{\partial u^0}[\mathfrak{C}^{0}u^0 + \varphi^{0}(u^0,u^{+}+F^{cs}(u^0,u^-),u^-)] \\
&\quad - \frac{\partial F^{cs}(u^0,u^-)}{\partial u^-}[\mathfrak{C}^{-}u^- + \varphi^{-}(u^0,u^{+}+F^{cs}(u^0,u^-),u^-)] \\
&= \mathfrak{C}^{+}u^{+} + [\varphi^{+}(u^0,u^{+}+F^{cs}(u^0,u^-),u^-) - \varphi^{+}(u^0,F^{cs}(u^0,u^-),u^-)] \\
&\quad - \frac{\partial F^{cs}(u^0,u^-)}{\partial u^0}[\varphi^{0}(u^0,u^{+}+F^{cs}(u^0,u^-),u^-) - \varphi^{0}(u^0,F^{cs}(u^0,u^-),u^-)] \\
&\quad - \frac{\partial F^{cs}(u^0,u^-)}{\partial u^-}[\varphi^{-}(u^0,u^{+}+F^{cs}(u^0,u^-),u^-) - \varphi^{-}(u^0,F^{cs}(u^0,u^-),u^-)]
\end{aligned}$$

Since

$$\begin{aligned}
&\varphi^{*}(u^0,u^{+}+F^{cs}(u^0,u^-),u^-) - \varphi^{*}(u^0,F^{cs}(u^0,u^-),u^-) \\
&\qquad = [\textstyle\int_0^1 \frac{\partial \varphi^{*}(u^0,tu^{+}+F^{cs}(u^0,u^-),u^-)}{\partial q^+}dt]u^{+},
\end{aligned}$$

where $* \in \{0,+,-\}$, it follows that (D.6) becomes

$$\begin{cases}
u'^{0} = \mathfrak{C}^{0}u^0 + \varphi^{0}(u^0,F^{cs}(u^0,u^-),u^-) + \psi^{0}(u)u^{+}, \\
u'^{+} = \mathfrak{C}^{+}u^{+} + \psi^{+}(u)u^{+}, \\
u'^{-} = \mathfrak{C}^{-}u^{-} + \varphi^{-}(u^0,F^{cs}(u^0,u^-),u^-) + \psi^{-}(u)u^{+},
\end{cases}$$

where

$$\psi^{0}(u) = \int_0^1 \frac{\partial \varphi^{0}(u^0,tu^{+}+F^{cs}(u^0,u^-),u^-)}{\partial q^+}dt,$$

$$\psi^{-}(u) = \int_0^1 \frac{\partial \varphi^{-}(u^0,tu^{+}+F^{cs}(u^0,u^-),u^-)}{\partial q^+}dt$$

are C^l-smooth, and

$$\begin{aligned}
\psi^{+}(u) = &\textstyle\int_0^1 \frac{\partial \varphi^{+}(u^0,tu^{+}+F^{cs}(u^0,u^-),u^-)}{\partial q^+}dt \\
&- \textstyle\int_0^1 \frac{\partial F^{cs}(u^0,u^-)}{\partial u^0}\frac{\partial \varphi^{0}(u^0,tu^{+}+F^{cs}(u^0,u^-),u^-)}{\partial q^+}dt \\
&- \textstyle\frac{\partial F^{cs}(u^0,u^-)}{\partial u^-}\frac{\partial \varphi^{-}(u^0,tu^{+}+F^{cs}(u^0,u^-),u^-)}{\partial q^+}dt
\end{aligned}$$

is C^{l-1}-smooth. Moreover, by virtue of $\varphi(0) = 0$ and $\frac{\partial \varphi(0)}{\partial q} = 0$, $\psi(0) = 0$ holds.

Therefore, we can write the system (D.2) in the simpler form (D.5) by the transformation (D.4). □

Similarly, we have

COROLLARY D.5 (**Reduction Theorem**). *One can introduce new variables:*

$$\begin{cases}
u^0 = q^0 \\
u^+ = q^+ \\
u^- = q^- - F^{cu}(q^0,q^+)
\end{cases}$$

so that we can write the system (D.2) *in the simpler form*

$$\begin{cases} u'^0 = \mathfrak{C}^0 u^0 + \varphi^0(u^0, u^+, F^{cu}(u^0, u^+)) + \psi^0(u)u^-, \\ u'^+ = \mathfrak{C}^+ u^+ + \varphi^+(u^0, u^+, F^{cu}(u^0, u^+)) + \psi^+(u)u^-, \\ u'^- = \mathfrak{C}^- u^- + \psi^-(u)u^-, \end{cases}$$

where the functions ψ^0, ψ^+ *are* C^l*-smooth and* ψ^- *is* C^{l-1}*-smooth; in addition, all the functions* ψ^0, ψ^+, ψ^- *are vanishing at the origin, i.e.,* $\psi(0) = 0$.

In fact, a stronger version of reduction theorem holds:

THEOREM D.6 (**Reduction Theorem**, [**20**, p.283]). *By a* C^{l-1}*-smooth transformation the system* (D.2) *can be locally reduced to the simpler form*

$$\begin{cases} u'^0 = \mathfrak{C}^0 u^0 + \psi^0_0(u)u^0 + \psi^0_+(u)u^+ + \psi^0_-(u)u^-, \\ u'^+ = \mathfrak{C}^+ u^+ + \psi^+(u)u^+, \\ u'^- = \mathfrak{C}^- u^- + \psi^-(u)u^-, \end{cases}$$

where the functions $\psi^0_0, \psi^0_+, \psi^0_-$ *are* C^{l-1}*-smooth and vanishing at the origin; the functions* ψ^+, ψ^- *are* C^l*-smooth and vanishing at the origin. Furthermore,* ψ^0_+ *vanishes identically at* $u^- = 0$*, and* ψ^0_- *vanishes identically at* $u^+ = 0$.

However, we will not utilize this stronger version of reduction theorem in this paper. Instead, we will directly utilize a version of theorem of center manifold.

When we investigate orbits on the center-unstable manifold, that is, orbits such that $q^- = F^{cu}(q^0, q^+)$ or $u^- = 0$, the system is reduced to a form

$$\begin{cases} u'^0 = \mathfrak{C}^0 u^0 + \varphi^0(u^0, u^+, F^{cu}(u^0, u^+)), \\ u'^+ = \mathfrak{C}^+ u^+ + \varphi^+(u^0, u^+, F^{cu}(u^0, u^+)). \end{cases} \tag{D.7}$$

Note that this system is of class C^l. For this kind of system, it is well known that the following theorem holds.

THEOREM D.7 (**Center Manifold**, [**20**, p.271]). *Consider the system* (D.7)*, in a small neighborhood of* 0 *there exists an* n_0*-dimensional invariant center manifold* $\mathcal{W}^c_{loc} : u^+ = F^c(u^0)$ *of class* C^l*, which contains* 0 *and which is tangent to the subspace* $u^+ = 0$ *at* 0*. The manifold* $\mathcal{W}^c_{loc}$ *contains all orbits which stay in a small neighborhood of* 0 *for all times. Though the center manifold is not defined uniquely, for any two manifolds* $\mathcal{W}^c_1$ *and* $\mathcal{W}^c_2$ *the functions* F^c_1 *and* F^c_2 *have the same Taylor expansion at* 0 *(and at each point whose orbit stays in a small neighborhood of* 0*). In the case where the system is* C^∞*-smooth, the center manifold has, in general, only finite smoothness.*

Note also that the condition of invariance of the manifold $\mathcal{W}^c_{loc}$ may be expressed as

$$u'^+ = \frac{\partial F^c(u^0)}{\partial u^0} u'^0 \quad \text{when } u^+ = F^c(u^0)$$

or

$$\begin{aligned} &\mathfrak{C}^+ F^c(u^0) + \varphi^+(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0))) \\ &= \tfrac{\partial F^c(u^0)}{\partial u^0}[\mathfrak{C}^0 u^0 + \varphi^0(u^0, F^c(u^0), F^{cu}(u^0, F^c(u^0)))]. \end{aligned}$$

The above relation yields an algorithm for computing the center manifolds which will be used in the following.

The existence of a center manifold allows some problems related to the nonhyperbolic equilibrium to be reduced to the study of an n_0-dimensional system

$$u'^0 = \mathfrak{C}^0 u^0 + \varphi^0\left(u^0, F^c(u^0), F^{cu}\left(u^0, F^c(u^0)\right)\right) \tag{D.8}$$

In particular, when an orbit on the center-unstable manifold approaches the origin for $\tau \to -\infty$, we have the following result:

Theorem D.8 ([**6**, p.4, Th.2]). *Let $(u^0(\tau), u^+(\tau))$ be a solution of the system* (D.7). *Suppose that*

$$(u^0(\tau), u^+(\tau)) \to 0 \quad as \quad \tau \to -\infty,$$

then there exists a solution $\upsilon(\tau)$ of the system (D.8) *such that as $\tau \to -\infty$*

$$\begin{cases} u^0(\tau) = \upsilon(\tau) + O(e^{\sigma\tau}), \\ u^+(\tau) = F^c(\upsilon(\tau)) + O(e^{\sigma\tau}). \end{cases} \tag{D.9}$$

where $\sigma > 0$ is a constant depending only on $\mathfrak{C}^+$.

A direct implication of Theorem D.8 is the well known result that an orbit on the unstable manifold of a hyperbolic equilibrium point exponentially approaches the hyperbolic equilibrium point, i.e., the following result holds.

Corollary D.9 ([**20**, p.66]). *Consider the system* (D.1), *assume that the origin $q = 0$ is a hyperbolic equilibrium point. Let $q(\tau)$ be a solution on the unstable manifold of the origin, i.e.,*

$$q(\tau) \to 0 \quad as \quad \tau \to -\infty,$$

then there are two positive constants c and σ such that

$$dist(q(\tau), 0) \le ce^{\sigma\tau}$$

for sufficiently small τ, and σ depends only on $\mathfrak{C}^+$.

APPENDIX E

Plane Equilibrium Points

In this section, we will discuss some aspects of plane equilibrium points. In particular, for those orbits tending to the equilibrium points, we estimate their speed of convergence.

Let us consider an autonomous system on the plane $\mathbb{R}^2$

$$\begin{cases} \zeta' = f(\zeta, \eta), \\ \eta' = g(\zeta, \eta), \end{cases} \tag{E.1}$$

where f, g are continuous for small ζ, η and

$$f(0,0) = g(0,0) = 0.$$

One can introduce polar coordinates

$$\zeta = \rho \cos\vartheta, \eta = \rho \sin\vartheta$$

to transform the system (E.1) into

$$\begin{cases} \rho' = f(\rho\cos\vartheta, \rho\sin\vartheta)\cos\vartheta + g(\rho\cos\vartheta, \rho\sin\vartheta)\sin\vartheta, \\ \rho\vartheta' = -f(\rho\cos\vartheta, \rho\sin\vartheta)\sin\vartheta + g(\rho\cos\vartheta, \rho\sin\vartheta)\cos\vartheta. \end{cases} \tag{E.2}$$

As in [**9**], a direction $\vartheta = \vartheta_0$ at the origin is called **characteristic** for the system (E.1), if there exists a sequence $(\rho_1, \vartheta_1), (\rho_2, \vartheta_2), \cdots$ such that:

1) $\rho_k \to 0+, \vartheta_k \to \vartheta_0 \quad \text{as} \quad k \to \infty$;
2) (f_k, g_k) is not $(0,0)$, and the angle $\pmod{\pi}$ between the vectors (f_k, g_k) and $(\cos\vartheta_k, \sin\vartheta_k)$ tends to zero as $k \to \infty$, i.e.,

$$\frac{g_k \cos\vartheta_k - f_k \sin\vartheta_k}{\sqrt{f_k^2 + g_k^2}} \to 0 \quad \text{as} \quad k \to \infty. \tag{E.3}$$

Where $\rho_k \to 0+$ denotes $\rho_k \to 0$ and $\rho_k > 0$; (f_k, g_k) is the vector field (f, g) evaluated at $(\zeta_k, \eta_k) = (\rho_k \cos\vartheta_k, \rho_k \sin\vartheta_k)$.

The following two lemmas in [**9**] are important for the investigation of plane equilibrium points.

Lemma E.1. *Let f, g be continuous for small ζ, η and $f^2(\zeta,\eta) + g^2(\zeta,\eta) > 0$ except at the origin, that is, the origin is an isolated equilibrium point of the system* (E.1). *Let the system* (E.1) *possess a solution $(\zeta(\tau), \eta(\tau))$ for $-\infty < \tau \le 0$ such that*

$$\zeta^2(\tau) + \eta^2(\tau) \to 0+ \quad as \quad \tau \to -\infty,$$

Let $\rho(\tau) = \sqrt{f^2(\zeta,\eta) + g^2(\zeta,\eta)} > 0$ and $\vartheta(\tau)$ be a continuous determination of the angle between the ζ-axis and the vector $(\zeta(\tau), \eta(\tau))$. Let $\vartheta = \vartheta_0$ be a noncharacteristic direction. Then either $\vartheta'(\tau) > 0$ or $\vartheta'(\tau) < 0$ for all τ near $-\infty$ for which $\vartheta(\tau) = \vartheta_0 \mod 2\pi$.

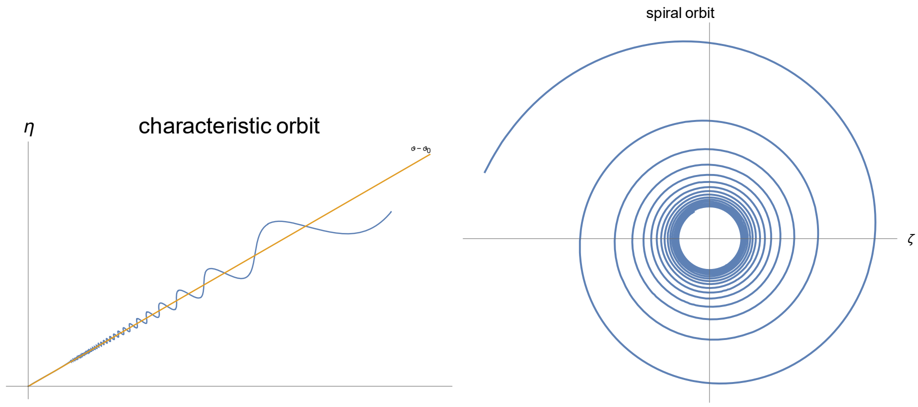

FIGURE E.1. Lemma E.2

LEMMA E.2. *Let f, g and $(\zeta(\tau), \eta(\tau))$ be as in Lemma* E.1. *Suppose that every ϑ-interval, $\alpha < \vartheta < \beta$, contains a noncharacteristic direction. Then either*

$$\vartheta_0 = \lim_{\tau\to-\infty} \vartheta(\tau) \quad \textit{exists (and is finite)} \tag{E.4}$$

or $(\zeta(\tau), \eta(\tau))$ is a spiral; i.e.,

$$|\vartheta(\tau)| \to \infty \quad \textit{as} \quad \tau \to -\infty. \tag{E.5}$$

In the case (E.4), *$\vartheta(\tau) = \vartheta_0$ is a characteristic direction.*

In the following, let us further assume that the functions f, g are

$$\begin{cases} f = P_m(\zeta, \eta) + p_m(\zeta, \eta), \\ g = Q_m(\zeta, \eta) + q_m(\zeta, \eta), \end{cases}$$

where P_m, Q_m are homogeneous polynomials of degree $m > 1$ and

$$p_m^2(\rho\cos\vartheta, \rho\sin\vartheta) + q_m^2(\rho\cos\vartheta, \rho\sin\vartheta) = o(\rho^{2m}) \quad \text{as} \quad \rho \to 0.$$

In terms of polar coordinates, define

$$\begin{cases} \Phi(\vartheta) = \rho^{-m}\left(P_m(\rho\cos\vartheta, \rho\sin\vartheta)\cos\vartheta + Q_m(\rho\cos\vartheta, \rho\sin\vartheta)\sin\vartheta\right), \\ \Psi(\vartheta) = \rho^{-m}\left(Q_m(\rho\cos\vartheta, \rho\sin\vartheta)\cos\vartheta - P_m(\rho\cos\vartheta, \rho\sin\vartheta)\sin\vartheta\right); \end{cases}$$

then Φ, Ψ are homogeneous polynomials of $\cos\vartheta, \sin\vartheta$ of degree $m + 1$.

In terms of polar coordinates, (E.2) can be written as

$$\begin{cases} \rho' = \rho^m \Phi(\vartheta) + o(\rho^m), \\ \vartheta' = \rho^{m-1}\Psi(\vartheta) + o(\rho^{m-1}). \end{cases} \tag{E.6}$$

THEOREM E.3. *Assume $\Psi(\vartheta) \not\equiv 0$ and m is an even number. Let f, g and $(\zeta(\tau), \eta(\tau))$ be as in Lemma* E.1. *Then*

$$\vartheta_0 = \lim_{\tau\to-\infty} \vartheta(\tau) \quad \textit{exists (and is finite)} \tag{E.7}$$

and $\Psi(\vartheta_0) = 0$.

PROOF. Our first goal is to show (E.7). By Lemma E.2, it suffices to exclude the possibility of (E.5).

By using reduction to absurdity, suppose that $(\zeta(\tau), \eta(\tau))$ is a spiral, i.e., (E.5) holds.

Without loss of generality, suppose

$$\vartheta(\tau) \to \infty \quad \text{as} \quad \tau \to -\infty. \tag{E.8}$$

According to $\Psi(\vartheta) \not\equiv 0$, it has only a finite number of zeros $\pmod{2\pi}$. Because Ψ are homogeneous polynomials of $\cos\vartheta, \sin\vartheta$ of degree $m+1$, we have

$$\Psi(\vartheta + \pi) = -\Psi(\vartheta). \tag{E.9}$$

As a result, the plane is split into several angular sectors, $\vartheta_k < \vartheta < \vartheta_{k+1}$ $(k = 1, \cdots, n)$, such that

$$n \geq 1, \vartheta_{n+1} = \vartheta_1 + 2\pi, \Psi(\vartheta_k) = 0$$

and

$$\Psi(\vartheta) \neq 0, \quad \text{for } \vartheta_k < \vartheta < \vartheta_{k+1}.$$

It follows from (E.9) that we can assume $\Psi(\vartheta) > 0$ for $\vartheta_1 < \vartheta < \vartheta_2$. Then there exists an angular sector, $\alpha_1 < \vartheta < \alpha_2$ included in $\vartheta_1 < \vartheta < \vartheta_2$, such that

$$\Psi(\vartheta) > \sigma, \qquad \text{for } \alpha_1 < \vartheta < \alpha_2,$$

where $\sigma > 0$ is a constant.

Note that $\frac{o(\rho^{m-1})}{\rho^{m-1}}$ tend to zero uniformly in ϑ as $\rho \to 0$. It follows that, there exists a sufficiently small $\rho_0 > 0$ such that

$$\rho^{m-1}\sigma + o(\rho^{m-1}) > 0 \quad \text{for any } 0 < \rho < \rho_0.$$

Since there exists a real number $\tau_0 < 0$ such that

$$0 < \rho(\tau) < \rho_0 \quad \text{for any } \tau \leq \tau_0.$$

Taking into consideration (E.8), we know that there exists a sequence of τ-intervals

$$(\beta_1, \gamma_1), (\beta_2, \gamma_2), \cdots$$

such that

$$\begin{cases} \gamma_{k+1} < \beta_k < \gamma_1 < \tau_0, & \\ \beta_k, \gamma_k \to -\infty & \text{as } k \to \infty; \\ \vartheta'(\tau) > 0 & \text{for } \tau \in [\beta_k, \gamma_k] \\ [\vartheta(\beta_k), \vartheta(\gamma_k)] \subset (\vartheta_1 - 2n_k\pi, \vartheta_2 - 2n_k\pi) & \text{for some } n_k \in \mathbb{N}. \end{cases} \tag{E.10}$$

We claim that

$$\vartheta(\tau) \leq \vartheta(\frac{\beta_1 + \gamma_1}{2}) < \vartheta(\gamma_1)$$

for any $\tau \leq \frac{\beta_1+\gamma_1}{2}$. Obviously, this is a contradiction with (E.8). So (E.7) will be proved if we can show the claim.

Set

$$\Omega = \{\tilde{\tau} | \vartheta(\tau) \leq \vartheta(\frac{\beta_1 + \gamma_1}{2}) \text{ for any } \tau \in (\tilde{\tau}, \gamma_1]\}.$$

Let $\tau_i = \inf \Omega$ be the infimum of above set. Following from (E.10), it is clear that $\tau_i < \beta_1$.

The above claim will be proved by showing that $\tau_i = -\infty$. Otherwise, τ_i is a certain negative number. It follows that

$$\vartheta(\beta_1) < \vartheta(\tau_i) = \vartheta(\frac{\beta_1+\gamma_1}{2}) < \vartheta(\gamma_1)$$

However, it is easy to prove that τ_i is not the infimum of Ω by above inequality. This leads to a contradiction.

Our task now is to show $\Psi(\vartheta_0) = 0$. Following from Lemma E.2, $\vartheta(\tau) = \vartheta_0$ is a characteristic direction. If $\Psi(\vartheta_0) \neq 0$, It is straightforward to show that (E.3) reduces to the following

$$\frac{\Psi(\vartheta_0)}{\sqrt{\Phi^2(\vartheta_0)+\Psi^2(\vartheta_0)}} = 0.$$

This leads to a contradiction.

The theorem is now evident from what we have proved. □

Using the same argument as in the proof of above theorem, we can prove the following more general result:

COROLLARY E.4. *Let f, g and $(\zeta(\tau), \eta(\tau))$ be as in Lemma* E.1. *Assume $\Psi(\vartheta)$ has both positive and negative values. Then*

$$\vartheta_0 = \lim_{\tau\to-\infty}\vartheta(\tau) \quad \textit{exists (and is finite)}$$

and $\Psi(\vartheta_0) = 0$.

So a spiral of the system (E.1) can occur only when Ψ is invariably nonnegative or nonpositive and m is an odd number.

THEOREM E.5. *Under the conditions in Theorem* E.3, *if $\Phi(\vartheta_0) \neq 0$, then $\Phi(\vartheta_0) > 0$ and*

$$\rho = (\frac{1}{(m-1)\Phi(\vartheta_0)})^{\frac{1}{m-1}}(\frac{1}{-\tau})^{\frac{1}{m-1}} + o((\frac{1}{-\tau})^{\frac{1}{m-1}}).$$

PROOF. Let us consider the first equation of (E.6):

$$\rho' = \rho^m\Phi(\vartheta) + o(\rho^m).$$

Since

$$\rho(\tau) \to 0+ \quad \text{as} \quad \tau \to -\infty,$$

it is easy to see that $\Phi(\vartheta_0) > 0$.

According to L'Hópital's rule, it follows that

$$\lim_{\tau\to-\infty}\frac{1}{\tau\rho^{m-1}} = (1-m)\Phi(\vartheta_0),$$

or

$$\rho = \left(\frac{1}{(1-m)\Phi(\vartheta_0)\tau}\right)^{\frac{1}{m-1}} + o((\frac{1}{-\tau})^{\frac{1}{m-1}}).$$

The proof of the theorem is now complete. □

Similarly, when m is an odd number, one can prove the following theorem:

THEOREM E.6. *Let f, g and $(\zeta(\tau), \eta(\tau))$ be as in Lemma* E.1. *If $\Phi(\vartheta) \neq 0$ for any ϑ, then there exists a positive number c such that*

$$\rho \leq c(\frac{1}{-\tau})^{\frac{1}{m-1}}.$$

PROOF. By $\Phi(\vartheta) \neq 0$ for any ϑ, it follows that there exists a positive number σ such that

$$|\Phi(\vartheta)| \geq \sigma.$$

According to

$$\rho(\tau) \to 0+ \quad \text{as} \quad \tau \to -\infty,$$

it follows that

$$\Phi(\vartheta) \geq \sigma.$$

Let us consider the first equation of (E.6):

$$\rho' = \rho^m \Phi(\vartheta) + o(\rho^m).$$

It is clear that there exists a real number $\tau_0 < 0$ such that

$$\rho^m \Phi(\vartheta) + o(\rho^m) \geq \frac{\sigma}{2}\rho^m \quad \text{for any } \tau \leq \tau_0.$$

Then

$$\rho^{1-m}(\tau) \geq \rho^{1-m}(\tau_0) + \frac{\sigma}{2}(m-1)(\tau_0 - \tau) \quad \text{for any } \tau \leq \tau_0.$$

As a result, it is evident to see that the theorem holds. □

Bibliography

[1] A. Albouy, H. E. Cabral, and A. A. Santos, *Some problems on the classical n-body problem*, Celestial Mech. Dynam. Astronom. **113** (2012), no. 4, 369–375, DOI 10.1007/s10569-012-9431-1. MR2970201

[2] A. Albouy and V. Kaloshin, *Finiteness of central configurations of five bodies in the plane*, Ann. of Math. (2) **176** (2012), no. 1, 535–588, DOI 10.4007/annals.2012.176.1.10. MR2925390

[3] V. I. Arnol′d, *Mathematical methods of classical mechanics*, 2nd ed., Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein, DOI 10.1007/978-1-4757-2063-1. MR997295

[4] V. I. Arnol′d, *Geometrical methods in the theory of ordinary differential equations*, 2nd ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 250, Springer-Verlag, New York, 1988. Translated from the Russian by Joseph Szücs [József M. Szűcs], DOI 10.1007/978-1-4612-1037-5. MR947141

[5] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical aspects of classical and celestial mechanics*, 3rd ed., Encyclopaedia of Mathematical Sciences, vol. 3, Springer-Verlag, Berlin, 2006. [Dynamical systems. III]; Translated from the Russian original by E. Khukhro. MR2269239

[6] J. Carr, *Applications of centre manifold theory*, Applied Mathematical Sciences, vol. 35, Springer-Verlag, New York-Berlin, 1981. MR635782

[7] J. Chazy. Sur certaines trajectoires du probleme des n corps. *Bull. Astron.*, 35:321–389, 1918.

[8] M. Hampton and R. Moeckel, *Finiteness of relative equilibria of the four-body problem*, Invent. Math. **163** (2006), no. 2, 289–312, DOI 10.1007/s00222-005-0461-0. MR2207019

[9] P. Hartman, *Ordinary differential equations*, Classics in Applied Mathematics, vol. 38, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Corrected reprint of the second (1982) edition [Birkhäuser, Boston, MA; MR0658490 (83e:34002)]; With a foreword by Peter Bates, DOI 10.1137/1.9780898719222. MR1929104

[10] C. Hassell and E. Rees, *The index of a constrained critical point*, Amer. Math. Monthly **100** (1993), no. 8, 772–778, DOI 10.2307/2324784. MR1237226

[11] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant manifolds*, Bull. Amer. Math. Soc. **76** (1970), 1015–1019, DOI 10.1090/S0002-9904-1970-12537-X. MR292101

[12] E. S. G. Leandro, *Finiteness and bifurcations of some symmetrical classes of central configurations*, Arch. Ration. Mech. Anal. **167** (2003), no. 2, 147–177, DOI 10.1007/s00205-002-0241-6. MR1971151

[13] R. McGehee, *Triple collision in the collinear three-body problem*, Invent. Math. **27** (1974), 191–227, DOI 10.1007/BF01390175. MR359459

[14] R. Moeckel, *On central configurations*, Math. Z. **205** (1990), no. 4, 499–517, DOI 10.1007/BF02571259. MR1082871

[15] R. Moeckel, *Celestial Mechanics (Especially Central Configurations)*, `http://www.math.umn.edu/~rmoeckel/notes/Notes.html`, 1994.

[16] R. Moeckel, *Central configurations*, Central configurations, periodic orbits, and Hamiltonian systems, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Basel, 2015, pp. 105–167. MR3469182

[17] J. I. Palmore, *Classifying relative equilibria. II*, Bull. Amer. Math. Soc. **81** (1975), 489–491, DOI 10.1090/S0002-9904-1975-13794-3. MR363076

[18] J. I. Palmore, *Measure of degenerate relative equilibria. I*, Ann. of Math. (2) **104** (1976), no. 3, 421–429, DOI 10.2307/1970964. MR420713

[19] D. G. Saari, *The manifold structure for collision and for hyperbolic-parabolic orbits in the n-body problem*, J. Differential Equations **55** (1984), no. 3, 300–329, DOI 10.1016/0022-0396(84)90072-X. MR766126

[20] L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, and L. O. Chua, *Methods of qualitative theory in nonlinear dynamics. Part I*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, vol. 4, World Scientific Publishing Co., Inc., River Edge, NJ, 1998. With the collaboration of Sergey Gonchenko (Sections 3.7 and 3.8), Oleg Sten′kin (Section 3.9 and Appendix A) and Mikhail Shashkov (Sections 6.1 and 6.2), DOI 10.1142/9789812798596. MR1691840

[21] M. Shub, *Appendix to Smale's paper: "Diagonals and relative equilibria"*, Manifolds–Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Math., Vol. 197, Springer, Berlin-New York, 1971, pp. 199–201. MR278700

[22] C. L. Siegel, *Lectures on the singularities of the three-body problem*, Tata Institute of Fundamental Research Lectures on Mathematics, No. 42, Tata Institute of Fundamental Research, Bombay, 1967. Notes by K. Balagangadharan and M. K. Venkatesha Murthy. MR267224

[23] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer **20** (1998), no. 2, 7–15, DOI 10.1007/BF03025291. MR1631413

[24] H. J. Sperling, *On the real singularities of the N-body problem*, J. Reine Angew. Math. **245** (1970), 15–40, DOI 10.1515/crll.1970.245.15. MR290630

[25] K. F. Sundman, *Mémoire sur le problème des trois corps* (French), Acta Math. **36** (1913), no. 1, 105–179, DOI 10.1007/BF02422379. MR1555085

[26] B. L. van der Waerden, *Algebra. Vol. I*, Springer-Verlag, New York, 1991. Based in part on lectures by E. Artin and E. Noether; Translated from the seventh German edition by Fred Blum and John R. Schulenberger, DOI 10.1007/978-1-4612-4420-2. MR1080172

[27] A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton Mathematical Series, vol. 5, Princeton University Press, Princeton, NJ, 1941. MR5824

[28] Z. Xia, *Central configurations with many small masses*, J. Differential Equations **91** (1991), no. 1, 168–179, DOI 10.1016/0022-0396(91)90137-X. MR1106123

Editorial Information

To be published in the *Memoirs*, a paper must be correct, new, nontrivial, and significant. Further, it must be well written and of interest to a substantial number of mathematicians. Piecemeal results, such as an inconclusive step toward an unproved major theorem or a minor variation on a known result, are in general not acceptable for publication.

Papers appearing in *Memoirs* are generally at least 80 and not more than 200 published pages in length. Papers less than 80 or more than 200 published pages require the approval of the Managing Editor of the Transactions/Memoirs Editorial Board. Published pages are the same size as those generated in the style files provided for $\mathcal{AMS}$-LaTeX.

Information on the backlog for this journal can be found on the AMS website starting from `http://www.ams.org/memo`.

A Consent to Publish is required before we can begin processing your paper. After a paper is accepted for publication, the Providence office will send a Consent to Publish and Copyright Agreement to all authors of the paper. By submitting a paper to the *Memoirs*, authors certify that the results have not been submitted to nor are they under consideration for publication by another journal, conference proceedings, or similar publication.

Information for Authors

Memoirs is an author-prepared publication. Once formatted for print and on-line publication, articles will be published as is with the addition of AMS-prepared frontmatter and backmatter. Articles are not copyedited; however, confirmation copy will be sent to the authors.

Initial submission. The AMS uses Centralized Manuscript Processing for initial submissions. Authors should submit a PDF file using the Initial Manuscript Submission form found at `www.ams.org/submission/memo`, or send one copy of the manuscript to the following address: Centralized Manuscript Processing, MEMOIRS OF THE AMS, 201 Charles Street, Providence, RI 02904-2294 USA. If a paper copy is being forwarded to the AMS, indicate that it is for *Memoirs* and include the name of the corresponding author, contact information such as email address or mailing address, and the name of an appropriate Editor to review the paper (see the list of Editors below).

The paper must contain a *descriptive title* and an *abstract* that summarizes the article in language suitable for workers in the general field (algebra, analysis, etc.). The *descriptive title* should be short, but informative; useless or vague phrases such as "some remarks about" or "concerning" should be avoided. The *abstract* should be at least one complete sentence, and at most 300 words. Included with the footnotes to the paper should be the 2020 *Mathematics Subject Classification* representing the primary and secondary subjects of the article. The classifications are accessible from `www.ams.org/msc/`. The Mathematics Subject Classification footnote may be followed by a list of *key words and phrases* describing the subject matter of the article and taken from it. Journal abbreviations used in bibliographies are listed in the latest *Mathematical Reviews* annual index. The series abbreviations are also accessible from `www.ams.org/msnhtml/serials.pdf`. To help in preparing and verifying references, the AMS offers MR Lookup, a Reference Tool for Linking, at `www.ams.org/mrlookup/`.

Electronically prepared manuscripts. The AMS encourages electronically prepared manuscripts, with a strong preference for $\mathcal{AMS}$-LaTeX. To this end, the Society has prepared $\mathcal{AMS}$-LaTeX author packages for each AMS publication. Author packages include instructions for preparing electronic manuscripts, samples, and a style file that generates the particular design specifications of that publication series.

Authors may retrieve an author package for *Memoirs of the AMS* from `www.ams.org/journals/memo/memoauthorpac.html`. The *AMS Author Handbook* is available in PDF format from the author package link. The author package can also be obtained free of charge by sending email to `tech-support@ams.org` or from the Publication Division, American Mathematical Society, 201 Charles St., Providence, RI 02904-2294, USA. When requesting an author package, please specify the publication in which your paper will appear. Please be sure to include your complete mailing address.

After acceptance. The source files for the final version of the electronic manuscript should be sent to the Providence office immediately after the paper has been accepted for publication. The author should also submit a PDF of the final version of the paper to the editor, who will forward a copy to the Providence office.

Accepted electronically prepared files can be submitted via the web at `www.ams.org/submit-book-journal/`, sent via FTP, or sent on CD to the Electronic Prepress Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA. TeX source files and graphic files can be transferred over the Internet by FTP to the Internet node `ftp.ams.org` (130.44.1.100). When sending a manuscript electronically via CD, please be sure to include a message indicating that the paper is for the *Memoirs*.

Electronic graphics. Comprehensive instructions on preparing graphics are available at `www.ams.org/authors/journals.html`. A few of the major requirements are given here.

Submit files for graphics as EPS (Encapsulated PostScript) files. This includes graphics originated via a graphics application as well as scanned photographs or other computer-generated images. If this is not possible, TIFF files are acceptable as long as they can be opened in Adobe Photoshop or Illustrator.

Authors using graphics packages for the creation of electronic art should also avoid the use of any lines thinner than 0.5 points in width. Many graphics packages allow the user to specify a "hairline" for a very thin line. Hairlines often look acceptable when proofed on a typical laser printer. However, when produced on a high-resolution laser imagesetter, hairlines become nearly invisible and will be lost entirely in the final printing process.

Screens should be set to values between 15% and 85%. Screens which fall outside of this range are too light or too dark to print correctly. Variations of screens within a graphic should be no less than 10%.

Any graphics created in color will be rendered in grayscale for the printed version unless color printing is authorized by the Managing Editor and the Publisher. In general, color graphics will appear in color in the online version.

Inquiries. Any inquiries concerning a paper that has been accepted for publication should be sent to `memo-query@ams.org` or directly to the Electronic Prepress Department, American Mathematical Society, 201 Charles St., Providence, RI 02904-2294 USA.

SELECTED PUBLISHED TITLES IN THIS SERIES

1520 **Guixiang Hong, Simeng Wang, and Xumin Wang,** Pointwise Convergence of Noncommutative Fourier Series, 2024

1519 **Katarzyna Mazowiecka, Michał Miśkiewicz, and Armin Schikorra,** On the Size of the Singular Set of Minimizing Harmonic Maps, 2024

1518 **L. Alexander Betts,** The Motivic Anabelian Geometry of Local Heights on Abelian Varieties, 2024

1517 **Changningphaabi Namoijam and Matthew A. Papanikolas,** Hyperderivatives of Periods and Quasi-Periods for Anderson t-Modules, 2024

1516 **Daniel Han-Kwan and David Michel,** On Hydrodynamic Limits of the Vlasov–Navier–Stokes System, 2024

1515 **Chenchen Mou and Jianfeng Zhang,** Wellposedness of Second Order Master Equations for Mean Field Games with Nonsmooth Data, 2024

1514 **Yohann Le Floch and Joseph Palmer,** Semitoric Families, 2024

1513 **Alcides Buss, Siegfried Echterhoff, and Rufus Willett,** Amenability and Weak Containment for Actions of Locally Compact Groups on C^*-Algebras, 2024

1512 **Alessandro Audrito and Susanna Terracini,** On the Nodal Set of Solutions to a Class of Nonlocal Parabolic Equations, 2024

1511 **Ermerson Araujo, Yuri Lima, and Mauricio Poletti,** Symbolic Dynamics for Nonuniformly Hyperbolic Maps with Singularities in High Dimension, 2024

1510 **C. Bleak, P. Cameron, Y. Maissel, A. Navas, and F. Olukoya,** The Further Chameleon Groups of Richard Thompson and Graham Higman: Automorphisms via Dynamics for the Higman-Thompson Groups $G_{n,r}$, 2024

1509 **Anthony Genevois,** Automorphisms of Graph Products of Groups and Acylindrical Hyperbolicity, 2024

1508 **Andrew J. Blumberg and Michael A. Mandell,** The Strong Künneth Theorem for Topological Periodic Cyclic Homology, 2024

1507 **Myoungjean Bae, Gui-Qiang G. Chen, and Mikhail Feldman,** Prandtl-Meyer Reflection Configurations, Transonic Shocks, and Free Boundary Problems, 2024

1506 **Radu Saghin, Pancho Valenzuela-Henríquez, and Carlos H. Vásquez,** Regularity with Respect to the Parameter of Lyapunov Exponents for Diffeomorphisms with Dominated Splitting, 2024

1505 **Deniz Bilman and Peter D. Miller,** Extreme Superposition: High-Order Fundamental Rogue Waves in the Far-Field Regime, 2024

1504 **Daniele Alessandrini, Olivier Guichard, Eugen Rogozinnikov, and Anna Wienhard,** Noncommutative Coordinates for Symplectic Representations, 2024

1503 **J. Adams, N. Arancibia Robert, and P. Mezo,** Equivalent Definitions of Arthur Packets for Real Classical Groups, 2024

1502 **R. M. Guralnick and R. Lawther,** Generic Stabilizers in Actions of Simple Algebraic Groups, 2024

1501 **Erasmo Caponio, Miguel Angel Javaloyes, and Miguel Sánchez,** Wind Finslerian Structures: From Zermelo's Navigation to the Causality of Spacetimes, 2024

1500 **Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono,** Exponential Decay Estimates and Smoothness of the Moduli Space of Pseudoholomorphic Curves, 2024

1499 **Stephen Theriault,** Homotopy Fibrations with a Section after Looping, 2024

1498 **Fabio Pusateri and Avraham Soffer,** Bilinear Estimates in the Presence of a Large Potential and a Critical NLS in 3D, 2024

1497 **Itay Glazer and Yotam I. Hendel,** On Singularity Properties of Word Maps and Applications to Probabilistic Waring Type Problems, 2024

For a complete list of titles in this series, visit the
AMS Bookstore at **www.ams.org/bookstore/memoseries/**.

ISBN 978-1-4704-7249-8
9 781470 472498
MEMO/304/1532

AMS
www.ams.org