ANALYTIC BESOV FUNCTIONS, PRE-SCHWARZIAN DERIVATIVES, AND INTEGRABLE TEICHMÜLLER SPACES

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Abstract. We study the embedding of integrable Teichmüller spaces T_p into analytic Besov spaces via pre-Schwarzian derivatives. In contrast to the Bers embedding by Schwarzian derivatives, a significant difference arises between the cases $p > 1$ and $p = 1$. In this paper we focus on the case $p = 1$ and extend previous results obtained for $p > 1$. This provides a unified framework for the complex-analytic theory of integrable Teichmüller spaces T_p for all $p \geq 1$.

1. Introduction

The integrable Teichmüller space has been extensively studied as a subspace of the universal Teichmüller space that carries the Weil–Petersson metric and parametrizes the family of Weil–Petersson curves. Bishop's recent characterization of Weil–Petersson curves [4] is closely related to this theory from the complex-analytic viewpoint. Wang [33] defined a Dirichlet energy arising from the Loewner ODE that generates SLE and showed that the finiteness of this energy forces the evolving arcs to be Weil–Petersson. Moreover, this Dirichlet energy coincides with the universal Liouville action on the integrable Teichmüller space, which serves as a Kähler potential for the Weil–Petersson metric.

The integrable Teichmüller space T_2 was introduced by Cui [5], and its Hilbert manifold structure and Weil–Petersson geometry were subsequently developed by Takhtajan and Teo [29]; foundational complex-analytic aspects were established by Shen [25]. This space is the quotient of $M_2(\mathbb{H}^+)$, the space of square-integrable Beltrami coefficients on $\mathbb{H}^+$ (where $\mathbb{H}^\pm$ denote the upper/lower half-planes), by Teichmüller equivalence, and is embedded homeomorphically into the Hilbert space $\mathcal{A}_2(\mathbb{H}^-)$ by the Bers embedding. We recall the definitions of these spaces and mappings below in the case $p = 2$. The Weil–Petersson metric is induced either from the inner product on $\mathcal{A}_2(\mathbb{H}^-)$ or equivalently from the pairing

$$\int_{\mathbb{H}^+} \mu(z)\overline{\nu(z)}\frac{dx\,dy}{|\mathrm{Im}\,z|^2}$$

for harmonic representatives $\mu, \nu \in M_2(\mathbb{H}^+)$ of tangent vectors of T_2 at the origin.

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For a parameter $p \geq 1$, the p-integrable Teichmüller space T_p is modeled on p-integrable Beltrami coefficients on the half-plane. Explicitly, we set

$$M_p(\mathbb{H}^+) = \Big\{\mu \in M(\mathbb{H}^+) \mid \|\mu\|_p = \Big(\int_{\mathbb{H}^+} |\mu(z)|^p \,\frac{dx\,dy}{|\mathrm{Im}\, z|^2}\Big)^{1/p} < \infty\Big\},$$

where $M(\mathbb{H}^+) = \{\mu \in L_\infty(\mathbb{H}^+) \mid \|\mu\|_\infty < 1\}$ is the space of Beltrami coefficients, and define $T_p = \{[\mu] \mid \mu \in M_p(\mathbb{H}^+)\}$ as the set of Teichmüller equivalence classes, which is included in the universal Teichmüller space $T = \{[\mu] \mid \mu \in M(\mathbb{H}^+)\}$ (see Section 5). The original theory concentrated on $p = 2$, was extended to $p \geq 2$ by Guo [11], Tang and Shen [30], and further to $p > 1$ by Wei and Matsuzaki [34]. In addition, it was proved in [36, 37] that the Bers embedding $\alpha : T_p \to \mathcal{A}_p(\mathbb{H}^-)$, defined by $\alpha([\mu]) = S_{F^\mu}$ via the Schwarzian derivative of the normalized conformal homeomorphism $F^\mu : \mathbb{H}^- \to \mathbb{C}$ with quasiconformal extension to the plane of dilatation $\mu \in M_p(\mathbb{H}^+)$, is a homeomorphism onto its image for all $p \geq 1$. Hence T_p inherits a natural complex structure modeled on the Banach space

$$\mathcal{A}_p(\mathbb{H}^-) = \Big\{\Phi \in \mathrm{Hol}(\mathbb{H}^-) \mid \|\Phi\|_{\mathcal{A}_p} = \Big(\int_{\mathbb{H}^-} |(\mathrm{Im}\, z)^2 \Phi(z)|^p \frac{dx\,dy}{|\mathrm{Im}\, z|^2}\Big)^{1/p} < \infty\Big\},$$

where $\mathrm{Hol}(\mathbb{H}^-)$ denotes the holomorphic functions on $\mathbb{H}^-$.

This paper develops a unified embedding theory for integrable Teichmüller spaces via the logarithm of derivative $\log(F^\mu)'$ and the pre-Schwarzian derivative $N_{F^\mu} = (\log(F^\mu)')'$, including the endpoint $p = 1$. For the universal Teichmüller space T, this model is intensively studied by Zhuravlev [40] on the unit disk $\mathbb{D}$. The target on the function side of T_p is the analytic Besov space on $\mathbb{H}^-$: for $p > 1$,

$$\mathcal{B}_p(\mathbb{H}^-) = \Big\{\Phi \in \mathrm{Hol}(\mathbb{H}^-) \mid \|\Phi\|_{\mathcal{B}_p} = \Big(\int_{\mathbb{H}^-} |(\mathrm{Im}\, z)\, \Phi'(z)|^p \,\frac{dx\,dy}{|\mathrm{Im}\, z|^2}\Big)^{1/p} < \infty\Big\},$$

while for $p \geq 1$ we also set

$$\mathcal{B}_p^{\#}(\mathbb{H}^-) = \Big\{\Phi \in \mathrm{Hol}(\mathbb{H}^-) \mid \|\Phi\|_{\mathcal{B}_p^{\#}} = \Big(\int_{\mathbb{H}^-} |(\mathrm{Im}\, z)^2\, \Phi''(z)|^p \,\frac{dx\,dy}{|\mathrm{Im}\, z|^2}\Big)^{1/p} < \infty\Big\}.$$

Then we define

$$\widehat{\mathcal{B}}_p(\mathbb{H}^-) = \mathcal{B}_p^{\#}(\mathbb{H}^-) \cap \mathrm{BMOA}(\mathbb{H}^-)$$

with norm $\|\Phi\|_{\widehat{\mathcal{B}}_p} = \|\Phi\|_{\mathcal{B}_p^{\#}} + \|\Phi\|_{\mathrm{BMOA}}$, where $\mathrm{BMOA}(\mathbb{H}^-)$ is the Banach space of holomorphic functions Φ on $\mathbb{H}^-$ that are given by the Poisson integral of BMO functions on the real line $\mathbb{R}$. BMOA can also be characterized by Carleson measures (see Section 2).

Since $\mathcal{B}_1(\mathbb{H}^-)$ collapses to constants, the appropriate target for the pre-Schwarzian at $p = 1$ is $\widehat{\mathcal{B}}_1(\mathbb{H}^-)$. Moreover, for $p > 1$, the norms $\|\Phi\|_{\mathcal{B}_p}$ and $\|\Phi\|_{\widehat{\mathcal{B}}_p}$ are equivalent. We also recall the Besov spaces defined on $\mathbb{D}$ and prove that the Cayley transformation yields a Banach space isomorphism between $\widehat{\mathcal{B}}_p(\mathbb{H}^-)$ and $\widehat{\mathcal{B}}_p(\mathbb{D})$ (Theorem 2.5).

Section 3 studies the pre-Schwarzian derivative map $L : M_p(\mathbb{H}^+) \to \widehat{\mathcal{B}}_p(\mathbb{H}^-)$ given by $L(\mu) = \log(F^\mu)'$. A direct adaptation of the Schwarzian argument shows the holomorphy of L under certain constraints on p; to remove these constraints, we exploit the Schwarzian derivative map $S : M_p(\mathbb{H}^+) \to \mathcal{A}_p(\mathbb{H}^-)$, $S(\mu) = S_{F^\mu}$, together with sharp norm estimates (Theorem 3.4). Using the existence of a local holomorphic right inverse to S, we prove that the canonical holomorphic map $J : L(M_p(\mathbb{H}^+)) \to S(M_p(\mathbb{H}^+))$, $J(\Psi) = \Psi'' - \frac{1}{2}(\Psi')^2$, is in fact biholomorphic (Theorem 3.10). Consequently, the three spaces $M_p(\mathbb{H}^+)$, $\widehat{\mathcal{B}}_p(\mathbb{H}^-)$, and $\mathcal{A}_p(\mathbb{H}^-)$ are uniformly linked for all $p \geq 1$ in a manner that extends earlier results (Theorem 3.12).

In Section 4 we revisit these results on $\mathbb{D}$ and the exterior unit disk $\mathbb{D}^*$. Although the Cayley transformation identifies $\widehat{\mathcal{B}}_p(\mathbb{H}^-)$ with $\widehat{\mathcal{B}}_p(\mathbb{D})$ as Banach spaces, the canonical map $J : L(M_p(\mathbb{D}^*)) \to S(M_p(\mathbb{D}^*))$ fails to be injective in this model. A modified statement shows that J is a holomorphic split submersion (Theorem 4.4). We analyze the fiber structure of $L(M_p(\mathbb{D}^*))$ over $S(M_p(\mathbb{D}^*))$, proving that $L(M_p(\mathbb{D}^*))$ is a real-analytic disk bundle (Theorem 4.6); for $p > 1$, a global real-analytic section identifies it with the product $S(M_p(\mathbb{D}^*)) \times \mathbb{D}^*$ (Corollary 4.7).

Section 5 discusses the complex Banach manifold structure, the topological group structure, and the Weil–Petersson metric on T_p for $p \geq 1$. In parallel with the Bers embedding $\alpha : T_p \to \mathcal{A}_p(\mathbb{H}^-)$ via S, we introduce the pre-Bers embedding $\beta : T_p \to \widehat{\mathcal{B}}_p(\mathbb{H}^-)$ via L, and prove that α and β induce biholomorphically equivalent complex structures (Theorem 5.1). Moreover, since the Weil–Petersson metric can be regarded as an invariant metric obtained by right translation of the norm on $\alpha(T_p)$, an analogous construction on $\beta(T_p)$ yields an alternative Weil–Petersson metric with similar properties (Theorem 5.4).

Finally, Section 6 compares T_p ($p \geq 1$) with the Teichmüller space T^γ ($0 < \gamma \leq 1$) of circle diffeomorphisms whose derivatives are Hölder–Zygmund continuous. These are defined by Beltrami coefficients on $\mathbb{D}^*$ satisfying $|\mu(z)| = O((|z| - 1)^\gamma)$ as $|z| \to 1$, corresponding to orientation-preserving circle diffeomorphisms h with $h' \in C^\gamma$ (for $\gamma = 1$, h' is continuous and satisfies the Zygmund condition). While $T^1 \subset T_p$ for every $p > 1$ and every $h \in T_1$ is known to be a C^1-diffeomorphism, there is no inclusion relation between T^1 and T_1 (Theorem 6.2).

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2. Analytic Besov functions

We denote by $\mathbb{H}$ either the upper or the lower half-plane. When necessary, we write $\mathbb{H}^+$ for the upper half-plane and $\mathbb{H}^-$ for the lower half-plane.

As a generalization of analytic Dirichlet functions (the case $p = 2$), we introduce the following classes of holomorphic functions on $\mathbb{H}$, which we call *analytic Besov functions*; see [39, Chapter 5], where these functions are defined on $\mathbb{D}$. As mentioned below, the seminorm $\|\cdot\|_{\mathcal{B}_p}$ in the following definition is conformally invariant. However, the treatment of $\|\cdot\|_{\mathcal{B}_p^\#}$ is more delicate.

Definition 1. For $p > 1$, define the seminorm

$$\|\Phi\|_{\mathcal{B}_p} = \Big(\int_{\mathbb{H}} |(\operatorname{Im} z)\, \Phi'(z)|^p \frac{dx\,dy}{|\operatorname{Im} z|^2}\Big)^{1/p}$$

for holomorphic functions Φ on $\mathbb{H}$. The set of all such Φ with $\|\Phi\|_{\mathcal{B}_p} < \infty$ is denoted by $\mathcal{B}_p(\mathbb{H})$. Moreover, for $p \geq 1$, define the seminorm

$$\|\Phi\|_{\mathcal{B}_p^\#} = \Big(\int_{\mathbb{H}} |(\operatorname{Im} z)^2\, \Phi''(z)|^p \frac{dx\,dy}{|\operatorname{Im} z|^2}\Big)^{1/p}.$$

The set of all such Φ with $\|\Phi\|_{\mathcal{B}_p^\#} < \infty$ is denoted by $\mathcal{B}_p^\#(\mathbb{H})$.

Remark 1. If one applies the seminorm $\|\cdot\|_{\mathcal{B}_p}$ with $p = 1$, then only constant functions Φ satisfy $\|\Phi\|_{\mathcal{B}_1} < \infty$; see [39, p. 132].

A holomorphic function Φ on $\mathbb{H}$ is called a *Bloch function* if the seminorm satisfies

$$\|\Phi\|_{\mathcal{B}_\infty} = \sup_{z \in \mathbb{H}} |(\operatorname{Im} z)\, \Phi'(z)| < \infty.$$

The set of all Bloch functions on $\mathbb{H}$ is denoted by $\mathcal{B}_\infty(\mathbb{H})$. Moreover, Φ is called a *BMOA function* if $|\operatorname{Im} z|\, |\Phi'(z)|^2\, dx\, dy$ is a Carleson measure on $\mathbb{H}$. In general, a (possibly infinite) measure m on $\mathbb{H}$ is said to be a Carleson measure if $\sup_{I \subset \mathbb{R}} m(\widehat{I})/|I| < \infty$, where the supremum is taken over all bounded intervals $I \subset \mathbb{R}$ and $\widehat{I} \subset \mathbb{H}$ denotes the Carleson box (the square in $\mathbb{H}$) above I. Accordingly, the BMOA seminorm of Φ is defined by

$$\|\Phi\|_{\mathrm{BMOA}} = \Big(\sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_{\widehat{I}} |(\operatorname{Im} z)\, \Phi'(z)|^2 \frac{dx\,dy}{|\operatorname{Im} z|}\Big)^{1/2}.$$

This definition of BMOA is equivalent to requiring that Φ be holomorphic and given by the Poisson integral of a BMO function on $\mathbb{R}$. On the unit disk $\mathbb{D}$, the corresponding equivalence is well known (see [9, Theorem 6.5]); on the half-plane $\mathbb{H}$, it also holds (see [8, p. 262]). The set of all BMOA functions on $\mathbb{H}$ is denoted by $\mathrm{BMOA}(\mathbb{H})$.

We next compare the above seminorms. For convenience, we include proofs of the standard estimates.

Proposition 2.1. (i) *For $1 < p \leq q \leq \infty$, there exists a constant $c_{p,q} > 0$ such that $\|\Phi\|_{\mathcal{B}_q} \leq c_{p,q} \|\Phi\|_{\mathcal{B}_p}$.* (ii) *There exists a constant $c > 0$ such that $\|\Phi\|_{\mathcal{B}_\infty} \leq c \|\Phi\|_{\mathrm{BMOA}}$.* (iii) *For $p > 1$, there exists a constant $c'_p > 0$ such that $\|\Phi\|_{\mathrm{BMOA}} \leq c'_p \|\Phi\|_{\mathcal{B}_p}$.*

Proof. (i) For $z \in \mathbb{H}$, let $\Delta(z, |\operatorname{Im} z|/2) \subset \mathbb{H}$ be the disk centered at z with radius $|\operatorname{Im} z|/2$. By the integral mean inequality for holomorphic functions and the Hölder inequality,

$$\begin{aligned}
|(\operatorname{Im} z)\Phi'(z)| &\leq \frac{4}{\pi|\operatorname{Im} z|}\int_{\Delta(z,|\operatorname{Im} z|/2)} |\Phi'(w)|\, du\, dv \\
&\leq \frac{4}{\pi|\operatorname{Im} z|}\Big(\frac{\pi|\operatorname{Im} z|^2}{4}\Big)^{1-1/p}\Big(\int_{\Delta(z,|\operatorname{Im} z|/2)} |\Phi'(w)|^p\, du\, dv\Big)^{1/p} \\
&= \Big(\frac{4}{\pi}\Big)^{1/p}|\operatorname{Im} z|^{1-2/p}\Big(\int_{\Delta(z,|\operatorname{Im} z|/2)} |\Phi'(w)|^p\, du\, dv\Big)^{1/p}. \qquad (1)
\end{aligned}$$

The last line is bounded by

$$\Big(\frac{4}{\pi}\Big)^{1/p}|\operatorname{Im} z|^{1-2/p}\Big(\frac{2^{p-2}}{|\operatorname{Im} z|^{p-2}}\int_{\Delta(z,|\operatorname{Im} z|/2)} |(\operatorname{Im} w)\Phi'(w)|^p \frac{du\, dv}{|\operatorname{Im} w|^2}\Big)^{1/p} \leq \frac{2}{\sqrt[p]{\pi}}\|\Phi\|_{\mathcal{B}_p},$$

whence $\|\Phi\|_{\mathcal{B}_\infty} \leq c_{p,\infty}\,\|\Phi\|_{\mathcal{B}_p}$ with $c_p = c_{p,\infty} = 2/\sqrt[p]{\pi}$.

For $p \leq q < \infty$, we have

$$\int_{\mathbb{H}}\Big(\frac{|(\operatorname{Im} z)\Phi'(z)|}{\|\Phi\|_{\mathcal{B}_\infty}}\Big)^q \frac{dx\, dy}{|\operatorname{Im} z|^2} \leq \int_{\mathbb{H}}\Big(\frac{|(\operatorname{Im} z)\Phi'(z)|}{\|\Phi\|_{\mathcal{B}_\infty}}\Big)^p \frac{dx\, dy}{|\operatorname{Im} z|^2},$$

i.e. $\|\Phi\|^q_{\mathcal{B}_q}/\|\Phi\|^q_{\mathcal{B}_\infty} \leq \|\Phi\|^p_{\mathcal{B}_p}/\|\Phi\|^p_{\mathcal{B}_\infty}$. It follows that

$$\frac{\|\Phi\|_{\mathcal{B}_q}}{\|\Phi\|_{\mathcal{B}_\infty}} \leq \Big(\frac{\|\Phi\|_{\mathcal{B}_p}}{\|\Phi\|_{\mathcal{B}_\infty}}\Big)^{p/q} = \Big(\frac{1}{c_p}\Big)^{p/q}\Big(\frac{c_p\|\Phi\|_{\mathcal{B}_p}}{\|\Phi\|_{\mathcal{B}_\infty}}\Big)^{p/q} \leq \Big(\frac{1}{c_p}\Big)^{p/q}\frac{c_p\|\Phi\|_{\mathcal{B}_p}}{\|\Phi\|_{\mathcal{B}_\infty}}.$$

Hence $\|\Phi\|_{\mathcal{B}_q} \leq c_{p,q}\,\|\Phi\|_{\mathcal{B}_p}$ with $c_{p,q} = c_p^{1-p/q}$.

(ii) This is sketched in [24, p. 92]; see also [9, Corollary 5.2]. From (2) with $p = 2$,

$$\begin{aligned}
|(\operatorname{Im} z)\Phi'(z)| &\leq \frac{2}{\sqrt{\pi}}\Big(\int_{\Delta(z,|\operatorname{Im} z|/2)} |\Phi'(w)|^2\, du\, dv\Big)^{1/2} \\
&\leq \frac{4}{\sqrt{\pi}}\Big(\frac{1}{2|\operatorname{Im} z|}\int_{I^2(z,|\operatorname{Im} z|)} |\operatorname{Im} w|\, |\Phi'(w)|^2\, du\, dv\Big)^{1/2},
\end{aligned}$$

where $I^2(z, |\operatorname{Im} z|)$ denotes the Carleson box square centered at z above the interval of length $2|\operatorname{Im} z|$ on $\mathbb{R}$. Taking the supremum over $z \in \mathbb{H}$ yields $\|\Phi\|_{\mathcal{B}_\infty} \leq c\,\|\Phi\|_{\mathrm{BMOA}}$ with $c = 4/\sqrt{\pi}$.

(iii) Suppose that $p > 2$. For any bounded interval $I \subset \mathbb{R}$, the Hölder inequality gives

$$\begin{aligned}
&\frac{1}{|I|}\int_{\widehat{I}} |\operatorname{Im} w||\Phi'(w)|^2 dudv \\
&\leq \frac{1}{|I|}\Big(\int_{\widehat{I}} |(\operatorname{Im} w)\Phi'(w)|^p \frac{dudv}{|\operatorname{Im} w|^2}\Big)^{2/p}\cdot\Big(\int_{\widehat{I}} |\operatorname{Im} w|^{p/(p-2)}\frac{dudv}{|\operatorname{Im} w|^2}\Big)^{1-2/p} \\
&\leq \Big(1-\frac{2}{p}\Big)^{1-2/p}\Big(\int_{\mathbb{H}} |(\operatorname{Im} w)\Phi'(w)|^p \frac{dudv}{|\operatorname{Im} w|^2}\Big)^{2/p}.
\end{aligned}$$

Taking the supremum over I yields $\|\Phi\|_{\mathrm{BMOA}} \le c'_p \|\Phi\|_{\mathcal{B}_p}$ with $c'_p = (1-\frac{2}{p})^{1/2-1/p}$. For $1 < p \le 2$, combine this with (i). □

Arguing as in (i) above with the definition $\|\Phi\|_{\mathcal{B}^\#_\infty} = \sup_{z\in\mathbb{H}} |(\operatorname{Im} z)^2\Phi''(z)|$ yields:

Proposition 2.2. *For $1 \le p \le q \le \infty$, there exists a constant $\tilde{c}_{p,q} > 0$ such that $\|\Phi\|_{\mathcal{B}^\#_q} \le \tilde{c}_{p,q} \|\Phi\|_{\mathcal{B}^\#_p}$.*

Definition 2. For $1 \le p < \infty$, set

$$\|\Phi\|_{\widehat{\mathcal{B}}_p} = \|\Phi\|_{\mathcal{B}^\#_p} + \|\Phi\|_{\mathrm{BMOA}}.$$

The collection of Φ with $\|\Phi\|_{\widehat{\mathcal{B}}_p} < \infty$ is denoted by $\widehat{\mathcal{B}}_p(\mathbb{H})$; equivalently,

$$\widehat{\mathcal{B}}_p(\mathbb{H}) = \mathcal{B}^\#_p(\mathbb{H}) \cap \mathrm{BMOA}(\mathbb{H}).$$

Hereafter, to suppress multiplicative constants, we use the notation $A(\tau) \lesssim B(\tau)$ to mean that there exists $C > 0$ such that $A(\tau) \le C\,B(\tau)$ uniformly in the relevant parameter τ; we write $A(\tau) \asymp B(\tau)$ when both $A(\tau) \lesssim B(\tau)$ and $A(\tau) \gtrsim B(\tau)$ hold.

By Proposition 2.2, we have $\|\Phi\|_{\widehat{\mathcal{B}}_q} \lesssim \|\Phi\|_{\widehat{\mathcal{B}}_p}$ for $1 \le p \le q$.

Proposition 2.3. *Let $1 < p < \infty$. Then $\|\Phi\|_{\mathcal{B}^\#_p} \lesssim \|\Phi\|_{\mathcal{B}_p}$ for $\Phi \in \mathcal{B}_p(\mathbb{H})$. Conversely, $\|\Phi\|_{\mathcal{B}_p} \lesssim \|\Phi\|_{\mathcal{B}^\#_p} + \|\Phi\|_{\mathcal{B}_\infty}$ for $\Phi \in \mathcal{B}^\#_p(\mathbb{H}) \cap \mathcal{B}_\infty(\mathbb{H})$. Hence, the seminorms $\|\Phi\|_{\mathcal{B}_p}$ and $\|\Phi\|_{\widehat{\mathcal{B}}_p}$ are equivalent.*

Proof. For the first inequality, we adapt the proof of [27, Lemma 3.3]. By the Cauchy integral formula,

$$|\Phi''(z)| \le \frac{1}{2\pi}\int_{|\zeta-z|=y/4} \frac{|\Phi'(\zeta)|}{|\zeta-z|^2}|d\zeta| \le \frac{4}{y}\max_{|\zeta-z|\le y/4} |\Phi'(\zeta)|$$

for $z = x+iy \in \mathbb{H}^+$. Moreover,

$$|\Phi'(\zeta)|^p \le \frac{16}{\pi y^2}\int_{|w-\zeta|\le y/4} |\Phi'(w)|^p\,du\,dv$$

for $w = u+iv$. Hence

$$y^{2p-2}|\Phi''(z)|^p \lesssim y^{p-4}\int_{|w-z|\le y/2} |\Phi'(w)|^p\,du\,dv \le y^{p-4}\int_{y/2}^{3y/2}\int_{x-y/2}^{x+y/2} |\Phi'(w)|^p\,du\,dv.$$

With the change of variables $(u,v) \mapsto (\xi,\eta)$ by $u = x+y\xi$ and $v = y\eta$, the right-hand side becomes

$$y^{p-2}\int_{1/2}^{3/2}\int_{-1/2}^{1/2} |\Phi'(x+y\xi+iy\eta)|^p\,d\xi\,d\eta.$$

Using the inequality of this form, we estimate $\|\Phi\|_{\mathcal{B}_p^\#}$ as

$$\|\Phi\|_{\mathcal{B}_p^\#}^p = \int_{\mathbb{H}} y^{2p-2}|\Phi''(z)|^p dxdy$$
$$\lesssim \int_{1/2}^{3/2}\int_{-1/2}^{1/2}\Big(\int_{\mathbb{H}} y^{p-2}|\Phi'(x+y\xi+iy\eta)|^p dxdy\Big)d\xi d\eta.$$

Again with the change of variables $(x,y)\mapsto(u,v)$ by $u=x+y\xi$ and $v=y\eta$, the last integral turns out to be

$$\int_{1/2}^{3/2}\int_{-1/2}^{1/2}\Big(\int_{\mathbb{H}}\Big(\frac{v}{\eta}\Big)^{p-2}|\Phi'(w)|^p\frac{dudv}{\eta}\Big)d\xi d\eta = \Big(\int_{1/2}^{3/2}\frac{d\eta}{\eta^{p-1}}\Big)\int_{\mathbb{H}} v^{p-2}|\Phi'(w)|^p dudv \asymp \|\Phi\|_{\mathcal{B}_p}^p.$$

Thus, $\|\Phi\|_{\mathcal{B}_p^\#}\lesssim\|\Phi\|_{\mathcal{B}_p}$ is verified.

For the converse, [15, Lemma 3.2] essentially gives

$$\|\Phi\|_{\mathcal{B}_p}\lesssim\|\Phi\|_{\mathcal{B}_p^\#}+\|\Phi''\|_{\mathcal{A}_\infty},$$

where $\|\Phi''\|_{\mathcal{A}_\infty}$ is defined later in (3) and satisfies $\|\Phi''\|_{\mathcal{A}_\infty}\asymp\|\Phi\|_{\mathcal{B}_\infty}$ by [26, Lemma 6.3]. This yields the stated bound. □

Remark 2. In the second statement, for $p>2$ one even has $\|\Phi\|_{\mathcal{B}_p}\lesssim\|\Phi\|_{\mathcal{B}_p^\#}$ for $\Phi\in\mathcal{B}_p^\#(\mathbb{H})\cap\mathcal{B}_\infty(\mathbb{H})$. Indeed, from $\Phi'(x+iy)=-i\int_y^{y_0}\Phi''(x+it)\,dt+\Phi'(x+iy_0)$ and letting $y_0\to\infty$, we obtain

$$\Phi'(x+iy)=-i\int_y^\infty\Phi''(x+it)\,dt \tag{2}$$

for $x+iy\in\mathbb{H}^+$ since $\lim_{y_0\to\infty}\Phi'(x+iy_0)=0$ when $\Phi\in\mathcal{B}_\infty(\mathbb{H})$. For $p>2$ and $1<q<2$ with $1/p+1/q=1$, this gives

$$|\Phi'(x+iy)|\le\Big(\int_y^\infty\frac{dt}{t^{2-q/p}}\Big)^{1/q}\Big(\int_y^\infty t^{2p-3}|\Phi''(x+it)|^p\,dt\Big)^{1/p},$$

hence

$$y^{p-2}|\Phi'(x+iy)|^p\lesssim\int_y^\infty t^{2p-3}|\Phi''(x+it)|^p\,dt,$$

and integrating over $\mathbb{H}$ and exchanging the order of integrals yield $\|\Phi\|_{\mathcal{B}_p}\lesssim\|\Phi\|_{\mathcal{B}_p^\#}$.

Identifying functions that differ by a constant, we may regard $\mathcal{B}_p(\mathbb{H})$ and $\widehat{\mathcal{B}}_p(\mathbb{H})$ as normed spaces with norms $\|\cdot\|_{\mathcal{B}_p}$ and $\|\cdot\|_{\widehat{\mathcal{B}}_p}$, respectively; under these norms they are complex Banach spaces.

For the unit disk $\mathbb{D}$, define $\mathcal{B}_p(\mathbb{D})$, $\mathcal{B}_p^\#(\mathbb{D})$, BMOA$(\mathbb{D})$, and $\widehat{\mathcal{B}}_p(\mathbb{D})$ analogously by replacing the hyperbolic density $1/|\mathrm{Im}\,z|$ on $\mathbb{H}$ with $2/(1-|z|^2)$ on $\mathbb{D}$. Let $K(z)=(z-i)/(z+i)$ be the Cayley transformation, which maps $\mathbb{H}^+$ conformally onto $\mathbb{D}$ with $K(i)=0$ (and $K(z)=(-z-i)/(-z+i)$ maps $\mathbb{H}^-$ onto $\mathbb{D}$ with $K(-i)=0$). For a function Φ on $\mathbb{H}$, write $K_*(\Phi)=\Phi\circ K^{-1}$ for the push-forward to $\mathbb{D}$. Then K_* is an isometric isomorphism from $\mathcal{B}_p(\mathbb{H})$ onto $\mathcal{B}_p(\mathbb{D})$ for $p > 1$ (including $p = \infty$), by conformal invariance. For the spaces $\widehat{\mathcal{B}}_p(\mathbb{H})$ and $\widehat{\mathcal{B}}_p(\mathbb{D})$, which involve Φ'', the situation is subtler.

To show that K_* gives a Banach isomorphism between $\widehat{\mathcal{B}}_p(\mathbb{H})$ and $\widehat{\mathcal{B}}_p(\mathbb{D})$, we prepare the following lemma. For a holomorphic function Φ on $\mathbb{H}^+$, the seminorm defined by its derivative in the Hardy space $\mathcal{H}_1$ is

$$\|\Phi\|_{\dot{\mathcal{H}}_1^1} = \sup_{y>0} \int_{-\infty}^{\infty} |\Phi'(x+iy)|\, dx, \tag{3}$$

and write $\dot{\mathcal{H}}_1^1(\mathbb{H})$ for the corresponding space. Similarly, for a holomorphic function Φ_* on $\mathbb{D}$, set

$$\|\Phi_*\|_{\dot{\mathcal{H}}_1^1} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |\Phi_*'(re^{i\theta})|\, d\theta, \tag{4}$$

and denote the space by $\dot{\mathcal{H}}_1^1(\mathbb{D})$.

Lemma 2.4. (i) *Every* $\Phi \in \mathcal{B}_\infty(\mathbb{H})$ *satisfies* $\|\Phi\|_{\dot{\mathcal{H}}_1^1} \le \|\Phi\|_{\mathcal{B}_1^\#}$. (ii) *Every holomorphic function* Φ_* *on* $\mathbb{D}$ *satisfies* $\|\Phi_*\|_{\dot{\mathcal{H}}_1^1} \le C\big(\|\Phi_*\|_{\mathcal{B}_1^\#} + \|\Phi_*\|_{\mathcal{B}_\infty}\big)$ *for some absolute constant* $C > 0$.

Proof. (i) From (2),

$$\int_{-\infty}^{\infty} |\Phi'(x+iy)|\, dx \le \int_{-\infty}^{\infty}\int_y^{\infty} |\Phi''(x+it)|\, dt\, dx \le \int_{\mathbb{H}} |\Phi''(z)|\, dx\, dy,$$

and taking the supremum over $y > 0$ gives the claim.

(ii) Likewise, $\Phi_*'(re^{i\theta}) = \int_\varepsilon^r \Phi_*''(te^{i\theta})\, dt + \Phi_*'(\varepsilon e^{i\theta})$ for $0 < \varepsilon < r < 1$. Hence

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |\Phi_*'(re^{i\theta})|\, d\theta &\le \frac{1}{2\pi}\int_0^{2\pi}\int_\varepsilon^r |\Phi_*''(te^{i\theta})|\, dt\, d\theta + \frac{2}{1-\varepsilon^2}\|\Phi_*\|_{\mathcal{B}_\infty} \\ &\le \frac{1}{2\pi\varepsilon}\|\Phi_*\|_{\mathcal{B}_1^\#} + \frac{2}{1-\varepsilon^2}\|\Phi_*\|_{\mathcal{B}_\infty}, \end{aligned}$$

which implies the claim with $C = \min_{0<\varepsilon<1} \max\{\frac{1}{2\pi\varepsilon}, \frac{2}{1-\varepsilon^2}\}$. □

We can now establish the expected correspondence between $\widehat{\mathcal{B}}_p(\mathbb{H})$ and $\widehat{\mathcal{B}}_p(\mathbb{D})$. An idea for its proof is in [22, Section 9].

Theorem 2.5. *The push-forward* K_* *by the Cayley transformation is a Banach isomorphism from* $\widehat{\mathcal{B}}_p(\mathbb{H})$ *onto* $\widehat{\mathcal{B}}_p(\mathbb{D})$ *for* $p \ge 1$.

Proof. First, by conformal invariance, $\|K_*(\Phi)\|_{\mathrm{BMOA}} \asymp \|\Phi\|_{\mathrm{BMOA}}$: BMO functions on $\mathbb{R}$ and $\mathbb{S}$ correspond under the Cayley transformation (see [8, Corollary. VI.1.3]), and BMOA functions are holomorphic functions obtained by the Poisson integral of those functions.

We estimate the $\mathcal{B}_p^{\#}$-seminorms. Let $\Phi_* = K_*(\Phi) = \Phi \circ K^{-1}$. Changing variables $\zeta = K(z)$ gives

$$\begin{aligned}
&\int_{\mathbb{H}} |(\operatorname{Im} z)^2\, \Phi''(z)|^p \frac{dx\,dy}{|\operatorname{Im} z|^2} \qquad (5)\\
&= \int_{\mathbb{H}} |(\operatorname{Im} z)^2\, (\Phi_*'' \circ K(z) \cdot K'(z)^2 + \Phi_*' \circ K(z) \cdot K''(z))|^p \frac{dx\,dy}{|\operatorname{Im} z|^2}\\
&\le 2^{p-1} \int_{\mathbb{D}} \Big| \Big(\frac{1-|\zeta|^2}{2}\Big)^2 \Phi_*''(\zeta) \Big|^p \frac{4\,d\xi\,d\eta}{(1-|\zeta|^2)^2}\\
&\quad + 2^{p-1} \int_{\mathbb{D}} \Big| \Big(\frac{1-|\zeta|^2}{2}\Big) \Phi_*'(\zeta) \Big|^p \Big(\frac{1-|\zeta|^2}{|1-\zeta|}\Big)^p \frac{4\,d\xi\,d\eta}{(1-|\zeta|^2)^2}.
\end{aligned}$$

Note that $1-|\zeta|^2 \le 2|1-\zeta|$.

Suppose $\Phi_* \in \widehat{\mathcal{B}}_p(\mathbb{D})$. For $p>1$, (2) yields

$$\|\Phi\|^p_{\mathcal{B}_p^{\#}} \le 2^{p-1} \|\Phi_*\|^p_{\mathcal{B}_p^{\#}} + 2^{2p-1} \|\Phi_*\|^p_{\mathcal{B}_p}.$$

Because $\|\Phi_*\|_{\mathcal{B}_p} \asymp \|\Phi_*\|_{\mathcal{B}_p^{\#}} + |\Phi_*'(0)| \lesssim \|\Phi_*\|_{\widehat{\mathcal{B}}_p}$ (see [38, p. 327]), we obtain $\|\Phi\|_{\widehat{\mathcal{B}}_p} \lesssim \|K_*(\Phi)\|_{\widehat{\mathcal{B}}_p}$. When $p=1$, the second integral in (2) becomes

$$\int_{\mathbb{D}} |\Phi_*'(\zeta)| \frac{2\,d\xi\,d\eta}{|1-\zeta|}. \tag{6}$$

By Lemma 2.4, $\Phi_* \in \dot{\mathcal{H}}_1^1(\mathbb{D})$. Moreover, $dm^* = 2\,d\xi\,d\eta/|1-\zeta|$ is a Carleson measure on $\mathbb{D}$. This can be verified by straightforward computation; indeed, it suffices to show that for a disk $\Delta(1,r)$ with center at 1 and radius $r>0$,

$$\frac{1}{r} \int_{\Delta(1,r)} \frac{d\xi d\eta}{|1-\zeta|} \le \pi.$$

Here, we apply the Carleson embedding theorem (see [7, Theorem 9.3], [8, Theorem II.3.9]). This in particular implies that for any holomorphic function Ψ in the Hardy space $\mathcal{H}_1(\mathbb{D})$ with norm $\|\cdot\|_{\mathcal{H}_1}$ and for any Carleson measure dm^* on $\mathbb{D}$, there exists a constant $c'>0$ depending only on dm^* such that $\int_{\mathbb{D}} |\Psi(\zeta)| dm^*(\zeta) \le c' \|\Psi\|_{\mathcal{H}_1}$. Thus, integral (2) is bounded by $c'\|\Phi_*\|_{\dot{\mathcal{H}}_1^1}$. Plugging this estimate into inequality (2) and using Lemma 2.4, we obtain that

$$\|\Phi\|_{\mathcal{B}_1^{\#}} \le \|K_*(\Phi)\|_{\mathcal{B}_1^{\#}} + c'\|K_*(\Phi)\|_{\dot{\mathcal{H}}_1^1} \lesssim \|K_*(\Phi)\|_{\widehat{\mathcal{B}}_1}.$$

This yields $\|\Phi\|_{\widehat{\mathcal{B}}_1} \lesssim \|K_*(\Phi)\|_{\widehat{\mathcal{B}}_1}$.

Conversely, assume $\Phi \in \widehat{\mathcal{B}}_p(\mathbb{H})$. Likewise to the above computation, we have

$$\int_{\mathbb{D}} \Big| \Big(\frac{1-|\zeta|^2}{2} \Big)^2 \Phi_*''(\zeta) \Big|^p \frac{4\, d\xi\, d\eta}{(1-|\zeta|^2)^2}$$
$$\leq 2^{p-1} \int_{\mathbb{H}} |(\operatorname{Im} z)^2\, \Phi''(z)|^p \frac{dx\, dy}{|\operatorname{Im} z|^2} + 2^{p-1} \int_{\mathbb{H}} |(\operatorname{Im} z)\, \Phi'(z)|^p \Big(\frac{2 \operatorname{Im} z}{|z+i|} \Big)^p \frac{dx\, dy}{|\operatorname{Im} z|^2}, \tag{7}$$

where $\operatorname{Im} z \leq |z+i|$. For $p > 1$, (2) implies

$$\|K_*(\Phi)\|^p_{\mathcal{B}_p^\#} \leq 2^{p-1} \|\Phi\|^p_{\mathcal{B}_p^\#} + 2^{2p-1} \|\Phi\|^p_{\mathcal{B}_p},$$

and with Proposition 2.3 we get $\|K_*(\Phi)\|_{\widehat{\mathcal{B}}_p} \lesssim \|\Phi\|_{\widehat{\mathcal{B}}_p}$. When $p = 1$, the second integral on the right of (2) equals

$$\int_{\mathbb{H}} |\Phi'(z)| \frac{2\, dx\, dy}{|z+i|}. \tag{8}$$

Since $\Phi \in \dot{\mathcal{H}}_1^1(\mathbb{H})$ by Lemma 2.4 and $dm = 2\, dx\, dy/|z+i|$ is a Carleson measure on $\mathbb{H}$, the Carleson embedding theorem implies that (2) is bounded by $c''\|\Phi\|_{\dot{\mathcal{H}}_1^1}$ where $c'' > 0$ depends only on dm. Using Lemma 2.4 again for this, we obtain from (2) that

$$\|K_*(\Phi)\|_{\mathcal{B}_1^\#} \leq \|\Phi\|_{\mathcal{B}_1^\#} + c''\|\Phi\|_{\dot{\mathcal{H}}_1^1} \leq (1 + c'')\|\Phi\|_{\mathcal{B}_1^\#}, \tag{9}$$

which implies $\|K_*(\Phi)\|_{\widehat{\mathcal{B}}_1} \lesssim \|\Phi\|_{\widehat{\mathcal{B}}_1}$. □

We conclude the section with equivalent norms for $\|\cdot\|_{\widehat{\mathcal{B}}_1}$ on $\widehat{\mathcal{B}}_1(\mathbb{H})$ and $\widehat{\mathcal{B}}_1(\mathbb{D})$.

Proposition 2.6. (i) *On $\widehat{\mathcal{B}}_1(\mathbb{D})$ the norm $\|\Phi_*\|_{\widehat{\mathcal{B}}_1}$ is equivalent to $\|\Phi_*\|_{\mathcal{B}_1^\#} + \|\Phi_*\|_{\mathcal{B}_\infty}$ and to $\|\Phi_*\|_{\mathcal{B}_1^\#} + \|\Phi_*\|_{\dot{\mathcal{H}}_1^1}$.* (ii) *On $\widehat{\mathcal{B}}_1(\mathbb{H})$ the norm $\|\Phi\|_{\widehat{\mathcal{B}}_1}$ is equivalent to $\|\Phi\|_{\mathcal{B}_1^\#} + \|\Phi\|_{\mathcal{B}_\infty}$ and to $\|\Phi\|_{\mathcal{B}_1^\#} + \|\Phi\|_{\dot{\mathcal{H}}_1^1}$.*

Proof. (i) By using the facts that $\|\Phi_*\|_{\mathcal{B}_p^\#} \lesssim \|\Phi_*\|_{\mathcal{B}_1^\#}$ for any $p > 1$ (which is the same as Proposition 2.2) and $|\Phi_*'(0)| \leq \|\Phi_*\|_{\dot{\mathcal{H}}_1^1}$, we obtain

$$\|\Phi_*\|_{\mathrm{BMOA}} \lesssim \|\Phi_*\|_{\mathcal{B}_p} \asymp \|\Phi_*\|_{\mathcal{B}_p^\#} + |\Phi_*'(0)| \lesssim \|\Phi_*\|_{\mathcal{B}_1^\#} + \|\Phi_*\|_{\dot{\mathcal{H}}_1^1}. \tag{10}$$

Hence $\|\Phi_*\|_{\widehat{\mathcal{B}}_1} \lesssim \|\Phi_*\|_{\mathcal{B}_1^\#} + \|\Phi_*\|_{\dot{\mathcal{H}}_1^1}$. The bound $\|\Phi_*\|_{\mathcal{B}_1^\#} + \|\Phi_*\|_{\mathcal{B}_\infty} \lesssim \|\Phi_*\|_{\widehat{\mathcal{B}}_1}$ is immediate. Finally, $\|\Phi_*\|_{\mathcal{B}_1^\#} + \|\Phi_*\|_{\dot{\mathcal{H}}_1^1} \lesssim \|\Phi_*\|_{\mathcal{B}_1^\#} + \|\Phi_*\|_{\mathcal{B}_\infty}$ follows from Lemma 2.4.

(ii) Transfer the estimate for $\Phi_* = K_*(\Phi)$ on $\mathbb{D}$ back to Φ on $\mathbb{H}$. From (2), $\|K_*(\Phi)\|_{\mathcal{B}_1^\#} \lesssim \|\Phi\|_{\mathcal{B}_1^\#}$. Moreover, $\|K_*(\Phi)\|_{\dot{\mathcal{H}}_1^1} \lesssim \|\Phi\|_{\dot{\mathcal{H}}_1^1}$. Indeed, the line integral along the horizontal line in (2) is transfered by K to the line integral along a horocycle in $\mathbb{D}$ tangent at 1, which dominates the integral along the circle in (2); see the argument in [7, Section 11.1]. Thus, by (2),

$$\|\Phi\|_{\mathrm{BMOA}} \asymp \|K_*(\Phi)\|_{\mathrm{BMOA}} \lesssim \|K_*(\Phi)\|_{\mathcal{B}_1^\#} + \|K_*(\Phi)\|_{\dot{\mathcal{H}}_1^1} \lesssim \|\Phi\|_{\mathcal{B}_1^\#} + \|\Phi\|_{\dot{\mathcal{H}}_1^1},$$

which implies $\|\Phi\|_{\widehat{\mathcal{B}}_1} \lesssim \|\Phi\|_{\mathcal{B}_1^\#} + \|\Phi\|_{\dot{\mathcal{H}}_1^1}$. The remaining implications are as in (i), again using Lemma 2.4. □

Remark 3. Proposition 2.6 implies in particular that

$$\widehat{\mathcal{B}}_1(\mathbb{H}) = \mathcal{B}_1^{\#}(\mathbb{H}) \cap \mathcal{B}_\infty(\mathbb{H}) = \mathcal{B}_1^{\#}(\mathbb{H}) \cap \dot{\mathcal{H}}_1^1(\mathbb{H}),$$

and in fact $\mathcal{B}_1^{\#}(\mathbb{H}) \not\subseteq \mathcal{B}_\infty(\mathbb{H})$ and $\mathcal{B}_1^{\#}(\mathbb{H}) \not\subseteq \dot{\mathcal{H}}_1^1(\mathbb{H})$. By contrast, $\mathcal{B}_1^{\#}(\mathbb{D}) \subset \mathcal{B}_\infty(\mathbb{D})$ since every function in $\mathcal{B}_1^{\#}(\mathbb{D})$ is bounded (see [39, Theorem 5.19]), hence $\mathcal{B}_1^{\#}(\mathbb{D}) \subset \dot{\mathcal{H}}_1^1(\mathbb{D})$ by Lemma 2.4. Consequently, $\mathcal{B}_1^{\#}(\mathbb{D}) = \widehat{\mathcal{B}}_1(\mathbb{D})$ by Proposition 2.6.

Remark 4. In defining $\widehat{\mathcal{B}}_p(\mathbb{H})$ we included the BMOA seminorm $\|\Phi\|_{\mathrm{BMOA}}$, but one could equally well use $\|\Phi\|_{\mathcal{B}_\infty}$ or $\|\Phi\|_{\dot{\mathcal{H}}_1^1}$. The specific choice is not essential; our goals are twofold: (1) to ensure that the seminorm on $\widehat{\mathcal{B}}_p(\mathbb{H})$ annihilates only constants, and (2) to preserve the Banach isomorphism between $\widehat{\mathcal{B}}_p(\mathbb{H})$ and $\widehat{\mathcal{B}}_p(\mathbb{D})$ under the Cayley transformation.

3. The pre-Schwarzian derivative map

We consider the properties of conformal mappings induced by integrable Beltrami coefficients. A measurable function μ on $\mathbb{H}$ with $\|\mu\|_\infty < 1$ is called a *Beltrami coefficient*. The set of all Beltrami coefficients on $\mathbb{H}$ is denoted by $M(\mathbb{H})$, which is the open unit ball of $L_\infty(\mathbb{H})$ with the supremum norm $\|\mu\|_\infty$.

Definition 3. For $p \geq 1$, the space of p-integrable Beltrami coefficients is defined by

$$M_p(\mathbb{H}) = \Big\{\mu \in M(\mathbb{H}) \mid \|\mu\|_p = \Big(\int_{\mathbb{H}} |\mu(z)|^p \frac{dxdy}{|\mathrm{Im}\, z|^2}\Big)^{1/p} < \infty\Big\}.$$

We equip $M_p(\mathbb{H})$ with the norm $\|\mu\|_p + \|\mu\|_\infty$.

For $\mu \in M_p(\mathbb{H}^+)$, we denote by F^μ the normalized conformal homeomorphism of $\mathbb{H}^-$ that extends quasiconformally to $\mathbb{C}$ with complex dilatation μ on $\mathbb{H}^+$. The normalization is given by fixing the three points 0, 1, and ∞. For a conformal homeomorphism $F : \mathbb{H}^- \to \mathbb{C}$, the pre-Schwarzian derivative N_F and the Schwarzian derivative S_F are defined by

$$N_F = (\log F')' \; ; \quad S_F = (N_F)' - \tfrac{1}{2}(N_F)^2.$$

For the conformal homeomorphism F^μ of $\mathbb{H}^-$ with $\mu \in M(\mathbb{H}^+)$, let $L(\mu) = \log(F^\mu)'$ and $S(\mu) = S_{F^\mu}$. We call the maps L and S on $M(\mathbb{H}^+)$ the *pre-Schwarzian* and the *Schwarzian derivative maps*.

For $p \geq 1$, we define the norm

$$\|\Phi\|_{\mathcal{A}_p} = \Big(\int_{\mathbb{H}} |(\mathrm{Im}\, z)^2\, \Phi(z)|^p \frac{dxdy}{|\mathrm{Im}\, z|^2}\Big)^{1/p} \tag{11}$$

for holomorphic functions Φ on $\mathbb{H}$. For $p = \infty$, we set $\|\Phi\|_{\mathcal{A}_\infty} = \sup_{z\in\mathbb{H}} |(\mathrm{Im}\, z)^2\, \Phi(z)|$. The set of all such Φ with $\|\Phi\|_{\mathcal{A}_p} < \infty$ is denoted by $\mathcal{A}_p(\mathbb{H})$, which is a complex Banach space with this norm.

The Schwarzian derivative map S on $M_p(\mathbb{H}^+)$ has been thoroughly studied. In addition, we obtain the following result; see [36, Lemma 3.2]. Remark 5 below outlines the proof of the first statement.

Proposition 3.1. *For $p \geq 1$, there exists a constant $\widetilde{C}_p > 0$ such that the Schwarzian derivative map S satisfies $\|S(\mu)\|_{\mathcal{A}_p} \leq \widetilde{C}_p \|\mu\|_p$ for every $\mu \in M_p(\mathbb{H}^+)$. Moreover, $S : M_p(\mathbb{H}^+) \to \mathcal{A}_p(\mathbb{H}^-)$ is holomorphic.*

We note that the holomorphy of S follows from its local boundedness in this situation. Since this result will be used repeatedly later, we state it in a more general form. Let X and Y be complex Banach spaces and let $W \subset X$ be a domain. A map $J : W \to Y$ is called *Gâteaux holomorphic* if, for any $w \in W$ and $x \in X$, the function $J(w + \zeta x)$ is holomorphic in $\zeta \in \mathbb{C}$ into Y in some neighborhood of the origin. The following result is known (see [6, Theorem 14.9]).

Proposition 3.2. *If $J : W \to Y$ is locally bounded and Gâteaux holomorphic, then J is holomorphic.*

Moreover, when Y is a complex Banach space of holomorphic functions, the Gâteaux holomorphy can be verified in several ways; see [14, Lemma V.5.1] and [35, Lemma 6.1].

We prove the same claim for the pre-Schwarzian derivative map L. First, we show it under a special assumption on p. This is mentioned without proof in the proof of [37, Theorem 6.10].

Lemma 3.3. *For $p > 2$, there exists a constant $C_p > 0$ depending only on p such that the pre-Schwarzian derivative map L satisfies $\|L(\mu)\|_{\mathcal{B}_p} \leq C_p \|\mu\|_p$ for every $\mu \in M_p(\mathbb{H}^+)$.*

Proof. We first represent the directional derivative $d_\mu L(\nu)$ of L at $\mu \in M_p(\mathbb{H}^+)$ in the direction of a tangent vector ν. Let $\Omega^+ = F(\mathbb{H}^+)$ and $\Omega^- = F(\mathbb{H}^-)$ for the quasiconformal extension F of F^μ to $\mathbb{C}$, and let ρ_+ and ρ_- denote their hyperbolic densities. For the normalized Riemann mapping $G : \mathbb{H}^+ \to \Omega^+$ associated with F, the push-forward of the Beltrami coefficient ν on $\mathbb{H}^+$ by G is defined by

$$G_*(\nu)(w) = \nu(G^{-1}(w)) \frac{(G^{-1})_{\bar{w}}}{(G^{-1})_w} \quad (w \in \Omega^+).$$

As in the case of the Schwarzian derivative map (see [12, Lemma 5] and [29, Theorem I.2.3]), we see that

$$d_\mu L'(\nu)(F^{-1}(\zeta))(F^{-1})'(\zeta) = -\frac{2}{\pi} \int_{\Omega^+} \frac{G_*(\nu)(w)}{(w-\zeta)^3} dudv \quad (\zeta \in \Omega^-). \tag{12}$$

Here, $d_\mu L'(\nu)$ stands for the derivative of the holomorphic function $d_\mu L(\nu)$ in $\mathcal{B}_p(\mathbb{H}^-)$.

We estimate the norm of $d_\mu L(\nu)$:

$$\begin{aligned} \|d_\mu L(\nu)\|_{\mathcal{B}_p}^p &= \int_{\mathbb{H}^-} |(\operatorname{Im} z)\, d_\mu L'(\nu)(z)|^p \frac{dxdy}{|\operatorname{Im} z|^2} \\ &= \int_{\Omega^-} |d_\mu L'(\nu)(F^{-1}(\zeta))(F^{-1})'(\zeta)|^p \rho_-^{2-p}(\zeta) d\xi d\eta \\ &= \left(\frac{2}{\pi}\right)^p \int_{\Omega^-} \left| \int_{\Omega^+} \frac{G_*(\nu)(w)}{(w-\zeta)^3} dudv \right|^p \rho_-^{2-p}(\zeta) d\xi d\eta. \end{aligned} \tag{13}$$

Then, applying the Hölder inequality to the absolute value of the inner integral, we obtain

$$\Big|\int_{\Omega^+} \frac{G_*(\nu)(w)}{(w-\zeta)^3} dudv\Big|^p \leq \Big(\int_{\Omega^+} \frac{1}{|w-\zeta|^{4-q} dudv}\Big)^{p/q}\Big(\int_{\Omega^+} \frac{|G_*(\nu)(w)|^p}{|w-\zeta|^4} dudv\Big) \tag{14}$$

for $1/p + 1/q = 1$. Here, we note the following inequalities for the hyperbolic densities (see [14, p.6]):

$$\rho_-(\zeta) \geq \frac{1}{2d(\zeta, \partial\Omega^-)}; \quad \rho_+(w) \geq \frac{1}{2d(w, \partial\Omega^+)}.$$

Then, by virtue of the condition $q < 2$, the first integral is bounded as follows:

$$\begin{aligned} \int_{\Omega^+} \frac{1}{|w-\zeta|^{4-q}} dudv &\leq \int_{|w-\zeta| \geq d(\zeta, \partial\Omega^-)} \frac{1}{|w-\zeta|^{4-q}} dudv \\ &= \int_0^{2\pi}\int_{d(\zeta,\partial\Omega^-)}^{\infty} \frac{1}{r^{3-q}} drd\theta \\ &= \frac{2\pi}{2-q}\frac{1}{d(\zeta, \partial\Omega^-)^{2-q}} \leq \frac{8\pi}{2-q}\rho_-^{2-q}(\zeta). \end{aligned} \tag{15}$$

In the same way, we also have

$$\int_{\Omega^-} \frac{1}{|w-\zeta|^4} d\xi d\eta \leq 4\pi\rho_+^2(w). \tag{16}$$

The substitution of the above inequalities (3), (3), and (3) into (3) yields

$$\begin{aligned} \|d_\mu L(\nu)\|_{\mathcal{B}_p}^p &\leq \Big(\frac{2}{\pi}\Big)^p \Big(\frac{8\pi}{2-q}\Big)^{p/q} \int_{\Omega^-}\int_{\Omega^+} \Big(\frac{|G_*(\nu)(w)|^p}{|w-\zeta|^4} dudv\Big) (\rho_-^{2-q}(\zeta))^{p/q} \rho_-^{2-p}(\zeta) d\xi d\eta \\ &\leq \Big(\frac{16}{2-q}\Big)^p \int_{\Omega^+}\Big(\int_{\Omega^-} \frac{1}{|w-\zeta|^4} d\xi d\eta\Big) |G_*(\nu)(w)|^p dudv \\ &\leq \Big(\frac{16}{2-q}\Big)^p \int_{\Omega^+} 4\pi\rho_+^2(w)|G_*(\nu)(w)|^p dudv \\ &= 4\pi\Big(\frac{16}{2-q}\Big)^p \int_{\mathbb{H}^+} |\nu(z)|^p \frac{dxdy}{|\operatorname{Im} z|^2} = 4\pi\Big(\frac{16}{2-q}\Big)^p \|\nu\|_p^p. \end{aligned} \tag{17}$$

For $\mu \in M_p(\mathbb{H}^+)$, let $L_\mu(t) = L(t\mu)$ for $t \in [0, 1]$. By the fundamental theorem of calculus, we have

$$L(\mu) = L_\mu(1) - L_\mu(0) = \int_0^1 \frac{dL_\mu}{dt}(t)dt,$$

where $\frac{dL_\mu}{dt}(t) = d_{t\mu}L(\mu)$. Inequality (3) proved above shows that

$$\|d_{t\mu}L(\mu)\|_{\mathcal{B}_p}^p \leq C_p^p \|\mu\|_p^p$$

for all $t \in [0,1]$, where $C_p > 0$ is the constant depending only on p. Hence,

$$\begin{aligned}\|L(\mu)\|_{\mathcal{B}_p}^p &= \int_{\mathbb{H}^-} \Big|\Big(\int_0^1 \frac{dL_\mu}{dt}(t)dt\Big)'(z)\Big|^p |\operatorname{Im} z|^{p-2} dxdy \\ &\le \int_{\mathbb{H}^-} \Big(\int_0^1 |d_{t\mu}L'(\mu)(z)|dt\Big)^p |\operatorname{Im} z|^{p-2} dxdy \\ &\le \int_0^1 \Big(\int_{\mathbb{H}^-} |d_{t\mu}L'(\mu)(z)|^p |\operatorname{Im} z|^{p-2} dxdy\Big) dt = \|d_{t\mu}L(\mu)\|_{\mathcal{B}_p}^p,\end{aligned}$$

which is bounded also by $C_p^p \|\mu\|_p^p$. □

Remark 5. In the case of the Schwarzian derivative map $S : M_p(\mathbb{H}^+) \to \mathcal{A}_p(\mathbb{H}^-)$, a similar argument can be applied. This has been done in Theorem 2.3 and Lemma 2.9 of [29, Chapter I]. The corresponding formula to (3) is

$$d_\mu S(\nu)(F^{-1}(\zeta))(F^{-1})'(\zeta)^2 = -\frac{6}{\pi}\int_{\Omega^+} \frac{G_*(\nu)(w)}{(w-\zeta)^4} dudv \quad (\zeta \in \Omega^-),$$

and (3) with the density $\rho_-^{2-2p}(\zeta)$ turns out to be

$$\begin{aligned}\Big|\int_{\Omega^+} \frac{G_*(\nu)(w)}{(w-\zeta)^4} dudv\Big|^p \rho_-^{2-2p}(\zeta) &\le \Big(\int_{\Omega^+} \frac{1}{|w-\zeta|^4 dudv}\Big)^{p/q} \Big(\int_{\Omega^+} \frac{|G_*(\nu)(w)|^p}{|w-\zeta|^4} dudv\Big) \rho_-^{2-2p}(\zeta) \\ &\le (4\pi)^{p/q} \int_{\Omega^+} \frac{|G_*(\nu)(w)|^p}{|w-\zeta|^4} dudv\end{aligned}$$

by using (3). This holds without any condition on $p \ge 1$. In the case $p = 1$, the usual modification is applied for $q = \infty$. The other parts of the proof are the same. This gives the first statement of Proposition 3.1.

We remove the condition $p > 2$ in the statement of Lemma 3.3 and show the required result in full generality with the aid of properties of the Schwarzian derivative map S.

Theorem 3.4. *For $p \ge 1$, the pre-Schwarzian derivative map L satisfies $\|L(\mu)\|_{\mathcal{B}_p^\#} \le C_p^\# \|\mu\|_p$ for every $\mu \in M_p(\mathbb{H}^+)$, where $C_p^\# > 0$ is a constant depending on p and $\|\mu\|_p$. Moreover, $L : M_p(\mathbb{H}^+) \to \widehat{\mathcal{B}}_p(\mathbb{H}^-)$ is holomorphic.*

Proof. For any $\mu \in M_p(\mathbb{H}^+)$, let $F = F^\mu$. Then, using $S_F = (N_F)' - \frac{1}{2}(N_F)^2$, we have

$$\begin{aligned}\|L(\mu)\|_{\mathcal{B}_p^\#}^p &= \int_{\mathbb{H}^-} |(\operatorname{Im} z)^2 (N_F)'(z)|^p \frac{dxdy}{|\operatorname{Im} z|^2} \\ &\le 2^{p-1}\int_{\mathbb{H}^-} |(\operatorname{Im} z)^2 S_F(z)|^p \frac{dxdy}{|\operatorname{Im} z|^2} + \frac{1}{2}\int_{\mathbb{H}^-} |(\operatorname{Im} z) N_F(z)|^{2p} \frac{dxdy}{|\operatorname{Im} z|^2} \\ &\le 2^{p-1}\|S(\mu)\|_{\mathcal{A}_p}^p + \tfrac{1}{2}\|L(\mu)\|_{\mathcal{B}_{2p}}^{2p}. \qquad (18)\end{aligned}$$

We first assume $p > 1$. By Proposition 3.1, Lemma 3.3, and $\|\mu\|_{2p} \le \|\mu\|_p$, inequality (3) implies that

$$\begin{aligned}\|L(\mu)\|^p_{\mathcal{B}^\#_p} &\le 2^{p-1}(\widetilde{C}_p\|\mu\|_p)^p + \tfrac{1}{2}(C_{2p}\|\mu\|_{2p})^{2p}\\ &\le 2^{p-1}(\widetilde{C}_p^{\,p} + C_{2p}^{\,2p}\|\mu\|^p_p)\|\mu\|^p_p.\end{aligned}$$

This yields $\|L(\mu)\|_{\mathcal{B}^\#_p} \le C^\#_p\|\mu\|_p$ for $p > 1$, where $C^\#_p > 0$ is a constant depending also on $\|\mu\|_p$.

In the case $p = 1$, we apply (3) again to have

$$\|L(\mu)\|_{\mathcal{B}^\#_1} \le \|S(\mu)\|_{\mathcal{A}_1} + \|L(\mu)\|^2_{\mathcal{B}_2}.$$

By using $\|\mu\|_2 \le \|\mu\|_1$, this implies that

$$\|L(\mu)\|_{\mathcal{B}^\#_1} \le \widetilde{C}_1\|\mu\|_1 + (C_2\|\mu\|_1)^2.$$

Hence, we can also find $C^\#_1 > 0$ depending on $\|\mu\|_1$ such that $\|L(\mu)\|_{\mathcal{B}^\#_1} \le C^\#_1\|\mu\|_1$. This completes the proof of the first statement of the theorem.

For the second statement, we note that $L : M(\mathbb{H}^+) \to \mathcal{B}_\infty(\mathbb{H}^-)$ satisfies $\|L(\mu)\|_{\mathcal{B}_\infty} \le 3\|\mu\|_\infty$ (see [10, Proposition 5.3]). Then, combined with the first statement and Remark 4, this yields that

$$\|L(\mu)\|_{\widehat{\mathcal{B}}_p} \asymp \|L(\mu)\|_{\mathcal{B}^\#_p} + \|L(\mu)\|_{\mathcal{B}_\infty} \le \max\{C^\#_p, 3\}(\|\mu\|_p + \|\mu\|_\infty)$$

for every $p \ge 1$. Hence, $L : M_p(\mathbb{H}^+) \to \widehat{\mathcal{B}}_p(\mathbb{H}^-)$ is in particular locally bounded. Under this condition, the standard argument implies that L is in fact holomorphic in virtue of Proposition 3.2. □

Remark 6. The continuity of $L : M_p(\mathbb{H}^+) \to \widehat{\mathcal{B}}_p(\mathbb{H}^-)$ can be proved directly as in [25, Theorem 2.4] and [30, Theorem 2.4], from which holomorphy also follows. Indeed, for any $\mu, \nu \in M_p(\mathbb{H})$, the same argument as above gives

$$\|L(\mu) - L(\nu)\|^p_{\mathcal{B}^\#_p} \le 2^{p-1}\{\|S(\mu) - S(\nu)\|^p_{\mathcal{A}_p} + (\|L(\mu)\|^p_{\mathcal{B}_{2p}} + \|L(\nu)\|^p_{\mathcal{B}_{2p}})\|L(\mu) - L(\nu)\|^p_{\mathcal{B}_{2p}}\}.$$

Remark 7. Theorem 3.4 improves the statement of [37, Theorem 6.10] by replacing the assumption $p > 2$ with $p \ge 1$.

Corollary 3.5. *For $p \ge 1$, the derivative of the pre-Schwarzian derivative map L at the origin satisfies $\|d_0L(\mu)\|_{\mathcal{B}^\#_p} \le C^\#_p\|\mu\|_p$ for every $\mu \in M_p(\mathbb{H}^+)$.*

Next, we link S and L by the canonical holomorphic map $J : \mathcal{B}_\infty(\mathbb{H}) \to \mathcal{A}_\infty(\mathbb{H})$ defined by $\Phi \mapsto \Phi'' - (\Phi')^2/2$ for $\Phi \in \mathcal{B}_\infty(\mathbb{H})$.

Lemma 3.6. *For each $p \ge 1$, every $\Phi \in \widehat{\mathcal{B}}_p(\mathbb{H})$ satisfies $\|J(\Phi)\|_{\mathcal{A}_p} \le c_p\|\Phi\|_{\widehat{\mathcal{B}}_p}$, where $c_p > 0$ is a constant depending on p and $\|\Phi\|_{\widehat{\mathcal{B}}_p}$. Moreover, J is holomorphic on $\widehat{\mathcal{B}}_p(\mathbb{H})$ with respect to $\|\cdot\|_{\widehat{\mathcal{B}}_p}$.*

Proof. We have

$$
\begin{aligned}
\|J(\Phi)\|_{\mathcal{A}_p}^p &= \int_{\mathbb{H}} |(\operatorname{Im} z)^2(\Phi''(z) - \tfrac{1}{2}\Phi'(z)^2)|^p \frac{dxdy}{|\operatorname{Im} z|^2} \\
&\le 2^{p-1}\int_{\mathbb{H}} |(\operatorname{Im} z)^2\Phi''(z)|^p \frac{dxdy}{|\operatorname{Im} z|^2} + \frac{1}{2}\int_{\mathbb{H}} |(\operatorname{Im} z)\Phi'(z)|^{2p} \frac{dxdy}{|\operatorname{Im} z|^2} \\
&= 2^{p-1}\|\Phi\|_{\mathcal{B}_p^{\#}}^p + \tfrac{1}{2}\|\Phi\|_{\mathcal{B}_{2p}}^{2p}.
\end{aligned}
$$

Since $\|\Phi\|_{\mathcal{B}_{2p}} \asymp \|\Phi\|_{\widehat{\mathcal{B}}_{2p}} \lesssim \|\Phi\|_{\widehat{\mathcal{B}}_p}$ by Propositions 2.2 and 2.3, this implies that $\|J(\Phi)\|_{\mathcal{A}_p} \le c_p\|\Phi\|_{\widehat{\mathcal{B}}_p}$ for some $c_p > 0$, and in particular, J is locally bounded. It is easy to see that $J : \widehat{\mathcal{B}}_p(\mathbb{H}) \to \mathcal{A}_p(\mathbb{H})$ is Gâteaux holomorphic, and hence holomorphic by Proposition 3.2. □

We consider the holomorphic map J on the image $L(M_p(\mathbb{H}^+))$ of the pre-Schwarzian derivative map. We note that J is injective on $L(M(\mathbb{H}^+))$. Since F^μ is normalized by fixing ∞, it is determined by $\mu \in M(\mathbb{H}^+)$ up to post-composition by affine transformations of $\mathbb{C}$. Therefore, for $\mu, \nu \in M(\mathbb{H}^+)$, $S_{F^\mu} = S_{F^\nu}$ if and only if $N_{F^\mu} = N_{F^\nu}$. This shows the injectivity of J on $L(M(\mathbb{H}^+))$, and hence on $L(M_p(\mathbb{H}^+))$.

The existence of a local holomorphic right inverse of the Schwarzian derivative map S is a crucial fact for the holomorphy of J^{-1}. The following claim has appeared in [36, Theorem 4.1]. Its proof omits the argument of approximating a given Schwarzian derivative by those extending holomorphically to the boundary; however, this part can be verified by using [28, Proposition 3].

Proposition 3.7. *Let $S : M_p(\mathbb{H}^+) \to \mathcal{A}_p(\mathbb{H}^-)$ be the Schwarzian derivative map for $p \ge 1$. For each Ψ_0 in $S(M_p(\mathbb{H}^+))$, there exists a neighborhood V_{Ψ_0} of Ψ_0 in $\mathcal{A}_p(\mathbb{H}^-)$ and a holomorphic map $\sigma : V_{\Psi_0} \to M_p(\mathbb{H}^+)$ such that $S \circ \sigma$ is the identity on V_{Ψ_0}.*

In addition, because the quasiconformal homeomorphism of $\mathbb{H}^+$ corresponding to $\Psi \in V_{\Psi_0}$ can be explicitly represented by using a real-analytic quasiconformal reflection and by solving the Schwarzian differential equation, it is a real-analytic diffeomorphism.

Proposition 3.8. *For the local holomorphic right inverse $\sigma : V_{\Psi_0} \to M_p(\mathbb{H}^+)$ of S given in Proposition 3.7, let $\mu = \sigma(\Psi)$ for any $\Psi \in V_{\Psi_0}$. Then, the quasiconformal homeomorphism $\widetilde{F}^\mu$ of $\mathbb{H}^+$ with $\widetilde{F}^\mu(\infty) = \infty$ whose complex dilatation is μ is a real-analytic diffeomorphism.*

Proof. For $\Psi_0 \in \mathcal{A}_p(\mathbb{H}^-)$, it is proved in [36, Lemma 4.3] that there exists $\nu \in M_p(\mathbb{H}^+)$ such that $S(\nu) = \Psi_0$ and $\widetilde{F}^\nu : \mathbb{H}^+ \to \Omega^+$ is a real-analytic bi-Lipschitz diffeomorphism with respect to the hyperbolic metrics on $\mathbb{H}^+$ and its image domain $\Omega^+ \subset \mathbb{C}$. Its conformal extension is $F^\nu : \mathbb{H}^- \to \Omega^- = \mathbb{C} \setminus \overline{\Omega^+}$. Then, the quasiconformal reflection $r : \Omega^+ \to \Omega^-$ with respect to $\partial\Omega^+ = \partial\Omega^-$ is defined by

$$
r(\zeta) = F^\nu\left(\overline{(\widetilde{F}^\nu)^{-1}(\zeta)}\right) \quad (\zeta \in \Omega^+),
$$

which is a real-analytic bi-Lipschitz diffeomorphism.

For any $\Psi \in V_{\Psi_0}$, we consider the push-forward $F^\nu_*(\Psi)$ by the conformal homeomorphism $F^\nu : \mathbb{H}^- \to \Omega^-$ and solve the differential equation $2w''(z) + F^\nu_*(\Psi)(z)w(z) = 0$ on Ω^-. Let w_1 and w_2 be linearly independent solutions so normalized that $w_1 w_2' - w_2 w_1' = 1$. Then, $S(w_1/w_2) = F^\nu_*(\Psi)$ on Ω^-, and the quasiconformal homeomorphism $\widetilde{F}^\mu$ of $\mathbb{H}^+$ whose complex dilatation is $\mu = \sigma(\Psi)$ is given by the composition of $\widetilde{F}^\nu : \mathbb{H}^+ \to \Omega^+$ with

$$\frac{w_1(r(\zeta)) + (\zeta - r(\zeta))w_1'(r(\zeta))}{w_2(r(\zeta)) + (\zeta - r(\zeta))w_2'(r(\zeta))},$$

which is a quasiconformal real-analytic diffeomorphism of Ω^+. We can prove this by [28, Lemma 4], together with its subsequent comment and remark. In particular, $\widetilde{F}^\mu$ is a real-analytic diffeomorphism of $\mathbb{H}^+$. □

Concerning a global right inverse of the Schwarzian derivative map S, the following result is proved in [34, Theorem 1.4] in the case $p > 1$.

Proposition 3.9. *For $p > 1$, there exists a real-analytic map $\Sigma : S(M_p(\mathbb{H}^+)) \to M_p(\mathbb{H}^+)$ such that $S \circ \Sigma$ is the identity on $S(M_p(\mathbb{H}^+))$. Moreover, every $\mu \in M_p(\mathbb{H}^+)$ in the image of Σ induces a quasiconformal real-analytic diffeomorphism $\widetilde{F}^\mu$ of $\mathbb{H}^+$.*

We are ready to prove the desired claim.

Theorem 3.10. *For $p \geq 1$, the holomorphic map $J : \widehat{\mathcal{B}}_p(\mathbb{H}^-) \to \mathcal{A}_p(\mathbb{H}^-)$ with $J \circ L = S$ is a biholomorphic homeomorphism between $L(M_p(\mathbb{H}^+))$ and $S(M_p(\mathbb{H}^+))$.*

Proof. Since $J \circ L = S$, the restriction $J|_{L(M_p(\mathbb{H}^+))}$ of the holomorphic map $J : \mathcal{A}_p(\mathbb{H}^-) \to \widehat{\mathcal{B}}_p(\mathbb{H}^-)$ given in Lemma 3.6 sends $L(M_p(\mathbb{H}^+))$ into $S(M_p(\mathbb{H}^+))$ injectively. Conversely, Proposition 3.7 shows that, for every $\Psi_0 \in S(M_p(\mathbb{H}^+))$, there is a local holomorphic map $\sigma : V_{\Psi_0} \to M_p(\mathbb{H}^+)$ such that $S \circ \sigma$ is the identity on $V_{\Psi_0} \subset S(M_p(\mathbb{H}^+))$. Then, $J \circ L \circ \sigma$ is the identity on V_{Ψ_0}, and hence $L \circ \sigma$ is a local holomorphic right inverse of J. This implies that J is a biholomorphic homeomorphism of $L(M_p(\mathbb{H}^+))$ onto $S(M_p(\mathbb{H}^+))$. □

Corollary 3.11. *For each Φ_0 in $L(M_p(\mathbb{H}^+))$ with $p \geq 1$, there exists a neighborhood U_{Φ_0} of Φ_0 in $\widehat{\mathcal{B}}_p(\mathbb{H}^-)$ and a holomorphic map $\tau : U_{\Phi_0} \to M_p(\mathbb{H}^+)$ such that $L \circ \tau$ is the identity on U_{Φ_0}.*

Proof. Let $\Psi_0 = J(\Phi_0)$. We choose V_{Ψ_0} and $\sigma : V_{\Psi_0} \to M_p(\mathbb{H}^+)$ as in Proposition 3.7. Then, $U_{\Phi_0} = J^{-1}(V_{\Psi_0})$ and $\tau = \sigma \circ J$ possess the required properties. □

By Proposition 3.9, we also have that $\Sigma \circ J$ is a global real-analytic right inverse of the pre-Schwarzian derivative map $L : M_p(\mathbb{H}^+) \to \widehat{\mathcal{B}}_p(\mathbb{H}^-)$ for $p > 1$.

As a by-product of the above arguments, we can also obtain a characterization of p-integrable Beltrami coefficients in terms of the pre-Schwarzian and Schwarzian derivative maps. This has been given in the case $p = 2$; see [27, Theorem 4.4]. We remark that the reasoning of (3) ⇒ (1) in [37, Theorem 7.1] should be read as given below.

Theorem 3.12. *Let $F : \mathbb{H}^- \to \mathbb{C}$ be a conformal map with $F(\infty) = \infty$ that extends to a quasiconformal homeomorphism of $\mathbb{C}$. Then, the following conditions are equivalent for every $p \geq 1$:*

1. *F extends quasiconformally to $\mathbb{H}^+$ so that its complex dilatation is in $M_p(\mathbb{H}^+)$;*
2. *$\log F'$ belongs to $\widehat{\mathcal{B}}_p(\mathbb{H}^-)$;*
3. *S_F belongs to $\mathcal{A}_p(\mathbb{H}^-)$.*

Proof. The implication (1) ⇒ (2) is obtained by Theorem 3.4, and (2) ⇒ (3) by Lemma 3.6. We may consider (3) ⇒ (1) on the unit disk because Schwarzian derivatives are invariant under Möbius transformations. We have to show that $S_F \in \mathcal{A}_p(\mathbb{D})$ implies that F has the desired quasiconformal extension. However, the same proof as in [5, Theorem 2], relying on the local quasiconformal extension by Becker and Pommerenke [3, Satz 4], applies for $p \geq 1$. □

4. Fiber spaces in the unit disk model

Let $S : M_p(\mathbb{D}^*) \to \mathcal{A}_p(\mathbb{D})$ be the Schwarzian derivative map and $L : M_p(\mathbb{D}^*) \to \widehat{\mathcal{B}}_p(\mathbb{D})$ the pre-Schwarzian derivative map for $p \geq 1$, defined in a similar way for $\mathbb{D}$ and $\mathbb{D}^* = \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$. Almost all statements in the previous section are also valid for these maps. The exception occurs for the holomorphic map $J : \widehat{\mathcal{B}}_p(\mathbb{D}) \to \mathcal{A}_p(\mathbb{D})$ with $J \circ L = S$. In fact, J maps $L(M_p(\mathbb{D}^*))$ onto $S(M_p(\mathbb{D}^*))$ surjectively but not injectively. While the statements up to Proposition 3.9 in the previous section can be translated directly to this case, Theorem 3.10 requires a modification regarding the injectivity of $J : L(M_p(\mathbb{D}^*)) \to S(M_p(\mathbb{D}^*))$. We will examine the structure of this map more closely.

First, we give the precise definition of the pre-Schwarzian derivative map $L : M(\mathbb{D}^*) \to \mathcal{B}_\infty(\mathbb{D})$ in the present setting. We impose the following normalization on F^μ. For $\mu \in M(\mathbb{D}^*)$, let F^μ be the conformal homeomorphism of $\mathbb{D}$ onto a bounded domain in $\mathbb{C}$ with $F^\mu(0) = 0$ and $(F^\mu)'(0) = 1$ that extends to a quasiconformal self-homeomorphism of $\mathbb{C}$ with complex dilatation μ on $\mathbb{D}^*$. We assume $F^\mu(\infty) = \infty$. This normalization uniquely determines F^μ by μ, and we use the same notation for its quasiconformal extension. Later, its restriction to $\mathbb{D}^*$ is denoted by $\widetilde{F}^\mu$ to distinguish it from the conformal mapping on $\mathbb{D}$. Then the pre-Schwarzian derivative map L is defined by $L(\mu) = \log(F^\mu)'$, which belongs to $\mathcal{B}_\infty(\mathbb{D})$. If $\mu \in M_p(\mathbb{D}^*)$, then $L(\mu) \in \widehat{\mathcal{B}}_p(\mathbb{D})$.

The fact that J is not injective on $L(M_p(\mathbb{D}^*))$ is seen from the following proposition, which can be verified easily (see [21, Proposition 3.1]).

Proposition 4.1. (i) *For $\mu, \nu \in M(\mathbb{D}^*)$, we have $S_{F^\mu} = S_{F^\nu}$ if and only if $F^\mu = W \circ F^\nu$ on $\mathbb{D}$ for some Möbius transformation W of $\widehat{\mathbb{C}}$ such that $W \circ F^\nu(\mathbb{D})$ is a bounded domain in $\mathbb{C}$. Moreover, $N_{F^\mu} = N_{F^\nu}$ if and only if $F^\mu = W \circ F^\nu$ on $\mathbb{D}$ for some affine transformation W of $\mathbb{C}$.* (ii) *For any $\nu \in M_p(\mathbb{D}^*)$ with $p \geq 1$ and any Möbius transformation W such that $W \circ F^\nu(\mathbb{D})$ is a bounded domain, there exists some $\mu' \in M_p(\mathbb{D}^*)$ such that $N_{W \circ F^\nu} = N_{F^{\mu'}}$.*

Furthermore, the above variations of F^ν by such Möbius transformations W with $W \circ F^\nu(\mathbb{D})$ bounded yield all $\Phi = \log(W \circ F^\nu)'$ ($\Phi' = N_{W\circ F^\nu}$) in $\widehat{\mathcal{B}}_p(\mathbb{D})$ with $J(\Phi) = S(\nu)$. This is a special case of the more general result shown in [21, Lemma 3.3].

Proposition 4.2. *The set of all holomorphic functions* $\Phi = \log(W \circ F^\mu)'$ *in* $\widehat{\mathcal{B}}_p(\mathbb{D})$*, given by Möbius transformations* W *of* $\widehat{\mathbb{C}}$ *and* $\mu \in M_p(\mathbb{D}^*)$*, coincides with* $L(M_p(\mathbb{D}^*))$ *for every* $p \geq 1$.

Let W_a be a Möbius transformation that sends $a \in \widetilde{F}^\nu(\mathbb{D}^*)$ to ∞. Here and in the sequel, $\widetilde{F}^\nu$ stands for the quasiconformal extension of F^ν to $\mathbb{D}^*$. Since $W_a \circ F^\nu$ is uniquely determined by $\nu \in M_p(\mathbb{D}^*)$ and $a \in \widetilde{F}^\nu(\mathbb{D}^*)$ up to post-composition by affine transformations of $\mathbb{C}$, we can define a map $\widetilde{L}(\nu, a) = \log(W_a \circ F^\nu)' \in L(M_p(\mathbb{D}^*))$ on the fiber space over $M_p(\mathbb{D}^*)$ given as a domain in the product manifold

$$\widetilde{M_p}(\mathbb{D}^*) = \{(\nu, a) \in M_p(\mathbb{D}^*) \times \widehat{\mathbb{C}} \mid a \in \widetilde{F}^\nu(\mathbb{D}^*)\}.$$

We note that $\widetilde{L}(\nu, \infty) = L(\nu)$.

The arguments and results in the rest of this section are applied also to different kinds of Teichmüller spaces (see [20, 21]).

Lemma 4.3. $\widetilde{L} : \widetilde{M_p}(\mathbb{D}^*) \to L(M_p(\mathbb{D}^*))$ *is holomorphic.*

Proof. Let $\Phi_0 = \widetilde{L}(\nu, \infty) = \log(F^\nu)'$ and $\Phi = \widetilde{L}(\nu, a) = \log(W_a \circ F^\nu)'$. Then a simple computation yields

$$\begin{aligned}\Phi'(z) &= N_{W_a \circ F^\nu}(z) = N_{W_a} \circ F^\nu(z) \cdot (F^\nu)'(z) + N_{F^\nu}(z) = \frac{-2(F^\nu)'(z)}{F^\nu(z) - a} + \Phi_0'(z);\\ \Phi''(z) &= \frac{2(F^\nu)'(z)^2}{(F^\nu(z) - a)^2} - \frac{2(F^\nu)''(z)}{F^\nu(z) - a} + \Phi_0''(z).\end{aligned}$$

When $a = \infty$, these read as $\Phi'(z) = \Phi_0'(z)$ and $\Phi''(z) = \Phi_0''(z)$. We may assume $a \neq \infty$. Since $a \in \widetilde{F}^\nu(\mathbb{D}^*)$, the denominator $F^\nu(z) - a$ with $z \in \mathbb{D}$ is bounded below by the distance $d(a, \partial F^\nu(\mathbb{D}))$, which is bounded away from 0 uniformly in z and locally uniformly in a. Hence, it suffices to estimate the norms of $((F^\nu)')^2$ and $(F^\nu)''$ for $\|\Phi - \Phi_0\|_{\widehat{\mathcal{B}}_p} \asymp \|\Phi - \Phi_0\|_{\mathcal{B}_p^\#} + \|\Phi - \Phi_0\|_{\mathcal{B}_\infty}$ (see Remark 4).

First, we consider the $\mathcal{B}_p^\#$-norm:

$$\|\Phi - \Phi_0\|_{\mathcal{B}_p^\#}^p \lesssim \int_{\mathbb{D}} |(1 - |z|^2)^2 (F^\nu)'(z)^2|^p \frac{dxdy}{(1 - |z|^2)^2} + \int_{\mathbb{D}} |(1 - |z|^2)^2 (F^\nu)''(z)|^p \frac{dxdy}{(1 - |z|^2)^2}.$$

The first term is estimated by

$$\int_{\mathbb{D}} |(1 - |z|^2)^2 (F^\nu)'(z)^2|^p \frac{dxdy}{(1 - |z|^2)^2} = \int_{F^\nu(\mathbb{D})} \delta(\zeta)^{2p-2} d\xi d\eta \lesssim (\mathrm{diam}(F^\nu(\mathbb{D})))^{2p}, \tag{19}$$

where δ is the inverse of half the hyperbolic density in $F^\nu(\mathbb{D})$, that is, $\delta(F^\nu(z)) = (1-|z|^2)|(F^\nu)'(z)|$. We note that $\delta(\zeta)$ is comparable to the distance $d(\zeta, \partial F^\nu(\mathbb{D}))$ from ζ to the boundary $\partial F^\nu(\mathbb{D})$. For the second term, we apply the Cauchy–Schwarz inequality:

$$\begin{aligned}
&\int_{\mathbb{D}} |(1-|z|^2)^2 (F^\nu)''(z)|^p \frac{dxdy}{(1-|z|^2)^2} \\
&= \int_{\mathbb{D}} \left|(1-|z|^2)\frac{(F^\nu)''(z)}{(F^\nu)'(z)}\right|^p \cdot |(1-|z|^2)(F^\nu)'(z)|^p \frac{dxdy}{(1-|z|^2)^2} \\
&\le \left(\int_{\mathbb{D}} |(1-|z|^2)N_{F^\nu}(z)|^{2p} \frac{dxdy}{(1-|z|^2)^2}\right)^{1/2} \left(\int_{\mathbb{D}} |(1-|z|^2)(F^\nu)'(z)|^{2p} \frac{dxdy}{(1-|z|^2)^2}\right)^{1/2} \\
&\lesssim \Big(\|\Phi_0\|_{\mathcal{B}_{2p}} \operatorname{diam}(F^\nu(\mathbb{D}))\Big)^p.
\end{aligned}$$

Here, we have applied (4) in the last line.

Next, we consider the $\mathcal{B}_\infty$-norm dominated by $\mathcal{B}_{2p}$-norm by Proposition 2.1 (i):

$$\begin{aligned}
\|\Phi - \Phi_0\|_{\mathcal{B}_\infty} &\lesssim \|\Phi - \Phi_0\|_{\mathcal{B}_{2p}} \\
&\lesssim \left(\int_{\mathbb{D}} |(1-|z|^2)(F^\nu)'(z)|^{2p} \frac{dxdy}{(1-|z|^2)^2}\right)^{1/(2p)} \lesssim \operatorname{diam}(F^\nu(\mathbb{D})),
\end{aligned}$$

where (4) is used again.

By the above computations, we see that $\|\widetilde{L}(\nu, a)\|_{\widehat{\mathcal{B}}_p}$ is bounded by a constant determined in terms of $d(a, \partial F^\nu(\mathbb{D}))$, $\|L(\nu)\|_{\widehat{\mathcal{B}}_p}$, $\|L(\nu)\|_{\mathcal{B}_{2p}}$, and $\operatorname{diam}(F^\nu(\mathbb{D}))$. For a given $(\nu_0, a_0) \in \widetilde{M}_p(\mathbb{D}^*)$, all these quantities vary within a bounded range when $\nu \in M_p(\mathbb{D}^*)$ and $a \in \widetilde{F}^{\nu_0}(\mathbb{D}^*)$ move slightly from (ν_0, a_0). This shows that $\widetilde{L}$ is locally bounded.

Under this local boundedness condition, if $\widetilde{L}$ is Gâteaux holomorphic, then it is holomorphic by Proposition 3.2. As shown in [35, Lemma 6.1], the Gâteaux holomorphy of $\widetilde{L}$ follows from the condition that for each fixed $z \in \mathbb{D}$, $\widetilde{L}(\nu, a)(z) = \log(W_a \circ F^\nu)'(z)$ is Gâteaux holomorphic as a complex-valued function. By the holomorphic dependence of quasiconformal mappings on the Beltrami coefficients (see [1, V. Theorem 5]), this can be verified. Thus, $\widetilde{L}$ is holomorphic on $\widetilde{M}_p(\mathbb{D}^*)$. □

Now we state the replacement of Theorem 3.10 as follows.

Theorem 4.4. *$J : L(M_p(\mathbb{D}^*)) \to S(M_p(\mathbb{D}^*))$ is a holomorphic split submersion for $p \ge 1$.*

Proof. For any $\Phi \in L(M_p(\mathbb{D}^*))$, let $\Psi_0 = J(\Phi) \in S(M_p(\mathbb{D}^*))$. Then there exists a neighborhood V_{Ψ_0} of Ψ_0 in $S(M_p(\mathbb{D}^*))$ and a holomorphic map $\sigma : V_{\Psi_0} \to M_p(\mathbb{D}^*)$ such that $S \circ \sigma$ is the identity on V_{Ψ_0}, as in the case of $\mathbb{H}$ in Proposition 3.7. Let $\Phi_0 = L \circ \sigma(\Psi_0)$, which may be different from Φ. Since Φ_0 can be represented as $\log(F^{\sigma(\Psi_0)})'$, we have $\Phi = \log(W_a \circ F^{\sigma(\Psi_0)})'$ for some $a \in \widetilde{F}^{\sigma(\Psi_0)}(\mathbb{D}^*)$ by Proposition 4.1. Namely, $\Phi = \widetilde{L}(\sigma(\Psi_0), a)$.

Fix this a and define a map $\widetilde{L}(\sigma(\cdot), a) : V_{\Psi_0} \to L(M_p(\mathbb{D}^*))$ after shrinking V_{Ψ_0} if necessary. By Lemma 4.3, this is a holomorphic map on V_{Ψ_0}. Since $J \circ \widetilde{L}(\sigma(\Psi), a) = \Psi$

for every $\Psi \in V_{\Psi_0}$, the map $\widetilde{L}(\sigma(\cdot), a)$ is a local holomorphic right inverse of J such that $\widetilde{L}(\sigma(V_{\Psi_0}), a)$ passes through the given point $\Phi = \widetilde{L}(\sigma(\Psi_0), a)$. This is equivalent to saying that J is a holomorphic split submersion. □

The *Bers fiber space* $\widetilde{T}_p$ over $S(M_p(\mathbb{D}^*))$ is defined as

$$\widetilde{T}_p = \{(\Psi, a) \in S(M_p(\mathbb{D}^*)) \times \widehat{\mathbb{C}} \mid \Psi = S(\nu),\ a \in \widetilde{F^\nu}(\mathbb{D}^*),\ \nu \in M_p(\mathbb{D}^*)\}.$$

Theorem 5.1 in the next section identifies $S(M_p(\mathbb{D}^*))$ with the Teichmüller space T_p. We note that the quasidisk $\widetilde{F^\nu}(\mathbb{D}^*)$ is determined by Ψ independently of the choice of $\nu \in M_p(\mathbb{D}^*)$ with $S(\nu) = \Psi$. We define a map $\lambda : \widetilde{T}_p \to L(M_p(\mathbb{D}^*))$ by $\lambda(\Psi, a) = \widetilde{L}(\nu, a)$ for $S(\nu) = \Psi$. This is well defined independently of the choice of ν.

Note that the condition $a \in \widetilde{F^\nu}(\mathbb{D}^*)$ is equivalent to requiring that $W_a \circ F^\nu$ maps $\mathbb{D}$ onto a bounded domain in $\mathbb{C}$, and that $a = \infty$ if and only if W_a is an affine transformation of $\mathbb{C}$. Hence, by Proposition 4.1, λ is bijective. In fact, λ is bijective on each fiber. That is, for each $\Psi \in S(M_p(\mathbb{D}^*))$ with $S(\nu) = \Psi$, $\lambda(\Psi, \cdot)$ maps $\widetilde{F^\nu}(\mathbb{D}^*)$ bijectively onto $J^{-1}(\Psi) \subset L(M_p(\mathbb{D}^*))$. Here, $J^{-1}(\Psi)$ is a 1-dimensional complex submanifold of $L(M_p(\mathbb{D}^*))$ since J is a holomorphic split submersion by Theorem 4.4.

Lemma 4.5. $\lambda : \widetilde{T}_p \to L(M_p(\mathbb{D}^*))$ *is a biholomorphic homeomorphism.*

Proof. Choose any $\Psi_0 \in S(M_p(\mathbb{D}^*))$, and take V_{Ψ_0} and σ as in the proof of Theorem 4.4. The restriction of λ to the domain

$$\widetilde{V}_{\Psi_0} = \{(\Psi, a) \in V_{\Psi_0} \times \widehat{\mathbb{C}} \mid a \in \widetilde{F^{\sigma(\Psi)}}(\mathbb{D}^*)\} \subset \widetilde{T}_p$$

is explicitly represented as $\lambda_\sigma(\Psi, a) = \widetilde{L}(\sigma(\Psi), a)$. Then λ_σ is holomorphic on $\widetilde{V}_{\Psi_0}$ by Lemma 4.3, and thus λ is a holomorphic bijection.

Moreover, for each fixed $\Psi \in V_{\Psi_0}$, the domain $\widetilde{F^{\sigma(\Psi)}}(\mathbb{D}^*)$ of complex dimension 1 is mapped by $\lambda_\sigma(\Psi, \cdot)$ holomorphically and bijectively onto the complex submanifold $J^{-1}(\Psi) \subset L(M_p(\mathbb{D}^*))$. Hence, $\lambda_\sigma(\Psi, \cdot)$ is a biholomorphic homeomorphism. It follows from this fiberwise property that λ^{-1} is holomorphic, and thus λ is biholomorphic. □

The structure of the space $L(M_p(\mathbb{D}^*))$ over $S(M_p(\mathbb{D}^*))$ can be described precisely as follows.

Theorem 4.6. $L(M_p(\mathbb{D}^*))$ *is a real-analytic disk bundle over* $S(M_p(\mathbb{D}^*))$ *with projection* J.

Proof. We have seen that $\lambda_\sigma(\Psi, a) = \widetilde{L}(\sigma(\Psi), a) = \log(W_a \circ F^{\sigma(\Psi)})'$ is a biholomorphic homeomorphism of $\widetilde{V}_{\Psi_0} \subset \widetilde{T}_p$. Using this, we provide the structure of a disk bundle over $S(M_p(\mathbb{D}^*))$ for $L(M_p(\mathbb{D}^*))$. For every $\Psi_0 \in S(M_p(\mathbb{D}^*))$, define

$$\ell_\sigma : V_{\Psi_0} \times \mathbb{D}^* \to J^{-1}(V_{\Psi_0}) \subset L(M_p(\mathbb{D}^*))$$

by $\ell_\sigma(\Psi, \zeta) = \lambda_\sigma(\Psi, \widetilde{F^{\sigma(\Psi)}}(\zeta))$. By Proposition 4.1, ℓ_σ is a bijection satisfying $J \circ \ell_\sigma(\Psi, \zeta) = \Psi$. Moreover, ℓ_σ is a real-analytic diffeomorphism since λ_σ is biholomorphic and $\widetilde{F^{\sigma(\Psi)}}$ is real-analytic by Proposition 3.8. Hence, ℓ_σ gives a local trivialization for the projection $J : L(M_p(\mathbb{D}^*)) \to S(M_p(\mathbb{D}^*))$. This implies that $L(M_p(\mathbb{D}^*))$ possesses the structure of a fiber bundle described in the statement. □

A global section of the bundle projection J can be obtained by using the global real-analytic right inverse Σ of the Schwarzian derivative map $S : M_p(\mathbb{D}^*) \to S(M_p(\mathbb{D}^*))$ for $p > 1$, which is given in Proposition 3.9 for the case of $\mathbb{H}$. Replacing the local right inverse σ in the proofs of Theorems 4.4 and 4.6 with this Σ, we define a bi-real-analytic map

$$\ell_\Sigma : S(M_p(\mathbb{D}^*)) \times \mathbb{D}^* \to L(M_p(\mathbb{D}^*))$$

by $\ell_\Sigma(\Psi, \zeta) = \widetilde{L}(\Sigma(\Psi), \widetilde{F}^{\Sigma(\Psi)}(\zeta))$. Then, in the real-analytic category, the total space $L(M_p(\mathbb{D}^*))$ has the product structure, and the bundle becomes trivial.

Corollary 4.7. *Let $p > 1$. Each $\zeta \in \mathbb{D}^*$ defines a global real-analytic section*

$$\ell_\Sigma(\cdot, \zeta) : S(M_p(\mathbb{D}^*)) \to L(M_p(\mathbb{D}^*))$$

for the holomorphic bundle projection $J : L(M_p(\mathbb{D}^*)) \to S(M_p(\mathbb{D}^*))$. Moreover, the total space $L(M_p(\mathbb{D}^*))$ is real-analytically equivalent to $S(M_p(\mathbb{D}^*)) \times \mathbb{D}^*$ under ℓ_Σ, with $J \circ \ell_\Sigma(\Psi, \zeta) = \Psi$.

Finally, we mention the characterization of $M_p(\mathbb{D}^*)$ in terms of $\widehat{\mathcal{B}}_p(\mathbb{D})$ and $\mathcal{A}_p(\mathbb{D})$. The difference in J from the case of $\mathbb{H}$ does not affect other statements for the disk model substantially, and the result parallel to Theorem 3.12 can be stated as follows.

Corollary 4.8 (to Theorem 3.12)**.** *Let $F : \mathbb{D} \to \mathbb{C}$ be a conformal map onto a bounded domain that extends to a quasiconformal homeomorphism of the extended complex plane $\widehat{\mathbb{C}}$. Then, the following conditions are equivalent for every $p \geq 1$:*

(1) *F extends quasiconformally to $\mathbb{D}^*$ so that its complex dilatation is in $M_p(\mathbb{D}^*)$;*
(2) *$\log F'$ belongs to $\widehat{\mathcal{B}}_p(\mathbb{D})$;*
(3) *S_F belongs to $\mathcal{A}_p(\mathbb{D})$.*

We note that the equivalence of (1) and (3) follows from that in Theorem 3.12 by the Möbius invariance of the Schwarzian derivative. However, despite the isomorphic relation between $\widehat{\mathcal{B}}_p(\mathbb{H})$ and $\widehat{\mathcal{B}}_p(\mathbb{D})$ as in Theorem 2.5, the equivalence involving (2) does not follow directly from Theorem 3.12. By preparing the disk versions of Theorem 3.4 and Lemma 3.6, we must repeat the same arguments as in Theorem 3.12 to obtain Corollary 4.8.

5. Structures of integrable Teichmüller spaces

The *universal Teichmüller space* T is the set of all normalized quasisymmetric homeomorphisms $h : \mathbb{R} \to \mathbb{R}$ that extend to quasiconformal homeomorphisms $H(\mu) : \mathbb{H} \to \mathbb{H}$ with complex dilatations $\mu \in M(\mathbb{H})$. Via the correspondence from Beltrami coefficients μ to quasisymmetric homeomorphisms h through $H(\mu)$, we obtain a map $\pi : M(\mathbb{H}) \to T$, called the Teichmüller projection. When $\pi(\mu) = \pi(\nu)$, we say that μ and ν are *Teichmüller equivalent*. An element $h(\mu) = H(\mu)|_{\mathbb{R}}$ of T can be represented by the Teichmüller equivalence class $[\mu]$ for $\mu \in M(\mathbb{H})$. We refer to [14, Chapter 3] for the basis of the universal Teichmüller space.

Let F^μ denote the normalized conformal homeomorphism of $\mathbb{H}^-$ that extends quasiconformally to $\mathbb{C}$ with complex dilatation μ on $\mathbb{H}^+$, for $\mu \in M(\mathbb{H}^+)$. The Schwarzian derivative map $S : M(\mathbb{H}^+) \to \mathcal{A}_\infty(\mathbb{H}^-)$, where $\mathcal{A}_\infty(\mathbb{H}^-)$ is the Banach space of all holomorphic functions Ψ on $\mathbb{H}^-$ with norm

$$\|\Psi\|_{\mathcal{A}_\infty} = \sup_{z\in\mathbb{H}^-} |\mathrm{Im}\, z|^2 |\Psi(z)| < \infty,$$

is defined by the correspondence $\mu \mapsto S_{F^\mu}$, and the pre-Schwarzian derivative map $L : M(\mathbb{H}^+) \to \mathcal{B}_\infty(\mathbb{H}^-)$ by the correspondence $\mu \mapsto \log(F^\mu)'$. It is well known that both S and L are holomorphic split submersions onto their images with respect to the norm $\|\cdot\|_\infty$ of $M(\mathbb{H}^+)$.

For μ and ν in $M(\mathbb{H}^+)$, we have $\pi(\mu) = \pi(\nu)$ if and only if $F^\mu|_{\mathbb{H}^-} = F^\nu|_{\mathbb{H}^-}$. This induces well-defined injections $\alpha : T \to \mathcal{A}_\infty(\mathbb{H}^-)$ with $\alpha \circ \pi = S$, and $\beta : T \to \mathcal{B}_\infty(\mathbb{H}^-)$ with $\beta \circ \pi = L$. We call α the *Bers embedding* and β the *pre-Bers embedding*. Then, it follows from the property of split submersion that both α and β are homeomorphisms onto their images.

For $p \geq 1$, the *p-integrable Teichmüller space* T_p is defined by $T_p = \pi(M_p(\mathbb{H}))$. The topology on T_p is the quotient topology induced by π from that on $M_p(\mathbb{H})$ with norm $\|\cdot\|_p + \|\cdot\|_\infty$. Under this stronger topology, the holomorphy of $S : M_p(\mathbb{H}) \to \mathcal{A}_p(\mathbb{H}^-)$ and $L : M_p(\mathbb{H}) \to \widehat{\mathcal{B}}_p(\mathbb{H}^-)$ claimed by Proposition 3.1 and Theorem 3.4 is still valid because $\mathcal{A}_p(\mathbb{H}^-) \subset \mathcal{A}_\infty(\mathbb{H}^-)$ and $\widehat{\mathcal{B}}_p(\mathbb{H}^-) \subset \mathcal{B}_\infty(\mathbb{H}^-)$ and the inclusions are continuous. Since $T_p \subset T$, α and β are also defined on T_p by the restriction of these maps. The complex Banach structure on T_p is induced by these embeddings α and β. Indeed, Proposition 3.7, Theorem 3.10, Corollary 3.11, and Theorem 3.12 imply the following.

Theorem 5.1. *Let $p \geq 1$. The Bers embedding α is a homeomorphism onto the open set $\alpha(T_p) = S(M_p(\mathbb{H}^+))$ in $\mathcal{A}_p(\mathbb{H}^-)$. The pre-Bers embedding β is a homeomorphism onto the open set $\beta(T_p) = L(M_p(\mathbb{H}^+))$ in $\widehat{\mathcal{B}}_p(\mathbb{H}^-)$. These sets are given by*

$$\alpha(T_p) = \alpha(T) \cap \mathcal{A}_p(\mathbb{H}^-), \quad \beta(T_p) = \beta(T) \cap \widehat{\mathcal{B}}_p(\mathbb{H}^-).$$

The topological embeddings α and β endow T_p with complex Banach structures that are biholomorphically equivalent.

Remark 8. Using $M_p(\mathbb{D}^*)$ and $\mathcal{A}_p(\mathbb{D})$, the Bers embedding α is defined in the same way and has the same properties as above. However, the pre-Bers embedding β cannot be defined in this setting, because the analogue of Theorem 3.10 fails with respect to the injectivity of J.

Next, we consider the metric structure of T_p. In the universal Teichmüller space T, the Teichmüller distance is defined using the L_∞-norm of Beltrami coefficients: the distance from the origin to $[\mu] \in T$ is the infimum of $\log\big((1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)\big)$ taken over all Beltrami coefficients μ in the Teichmüller equivalence class $[\mu]$, and this is extended to every point of T by right translations. We can provide a similar distance for T_p; in particular, its underlying topological structure is defined in [36] as follows.

Definition 4. A sequence $[\mu_n]$ in T_p for $p \geq 1$ converges to $[\nu] \in T_p$ if

$$\inf \{\|\mu_n * \nu^{-1}\|_p \mid \mu_n \in [\mu_n],\ \nu \in [\nu]\} \to 0 \quad (n \to \infty),$$

where $\mu * \nu^{-1}$ denotes the complex dilatation of the quasiconformal self-homeomorphism $H(\mu) \circ H(\nu)^{-1}$ of $\mathbb{H}$. We call this the *Teichmüller topology.*

We first show the following.

Proposition 5.2. *For $p \geq 1$, the Teichmüller topology $\mathcal{O}_p$ on T_p coincides with the quotient topology $\mathcal{Q}_{p,\infty}$ induced from $M_p(\mathbb{H})$ with norm $\|\cdot\|_p + \|\cdot\|_\infty$.*

Proof. To see that the quotient topology $\mathcal{Q}_{p,\infty}$ is stronger than $\mathcal{O}_p$, we show that $\pi : M_p(\mathbb{H}^+) \to (T_p, \mathcal{O}_p)$ is continuous. For each $[\nu] \in T_p$, there is a representative $\nu \in M_p(\mathbb{H}^+)$ such that F^ν is a bi-Lipschitz self-diffeomorphism of $\mathbb{H}^+$ by [36, Lemma 3.4], and for such ν the convergences $\|\mu_n - \nu\|_p \to 0$ and $\|\mu_n * \nu^{-1}\|_p \to 0$ as $n \to \infty$ are equivalent by [36, Lemma 3.1]. Hence, the projection π is continuous.

To see that $\mathcal{O}_p$ is stronger than $\mathcal{Q}_{p,\infty}$, we show that the identity map $\iota : (T_p, \mathcal{O}_p) \to (T_p, \mathcal{Q}_{p,\infty})$ is continuous. The fact that $\alpha : (T_p, \mathcal{O}_p) \to \mathcal{A}_p(\mathbb{H}^-)$ is continuous can be verified by an analogues argument of [29, I, Lemma 2.9] with the bi-Lipschitz representative as above. Since $\alpha(T_p) \subset \mathcal{A}_p(\mathbb{H}^-)$ is homeomorphic to $(T_p, \mathcal{Q}_{p,\infty})$ by Theorem 5.1, the identity map ι is continuous. □

Remark 9. In [36], a different Teichmüller topology $\mathcal{O}_{p,\infty}$ is used, defined by replacing $\|\mu_n * \nu^{-1}\|_p$ with $\|\mu_n * \nu^{-1}\|_p + \|\mu_n * \nu^{-1}\|_\infty$ in the above definition. Obviously, $\mathcal{O}_{p,\infty}$ is stronger than $\mathcal{O}_p$. However, since the continuity of $\pi : M_p(\mathbb{H}^+) \to (T_p, \mathcal{O}_{p,\infty})$ can be proved in the same way, the two topologies coincide.

We now mention the topological group structure of T_p. The Teichmüller space T_p (as well as T) carries a group structure under the composition of quasisymmetric homeomorphisms. For $h(\mu) = \pi(\mu)$ and $h(\nu) = \pi(\nu)$ in T_p, the Teichmüller equivalence class of the composition $h(\mu) \circ h(\nu)$ is denoted by $[\mu] * [\nu]$, and the inverse $h(\mu)^{-1}$ by $[\mu]^{-1}$. For every $[\nu] \in T_p$, the right translation $r_{[\nu]} : T_p \to T_p$ is defined by $[\mu] \mapsto [\mu] * [\nu]$.

The following topological-group property is proved in [29, Theorem I.3.8] and [37, Theorem 6.1]. The biholomorphic property is shown in [36, Section 4].

Proposition 5.3. *For $p \geq 1$, T_p is a topological group. Moreover, every right translation $r_{[\nu]}$ is a biholomorphic automorphism of T_p.*

The Weil–Petersson metric on T_2 is studied in [5] and [29]. This metric was generalized to T_p for $p \geq 2$ in [17]. In fact, the same definition also works for $p \geq 1$. The *p-Weil–Petersson metric* on the tangent bundle of T_p is easily defined by embedding T_p into $\mathcal{A}_p(\mathbb{H})$ via the Bers embedding α and assuming that the tangent space $\mathscr{T}_{[\nu]}(T_p)$ of $T_p \cong \alpha(T_p)$ at any point $[\nu] \in T_p$ is $\mathcal{A}_p(\mathbb{H})$. Then, at the origin of $\alpha(T_p)$, the norm of a tangent vector v

in $\mathscr{T}_{[0]}(T_p) \cong \mathcal{A}_p(\mathbb{H})$ is defined to be $\|v\|_{\mathcal{A}_p}$ (or the norm of the adjoint operator v^* acting on $\mathcal{A}_q(\mathbb{H})$ for $1/p + 1/q = 1$); see [16, Section 6.5]. For an arbitrary point $[\nu] \in T_p$ with $\alpha([\nu]) = \Psi$, consider the conjugate of the right translation $r_{[\nu]}^{-1}$ by α. Then $\alpha \circ r_{[\nu]}^{-1} \circ \alpha^{-1}$ is a biholomorphic automorphism of $\alpha(T_p)$ sending Ψ to 0. The norm of a tangent vector v in $\mathscr{T}_{[\nu]}(T_p) \cong \mathcal{A}_p(\mathbb{H})$ is defined to be $\|d_\Psi(\alpha \circ r_{[\nu]}^{-1} \circ \alpha^{-1})(v)\|_{\mathcal{A}_p}$. This yields a Finsler metric on the tangent bundle of T_p in a broad sense. From the definition, the p-Weil–Petersson metric is invariant under the right translations of T_p. The distance induced by this metric is called the *p-Weil–Petersson distance*, which dominates the Teichmüller topology on T_p.

We can also introduce a different invariant Finsler metric using the pre-Bers embedding $\beta : T_p \to \widehat{\mathcal{B}}_p(\mathbb{H})$.

Definition 5. For any tangent vector $u \in \mathscr{T}_{[\nu]}(T_p) \cong \widehat{\mathcal{B}}_p(\mathbb{H})$ at $[\nu] \in T_p$ with $\beta([\nu]) = \Phi$ for $p \geq 1$, the *p-pre-Weil–Petersson metric* is the Finsler metric on the tangent bundle of T_p modeled on $\widehat{\mathcal{B}}_p(\mathbb{H})$ given by $\|d_\Phi(\beta \circ r_{[\nu]}^{-1} \circ \beta^{-1})(u)\|_{\widehat{\mathcal{B}}_p}$.

Theorem 5.4. *The integrable Teichmüller space T_p for $p \geq 1$ is complete with respect to the p-pre-Weil–Petersson distance. Moreover, the p-pre-Weil–Petersson metric is continuous on the tangent bundle of T_p and invariant under the right translations of T_p.*

Proof. The proof can be reproduced by mimicking that for the Weil–Petersson metric in [5, Theorem 5] and [17, Section 8]. The only gap for the pre-Bers embedding case is the analogue of the Ahlfors–Weill section for the Schwarzian derivative map. However, this is successfully filled by the following claim obtained in [10, Theorem 5.1] via the theory of chordal Loewner chains on the half-plane. Alternatively for the latter statement, since $J : \beta(T_p) \to \alpha(T_p)$ is biholomorphic by Theorem 3.10, the results for the Bers embedding transfer directly to the present case. □

Lemma 5.5. *If $\Phi \in \mathcal{B}_\infty(\mathbb{H}^-)$ satisfies $\|\Phi\|_{\mathcal{B}_\infty} < \frac{1}{2}$, then $\mu(z) = -2\,\mathrm{Im}(z)|\Phi'(\bar{z})|$ for $z \in \mathbb{H}^+$ belongs to $M(\mathbb{H}^+)$ and satisfies $\beta([\mu]) = \Phi$.*

6. Relationship with Teichmüller spaces of diffeomorphisms

In this section, we study the relationship between the integrable Teichmüller spaces T_p ($p \geq 1$) and the Teichmüller spaces T^γ ($0 < \gamma \leq 1$) of orientation-preserving self-diffeomorphisms of $\mathbb{R}$ and $\mathbb{S}$, scaled by the regularity of their derivatives. Since T^γ can be characterized by the decay order of the supremum norm of Beltrami coefficients μ (see [18], [19], [31], and [32]), we use this characterization of T^γ. Moreover, because the degeneration of the norm toward $\mathbb{R}$ and $\mathbb{S}$ leads to a discrepancy between the Teichmüller spaces modeled on $\mathbb{H}$ and on $\mathbb{D}$, we restrict attention here to the disk model.

For $0 < \gamma \leq 1$, the space $M^\gamma(\mathbb{D}^*)$ of γ-decay Beltrami coefficients consists of all $\mu \in M(\mathbb{D}^*)$ such that

$$\operatorname*{ess\,sup}_{z \in \mathbb{D}^*} \left(((|z|^2 - 1)^{-\gamma} \vee 1)|\mu(z)|\right) < \infty.$$

Then the Teichmüller space T^γ of circle diffeomorphisms $h : \mathbb{S} \to \mathbb{S}$ whose derivatives h' are γ-Hölder continuous turns out to be $\pi(M^\gamma(\mathbb{D}^*))$. For $0 < \gamma < 1$, this is revealed in [18, Theorems 1.1, 6.7], summarizing existing results. When $\gamma = 1$, the corresponding circle diffeomorphisms h have continuous derivatives h' satisfying the Zygmund condition:

$$|h'(e^{i(\theta+t)}) - 2h'(e^{i\theta}) + h'(e^{i(\theta-t)})| = O(t) \quad (t \to 0).$$

The correspondence with $M^1(\mathbb{D}^*)$ is shown in [31, Theorem 1.1].

For the image of $M^\gamma(\mathbb{D}^*)$ under the pre-Schwarzian derivative map L, we introduce the space $\mathcal{B}^\gamma(\mathbb{D})$ of γ-decay Bloch functions $\Phi \in \mathcal{B}_\infty(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{2-\gamma}|\Phi''(z)| < \infty.$$

When $0 < \gamma < 1$, this is equivalent to $\sup_{z\in\mathbb{D}}(1-|z|^2)^{1-\gamma}|\Phi'(z)| < \infty$. As before, L is defined by $L(\mu) = \log(F^\mu)'$ on $\mathbb{D}$, where F^μ is the normalized conformal homeomorphism of $\mathbb{D}$ onto a bounded domain that extends quasiconformally to $\mathbb{C}$ with complex dilatation μ on $\mathbb{D}^*$.

It is proved in [18, Theorem 4.6] and [31, Theorem 1.1] that $L(M^\gamma(\mathbb{D}^*)) \subset \mathcal{B}^\gamma(\mathbb{D})$. Moreover,

$$L(M^\gamma(\mathbb{D}^*)) = L(M(\mathbb{D}^*)) \cap \mathcal{B}^\gamma(\mathbb{D}), \tag{20}$$

and, in particular, there exists a neighborhood of the origin in $\mathcal{B}^\gamma(\mathbb{D})$ contained in $L(M^\gamma(\mathbb{D}^*))$. This is the unique component in $\mathcal{B}^\gamma(\mathbb{D})$ arising from pre-Schwarzian derivative maps with different normalizations of F^μ; that is, the analogue of Proposition 4.2 holds. See [31, Theorem 1.3] and [32, Theorem 1.1].

A basic relation between T^γ and T_p is as follows.

Proposition 6.1. *If $\gamma p > 1$, then $T^\gamma \subset T_p$. In particular, $T^1 \subset T_p$ for all $p > 1$.*

Proof. This follows from the inclusion $M^\gamma(\mathbb{D}^*) \subset M_p(\mathbb{D}^*)$ when $\gamma p > 1$, which is verified by a direct estimate. □

Thus, for $\gamma p > 1$ we have the inclusion diagram

$$\begin{array}{ccc} T^1 \subset T^\gamma \subset \cdots \subset & \lim\limits_{\gamma\searrow 0} T^\gamma & \text{(decay order)} \\ \cap & \cap & \\ T_1 \subset T_p \subset \cdots \subset & \lim\limits_{p\nearrow\infty} T_p & \text{(integrability).} \end{array}$$

We focus on the relation between T^1 and T_1. It is shown in [2] that every quasisymmetric homeomorphism in the 1-integrable Teichmüller space T_1 is a C^1-diffeomorphism of $\mathbb{S}$ onto itself with nonvanishing derivative. One might expect $T^1 \subset T_1$, but this is not the case.

Theorem 6.2. *There is no inclusion relation between T^1 and T_1.*

Proof. It is shown in [13, p.17] that $\mathcal{B}^1(\mathbb{D})$ and $\mathcal{B}_1^\#(\mathbb{D})$ are incomparable. More explicitly, $\Phi^1(z) = a\sum_{n=0}^\infty 2^{-n}z^{2^n}$ belongs to $\mathcal{B}^1(\mathbb{D}) \setminus \mathcal{B}_1^\#(\mathbb{D})$, while $\Phi_1(z) = a(1-z)(\log 1/(1-z))^2$ belongs to $\mathcal{B}_1^\#(\mathbb{D}) \setminus \mathcal{B}^1(\mathbb{D})$ for any constant $a \in \mathbb{C}$. In Remark 3, we observed that $\mathcal{B}_1^\#(\mathbb{D}) = \widehat{\mathcal{B}}_1(\mathbb{D})$.

For the pre-Schwarzian derivative map L defined on $M_1(\mathbb{D}^*)$, we have

$$L(M_1(\mathbb{D}^*)) = L(M(\mathbb{D}^*)) \cap \widehat{\mathcal{B}}_1(\mathbb{D}),$$

which follows from Corollary 4.8. Combining this with (6), we see that by choosing $a > 0$ sufficiently small, both Φ^1 and Φ_1 lie in $L(M(\mathbb{D}^*))$ and satisfy

$$\Phi^1 \in L(M^1(\mathbb{D}^*)) \setminus L(M_1(\mathbb{D}^*)), \qquad \Phi_1 \in L(M_1(\mathbb{D}^*)) \setminus L(M^1(\mathbb{D}^*)).$$

Applying $J : L(M(\mathbb{D}^*)) \to S(M(\mathbb{D}^*))$, which is not injective, we claim that

$$J(\Phi^1) \in S(M^1(\mathbb{D}^*)) \setminus S(M_1(\mathbb{D}^*)), \qquad J(\Phi_1) \in S(M_1(\mathbb{D}^*)) \setminus S(M^1(\mathbb{D}^*)). \tag{21}$$

These two conditions yield the theorem, because $S(M^1(\mathbb{D}^*))$ is identified with T^1 via the Bers embedding $\alpha : T^1 \to S(M^1(\mathbb{D}^*))$ by [19, Theorem 3], while $S(M_1(\mathbb{D}^*))$ is identified with T_1 by Theorem 5.1 and Remark 8.

It remains to prove (6). Set $\Phi^1 = \log(F^\mu)'$ with $\mu \in M^1(\mathbb{D}^*)$ and $J(\Phi^1) = S(\mu)$. Suppose, toward a contradiction, that $J(\Phi^1) \in S(M_1(\mathbb{D}^*))$. Then there exists $\nu \in M_1(\mathbb{D}^*)$ such that $S(\nu) = S(\mu)$. Proposition 4.1 (i) yields a Möbius transformation W of $\widehat{\mathbb{C}}$ such that $\Phi^1 = \log(W \circ F^\nu)'$ and $W \circ F^\nu(\mathbb{D})$ is bounded, and then (ii) implies $\Phi^1 \in L(M_1(\mathbb{D}^*))$. However, this contradicts $\Phi^1 \notin L(M_1(\mathbb{D}^*))$. Thus $J(\Phi^1) \notin S(M_1(\mathbb{D}^*))$, proving the first inclusion in (6). The second follows by the same argument. □

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