The Ramsey number of generalized loose paths in uniform hypergraphs

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Abstract

Let \( H = (V, E) \) be an \( r \)-uniform hypergraph. For each \( 1 \leq s \leq r - 1 \), an \( s \)-path \( P_n^{r,s} \) in \( H \) of length \( n \) is a sequence of distinct vertices \( v_1, v_2, \ldots, v_{s+n(r-s)} \) such that \( \{v_1+i(r-s), \ldots, v_{i+1}(r-s)\} \in E(H) \) for each \( 0 \leq i \leq n-1 \). Recently, the Ramsey number of \( s \)-paths and \( 1 \)-cycles in uniform hypergraphs attracted a lot of attention. The asymptotic value of \( R(P_n^{3,1}, P_n^{3,1}) \) was first determined. The exact values of \( R(P_n^{3,1}, P_n^{4,1}) \) and \( R(P_n^{4,1}, P_n^{4,1}) \) are known; for \( n \geq m \geq 1 \), \( R(P_n^{3,1}, C_n^m) \), \( R(P_n^{3,1}, C_m^1) \), and \( R(C_n^m, C_m^1) \) are also proved. In this paper, we investigate the Ramsey number of \( r/2 \)-paths for even \( r \). We prove the following exact results:

\[
R(P_n^{r/2}, P_n^{r/2}) = \frac{(n+1)r}{2} + 1 \quad \text{and} \quad R(P_n^{r/2}, P_n^{r/2}) = \frac{(n+1)r}{2} + 1.
\]

All approaches dealing with \( 1 \)-path can not be applied to the studying of \( r/2 \)-path for even \( r \). The main ingredients of the proofs are the parity of different types of edges and the analysis how does the color of one type of edges forces the color of the other type of edges.

1 Introduction

An \( r \)-uniform hypergraph \( H \) is a pair \( H = (V, E) \), where \( V \) is a set of vertices and \( E \) is a collection of \( r \)-subsets of \( V \). For two \( r \)-uniform hypergraphs \( H_1 \) and \( H_2 \), the Ramsey number \( R(H_1, H_2) \) is the minimum value of \( N \) such that each red-blue coloring of edges in the complete \( r \)-uniform hypergraph \( K_N^r \) on \( N \) vertices contains either a red \( H_1 \) or a blue \( H_2 \). Let \( H \) be an \( r \)-uniform hypergraph. For each \( 1 \leq s \leq r - 1 \), an \( s \)-path \( P_n^{r,s} \) in \( H \) with length \( n \) is a sequence of distinct vertices \( v_1, v_2, \ldots, v_{s+n(r-s)} \) such that \( \{v_1+i(r-s), \ldots, v_{i+1}(r-s)\} \) is an edge of \( H \) for each \( 0 \leq i \leq n-1 \). Similarly, an \( s \)-cycle \( C_n^r \) of length \( n \) is a sequence of vertices \( v_1, v_2, \ldots, v_{s+n(r-s)} \) such that \( \{v_1+i(r-s), \ldots, v_{i+1}(r-s)\} \) is an edge of \( H \) for each \( 0 \leq i \leq n-1 \), \( v_1, \ldots, v_{n(r-s)} \) are distinct, and \( v_{n(r-s)+j} = v_j \) for each \( 1 \leq j \leq s \). An \( s \)-path (and an \( s \)-cycle) is loose if \( 1 \leq s \leq r/2 \) and an \( s \)-path (and an \( s \)-cycle) is tight if \( r/2 < s \leq r - 1 \).

When \( r = 2 \) and \( s = 1 \), we get the definition of paths and cycles in graphs. A classical result from Ramsey theory [3] says \( R(P_m, P_m) = n + \left\lceil \frac{n+1}{2} \right\rceil \) for \( n \geq m \geq 1 \); it is also known [1][2] that \( R(P_n, C_m) = R(P_n, P_m) = n + \frac{m}{2} \) for \( n \geq m \) and \( m \) even. One may ask what is the Ramsey number of paths and cycles in uniform hypergraphs?

The following construction [6] was used to show a lower bound on \( R(P_n^{3,1}, P_n^{3,1}) \) for \( n \geq 1 \); we can adapt it to show that \( N = s + n(r-s) + \left\lceil \frac{n+1}{2} \right\rceil - 2 \) is a lower bound on \( R(P_n^{r,s}, P_n^{r,s}) \)

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for $n \geq m \geq 1$ and $1 \leq s \leq r - 1$. To see this, we partition the vertex set of $K_N^r$ into two subsets $A$ and $B$, where $|A| = s + n(r - s) - 1$ and $|B| = \lfloor \frac{m+1}{2} \rfloor - 1$; we color all edges $f$ satisfying $V(f) \subseteq A$ or $V(f) \subseteq B$ red and the remaining edges blue. Observe that the number of vertices in an $s$-path with length $n$ equals $s + n(r - s)$, so there is no red $P_{r,s}$. Obviously, there is also no blue $s$-path of length $m$ and we proved the lower bound.

We have the following interesting question which asks whether the construction above gives the true values of $R(P_{r,s}, P_{m,s})$.

**Question 1** Is $R(P_{r,s}, P_{m,s}) = s + n(r - s) + \lfloor \frac{m+1}{2} \rfloor - 1$ for $n \geq m \geq 1$ and $1 \leq s \leq r - 1$?

There are some positive answers to this question for the case $s = 1$. It was shown by Haxell et al. [6] that $R(P_{3,1}^3, P_{3,1}^3), R(C_{3,1}^3, C_{3,1}^3)$, and $R(P_{1,1}^3, C_{3,1}^3)$, equal $\frac{9n}{2}$ asymptotically. Later, Gyárfás and Raeisi [2] extended this result to all $r \geq 3$; namely they proved that $R(P_{r,1}^r, P_{r,1}^r), R(P_{r,1}^r, C_{r,1}^r)$, and $R(C_{r,1}^r, C_{r,1}^r)$ are asymptotically equal to $\left(\frac{2r-1}{2}\right)n$. There are some exact results on short paths and cycles. Gyárfás, Sárközy, and Szemerédi [2] first showed

$$R(P_{3,1}^3, P_{3,1}^3) = R(P_{3,1}^3, C_{3,1}^3) = R(C_{3,1}^3, C_{3,1}^3) + 1 = 3r - 1;$$

they also proved

$$R(P_{4,1}^4, P_{4,1}^4) = R(P_{4,1}^4, C_{4,1}^4) = R(C_{4,1}^4, C_{4,1}^4) + 1 = 4r - 2.$$

For $r = 3$ and $s = 1$, the exact value of long path is determined. Maherani et al. [8] first proved for $n \geq \lfloor \frac{5m}{2} \rfloor$, we have

$$R(P_{n,1}^3, P_{m,1}^3) = 2n + \lfloor \frac{m+1}{2} \rfloor.$$

Recently, Meherani and Shahsihya [9] showed for $n \geq m \geq 1$, we have

$$R(P_{n,1}^3, P_{m,1}^3) = R(P_{n,1}^3, C_{m,1}^3) = R(C_{n,1}^3, C_{m,1}^3) + 1 = 2n + \lfloor \frac{m+1}{2} \rfloor$$

and

$$R(P_{m,1}^3, C_{n,1}^3) = 2n + \lfloor \frac{m-1}{2} \rfloor.$$

For more details on small Ramsey numbers, the reader is referred to the dynamic survey paper [10].

As the author’s best knowledge, there is no attempt to study the Ramsey number of other types of paths in hypergraphs. In this paper, we will show some exact results for $s = r/2$ and $r$ even. Before we state the theorems, we have the following lemma.

**Lemma 1** For each $s \geq 1$ and $n \geq 2$, we have

$$R(P_{n,s}^{2s}, P_{2s}^{2s}) = (n+1)s.$$

The proof of this lemma is simple and it is omitted here. We will prove the following two main theorems.

**Theorem 1** For each $s \geq 1$ and $n \geq 3$, we have

$$R(P_{n,s}^{2s}, P_{3s}^{2s}) = (n+1)s + 1.$$

**Theorem 2** For each $s \geq 1$ and $n \geq 4$, we have

$$R(P_{n,s}^{2s}, P_{4s}^{2s}) = (n+1)s + 1.$$

Notice that theorems above provide partial positive answer to Question[1] for $s = r/2$ and $r$ even. To prove Theorem[1] and Theorem[2] we will need only to prove the upper bound.

Throughout this paper, for a red-blue coloring of a uniform hypergraph, we use $F_{red}$ (and $F_{blue}$) to denote the subhypergraph induced by all red (and blue) edges respectively. Since we
will work on a fixed type path \( P_n^{2,s} \) in section 2 and section 3, we will drop the superscripts and write \( P_n \) for \( P_n^{2,s} \). Fix an \( m \geq 2 \), let \( f_i = \{A_i, A_{i+1}\} \) for \( 1 \leq i \leq m - 1 \). We will write \( f_1, f_2, \ldots, f_{m-1} \) as an \( s \)-path of length \( m - 1 \); we also write the path as \( A_1, A_2, \ldots, A_m \) on some occasions. We will refer \( A_1 \) and \( A_m \) as ending \( s \)-sets of \( P_n \). If \( g_1 = \{A_i, C, C'\} \) and \( g_2 = \{C, C', A_{i+2}\} \) for some disjoint sets \( C \) and \( C' \), then we get a new path of length \( m - 1 \) by replacing the edges \( f_i \) and \( f_{i+1} \) by \( g_1 \) and \( g_2 \) respectively; we will write this new path as \( A_1, A_2, \ldots, A_i, g_1, g_2, A_{i+2}, \ldots, A_m \) without arising any confusion.

The paper is organized as follows. The proof of Theorem 2 will need the truth of Theorem 1, so we will prove Theorem 1 in section 2 and Theorem 2 will be proved in section 3. We will give some concluding remarks in the last section.

## 2 Proof of Theorem 1

For a fixed \( s \), Theorem 1 will be proved by induction on \( n \). The idea for proving the base step and the inductive step are similar; we give an outline for the inductive step here. Suppose Theorem 1 holds for all \( 3 \leq k \leq m - 1 \). Let \( c \) be a red-blue coloring of edges in \( K_{(m+1)s+1} \). As \((m+1)s+1 \geq R(P_{m-1}, P_3)\) by the inductive hypothesis, either there is a red \( P_{m-1} \), or there is a blue \( P_m \). We need only to consider the former case and also assume that there is no red \( P_m \). Let \( \{A_1, A_2, \ldots, A_m\} \) be mutually disjoint \( s \)-sets of \([ms]\) and \( B \) be the remaining \( s + 1 \) vertices. We fix a red path \( A_1, A_2, \ldots, A_m \) and aim to find a blue \( P_m \). For each \( 0 \leq l \leq s \), we say an edge \( f \) is of type \((l, s, s-l)\) if \(|f \cap B| = l\), \(|f \cap A_i| = s\) for some \( 1 \leq p \leq m \), and \(|f \cap A_q| = s-l\) for some \( 1 \leq q \leq m \) with \( q \neq p \). We will pair edges of types \((s+1-l, s, l-1)\) and \((l, s, s-l)\) as well as edges of types \((s-l, s, l)\) and \((s-1-l, s, l-1)\). Lemma 2 and Lemma 3 will show how does the color of edges of the first type forces the color of edges of the second type under some assumptions. Note \( m \) is fixed and \( m \geq 3 \).

**Lemma 2** Assume \( \{A_i, A_j\} \subseteq F_{\text{red}} \) for each \( 1 \leq i \neq j \leq m \) and there is no \( P_m \subseteq F_{\text{red}} \). For each \( 1 \leq l \leq \left\lfloor \frac{s}{2} \right\rfloor \), if all edges of the type \((s+1-l, s, l-1)\) are blue, then the existence of a blue edge of the type \((l, s, s-l)\) implies the existence a blue \( P_m \).

**Proof:** Suppose that there is some blue edge \( g_1 \) of the type \((l, s, s-l)\). Without loss of generality, we can assume \( g_1 = \{B', A_1, A'_1\} \), where \( B' \) is an \( l \)-subset of \( B \) and \( A'_i \subseteq A_i \) with \(|A'_i| = s-l\) for some \( 2 \leq i \leq m \). Choose \( A''_i \) to be an \((l-1)\)-subset of \( A_i \setminus A'_i \). Since we assume \( m \geq 3 \), let \( A_j \) be an \( s \)-set which is different from \( A_1 \) and \( A_i \). We define

\[
g_2 = \{A_1, B \setminus B', A''_1\} \quad \text{and} \quad g_3 = \{B \setminus B', A''_i, A_j\}.
\]

By the assumption, both \( g_2 \) and \( g_3 \) are blue. Now \( g_1, g_2, \) and \( g_3 \) induce a blue \( P_m \). We proved the lemma.

**Lemma 3** Assume \( \{A_i, A_j\} \subseteq F_{\text{red}} \) for each \( 1 \leq i \neq j \leq m \) and there is no \( P_m \subseteq F_{\text{red}} \). For each \( 1 \leq l \leq \left\lfloor \frac{s}{2} \right\rfloor \), if all edges of the type \((l, s, s-l)\) are red, then all edges of the type \((s-l, s, l)\) are blue.

**Proof:** Suppose indirectly that there is some red edge \( g_1 \) of the type \((s-l, s, l)\). Without loss of generality, we can assume \( g_1 = \{B', A_1, A'_1\} \) for some \( B' \subseteq B \) with \(|B'| = s-l\) and \( A'_i \subseteq A_i \) with \(|A'_i| = l\). Pick an arbitrary \( l \)-subset \( B'' \) of \( B \setminus B' \) and define

\[
g_j = \begin{cases} 
\{A_{j-1}, A_j\} & \text{if } 2 \leq j \leq i-1; \\
\{A_{i-1}, A_m\} & \text{if } j = i; \\
\{A_{m-(j-i-1)}, A_{m-(j-i)}\} & \text{if } i+1 \leq j \leq m-1; \\
\{A_{i+1}, A_i \setminus A'_i, B''\} & \text{if } j = m.
\end{cases}
\]
By the assumption, \( g_j \subseteq F_{\text{red}} \) for each \( 2 \leq j \leq m \). Now \( g_1, g_2, \ldots, g_m \) form a red \( P_m \), which is a contradiction to the assumption. We completed the proof of the lemma.

The next lemma will tell us that the combination of two Lemmas above forces a blue \( P_3 \) under the conditions.

**Lemma 4** Assume \( \{A_i, A_j\} \subseteq F_{\text{red}} \) for each \( 1 \leq i \neq j \leq m \) and there is no \( P_m \subseteq F_{\text{red}} \). Then there must be a blue \( P_3 \).

**Proof:** Since there is no red \( P_m \), all edges of the type \((s, s, 0)\) must be blue. We start to apply Lemma 2 and Lemma 3 alternatively. For each \( 1 \leq j \leq \left\lfloor \frac{m+1}{2} \right\rfloor \), we first apply Lemma 2 with \( l = j \); and we stop it if it succeeds to give us a blue \( P_3 \); otherwise, we get all edges of the type \((j, s, s-j)\) are red and we apply Lemma 3 with \( l = j \). If we stop for some \( 1 \leq j \leq \left\lfloor \frac{m+1}{2} \right\rfloor \), then we find a blue \( P_3 \) and we complete the proof. Otherwise, we assume Lemma 2 fails to produce a blue \( P_3 \) for each \( 1 \leq j \leq \left\lfloor \frac{m+1}{2} \right\rfloor \).

If \( s \) is odd, then we obtain that all edges of the type \(\left(\frac{s+1}{2}, s, \frac{s-1}{2}\right)\) are blue by Lemma 3 with \( l = \frac{s-1}{2} \). We choose \( B' \) to be a subset of \( B \) with size \( \frac{s+1}{2} \), \( A_1' \) and \( A_1'' \) to be disjoint subsets of \( A_1 \) with size \( \frac{s-1}{2} \). We define

\[
g_1 = \{B', A_1', A_2\}, \quad g_2 = \{A_2, A_1'', B \setminus B'\}, \quad \text{and} \quad g_3 = \{A_1'', B \setminus B', A_3\}.
\]

We get a blue \( P_3 \) with edges \( g_1, g_2, \) and \( g_3 \).

Similarly, if \( s \) is even, then we get that all edges of the type \(\left(\left\lfloor \frac{s+1}{2} \right\rfloor, s, \left\lfloor \frac{s-1}{2} \right\rfloor\right)\) are blue by Lemma 3 with \( l = \left\lfloor \frac{s-1}{2} \right\rfloor \). We need to apply Lemma 2 with \( l = \left\lfloor \frac{s-1}{2} \right\rfloor \) (note \( \left\lfloor \frac{s-1}{2} \right\rfloor = \frac{s}{2} \)) again. If Lemma 2 does not yield a blue \( P_3 \), then all edges of the type \(\left(\frac{s}{2}, s, \frac{s}{2}\right)\) are red. We pick two disjoint subsets \( B' \) and \( B'' \) from \( B \) with size \( \frac{s}{2} \), and a subset \( A_1' \) from \( A_1 \) with size \( \frac{s}{2} \). We define

\[
g_j = \begin{cases} 
\{B', A_1', A_2\} & \text{if } j = 1; \\
\{A_2, A_1'', B \setminus B'\} & \text{if } 2 \leq j \leq m-1; \\
\{A_m, A_1 \setminus A_1', B''\} & \text{if } j = m.
\end{cases}
\]

Clearly, \( g_j \) is red for each \( 1 \leq j \leq m \) and we obtain a red \( P_m \), which is a contradiction to the initial assumption. Therefore, there must be a blue \( P_3 \) when we apply Lemma 2 with \( l = \left\lfloor \frac{s-1}{2} \right\rfloor \). We finished the proof of the lemma.

With all lemmas in hand, we are ready to prove Theorem 1.

**Proof of Theorem 1** We will prove the theorem by induction on \( n \). The base step is \( n = 3 \). Let \( c \) be a 2-coloring of \( K_{4+1} \). Since \( 4s + 1 \geq R(P_3, P_2) \), either there is some red \( P_3 \), or there is some blue \( P_2 \). If we are in the previous case, then there is nothing to show. Thus we assume a maximum blue path is \( A_1, A_2, A_3 \). Let \( B \) be the remaining \( s + 1 \) vertices. Observe that the edges \( \{B', A_1\} \) and \( \{B', A_3\} \) must be red for each \( s \)-subset \( B' \) of \( B \). If \( \{A_1, A_3\} \) is a blue edge, then a red \( P_3 \) follows from Lemma 3 by swapping colors. If \( \{A_1, A_3\} \) is red, then \( \{B', A_1\}, \{A_1, A_3\}, \{A_3, B'\} \) be a red \( P_3 \) for some \( B' \subseteq B \). If there is no red \( P_3 \), then there has to be a blue \( P_3 \) by Lemma 3, which is a contradiction. In either case, we are able to find a red \( P_3 \) and we completed the proof for the base step.

Assume Theorem 1 holds for all \( 3 \leq k \leq m - 1 \) with \( m \geq 4 \). Consider a 2-coloring \( c \) of \( K_{(m+1)s+1} \). Since \( (m+1)s + 1 \geq R(P_{m-1}, P_3) = ms + 1 \) by the inductive assumption, either there is a red \( P_{m-1} \), or there is a blue \( P_3 \). We need only to consider the case that the maximum length of a red path is \( m - 1 \). Let \( f_1, f_2, \ldots, f_{m-1} \) be a red \( P_{m-1} \), where \( f_i = \{A_i, A_{i+1}\} \) for \( 1 \leq i \leq m - 1 \). Let \( B \) be the remaining \( s + 1 \) vertices. Since there is no red \( P_m \), the edges \( \{B', A_1\} \) and \( \{B', A_n\} \) must be blue for each subset \( B' \) of \( B \) with size \( s \).

We have the following mutually disjoint cases.

**Case 1:** Either \( \{A_1, A_j\} \subseteq F_{\text{blue}} \) for some \( 3 \leq j \leq m - 1 \) or \( \{A_k, A_m\} \subseteq F_{\text{blue}} \) for some \( 2 \leq k \leq m - 2 \). We observe that \( \{A_m, B', \{B', A_1\}, \{A_1, A_j\} \) form a blue \( P_3 \) in the previous case, and \( \{A_1, B', \{B', A_m\}, \{A_m, A_k\} \) form a blue \( P_3 \) in the later case.
Case 2: We have \( \{A_1, A_i\} \subseteq \mathcal{F}_{\text{red}} \) for each \( 3 \leq i \leq m - 1 \) and \( \{A_i, A_m\} \subseteq \mathcal{F}_{\text{red}} \) for each \( 2 \leq i \leq m - 2 \). Moreover, there are \( 2 \leq j < k \leq m - 1 \) such that \( \{A_j, A_k\} \subseteq \mathcal{F}_{\text{blue}} \). We define

\[
g_q = \begin{cases} 
  f_q & \text{if } 1 \leq q \leq k - 2; \\
  \{A_{k-1}, A_m\} & \text{if } q = k - 1; \\
  \{A_{m-(q-k)}, A_{m-(q-k)-1}\} & \text{if } k \leq q \leq m - 1.
\end{cases}
\]

Notice that \( g_1, g_2, \ldots, g_{m-1} \) is a new red \( \mathcal{P}_{m-1} \). Now \( A_k \) is an ending \( s \)-set of this new path and we can find a blue \( \mathcal{P}_3 \) by the same way as Case 1.

Case 3: We have \( \{A_1, A_j\} \subseteq \mathcal{F}_{\text{red}} \) for all \( 1 \leq i \neq j \leq m \) such that \( \{i, j\} \neq \{1, m\} \). Now if \( \{A_1, A_m\} \) is blue, then we can find a blue \( \mathcal{P}_3 \) by the same argument as Case 2; namely by finding a new red \( \mathcal{P}_{m-1} \) with one of \( A_1 \) and \( A_m \) as an ending \( s \)-set but not the other one. If \( \{A_1, A_m\} \subseteq \mathcal{F}_{\text{red}} \), then a blue \( \mathcal{P}_3 \) is ensured by Lemma 4.

We finished the proof of the inductive step and completed the proof the theorem. \( \square \)

3 Proof of Theorem 2

For a fixed \( s \geq 1 \), we will also prove Theorem 2 by induction on \( n \). The main work is on the inductive step. We assume \( R(\mathcal{P}_k, \mathcal{P}_4) = (k+1)s + 1 \) for all \( 4 \leq k \leq m - 1 \). For the inductive step, let \( c \) be a red-blue coloring of edges of \( K_{(m+1)s+1} \). Since \( (m+1)s + 1 \geq R(\mathcal{P}_{m-1}, \mathcal{P}_4) = ms + 1 \) by the inductive hypothesis, either there is some red \( \mathcal{P}_{m-1} \) or there is some blue \( \mathcal{P}_4 \).

There is nothing to show if either there is some \( \mathcal{P}_m \subseteq \mathcal{F}_{\text{red}} \) or \( \mathcal{P}_4 \subseteq \mathcal{F}_{\text{blue}} \). Thus we assume that the maximum length of a red path is \( m - 1 \); our goal is to find a blue \( \mathcal{P}_4 \) under this condition. Let \( A_1, A_2, \ldots, A_m \) be a fixed red \( \mathcal{P}_{m-1} \) induced by \( c \), where \( \{A_1, A_2, \ldots, A_m\} \) is a collection of mutually disjoint \( s \)-sets of \( [ms] \). Let \( B = [(m+1)s + 1] \setminus [ms] \). We will frequently replace some edges of the existed red \( \mathcal{P}_{m-1} \) to obtain a new red \( \mathcal{P}_{m-1} \) with new ending \( s \)-sets. To get a blue \( \mathcal{P}_4 \), a blue edge \( f \) with vertices from \( \cup_{i=1}^{m} A_i \) will help us a lot. There are many possible arrangements of the vertices of \( f \). The simplest case is \( f = \{A_i, A_j\} \) for some \( 1 \leq i \neq j \leq m \); we will show that we can always reduce the case \( f = \{A_i, A_j\} \) to the case \( f = \{A_1, A_p\} \) for some \( 3 \leq p \leq m - 1 \). If \( f = \{A_1, A_p\} \subseteq \mathcal{F}_{\text{red}} \), then the following lemmas tells us how can we find the desired blue \( \mathcal{P}_4 \) under some conditions. We will frequently utilize the following fact.

**Fact 1** Let \( A_1, \ldots, A_m \) be a maximum red \( \mathcal{P}_{m-1} \) induced by \( c \). Then \( \{A_1, B\}, \{A_m, B\} \subseteq \mathcal{F}_{\text{blue}} \) for each \( s \)-subset \( B \) of \( B \).

The fact follows from the maximality of the red path \( \mathcal{P}_{m-1} \).

Fix a fixed red path \( A_1, A_2, \ldots, A_m \), we say an edge \( f \) is of type \((l, s-l)\) if \( |f \cap B| = l \), \( |f \cap A_j| = s \) for some \( 1 \leq j \leq m \) with \( j \neq 2 \), and \( |f \cap A_2| = s - l \). Lemma 5 to Lemma 7 play the same role as Lemma 2 to Lemma 4.

**Lemma 5** Assume \( \{A_1, A_p\} \subseteq \mathcal{F}_{\text{blue}} \) for some \( 3 \leq p \leq m - 1 \), \( \{A_1, A_i\} \subseteq \mathcal{F}_{\text{red}} \) for \( 3 \leq i \neq p \leq m - 1 \), and \( \{A_j, A_m\} \subseteq \mathcal{F}_{\text{red}} \) for \( 2 \leq j \leq m - 2 \). Furthermore, there is no red \( \mathcal{P}_m \). Fix \( 1 \leq l \leq \lfloor \frac{m}{2} \rfloor \). If all edges of the type \((s + 1-l, s-l-1)\) are blue, then the existence of a blue edge of the type \((l, s-l)\) implies the existence of a blue \( \mathcal{P}_4 \).

**Proof:** We assume that there is some blue edge of the type \((l, s-l)\), say \( g_1 = \{B', A_j, A_2', A_2''\} \), where \( j \neq 2 \), \( B' \) is an \( l \)-subset of \( B \), and \( A_2'' \) is an \((s-l)\)-subset of \( A_2 \). We define \( A_2'' \) to be an arbitrary \((l-1)\)-subset of \( A_2 \). We have two cases.

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**Case 1:** \( j \in \{1, p\} \). Without loss of generality, we assume \( j = p \). We define
\[
g_2 = \{A_p, A_1\}, \quad g_3 = \{A_1, B \setminus B', A'_2\}, \quad \text{and} \quad g_4 = \{B \setminus B', A'_2, A_m\}.
\]

By the assumption, we get that \( g_1, g_2, g_3, \) and \( g_4 \) form a blue \( P_4 \). We can obtain a blue \( P_3 \) similarly when \( j = 1 \).

**Case 2:** \( j \not\in \{1, p\} \). We define
\[
g_2 = \{A_j, B \setminus B', A'_2\}, \quad g_3 = \{B \setminus B', A'_2, A_1\}, \quad \text{and} \quad g_4 = \{A_1, A_p\}.
\]

By the assumption, we obtain a blue \( P_4 \) with edges \( g_1, g_2, g_3, \) and \( g_4 \).

We proved the lemma. \( \square \)

We also have the following lemma which is similar to Lemma 3.

**Lemma 6** Assume \( \{A_1, A_p\} \subseteq F_{\text{blue}} \) for some \( 3 \leq p \leq m - 1 \), \( \{A_1, A_i\} \subseteq F_{\text{red}} \) for \( 3 \leq i \neq p \leq m - 1 \), and \( \{A_j, A_m\} \subseteq F_{\text{red}} \) for \( 2 \leq j \leq m - 2 \). Furthermore, there is no red \( P_m \). Fix \( 1 \leq l \leq \left\lceil \frac{m - 1}{2} \right\rceil \). If all edges of the type \((l, s, s - l)\) are red, then all edges of the type \((s - l, s, l)\) are blue.

**Proof:** Suppose that there is some red edge \( g_1 \) of the type \((s - l, s, l)\). We can assume \( g_1 = \{B', A_j, A'_2\} \) for some \( j \neq 2 \), where \( B' \) is a subset of \( B \) with size \( s - l \) and \( A'_2 \) is a subset of \( A_2 \) with size \( l \). We first assume \( j \neq 1 \). Let \( B'' \) be an \( l \)-subset of \( B \setminus B' \). We define
\[
g_q = \begin{cases} 
\{A_j + q - 1, A_j + q - 2\} & \text{if } 2 \leq q \leq m - j + 1; \\
\{A_m, A_j - 1\} & \text{if } q = m - j + 2; \\
\{A_m - q + 2, A_m - q - 1\} & \text{if } m - j + 3 \leq q \leq m - 2; \\
\{A_3, B'', A_2 \setminus A'_2\} & \text{if } q = m - 1; \\
\{B'', A_2 \setminus A'_2, A_1\} & \text{if } q = m.
\end{cases}
\]

Observe that \( g_1, \ldots, g_m \) induce a red \( P_m \), which is a contradiction to the assumption. For \( j = 1 \), we can find a red \( P_m \) similarly. Therefore, all edges of the type \((s - l, s, l)\) must be blue and we completed the proof of the lemma. \( \square \)

The next lemma show how can we get a blue \( P_4 \) under conditions above.

**Lemma 7** Assume \( \{A_1, A_p\} \subseteq F_{\text{blue}} \) for some \( 3 \leq p \leq m - 1 \), \( \{A_1, A_i\} \subseteq F_{\text{red}} \) for \( 3 \leq i \neq p \leq m - 1 \), and \( \{A_j, A_m\} \subseteq F_{\text{red}} \) for \( 2 \leq j \leq m - 2 \). Furthermore, there is no red \( P_m \). Then there must be a blue \( P_4 \).

**Proof:** The proof of this lemma uses the same idea as the one in the proof of Lemma 4.

For each \( 1 \leq j \leq \left\lceil \frac{m - 1}{2} \right\rceil \), we first apply Lemma 5 with \( l = j \); if Lemma 5 succeeds to give us a blue \( P_4 \), then we stop. Otherwise, we get all edges of the type \((j, s, s - j)\) are red and we apply Lemma 6 with \( l = j \). Thus, we need only to take care of the case where Lemma 5 fails to give a blue \( P_4 \) for each \( 1 \leq j \leq \left\lceil \frac{m - 1}{2} \right\rceil \).

If \( s \) is odd, then all edges of the type \(\left(\frac{s + 1}{2}, s, \frac{s - 1}{2}\right)\) are blue followed from Lemma 6 with \( l = \frac{s - 1}{2} \). Let \( B' \subseteq B \) with \( |B'| = \frac{s + 1}{2} \). \( A'_2 \) and \( A''_2 \) be two disjoint subsets of \( A_2 \) with size \( \frac{s - 1}{2} \). We define
\[
g_1 = \{A_p, A_1\}, \quad g_2 = \{A_1, B', A'_2\}, \quad g_3 = \{B', A'_2, A_m\}, \quad \text{and} \quad g_4 = \{A_m, A'', B/B'\}.
\]

Now, we observe that \( g_1, g_2, g_3, \) and \( g_4 \) form a blue \( P_4 \).

If \( s \) is even, then we have that all edges of the type \(\left(\frac{s + 1}{2}, s, \frac{s - 1}{2}\right)\) are blue by Lemma 6 with \( l = \frac{s + 1}{2} \). Now, we appeal to Lemma 5 with \( l = \frac{s + 1}{2} \) (note \( \frac{s + 1}{2} = \frac{m - 1}{2} \)). If Lemma
can not give us a blue $P_4$, then all edges of the type $\left(\frac{s}{2}, s, \frac{t}{2}\right)$ are red. Choose two disjoint subsets $B', B'' \subseteq B$ with size $\frac{s}{2}$ and a subset $A_2 \subseteq A_1$ with size $\frac{t}{2}$. We define

$$g_j = \begin{cases} 
\{A_1, B', A_2\} & \text{if } j = 1; \\
\{B', A_2, A_3\} & \text{if } j = 2; \\
\{A_j, A_{j+1}\} & \text{if } 3 \leq j \leq m - 1; \\
\{A_m, B'', A_2 \setminus A_2'\} & \text{if } j = m.
\end{cases}$$

Clearly, $g_1, \ldots, g_m$ form a red $P_m$, which is a contradiction to the assumption. Therefore, when we apply Lemma 5 with $l = \left[\frac{s}{2}\right]$, it must produce a blue $P_4$. We completed the proof of this lemma.

We have the following remark on the case $m = 5$.

**Remark 1** If $m = 5$, $\{A_1, A_3\}, \{A_3, A_5\} \subseteq F_{\text{blue}}$, and $\{A_1, A_4\}, \{A_2, A_5\} \subseteq F_{\text{red}}$, then there is a blue $P_4$.

The proof of this remark follows exactly the same lines as Lemma 4 to Lemma 6 and it is omitted here.

As we mentioned before, a blue edge $f = \{A_i, A_j\}$ is helpful for finding a blue $P_4$. The next lemma will show the case $f = \{A_1, A_p\}$ for some $3 \leq p \leq m - 1$.

**Lemma 8** If $\{A_1, A_p\}$ is blue for some $3 \leq p \leq m - 1$, then there is a blue $P_4$.

**Proof:** If their is some $2 \leq j \neq p \leq m - 2$ such that $\{A_j, A_m\}$ is blue, then let $g_3 = \{A_1, B'\}$ and $g_4 = \{B', A_m\}$, where $B'$ is an $s$-subset of $B$. Fact 1 implies that both $g_3$ and $g_4$ are blue. Note that $f, g_3, g_4, \{A_m, A_j\}$ form a blue $P_4$. In the remaining proof, we assume $\{A_j, A_m\}$ is blue for each $2 \leq j \neq p \leq m - 2$. Note that the above argument gives us the assumptions in Lemma 7 for $m = 4$; thus a desired blue $P_4$ is ensured by Lemma 7 for $m = 4$.

We leave the case $m = 5$ for a while. For $m \geq 6$, we get that either $p - 1 \geq 3$ or $m - p \geq 3$. The main idea is that we find a new red path with length $m - 1$ which contains $A_q$ as an ending $s$-set for some $q \notin \{1, p, m\}$. We assume $p - 1 \geq 3$ and the case $m - p \geq 3$ can be proved similarly. If $\{A_p, A_m\} \subseteq F_{\text{blue}}$, then we consider a new red path $A_1, A_2, A_m, A_3, \ldots, A_p, \ldots, A_{m-1}$. Fact 1 implies $\{A_1, B'\}, \{A_{m-1}, B'\} \subseteq F_{\text{blue}}$ for each $s$-subset $B'$ of $B$. Now, $A_m, A_p, A_1, B', A_{m-1}$ is a blue $P_4$. Thus, we can assume $\{A_p, A_m\} \subseteq F_{\text{red}}$. By the same argument, we can also assume $\{A_1, A_j\} \subseteq F_{\text{red}}$ for each $3 \leq j \neq p \leq m - 1$; otherwise we can find a blue $P_4$ easily. Under the assumptions above, Lemma 7 gives a desired blue $P_4$.

We are left to prove the case $m = 5$. If either $p = 4$ or $p = 3$, a blue $P_4$ is given by Lemma 4. If $p = 3$ and $\{A_3, A_5\} \in F_{\text{blue}}$, then a blue $P_4$ is given by Remark 1. We proved the lemma.

The next lemma will tell us that we can reduce the general case $f = \{A_i, A_j\}$ to the case $f = \{A_1, A_p\}$.

**Lemma 9** If there is some blue edge $f = \{A_i, A_j\}$ for some $1 \leq i \neq j \leq m$, then there is a blue $P_4$.

**Proof:** We have the following mutually disjoint cases.

**Case 1:** $|\{i, j\} \cap \{1, m\}| = 1$. Note that the case $f = \{A_j, A_m\}$ is the same as $f = \{A_1, A_p\}$ by the symmetry, so the case is proved by Lemma 8.

**Case 2:** $2 \leq i < j \leq m$ and $\{A_p, A_q\} \subseteq F_{\text{red}}$ for all $|\{p, q\} \cap \{1, m\}| = 1$. We observe that $A_1, \ldots, A_m, A_{j-1}, \ldots, A_i, A_1$ is a new red $P_{m-1}$ and we can reduce it to Case 1.

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Case 3: \( \{i,j\} = \{1,m\} \) and \( \{A_p,A_q\} \subseteq F_{\text{red}} \) for all \( \{p,q\} \neq \{1,m\} \). We form a new blue \( P_{m-1} \) which is \( A_1,A_2,A_m,\ldots,A_3 \), and we reduce it to Case 1.

We finished the proof of the lemma.

We already showed how a blue edge \( \{A_1,A_2\} \) helps us to get a blue \( P_4 \). In general, \( f \) could intersect more than two \( A_i \)'s. The next lemma will give us a \( P_4 \subseteq F_{\text{blue}} \) for other possible intersections between \( f \) and \( A_i \)'s. We need some notations. Given a red path \( P_{m-1} = A_1,A_2,\ldots,A_m \) and an edge \( f \) with \( V(f) \subseteq \cup_{i=1}^{m} A_i \), let \( S(P_{m-1}, f) = \{1 \leq i \leq m: f \cap A_i \neq \emptyset\} \). We say the fixed coloring \( c \) has Property(i) if the existence of some edge \( f \) and a red path \( P_{m-1} \) satisfying \( S(P_{m-1}, f) = i \) implies that \( c \) induces a blue \( P_4 \). We have the following lemma.

**Lemma 10** For a fixed red-blue coloring \( c \) of edges of \( K_{(m+1)s+1} \) without \( P_m \subseteq F_{\text{red}} \), the coloring \( c \) has Property(i) for each \( 2 \leq i \leq \min\{m,s\} \).

**Proof:** We proceed the proof by induction on \( i \). The base step \( i = 2 \) is given by Lemma 1. For the inductive step, let us fix a red \( P_{m-1} \) and a blue edge \( f \) satisfying \( |S(P_{m-1}, f)| = k \). We can assume all edges \( f' \) with \( |S(P_{m-1}, f')| < k \) are red, otherwise a blue \( P_4 \) is ensured by the inductive hypothesis. Without loss of generality, we assume \( S(P_{m-1}, f) = \{1, \ldots, k\} \). Let \( A'_i = f \cap A_i \) for each \( 1 \leq i \leq k \).

If \( k \geq 4 \), then we assume \( |A'_1| \leq |A'_2| \leq \cdots \leq |A'_k| \). Clearly, \( |A'_1 \cup A'_2| \leq s \). Let \( C \) be a subset of \( A_1 \cup A_2 \) such that \( A'_1 \cup A'_2 \subseteq C \) and \( |C| = s \). Observe \( \{A_3,C\} \subseteq F_{\text{red}} \); otherwise, note \( |S(P_{m-1}, \{A_3,C\})| = 3 \) and the blue \( \{A_3,C\} \) is given by the inductive hypothesis. Let \( C' = (A_1 \cup A_2) \setminus C \) and we consider a new red path \( P'_{m-1} = A_m, \ldots, A_3, C, C' \). Observe \( |S(P'_{m-1}, f)| = k - 1 \), we get a blue \( P_4 \) by the inductive hypothesis.

If \( k = 3 \), then we need more argument. We need only to prove the case \( |A'_i \cup A'_j| > s \) for \( 1 \leq i \neq j \leq 3 \). We can also assume \( \{A_1,A_2\} \subseteq F_{\text{red}} \) for all \( 1 \leq i \neq j \leq m \); otherwise the base step produces a blue \( P_4 \).

We first consider that there is some \( 1 \leq i \leq 3 \) such that \( A_i \subseteq f \). Without loss of generality, we assume \( A_i \subseteq f \). Let \( g_2 = \{A_2 \setminus A'_2, A_3 \setminus A'_3, A_4\} \). If \( g_2 \subseteq F_{\text{blue}} \), then let \( g_3 = \{A_4, B'\} \) and \( g_4 = \{B', B\} \), where \( B' \subseteq B \) and \( |B'| = s \). Since we can view \( A_4 \) as one of ending \( s \)-sets of a red \( P_{m-1} \). Fact 1 implies \( g_3,g_4 \subseteq F_{\text{blue}} \). Now, \( g_2,g_3,g_4 \) form a blue \( P_4 \). If \( g_2 \not\subseteq F_{\text{red}} \), then we form a new red path \( P'_{m-1} = g_1,A_4, \ldots, A_m, A_1, f \). Note \( |f \cap P'_{m-1}| = 2 \) and the base case gives us a blue \( P_4 \). We are through in this case.

If \( |A_i \cap f| < s \) for each \( 1 \leq i \leq 3 \), then we pick a subset \( A''_2 \) from \( A_2 \) such that \( |A''_2 \cup A'_2| = s \). Let \( C = (A_2 \setminus A''_2) \cup (A_3 \setminus A'_3) \). We need only to consider the case \( \{A''_2, A'_3, A_4\}, \{A_1,C\} \subseteq F_{\text{red}} \). If \( f' = \{A''_2, A'_3, A_4\} \subseteq F_{\text{blue}} \), then a blue \( P_4 \) is given by the previous case by observing \( |f' \cap P_{m-1}| = 3 \) and \( A_4 \subseteq f' \). We have a similar argument for \( g_2 = \{A_1,C\} \subseteq F_{\text{red}} \). When \( \{A''_2, A'_3, A_4\}, \{A_1,C\} \subseteq F_{\text{red}} \), we observe \( P'_{m-1} = g_1,A_4, \ldots, A_m, A_1, C \) is a red path \( P'_{m-1} \), \( |S(P'_{m-1}, f)| = 3 \), and \( C \subseteq f \); we reduce this case to the previous case.

We proved the lemma.

We already know how to find a blue \( P_4 \) if there is some blue \( f \) such that \( f \subseteq \cup_{i=1}^{m-1} A_i \). Next, we assume \( f \) is red for all \( f \subseteq \cup_{i=1}^{m} A_i \) and show how can we find a blue \( P_4 \) under this assumption. We need one more definition. Fix a red path \( A_1,A_2, \ldots, A_m \) and let \( B \) be the remaining \( s + 1 \) vertices. For each \( 1 \leq l \leq s \), we say \( f \) is of type \((s-l,s+l)\) if \( |f \cap B| = s-l \) and \( |f \cap (\cup_{i=1}^{m-1} A_i)| = s+l \).

**Lemma 11** Let \( A_1,A_2, \ldots, A_m \) be a red \( P_{m-1} \). Assume all edges \( f \subseteq \cup_{i=1}^{m} A_i \) are red and there is no red \( P_m \). For each \( 1 \leq l \leq \left\lfloor \frac{s}{2} \right\rfloor \), if all edges of the type \((s-l+1,s+l-1)\) are blue, then the existence of a red edge of the type \((s-l,s+l)\) implies the existence of a blue \( P_4 \).
Lemma 11 with $l$ of the type $P$ proved the lemma. □

Let $m$ be a 2-coloring of edges in $G$. Similarly. Let $m$ assume $g \in \{1, 2\}$. We get both $g_1$ and $g_2$ are blue. Otherwise, if $g_1$ is red, then $g_1, A_3, \ldots, A_m, A_1, f$ is a red $P_m$, which is a contradiction to the assumption. If $g_2$ is red, then we can find a contradiction similarly. Let $A'_2$ be an $(l-1)$-subset of $A_2$. We define

$$g_3 = \{A_1, A'^2_2, B \setminus B''\} \text{ and } g_4 = \{A''_2, B \setminus B'', A_3\}.$$  

We get that both $g_3$ and $g_3$ are blue by the assumption. Now, $g_3, g_4, g_1, g_2$ is a blue $P_4$. We proved the lemma.

The next lemma will show how does Lemma 11 guarantee a blue $P_4$ under the assumption.

**Lemma 12** Let $A_1, A_2, \ldots, A_m$ be a red $P_{m-1}$. Assume all edge $f$ satisfying $f \subseteq \cup_{i=1}^m A_i$ are red and there is no red $P_m$. Then we have a blue $P_4$.

**Proof:** There is no red $P_m$ implies all edges of the type $(s, s)$ are blue. We start to apply Lemma 11 with $l = 1$. If there is some $1 \leq l \leq \lfloor \frac{s}{2} \rfloor$ such that Lemma 11 succeeds to give us a blue $P_4$, then we are through. Otherwise, Lemma 11 with $l = \lfloor \frac{s}{2} \rfloor$ tells us that all edges of the type $(s - \lfloor \frac{s}{2} \rfloor, s + \lfloor \frac{s}{2} \rfloor)$ are blue.

When $s$ is odd. Let $B'$ be a subset $B$ with size $\frac{s+1}{2}$, and $A'_1$ and $A''_2$ be two disjoint subsets of $A_1$ with size $\frac{s-1}{2}$. We define

$$g_1 = \{A_2, A'_1, B'\}, \quad g_2 = \{A'_1, B'', A_3\}, \quad g_3 = \{A_3, A''_2, B \setminus B'\} \text{ and } g_4 = \{A''_2, B \setminus B'', A_4\}.$$  

It is easy to see that $g_1, g_2, g_3, g_4$ form a blue $P_4$.

When $s$ is even. Let $B''$ be two disjoint subsets $B$ with size $\frac{s}{2}$ and $A'_1$ be subsets of $A_1$ with size $\frac{s}{2}$. We define

$$g_1 = \{A_2, A'_1, B'\}, \quad g_2 = \{A'_1, B''_1, A_3\}, \quad g_3 = \{A_3, A_1 \setminus A'_1, B''\} \text{ and } g_4 = \{A_1 \setminus A'_1, B'', A_4\}.$$  

It is easy to see that $g_1, g_2, g_3, g_4$ form a blue $P_4$. In either case, we are able to find a blue $P_4$ and we completed the proof. □

We are now ready to prove Theorem 2.

**Proof of Theorem 2** We prove the theorem by induction on $n$. For the base case, let $c$ be a 2-coloring of edges in $K_{5s+1}$. As $5s + 1 \geq R(P_4, P_3)$ by Theorem 1, either we have a red $P_4$ or we have a blue $P_3$. There is nothing to show in the previous case. Thus we assume $A_1, A_2, A_3, A_4$ is a maximum blue path. If there is an red edge with vertices from $\cup_{i=1}^4 A_i$, then we have a red $P_4$ by Lemma 11 with colors swapped. Otherwise, switch colors in Lemma 11 and it gives us a red $P_4$.

The inductive step is given by Lemma 11 and Lemma 12. We finished the proof of the theorem. □

4 Concluding remarks

In this paper, we give a partial affirmative answer to Question 1 for $s = r/2$, $r$ even, and $m \in \{3, 4\}$. However, unlike the paper [5], we are not able to determine the Ramsey number of small $r/2$-cycles for even $r$. A possible reason is following. In [5], they proved the following statement. Let $c$ be a red-blue coloring of edges in $K_N$, here $N = (r - 1)n + \lfloor \frac{2n+1}{r} \rfloor$. If
$C_{r,n}^1 \subseteq F_{\text{red}}$, then either $P_{r,1,n}^1 \subseteq F_{\text{red}}$ or $P_{m,1}^1 \subseteq F_{\text{blue}}$. Also, if $C_{r,n}^1 \subseteq F_{\text{red}}$, then either $P_{r,1,n}^1 \subseteq F_{\text{red}}$ or $C_{r,m}^1 \subseteq F_{\text{blue}}$. The statement above is a very important fact for $s = 1$; it helps to determine the values of $R(P_{r,1,n}^1, P_{r,1,m}^1)$, $R(P_{r,1,n}^1, C_{r,m}^1)$, and $R(C_{r,n}^1, C_{r,m}^1)$. We can not prove a similar lemma for $s = r/2$ and $r$ even since we are in short of vertices after we fix a red $C_{n,r/2}^r$. It would be helpful to prove some lemma which connects $R(P_{r,r/2,n}^1, P_{r,r/2,m}^1)$ to $R(C_{n,r/2}^r, C_{m,r/2}^r)$.

To answer Question 1, we need to determine the exact values of the Ramsey number of each type of paths; it is very possible that we need different techniques to deal with different types of paths. The author strongly believe the following conjecture holds.

**Conjecture 1** For fixed $r \geq 2$ and $n \geq m \geq 3$, we have

$$R(P_{n,r}^1, P_{m,r}^1) > R(P_{n,2}^r, P_{m,2}^r) > \ldots > R(P_{n,\lceil r/2 \rceil}^r, P_{m,\lceil r/2 \rceil}^r).$$

There are many other interesting questions on Ramsey number of paths and cycles in hypergraphs. The only known results addressing the tight cycles is due to Haxell et.al [7] who proved the asymptotic value of $R(C_{3,2}^3, C_{3,2}^3)$. A natural question is to determine the exact values of the Ramsey number of tight paths and cycles; the author has no inclination to whether the natural lower bound gives the true value of them.

**References**


