Verified Error Bounds for Isolated Singular Solutions of Polynomial Systems

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Let $F = \{f_1, \ldots, f_n\}$ be a polynomial system with $f_i \in \mathbb{C}[x_1, \ldots, x_n]$. 
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**Isolated Solutions**

An **isolated** solution of $F(x) = 0$ is a point $\hat{x} \in \mathbb{C}^n$ which satisfies:

$$\exists \ 0 < \varepsilon \ll 1 : \left\{ y \in \mathbb{C}^n : \|y - \hat{x}\| < \varepsilon \right\} \cap F^{-1}(0) = \{\hat{x}\}.$$
Let $F = \{f_1, \ldots, f_n\}$ be a polynomial system with $f_i \in \mathbb{C}[x_1, \ldots, x_n]$.

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### Singular Solutions

We call $\hat{x}$ a singular solution of $F(x) = 0$ if and only if

$$\text{rank}(F_x(\hat{x})) < n,$$

where $F_x$ denotes the Jacobian matrix of $F$ with respect to $x$. 

Nan Li (KLMM)
Let $d^\alpha_{\hat{x}} : \mathbb{C}[x] \rightarrow \mathbb{C}$ denote the differential functional defined by

$$d^\alpha_{\hat{x}}(g) = \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^{\alpha}|g|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(\hat{x}), \quad \forall g(x) \in \mathbb{C}[x].$$
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**Local Dual Space**

The local dual space of $l = (f_1, \ldots, f_n)$ at $\hat{x}$ is defined by

$$\mathcal{D}_{\hat{x}} := \{ \Lambda \in \mathcal{D}_{\hat{x}} \mid \Lambda(f) = 0, \forall f \in l \},$$

where $\mathcal{D}_{\hat{x}} = \text{Span}_\mathbb{C}\{d^\alpha_{\hat{x}}, \alpha \in \mathbb{N}^n\}$ and $\dim(\mathcal{D}_{\hat{x}}) = \mu.$
Let \( d_{\hat{x}}^\alpha : \mathbb{C}[x] \to \mathbb{C} \) denote the differential functional defined by

\[
d_{\hat{x}}^\alpha(g) = \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^{\mid\alpha\mid} g}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(\hat{x}), \quad \forall g(x) \in \mathbb{C}[x].
\]

The local dual space of \( I = (f_1, \ldots, f_n) \) at \( \hat{x} \) is defined by

\[
\mathcal{D}_{\hat{x}} := \{ \Lambda \in \mathcal{D}_{\hat{x}} \mid \Lambda(f) = 0, \forall f \in I \},
\]

where \( \mathcal{D}_{\hat{x}} = \text{Span}_\mathbb{C}\{d_{\hat{x}}^\alpha, \alpha \in \mathbb{N}^n\} \) and \( \text{dim}(\mathcal{D}_{\hat{x}}) = \mu \).

\( \hat{x} \) is an isolated singular solution of \( F(x) = 0 \) \( \iff \) \( 1 < \mu < \infty \).
Let $F = \{f_1, \ldots, f_n\}$ be a polynomial system in $\mathbb{R}[x]$ and $\bar{x} \in \mathbb{R}^n$. 
Let $F = \{f_1, \ldots, f_n\}$ be a polynomial system in $\mathbb{R}[x]$ and $\tilde{x} \in \mathbb{R}^n$.

**Theorem (Krawczyk69, Moore77, Rump83)**

Given $0 \in X \subseteq \mathbb{R}^n$, and $M \subseteq \mathbb{R}^{n \times n}$ satisfies $\nabla f_i(\tilde{x} + X) \subseteq M_{i,:}$. If

$$-F_x^{-1}(\tilde{x})F(\tilde{x}) + (I - F_x^{-1}(\tilde{x})M)X \subseteq \text{int}(X),$$

- there is a unique $\hat{x} \in X$ with $F(\hat{x}) = 0$.
- every matrix $\tilde{M} \in M$ is nonsingular.
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**Requirements:**
- square systems: $F(x) : \mathbb{R}^n \to \mathbb{R}^n$.
- regular solutions: $F_x(\hat{x})$ is nonsingular.
A singular solution $\hat{\mathbf{x}}$ of a square system $F(\mathbf{x}) = 0$ satisfies

$$\begin{cases} F(\mathbf{x}) = 0, \\ \det(F_{\mathbf{x}}(\mathbf{x})) = 0, \end{cases}$$

where $\det$ denotes the determinant.
Deflation Techniques

**Determinant**

A singular solution \( \hat{x} \) of a square system \( F(x) = 0 \) satisfies

\[
\begin{align*}
F(x) &= 0, \\
\det(F_x(x)) &= 0,
\end{align*}
\]

where \( \det \) denotes the determinant.

**Null Space**

Let \( r = \text{rank}(F_x(\hat{x})) \), then there exists a unique \( \hat{\lambda} \) such that \( (\hat{x}, \hat{\lambda}) \) is an isolated solution of

\[
\begin{align*}
F(x) &= 0, \\
F_x(x)B\lambda &= 0, \\
h^T\lambda &= 1,
\end{align*}
\]

where \( B \in \mathbb{C}^{n \times (r+1)} \), \( h \in \mathbb{C}^{r+1} \).
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Theorem (LVZ06)

*Deflation $\notin$ to derive a regular solution is strictly less than $\mu$.***
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Related Works

- Yuchi and Shin’ichi, 1999;

- Rump and Graillat, 2009;

- Mantzaflaris and Mourrain, 2011;

- Li and Zhi, 2012.
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- Yuchi and Shin’ichi, 1999;
  - the existence of “imperfect singular solutions”.

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- Yuchi and Shin’ichi, 1999;
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- Li and Zhi, 2012.
  - the existence of breadth-one solutions ($\text{rank}(F_x(\hat{x})) = n - 1$).
Introduce $\mu - 1$ smoothing parameters $b_0, b_1, \ldots, b_{\mu - 2}$ and consider

\[
G(x, \lambda, b) = \begin{pmatrix}
\tilde{F}(x, b) = F(x) - p(x_1, b)e_1 \\
F_1(x, \lambda_1, b) \\
\vdots \\
F_{\mu - 1}(x, \lambda_1, \ldots, \lambda_{\mu - 1}, b)
\end{pmatrix},
\]

where $p(x_1, b) = \sum_{\nu=0}^{\mu-2} \frac{b_{\nu} x_1^\nu}{\nu!}$ and

\[
F_k(x, \lambda_1, \ldots, \lambda_k, b) = L_k \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \lambda_1, \ldots, \lambda_k \right) \left[ \tilde{F}(x, b) \right].
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where $p(x_1, b) = \sum_{\nu=0}^{\mu-2} \frac{b_\nu x_1^\nu}{\nu!}$ and

$$F_k(x, \lambda_1, \ldots, \lambda_k, b) = L_k \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \lambda_1, \ldots, \lambda_k \right) \left[ \tilde{F}(x, b) \right].$$

$$n_x + \mu - 1 + (\mu - 1)(n - 1) = \mu n.$$
Theorem (LiZhi12)

Suppose \( \hat{x} \) is an isolated solution of \( F(x) = 0 \) with \( \text{rank}(F_x(\hat{x})) = n - 1 \) and multiplicity \( \mu \). Assume

\[
\text{rank}(F_{x_2, \ldots, x_n}(\hat{x}), e_1) = n,
\]

then \( G_{x, \lambda, b}(\hat{x}, \hat{\lambda}, 0) \) is nonsingular.

After \( \mu - 1 \) deflations, we derive a regular solution of a square system.
Theorem (LiZhi12)

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After \( \mu - 1 \) deflations, we derive a regular solution of a square system.

Theorem (LiZhi12)

Suppose Theorem KMR is applicable to the \( \mu n \times \mu n \) system \( G(x, \lambda, b) \) and yields inclusions for \( \hat{x}, \hat{\lambda} \) and \( \hat{b} \) such that \( G(\hat{x}, \hat{\lambda}, \hat{b}) = 0 \). Then \( \hat{x} \) is an isolated breadth-one solution of \( \tilde{F}(x, \hat{b}) \) with multiplicity \( \mu \).
Consider a polynomial system

\[ F = \{ x_1^2 x_2 - x_1 x_2^2, x_1 - x_2^2 \} \].

The system \( F \) has \((0, 0)\) as a 4-fold isolated zero.
Consider a polynomial system

\[ F = \{ x_1^2 x_2 - x_1 x_2^2, x_1 - x_2^2 \}. \]

The system \( F \) has \((0, 0)\) as a 4-fold isolated zero.

Add \( p(x_2, b) = b_0 + b_1 x_2 + \frac{b_2}{2} x_2^2 \) to \( x_1^2 x_2 - x_1 x_2^2 \) to construct

\[
G(x, b, \lambda) = \begin{pmatrix}
    x_1^2 x_2 - x_1 x_2^2 - b_0 - b_1 x_2 - \frac{b_2}{2} x_2^2 \\
    x_1 - x_2^2 \\
    2\lambda_1 x_1 x_2 - \lambda_1 x_2^2 + x_1^2 - 2x_1 x_2 - b_1 - b_2 x_2 \\
    \lambda_1 - 2x_2 \\
    \lambda_1^2 x_2 + 2\lambda_1 x_1 - 2\lambda_1 x_2 + 2\lambda_2 x_1 x_2 - \lambda_2 x_2^2 - x_1 - b_2 \\
    \lambda_2 - 1 \\
    \lambda_1^2 + \lambda_1 \lambda_2 x_2 - \lambda_1 + 2\lambda_2 x_1 - 2\lambda_2 x_2 + 2\lambda_3 x_1 x_2 - \lambda_3 x_2^2 \\
    \lambda_3
\end{pmatrix}.
\]
Apply INTLAB function `verifynlss` with

\[
(0.002, 0.003, 0.002, 1.001, -0.01, 0, 0, 0)
\]

and yields inclusions

\[-10^{-14} \leq \hat{x}_i \leq 10^{-14}, \text{ for } i = 1, 2,
\]
\[-10^{-14} \leq \hat{b}_i \leq 10^{-14}, \text{ for } i = 0, 1, 2,
\]

which proves \( \tilde{F}(x, \hat{b}) \) \((|\hat{b}_i| \leq 10^{-14}, i = 0, 1, 2)\) has a 4-fold root \( \hat{x} \) within \(|\hat{x}_i| \leq 10^{-14}, i = 1, 2\), where

\[
\tilde{F}(x, \hat{b}) = \begin{pmatrix}
  x_1^2 x_2 - x_1 x_2^2 - \hat{b}_0 - \hat{b}_1 x_2 - \frac{\hat{b}_2}{2} x_2^2 \\
  x_1 - x_2^2
\end{pmatrix}.
\]
Let $\hat{x} \in \mathbb{C}^n$ be an isolated singular solution of $F(x) = 0$, then

$$\text{rank}(F_x(\hat{x})) = n - d, \ (1 < d \leq n).$$
Let $\hat{x} \in \mathbb{C}^n$ be an isolated singular solution of $F(x) = 0$, then
\[
\text{rank}(F_x(\hat{x})) = n - d, \quad (1 < d \leq n).
\]

Let $F_c(\hat{x})$ be obtained from $F_x(\hat{x})$ by deleting its $c$-th columns, s.t.
\[
\text{rank}(F_c(\hat{x})) = n - d, \quad \text{for } c = \{j_1, j_2, \ldots, j_d\}.
\]
Let $\hat{x} \in \mathbb{C}^n$ be an isolated singular solution of $F(x) = 0$, then
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\]

Let $r = \{k_1, k_2, \ldots, k_d\}$ be an integer set,
\[
\text{s.t. } \text{rank}(F_c(\hat{x}), e_{k_1}, e_{k_2}, \ldots, e_{k_d}) = n.
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Let $r = \{k_1, k_2, \ldots, k_d\}$ be an integer set,

s.t. $\text{rank}(F_c(\hat{x}), e_{k_1}, e_{k_2}, \ldots, e_{k_d}) = n$.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{c} d = 2, \\ c = \{1, 2\}, \\ r = \{1, 2\}. \end{array}$$
Introduce $d$ smoothing parameters $b_1, b_2, \ldots, b_d$ and consider

$$G(x, \lambda, b) = \begin{cases} 
F(x) - \sum_{i=1}^{d} b_i e_{k_i} = 0, \\
F_x(x)v = 0,
\end{cases}$$

where $v$ consists of $n - d$ parameters $\lambda$ and entries 1 at $j_1, j_2, \ldots, j_d$-th.
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where $v$ consists of $n - d$ parameters $\lambda$ and entries 1 at $j_1, j_2, \ldots, j_d$-th.

- $F_x(\hat{x})v = 0$ has a unique solution $\hat{\lambda}$.
- $(\hat{x}, \hat{\lambda}, 0)$ is an isolated solution of $G(x, \lambda, b)$.
- If $(\hat{x}, \hat{\lambda}, 0)$ is singular, regard $G(x, \lambda, b)$ as $F(x)$ and repeat.
- Similar to Yamamoto84.
\[ \begin{align*}
F(x) - l_r b_0 &= 0, \\
F_x(x)v_1 - l_r' b_1 &= 0, \\
G_{x,\lambda_1,b_0}(x, \lambda_1, b_0)v_2 &= 0,
\end{align*} \]
\[
\begin{align*}
F(x) - I_r b_0 &= 0, \\
F_x(x)v_1 - I_{r'} b_1 &= 0, \\
G_{x, \lambda_1, b_0}(x, \lambda_1, b_0)v_2 &= 0,
\end{align*}
\]
\(\Rightarrow\) “imperfect singular solutions”
\[
\begin{aligned}
\begin{cases}
  F(x) - l_r b_0 &= 0, \\
  F_x(x)v_1 - l_{r'} b_1 &= 0, \\
  G_{x,\lambda_1,b_0}(x, \lambda_1, b_0)v_2 &= 0,
\end{cases}
\end{aligned}
\]

\[\Rightarrow \text{“imperfect singular solutions”}\]
2nd Deflation

\[
\begin{align*}
  F(x) - l_r b_0 &= 0, \\
  F_x(x)v_1 - l_{r'} b_1 &= 0, \\
  G_{x, \lambda_1, b_0}(x, \lambda_1, b_0)v_2 &= 0,
\end{align*}
\]

⇒ “imperfect singular solutions”

\[
\begin{align*}
  F(x) - l_r b_0 - X b_1 &= 0, \\
  F_x(x)v_1 - l_{r'} b_1 &= 0, \\
  \tilde{G}_{x, \lambda_1, b_0}(x, \lambda_1, b_0, b_1)v_2 &= 0,
\end{align*}
\]

\(X\) consists of \(x_{c'(i)}e_{r'(i)}, \ i = 1, \ldots, d'\)
2nd Deflation

\[
\begin{align*}
F(x) - l_r b_0 &= 0, \\
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\tilde{G}_{x, \lambda_1, b_0}(x, \lambda_1, b_0, b_1)v_2 &= 0,
\end{align*}
\]
\[
\Rightarrow \quad X \text{ consists of } x_{c'(i)}e_{r'(i)}, \ i = 1, \ldots, d'
\]

\[
\tilde{F}(x, b) = F(x) - l_r b_0 - X b_1
\]
2nd Deflation

\[
\begin{cases}
F(x) - l_r b_0 = 0, \\
F_x(x)v_1 - l_{r'} b_1 = 0, \\
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\tilde{G}_{x,\lambda_1,b_0}(x, \lambda_1, b_0, b_1)v_2 = 0,
\end{cases}
\]

\[
\tilde{F}(x, b) = F(x) - l_r b_0 - X b_1 \\
F_x(x)v_1 - l_{r'} b_1 = 0 \iff \tilde{F}_x(x, b)v_1 = 0
\]

\[X \text{ consists of } x_{c'(i)} e_{r'(i)}, i = 1, \ldots, d'\]
Modified Deflation

Theorem

After \( s \) deflations, we obtain

\[
\begin{align*}
\tilde{F}(\mathbf{x}, \mathbf{b}) &= 0, \\
\tilde{F}_x(\mathbf{x}, \mathbf{b})\mathbf{v}_1 &= 0, \\
\tilde{G}_{x, \lambda_1, \mathbf{b}_0}(\mathbf{x}, \lambda_1, \mathbf{b})\mathbf{v}_2 &= 0, \\
&\quad \cdots = 0,
\end{align*}
\]

and an isolated solution \((\hat{\mathbf{x}}, \hat{\lambda}, \mathbf{0})\), where

\[
\tilde{F}(\mathbf{x}, \mathbf{b}) = F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 - \cdots - X_s\mathbf{b}_s,
\]

and \( X_k \) consists of \( \frac{1}{k!} \cdot x^{k}_{c(k)(i)} \cdot e_{r(k)(i)}, i = 1, \ldots, d^{(k)} \).
After $s$ deflations, we obtain

\[
\begin{align*}
\tilde{F}(x, b) &= 0, \\
\tilde{F}_x(x, b)v_1 &= 0, \\
\tilde{G}_{x, \lambda_1, b_0}(x, \lambda_1, b)v_2 &= 0, \\
&\vdots &= 0,
\end{align*}
\]

and an isolated solution $(\hat{x}, \hat{\lambda}, 0)$, where

\[
\tilde{F}(x, b) = F(x) - X_0 b_0 - X_1 b_1 - \cdots - X_s b_s,
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and $X_k$ consists of \( \frac{1}{k!} \cdot x^{k}_{c(k)(i)} \cdot e^{r(k)(i)} \), \( i = 1, \ldots, d^{(k)} \).

Deflation $\#$ to derive a regular solution is strictly less than $\mu$. 
Suppose Theorem KMR is applicable to the modified system, and yields inclusions for $\hat{x}$ and $\hat{b}$. Then the perturbed system $\tilde{F}(x, b)$ has an isolated singular solution at $\hat{x}$. 
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$(\hat{x}, \hat{\lambda}, \hat{b})$ is the unique regular solution of

$$
\begin{align*}
\tilde{F}(x, b) & = 0, \\
\tilde{F}_x(x, b)v_1 & = 0, \\
\tilde{G}_{x,\lambda_1,b_0}(x, \lambda_1, b)v_2 & = 0, \\
\vdots & = 0,
\end{align*}
$$

inside the inclusions.

$$
\tilde{F}_x(\hat{x}, \hat{b})\hat{v}_1 = 0 \quad \text{and} \quad \hat{v}_1 \neq 0 \Rightarrow \text{rank}(\tilde{F}_x(\hat{x}, \hat{b})) < n.
$$

$\hat{x}$ is an isolated singular solution of $\tilde{F}(x, \hat{b})$. 

Example DZ1

**Example (DZ1, DZ05)**

Consider a polynomial system

\[ F = \{ x_1^4 - x_2 x_3 x_4, x_2^4 - x_1 x_3 x_4, x_3^4 - x_1 x_2 x_4, x_4^4 - x_1 x_2 x_3 \}. \]

The system \( F \) has \((0, 0, 0, 0)\) as a **131-fold** isolated zero.
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Consider a polynomial system

\[ F = \{ x_1^4 - x_2x_3x_4, x_2^4 - x_1x_3x_4, x_3^4 - x_1x_2x_4, x_4^4 - x_1x_2x_3 \}. \]

The system \( F \) has \((0, 0, 0, 0)\) as a 131-fold isolated zero.

\( F_x(\hat{x}) \) is a zero matrix, we derive \( d = 4, c = r = \{1, 2, 3, 4\} \) and

\[
G(x, b_0) = \begin{cases} 
F(x) - l_r b_0 = 0, \\
F_x(x)v_1 = 0, \\
v_1 = (1, 1, 1, 1)^T.
\end{cases}
\]

Its Jacobian matrix computes to

\[
G_{x,b_0}(0) = \begin{pmatrix}
O_{4,4} & -l_r \\
O_{4,4} & O_{4,4}
\end{pmatrix},
\]

so that \( d' = 4, c' = r' = \{1, 2, 3, 4\} \).
Example DZ1

\[
H(x, \lambda, b) = \begin{cases} 
F(x) - l_r b_0 - X_1 b_1 = 0, \\
F_x(x)v_1 - l_r b_1 = 0, \\
\tilde{G}_{x,b_0}(x, b_0, b_1)v_2 = 0,
\end{cases}
\]

\( (0, 0, 0, 0) \) is the unique solution of \( \tilde{G}_{x,b_0}(0)v_2 = 0 \).

\[
H_{x,\lambda,b}(0) = \begin{pmatrix}
O_{4,4} & O_{4,4} & O_{4,4} & -l_r \\
O_{4,4} & O_{4,4} & -l_{r'} & O_{4,4} \\
O_{4,4} & -l_{r'} & -l_{r'} & O_{4,4} \\
A & O_{4,4} & O_{4,4} & O_{4,4}
\end{pmatrix},
A = \begin{pmatrix}
0 & -2 & -2 & -2 \\
-2 & 0 & -2 & -2 \\
-2 & -2 & 0 & -2 \\
-2 & -2 & -2 & 0
\end{pmatrix}.
\]

\( H_{x,\lambda,b}(0) \) is nonsingular now, so we obtain \( H(x, \lambda, b) \) and

\[
\tilde{F}(x, b) = \begin{cases} 
x_1^4 - x_2 x_3 x_4 - b_1 - b_5 x_1 \\
x_2^4 - x_1 x_3 x_4 - b_2 - b_6 x_2 \\
x_3^4 - x_1 x_2 x_4 - b_3 - b_7 x_3 \\
x_4^4 - x_1 x_2 x_3 - b_4 - b_8 x_4
\end{cases}.
\]
Apply INTLAB function `verifynlss` with

\[(0.003, 0.010, 0.003, 0.007, 0, 0, 0, 0, 0, \ldots, 0)\]

and yields inclusions

\[-10^{-321} \leq \hat{x}_i \leq 10^{-321}, \text{ for } i = 1, 2, 3, 4,\]

\[-10^{-321} \leq \hat{b}_i \leq 10^{-321}, \text{ for } i = 1, 2, \ldots, 8,\]

which proves that \(\tilde{F}(x, \hat{b}) (|\hat{b}_i| \leq 10^{-321}, i = 1, 2, \ldots, 8)\) has an isolated singular solution \(\hat{x}\) within \(|\tilde{x}_i| \leq 10^{-321}, i = 1, 2, 3, 4,\) where

\[
\tilde{F}(x, \hat{b}) = \left\{ \begin{array}{c}
  x_1^4 - x_2x_3x_4 - \hat{b}_1 - \hat{b}_5x_1 \\
  x_2^4 - x_1x_3x_4 - \hat{b}_2 - \hat{b}_6x_2 \\
  x_3^4 - x_1x_2x_4 - \hat{b}_3 - \hat{b}_7x_3 \\
  x_4^4 - x_1x_2x_3 - \hat{b}_4 - \hat{b}_8x_4 \\
\end{array} \right\}.
\]
Add two small perturbations to DZ1:

\[ \{ x_1^4 - x_2 x_3 x_4 + 10^{-12}, x_2^4 - x_1 x_3 x_4, x_3^4 - x_1 x_2 x_4, x_4^4 - x_1 x_2 x_3 - 10^{-12} x_4 \} \]

This "approximate" system has no isolated singular solutions, but a "cluster" of roots near the original. Apply `verifynlss`, it yields

\[-10^{-25} \leq \hat{x}_i \leq 10^{-25}, \text{ for } i = 1, 2, 3, 4,\]

\[10^{-12} - 10^{-25} \leq \hat{b}_1 \leq 10^{-12} + 10^{-25},\]

\[-10^{-25} \leq \hat{b}_i \leq 10^{-25}, \text{ for } i = 2, 3, \ldots, 7,\]

\[-10^{-12} - 10^{-25} \leq \hat{b}_8 \leq -10^{-12} + 10^{-25},\]

which proves \( \tilde{F}(\mathbf{x}, \mathbf{b}) \) has an isolated singular solution \( \hat{\mathbf{x}} \) within above computed bounds.
### Experiment

| System  | $n$ | $\mu$ | Breadth   | $|\hat{x}|$ | $|X|$ | $|B|$        |
|---------|-----|-------|-----------|------------|-----|-------------|
| DZ1     | 4   | 131   | $4 \rightarrow 4 \rightarrow 0$ | e-3        | e-321 | e-321       |
| DZ2     | 3   | 16    | $2 \rightarrow 2 \rightarrow 1 \rightarrow 0$ | e-3        | e-14  | e-14        |
| cbms1   | 3   | 11    | $3 \rightarrow 0$ | e-3        | e-321 | e-321       |
| cbms2   | 3   | 8     | $3 \rightarrow 0$ | e-3        | e-321 | e-321       |
| mth191  | 3   | 4     | $2 \rightarrow 0$ | e-3        | e-14  | e-14        |
| KSS     | 10  | 638   | $9 \rightarrow 0$ | e-3        | e-14  | e-14        |
| Caprассе | 4   | 4     | $2 \rightarrow 0$ | e-3        | e-14  | e-14        |
| RuGr09  | 2   | 4     | $1 \rightarrow 1 \rightarrow 1 \rightarrow 0$ | e-3        | e-14  | e-14        |
| LiZhi12 | 1000 | 3    | $1 \rightarrow 1 \rightarrow 0$ | e-4        | e-12  | e-12        |

$n$ is the size of System, $\mu$ is the multiplicity, $|\hat{x}|$ is the number of correct digits for initial guess, $|X|$ and $|B|$ list the length of inclusions for $\hat{x}$ and $\hat{b}$.

- $(2, -i\sqrt{3}, 2, i\sqrt{3})$ is a 4-fold isolated solution of Caprассе.
References


Thanks 谢谢