Bressoud's Conjecture on the Rogers-Ramanujan Identities

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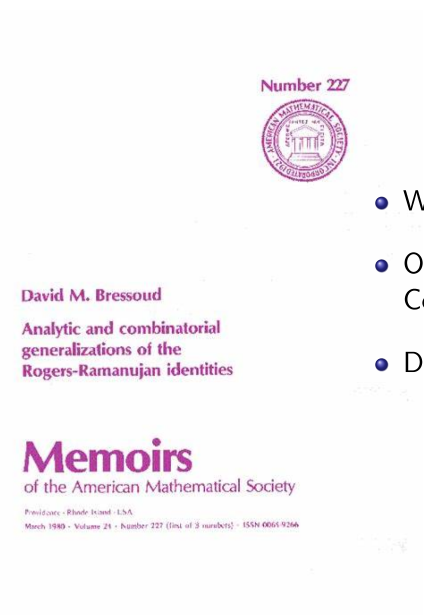
The outline of this talk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David M. Bressoud

Analytic and combinatorial generalizations of the Rogers-Ramanujan identities

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- What's Bressoud's conjecture?
- Our Proof: Unraveling Bressoud's Conjecture.
- Discoveries in the Proof of this Conjecture

The Rogers-Ramanujan identities

Theorem (Rogers-Ramanujan)

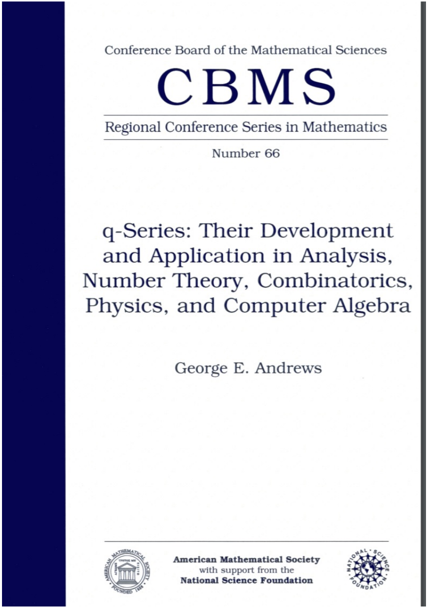
For $|q| < 1$,

$$\sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} \quad \textit{(RR1)}$$

$$\sum_{n \geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})} \quad \textit{(RR2)}.$$

These two identities were first proved by Rogers in 1894 and rediscovered by Ramanujan a few years later.

- Ramanujan's comment: It would be difficult to find more beautiful formulas than the Rogers-Ramanujan' identities.
- Hans Rademacher clearly was in agreement with Ramanujan's comments.



- There is a beautiful story about the discovery of the Rogers-Ramanujan identities.
- Rogers' reputation as a mathematician rests almost entirely on the discovery of these two identities.
- how they found these two identities have appeared in Baxter's exact solution to the hard hexagon model in statistical mechanics.

- The connection between RR identities and the representation theory of Lie algebras was initiated by J. Lepowsky, R.L. Wilson, S. Milne.

J. Lepowsky and S. Milne, Lie algebraic approaches to classical partition identities, Adv. Math. 29 (1978), 15–59

J. Lepowsky and R. L. Wilson, A new family of algebras underlying the Rogers-Ramanujan identities, Proc. Nat. Acad. Sci. USA 78 (1981) 7254–7258.

J. Lepowsky and R. L. Wilson, The structure of standard modules I: universal algebras and the Rogers–Ramanujan identities, Invent. Math. 77 (1984), 199–290.

J. Lepowsky and R. L. Wilson, The structure of standard modules II: the case $A_1^{(1)}$, principal gradation, Invent. Math. 79 (1985), 417–442.

Integer partitions

Definition

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\pi_1, \pi_2, \ldots, \pi_\ell)$ such that

$$\pi_1 + \pi_2 + \cdots + \pi_\ell = n.$$

Example: There are five partitions of 4, which are

$$(4), (3,1), (2,2), (2,1,1), (1,1,1,1).$$

Let $p(n)$ denote the number of partitions of n, we see that $p(4) = 5$.

Generating functions

Theorem (Euler)

For $|q| < 1$,

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}$$

The sketch of the proof: Consider the following expansion

$$\begin{aligned}
\prod_{j=1}^{\infty} \frac{1}{1-q^j} =& (1 + q^1 + q^{1+1} + q^{1+1+1} + \cdots) \quad (\frac{1}{1-q}) \\
& \times (1 + q^2 + q^{2+2} + q^{2+2+2} + \cdots) \quad (\frac{1}{1-q^2}) \\
& \times (1 + q^3 + q^{3+3} + q^{3+3+3} + \cdots) \quad (\frac{1}{1-q^3}) \\
& \times \cdots\cdots \\
& = \sum_{a_1 \geq 0} \sum_{a_2 \geq 0} \sum_{a_3 \geq 0} \cdots q^{1 \cdot a_1 + 2 \cdot a_2 + 3 \cdot a_3 + \cdots},
\end{aligned}$$

Observe that the exponent of q is just the partition $(1^{a_1} 2^{a_2} 3^{a_3} \cdots)$.

Euler's partition identity

Theorem (Euler)

For $n \geq 1$, the number of partitions of n into parts $\equiv 1 \pmod 2$ equals the number of $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 1$ for $1 \leq i \leq \ell - 1$.

Example: There are five partitions of 4, which are

$$(4), (3,1), (2,2), (2,1,1), (1,1,1,1).$$

where $(3,1)$, $(1,1,1,1)$ are partitions into odd parts, and $(4), (3,1)$ are partitions with distinct parts.

$$\prod_{j=1}^{\infty} \frac{1}{(1-q^{2j-1})} = \prod_{j=1}^{\infty} \frac{(1-q^{2j})}{(1-q^j)} = \prod_{j=1}^{\infty} \frac{(1+q^j)(1-q^j)}{(1-q^j)} = \prod_{j=1}^{\infty} (1+q^j).$$

Theorem (MacMahon)

- *Let* $A_1(-;1,2,2;n)$ *denote the number of partitions of* n *into parts* $\equiv 1,4 \pmod 5$ *(equivalently,* $\not\equiv 0, \pm 2 \pmod 5$*);*
- *Let* $B_1(-;1,2,2;n)$ *denote the number of partitions* $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ *of* n *with* $\pi_i - \pi_{i+1} \geq 2$ *for* $1 \leq i \leq \ell - 1$*.*

Then for $n \geq 0$*,*

$$A_1(-;1,2,2;n) = B_1(-;1,2,2;n).$$

$$\sum_{n\geq 0} A_1(-;1,2,2;n)q^n = \frac{1}{(q,q^4;q^5)_\infty}$$

$$\overset{(RR1)}{=} \sum_{n=0}^{\infty} \frac{q^{n^2} \ (=1+3+\cdots+2n-1)}{(q;q)_n} = \sum_{n\geq 0} B_1(-;1,2,2;n)q^n.$$

Notation:

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad (a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

$$(a_1, a_2, \ldots, a_k; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_k;q)_n.$$

MacMahon's combinatorial interpretation

Theorem (MacMahon)

- *Let* $A_1(-;1,2,1;n)$ *denote the number of partitions of* n *into parts* $\equiv 2,3 \pmod 5$ *(equivalently,* $\not\equiv 0,\pm 1 \pmod 5$*);*
- *Let* $B_1(-;1,2,1;n)$ *denote the number of partitions* $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ *of* n *with* $\pi_i - \pi_{i+1} \geq 2$ *for* $1 \leq i \leq \ell - 1$ *and* $\pi_\ell \geq 2$.

Then for $n \geq 0$,

$$A_1(-;1,2,1;n) = B_1(-;1,2,1;n).$$

$$\sum_{n\geq 0} A_1(-;1,2,1;n)q^n = \frac{1}{(q^2,q^3;q^5)_\infty}$$

$$\overset{(RR2)}{=} \sum_{n=0}^{\infty} \frac{q^{n^2+n \ (=2+4+\cdots+2n)}}{(q;q)_n} = \sum_{n\geq 0} B_1(-;1,2,1;n)q^n.$$

Gordon's combinatorial generalization

Theorem (Rogers-Ramanujan-Gordon, 1961)

- *For* $k \geq r \geq 1$, *let* $A_1(-;1,k,r;n)$ *denote the number of partitions of* n *into parts* $\not\equiv 0, \pm r \pmod{2k+1}$.
- *For* $k \geq r \geq 1$, *let* $B_1(-;1,k,r;n)$ *denote the number of partitions of* n *of the form* $(\pi_1, \ldots, \pi_s)$, *where* $\pi_j - \pi_{j+k-1} \geq 2$, *and at most* $r-1$ *of the* π_j *are equal to* 1.

Then, for $k \geq r \geq 1$ *and* $n \geq 0$,

$$A_1(-;1,k,r;n) = B_1(-;1,k,r;n).$$

When $k = 2$ and $r = 2, 1$, this identity reduces to (RR1) or (RR2).
Note: Gordon's theorem was independently discovered by G.E. Andrews, see G.E. Andrews, Some debts I owe.

B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, Amer. J. Math 83 (1961) 393–399.

The corresponding q-identity due to Andrews

Andrews found the following q-identity, which can be viewed as a companion to Gordon's partition identity.

Theorem (Andrews, 1974)

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q;q)_{N_{k-1}}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm r \pmod{2k+1}}}^{\infty} \frac{1}{1-q^n}.$$

When $k = 2$ and $r = 2, 1$ this identity reduces to (RR1) and (RR1).

G.E. Andrews, An analytic generalization of the Rogers-Ramanujan identities for odd moduli, Proc. Nat. Acad. Sci. USA 71 (1974) 4082–4085.

The combinatorial interpretation

- It is immediate to see that

$$\sum_{n\geq 0} A_1(-;1,k,r;n)q^n = \prod_{\substack{n=1 \\ n\not\equiv 0,\pm r \pmod{2k+1}}}^{\infty} \frac{1}{1-q^n}$$

- But the combinatorial proof of the generating function for $B_1(-;1,k,r;n)$ was given by Kurşungöz in 2010 by introducing the notion of the Gordon marking of a partition, and defining the forward move and the backward move, Kurşungöz gave a combinatorial proof of the following generating function.

$$\sum_{n\geq 0} B_1(-;1,k,r;n)q^n$$

$$= \sum_{N_1\geq N_2\geq \ldots \geq N_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}}}{(q;q)_{N_1-N_2}(q;q)_{N_2-N_3}\cdots(q;q)_{N_{k-1}}}.$$

K. Kurşungöz, Parity considerations in Andrews-Gordon identities, European J. Combin. 31 (2010) 976–1000.

Bressoud's combinatorial generalization

Bressoud observed that Gordon's partition identity only focused on the parts modular the odd numbers. He addressed the case involving the even numbers and derived the following partition identity.

Theorem (Bressoud-Rogers-Ramanujan, 1979)

- *For $k \geq r \geq 1$, let $A_0(-;1,k,r;n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm r \pmod{2k}$.*
- *For $k \geq r \geq 1$, let $B_0(-;1,k,r;n)$ denote the number of partitions of n of the form $(\pi_1, \ldots, \pi_s)$, where $\pi_j - \pi_{j+k-1} \geq 2$, at most $r-1$ of the π_j are equal to 1.*

  If $\pi_j - \pi_{j+k-2} \leq 1$, then $\pi_j + \cdots + \pi_{j+k-2} \equiv r-1 \pmod{2}$.

Then, for $k \geq r \geq 1$ and $n \geq 0$,

$$A_0(-;1,k,r;n) = B_0(-;1,k,r;n).$$

D.M. Bressoud, A generalization of the Rogers-Ramanujan identities for all moduli, J. Combin. Theory, Ser. A 27 (1979) 64–68.

Bressoud obtained the following q-identity, which can be viewed as a companion to his partition identity.

Theorem (Bressoud, 1980)

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm r \pmod{2k}}}^{\infty} \frac{1}{1-q^n}.$$

D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

A general theme

$$A(n; \text{congruence conditions on parts})$$

$$\Updownarrow$$

$$B(n; \text{ gap conditions on parts})$$

Some classical partition identities

- Euler's partition theorem: The number of partitions of n into parts $\equiv 1 \pmod 2$ is equal to the number of partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 1$ for $1 \leq i \leq \ell - 1$.

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}.$$

- Schur's theorem: The number of partitions of n into parts $\equiv \pm 1 \pmod 6$ is equal to the number of partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 3$ for $1 \leq i \leq \ell - 1$ with strict inequality if $\pi_i \equiv 0 \pmod 3$.

$$\sum_{n\geq 0} \frac{q^{3n^2}}{(q^3;q^3)_n} (-q^{2-3n};q^3)_n (-q^{1+3n};q^3)_\infty = \frac{1}{(q,q^5;q^6)_\infty}.$$

Some classical partition identities

- The Göllnitz-Gordon theorem I: The number of partitions of n into parts $\equiv 1, 4,$ or $7 \pmod 8$ is equal to the number of partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 2$ for $1 \leq i \leq \ell - 1$ with strict inequality if $\pi_i \equiv 0 \pmod 2$.

$$\sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q,q^4,q^7;q^8)_\infty}.$$

- The Göllnitz-Gordon theorem II: The number of partitions of n into parts $\equiv 3, 4,$ or $5 \pmod 8$ is equal to the number of partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 2$ for $1 \leq i \leq \ell - 1$ with strict inequality if $\pi_i \equiv 0 \pmod 2$. Furthermore, $\pi_\ell \geq 3$.

$$\sum_{n\geq 0} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty}.$$

Andrews' combinatorial generalization

Theorem (Andrews-Göllnitz-Gordon, 1967)

- *For $k \geq r \geq 1$, let $A_1(1;2,k,r;n)$ denote the number of partitions of n into parts $\not\equiv 2 \pmod 4$ and $\not\equiv 0, \pm(2r-1) \pmod{4k}$.*
- *For $k \geq r \geq 1$, let $B_1(1;2,k,r;n)$ denote the number of partitions of n of the form $(\pi_1, \ldots, \pi_s)$ such that no odd part is repeated, $\pi_j - \pi_{j+k-1} \geq 2$ with strict inequality if π_j is even, and at most $r-1$ of the π_j are less than or equal to 2.*

Then, for $k \geq r \geq 1$ and $n \geq 0$,

$$A_1(1;2,k,r;n) = B_1(1;2,k,r;n).$$

When $k = 2$ and $r = 1$, this theorem reduces to GGT-II. When $k = 2$ and $r = 2$, this theorem reduces to GGT-I.

G.E. Andrews, A generalization of the Göllnitz-Gordon partition theorems, Proc. Amer. Math. Soc. 18 (1967) 945–952.

The corresponding q-identity

Theorem

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \cdots + N_{k-1}^2 + N_r + \cdots + N_{k-1})}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}$$

$$= \frac{(q^2; q^4)_\infty (q^{2r-1}, q^{4k-2r+1}, q^{4k}; q^{4k})_\infty}{(q; q)_\infty}.$$

A further generalization

Assume that $\alpha_1, \alpha_2, \ldots, \alpha_\lambda$ and η are integers such that for $1 \le i \le \lambda$,
$0 < \alpha_1 < \cdots < \alpha_\lambda < \eta$, and $\alpha_i = \eta - \alpha_{\lambda+1-i}$.

Theorem (Bressoud, Mem. Amer. Math. Soc., 1980)

Let λ, k, r and $j = 0$ or 1 be the integers such that $(2k+j)/2 > r \ge \lambda \ge 0$. Then

$$\sum_{N_1 \ge \cdots \ge N_{k-1} \ge 0} \frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^{(2-j)\eta};q^{(2-j)\eta})_{N_{k-1}}}$$

$$\times \prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s}\prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty$$

$$= \frac{(-q^{\alpha_1},\ldots,-q^{\alpha_\lambda};q^\eta)_\infty(q^{\eta(r-\frac{\lambda}{2})},q^{\eta(2k-r-\frac{\lambda}{2}+j)},q^{\eta(2k-\lambda+j)};q^{\eta(2k-\lambda+j)})_\infty}{(q^\eta;q^\eta)_\infty}.$$

The A_j-partition function

Definition

Let λ, k, r and $j = 0$ or 1 be the integers such that $(2k + j)/2 > r \geq \lambda \geq 0$. Define the partition function $A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ to be the number of partitions of n into parts congruent to $0, \alpha_1, \ldots, \alpha_\lambda \pmod{\eta}$ such that

- *If λ is even, then only multiples of η may be repeated and no part is congruent to $0, \pm\eta(r - \lambda/2) \pmod{\eta(2k - \lambda + j)}$;*
- If λ is odd and $j = 1$, then only multiples of $\eta/2$ may be repeated, no part is congruent to $\eta \pmod{2\eta}$, and no part is congruent to $0, \pm\eta(2r - \lambda)/2 \pmod{\eta(2k - \lambda + 1)}$;
- If λ is odd and $j = 0$, then only multiples of $\eta/2$ which are not congruent to $\eta(2k - \lambda)/2 \pmod{\eta(2k - \lambda)}$ may be repeated, no part is congruent to $\eta \pmod{2\eta}$, no part is congruent to $0 \pmod{2\eta(2k - \lambda)}$, and no part is congruent to $\pm\eta(2r - \lambda)/2 \pmod{\eta(2k - \lambda)}$.

Remark: Recall that $\alpha_1, \alpha_2, \ldots, \alpha_\lambda$ and η are integers such that for $1 \leq i \leq \lambda$, $0 < \alpha_1 < \cdots < \alpha_\lambda < \eta$, and $\alpha_i = \eta - \alpha_{\lambda+1-i}$. When λ is odd, observing that

$$\eta = \alpha_{(\lambda+1)/2} + \alpha_{\lambda+1-(\lambda+1)/2} = 2\alpha_{(\lambda+1)/2},$$

we see that η must be even in such case.

By the definition, it's not difficult to see that for $k \geq r \geq \lambda \geq 0$, $k + j - 1 \geq \lambda$ and $j = 0$ or 1,

$$\sum_{n \geq 0} A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n$$

$$= (-q^{\alpha_1}, \ldots, -q^{\alpha_\lambda}; q^\eta)_\infty$$

$$\times \frac{(q^{\eta(r-\frac{\lambda}{2})}, q^{\eta(2k-r-\frac{\lambda}{2}+j)}, q^{\eta(2k-\lambda+j)}; q^{\eta(2k-\lambda+j)})_\infty}{(q^\eta; q^\eta)_\infty}.$$

The B_j-partition function

Definition

Let λ, k, r *and* $j = 0$ *or* 1 *be the integers such that* $(2k+j)/2 > r \geq \lambda \geq 0$. *Define* $B_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ *to be the number of partitions of n of the form* $(\pi_1, \ldots, \pi_s)$ *where* $\pi_i \geq \pi_{i+1}$ *satisfying the following conditions:*

(1) $\pi_i \equiv 0, \alpha_1, \ldots, \alpha_\lambda \pmod{\eta}$;

(2) Only multiples of η may be repeated;

(3) $\pi_i - \pi_{i+k-1} \geq \eta$ with strict inequality if $\eta \mid \pi_i$;

(4) At most $r-1$ of the π_i are less than or equal to η;

(5) If $\pi_i \leq \pi_{i+k-2} + \eta$ with strict inequality if $\eta \nmid \pi_i$, then

$$[\pi_i/\eta] + \cdots + [\pi_{i+k-2}/\eta] \equiv r - 1 + V_\pi(\pi_i) \pmod{2-j},$$

where $V_\pi(N)$ (or $V(N)$ for short) denotes the number of parts not exceeding N which are not divided by η in π and $[\,]$ denotes the greatest integer function.

Bressoud's conjecture

Conjecture (Bressoud)

Let λ, k, r and $j = 0$ or 1 be the integers such that $(2k + j)/2 > r \geq \lambda \geq 0$. Then

$$\sum_{n \geq 0} B_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n$$

$$= \sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{\eta(N_1^2 + \cdots + N_{k-1}^2 + N_r + \cdots + N_{k-1})}}{(q^\eta; q^\eta)_{N_1 - N_2} \cdots (q^\eta; q^\eta)_{N_{k-2} - N_{k-1}} (q^{(2-j)\eta}; q^{(2-j)\eta})_{N_{k-1}}}$$

$$\times \prod_{s=1}^{\lambda} (-q^{\eta - \alpha_s - \eta N_s}; q^\eta)_{N_s} \prod_{s=2}^{\lambda} (-q^{\eta - \alpha_s + \eta N_{s-1}}; q^\eta)_\infty.$$

D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

Bressoud's conjecture II

Conjecture (Bressoud)

Let λ, k, r and $j = 0$ or 1 be the integers such that $(2k + j)/2 > r \geq \lambda \geq 0$. Then for $n \geq 0$,

$$A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) = B_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n).$$

D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

Special cases

- Euler's partition theorem: $\lambda = 0, \eta = 1, j = 0, k = 3, r = 2$.
- Shur's theorem: $\lambda = k = r = 2, \alpha_1 = 1, \alpha_2 = 2, \eta = 3, j = 1$.
- Rogers-Ramanujan identities: $\lambda = 0, \eta = 1, j = 1, k = 2, r = 1$ or 2.
- Rogers-Ramanujan-Gordon identity: $\lambda = 0, \eta = 1, j = 1$.
- Rogers-Ramanujan-Bressoud identity: $\lambda = 0, \eta = 1, j = 0$.
- The Göllnitz-Gordon identity I, II: $\lambda = 1, \alpha_1 = 1, \eta = 2, j = 1, k = 2, r = 1$ or 2.
- Andrews' generalization of the Göllnitz-Gordon identity: $\lambda = 1, \alpha_1 = 1, \eta = 2, j = 1$.
- Bressoud's generalization of the Göllnitz-Gordon identity: $\lambda = 1, \alpha_1 = 1, \eta = 2, j = 0$.

Bressoud-Göllnitz-Gordon theorem

Theorem (Bressoud-Göllnitz-Gordon)

- *For* $k \geq r \geq 1$, *let* $A_0(1;2,k,r;n)$ *denote the number of partitions of* n *into parts* $\not\equiv 2 \pmod 4$, $\not\equiv 2k-1 \pmod{4k-2}$ *may be repeated, no part is multiples of* $8k-4$, *and* $\not\equiv \pm(2r-1) \pmod{4k-2}$.
- *For* $k \geq r \geq 1$, *let* $B_0(1;2,k,r;n)$ *denote the number of partitions of* n *of the form* $(\pi_1,\ldots,\pi_s)$ *such that no odd part is repeated,* $\pi_j - \pi_{j+k-1} \geq 2$ *with strict inequality if* π_j *is even, at most* $r-1$ *of the* π_j *are less than or equal to* 2, *and if* $\pi_j - \pi_{j+k-2} \leq 2$ *with strict inequality if* π_j *is odd, then*
$$\pi_j + \cdots + \pi_{j+k-2} \equiv r - 1 + V_\pi(\pi_j) \pmod 2,$$
where $V_\pi(N)$ (or $V(N)$ for short) denotes the number of odd parts not exceeding N in π.

For $k \geq r \geq 1$ *and* $n \geq 0$,
$$A_0(1;2,k,r;n) = B_0(1;2,k,r;n).$$

D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

The corresponding q-identities

Theorem

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}}{(q^2;q^2)_{N_1-N_2} \cdots (q^2;q^2)_{N_{k-2}-N_{k-1}} (q^4;q^4)_{N_{k-1}}}$$

$$= \frac{(q^2;q^4)_\infty (q^{2r-1}, q^{4k-2r-1}, q^{4k-2}; q^{4k-2})_\infty}{(q;q)_\infty}.$$

D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

Some progress on Bressoud's conjecture

- S. Kim and A.J. Yee (2014) gave a proof of Bressoud's conjecture for $j = 1$ and $\lambda = 2$.

S. Kim and A.J. Yee, Partitions with part difference conditions and Bressoud's conjecture, J. Combin. Theory Ser. A 126 (2014) 35–69.

Let $\lambda,\ k,\ r$ be the integers such that $k \geq r \geq \lambda \geq 0$. Then

$$\sum_{n\geq 0} B_1(\alpha_1,\alpha_2;\eta,k,r;n)q^n$$

$$= \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^\eta;q^\eta)_{N_{k-1}}}$$

$$\times (-q^{\eta-\alpha_1-\eta N_1};q^\eta)_{N_1}(-q^{\eta-\alpha_2-\eta N_2};q^\eta)_{N_2}(-q^{\eta-\alpha_2+\eta N_1};q^\eta)_\infty$$

Some progress on Bressoud's conjecture

- S. Kim (2018) proved Bressoud's conjecture holds for $j = 1$.

S. Kim, Bressoud's conjecture, Adv. Math. 325 (2018) 770–813.

$$\sum_{n\geq 0} B_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$
$$= (-q^{\alpha_1},\ldots,-q^{\alpha_\lambda};q^\eta)_\infty$$
$$\times \frac{(q^{\eta(r-\frac{\lambda}{2})},q^{\eta(2k-r-\frac{\lambda}{2}+1)},q^{\eta(2k-\lambda+1)};q^{\eta(2k-\lambda+1)})_\infty}{(q^\eta;q^\eta)_\infty}$$
$$= \sum_{n\geq 0} A_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n.$$

- Recently, we proved Bressoud's conjecture holds for $j = 0$.

Thomas Y. He, Kathy Q. Ji and Alice X.H. Zhao, Overpartitions and Bressoud's conjecture, I, Adv. Math. 404 (2022), Paper No. 108449, 81 pp.

Thomas Y. He, Kathy Q. Ji and Alice X.H. Zhao, Overpartitions and Bressoud's conjecture, II, submitted. pp.53.

Definition (Corteel and Lovejoy, 2004)

An overpartition π of n is a partition of n in which the first occurrence of a number can be overlined.

For example: There are fourteen overpartitions of 4.

(4)	$(\bar{4})$		
$(3,1)$	$(\bar{3},1)$	$(3,\bar{1})$	$(\bar{3},\bar{1})$
$(2,2)$	$(\bar{2},2)$		
$(2,1,1)$	$(\bar{2},1,1)$	$(2,\bar{1},1)$	$(\bar{2},\bar{1},1)$
$(1,1,1,1)$	$(\bar{1},1,1,1)$		

The overpartition $(\bar{2},\bar{1},1)$ can be viewed as $((\bar{2},\bar{1}),(1))$
Let $\bar{p}(n)$ denote the number of overpartitions of n. Then

$$\sum_{n\geq 0}\bar{p}(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}.$$

S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356

Theorem (Chen-Shi-Sang, 2013)

- *For $k > r \geq 1$, let $\overline{A}_1(-;1,k,r;n)$ denote the number of overpartitions of n such that non-overlined parts $\not\equiv 0, \pm r \pmod{2k}$, and for $k = r$, let $\overline{A}_1(-;1,k,k;n)$ denote the number of overpartitions of n into parts not divided by k.*
- *For $k \geq r \geq 1$, let $\overline{B}_1(-;1,k,r;n)$ denote the number of overpartitions π of n of the form $(\pi_1,\ldots,\pi_s)$, where $\pi_i - \pi_{i+k-1} \geq 1$ with strict inequality if π_i is non-overlined, and at most $r-1$ of the π_i are equal to 1 .*

Then for $k \geq r \geq 1$ and $n \geq 0$,

$$\overline{A}_1(-;1,k,r;n) = \overline{B}_1(-;1,k,r;n).$$

W.Y.C. Chen, D.D.M. Sang and D.Y.H. Shi, The Rogers-Ramanujan-Gordon thoerem for overpartitions, Proc. London Math. Soc. 106 (3) (2013) 1371–1393.

The corresponding q-identities

Theorem (Chen-Shi-Sang, 2013)

$$\sum_{N_1 \geq \cdots \geq N_{k-1}} \frac{(-q^{1-N_1};q)_{N_1-1}(1+q^{N_r})q^{N_1^2+\cdots+N_{k-1}^2+N_{r+1}+\cdots+N_{k-1}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q;q)_{N_{k-1}}}$$

$$= \frac{(-q;q)_\infty (q^r, q^{2k-r}, q^{2k}; q^{2k})_\infty}{(q;q)_\infty}.$$

W.Y.C. Chen, D.D.M. Sang and D.Y.H. Shi, The Rogers-Ramanujan-Gordon thoerem for overpartitions, Proc. London Math. Soc. 106 (3) (2013) 1371–1393.

An overpartition analogue of Bressoud-Gordon-identities

Theorem (Chen-Sang-Shi)

- *For $k \geq r \geq 1$, let $\overline{A}_0(-;1,k,r;n)$ denote the number of overpartitions of n such that non-overlined parts $\not\equiv 0, \pm r \pmod{2k-1}$;*
- *For $k \geq r \geq 1$, let $\overline{B}_0(-;1,k,r;n)$ denote the number of overpartitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n, where $\pi_i \geq \pi_{i+k-1} + 1$ with strict inequality if π_i is non-overlined for $1 \leq i \leq \ell - k + 1$, at most $r-1$ of the π_i are equal to 1, and for $1 \leq i \leq \ell - k + 2$, if $\pi_i \leq \pi_{i+k-2} + 1$ with strict inequality if π_i is overlined, then* $\pi_i + \cdots + \pi_{i+k-2} \equiv r - 1 + \overline{V}_\pi(\pi_i) \pmod 2$,

  where $\overline{V}_\pi(N)$ (or $\overline{V}_\pi(N)$ for short) denotes the number of overlined parts not exceeding N in π.

Then, for $k \geq r \geq 1$ and $n \geq 0$,

$$\overline{A}_0(-;1,k,r;n) = \overline{B}_0(-;1,k,r;n).$$

W.Y.C. Chen, D.D.M. Sang and D.Y.H. Shi, An overpartition analogue of Bressoud's theorem of Rogers-Ramanujan-Gordon type, Ramanujan J. 37 (2015) 653–679.

The corresponding analytic identities

Theorem (Sang-Shi, 2015)

$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}}(1+q^{-N_r})(-q^{1-N_1};q)_{N_1-1}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}$$

$$=\frac{(-q;q)_\infty(q^r,q^{2k-r-1},q^{2k-1};q^{2k-1})_\infty}{(q;q)_\infty}.$$

D.D.M. Sang and D.Y.H. Shi, An Andrews-Gordon type identity for overpartitions, Ramanujan J. 37 (2015) 653–679.

The overpartition analogue of A_j-partition function

Definition ($\overline{A}_j$-partition function)

Let λ, k, r and $j = 0$ or 1 be the integers such that $(2k+1-j)/2 > r \geq \lambda \geq 0$ and $k+j-1 > \lambda$. Define $\overline{A}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ to be the number of overpartitions of n satisfying $\pi_i \equiv 0, \alpha_1, \ldots, \alpha_\lambda \pmod{\eta}$ such that

- *If λ is even, then only multiples of η may be non-overlined and there is no non-overlined part congruent to $0, \pm\eta(r-\lambda/2) \pmod{\eta(2k-\lambda+j-1)}$;*
- *If λ is odd and $j=1$, then only multiples of $\eta/2$ may be non-overlined, no non-overlined part is congruent to $\eta(2k-\lambda)/2 \pmod{\eta(2k-\lambda)}$, no non-overlined part is congruent to $\eta \pmod{2\eta}$, no non-overlined part is congruent to $0 \pmod{2\eta(2k-\lambda)}$, no non-overlined part is congruent to $\pm\eta(2r-\lambda)/2 \pmod{\eta(2k-\lambda)}$, and no overlined part is congruent to $\eta/2 \pmod{\eta}$ and not congruent to $\eta(2k-\lambda)/2 \pmod{\eta(2k-\lambda)}$;*
- *If λ is odd and $j=0$, then only multiples of $\eta/2$ may be non-overlined, no non-overlined part is congruent to $\eta \pmod{2\eta}$, no non-overlined part is congruent to $0, \pm\eta(2r-\lambda)/2 \pmod{\eta(2k-\lambda-1)}$, and no overlined part is congruent to $\eta/2 \pmod{\eta}$.*

Thomas Y. He, Kathy Q. Ji and Alice X.H. Zhao, Overpartitions and Bressoud's conjecture, I, Adv. Math. 404 (2022), Paper No. 108449, 81 pp.

The overpartition analogue of B_j-partition function

Definition ($\overline{B}_j$-partition function)

Let λ, k, r and $j = 0$ or 1 be the integers such that $k \geq r \geq \lambda \geq 0$ and $k - 1 + j > \lambda$. Define $\overline{B}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ to be the number of overpartitions of n of the form $(\pi_1, \ldots, \pi_s)$ where $\pi_i \geq \pi_{i+1}$ satisfying the following conditions:

(1) $\pi_i \equiv 0, \alpha_1, \ldots, \alpha_\lambda \pmod{\eta}$;

(2) Only multiples of η may be non-overlined;

(3) $\pi_i \geq \pi_{i+k-1} + \eta$ with strict inequality if π_i is non-overlined;

(4) At most $r - 1$ of the π_i are less than or equal to η;

(5) If $\pi_i \leq \pi_{i+k-2} + \eta$ with strict inequality if π_i is overlined, then $[\pi_i/\eta] + \cdots + [\pi_{i+k-2}/\eta] \equiv r - 1 + \overline{V}(\pi_i) \pmod{2 - j}$.

The generating function of $\overline{A}_j$

By definition, it is easy to see that for $k \geq r \geq \lambda \geq 0$, $k+j-1 > \lambda$ and $j = 0$ or 1,

$$\sum_{n\geq 0} \overline{A}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n$$

$$= (-q^{\alpha_1}, \ldots, -q^{\alpha_\lambda}; q^\eta)_\infty (-q^\eta; q^\eta)_\infty$$

$$\times \frac{(q^{\eta(r-\frac{\lambda}{2})}, q^{\eta(2k-r-\frac{\lambda}{2}+j-1)}, q^{\eta(2k-\lambda+j-1)}; q^{\eta(2k-\lambda+j-1)})_\infty}{(q^\eta; q^\eta)_\infty}.$$

Hence, we have

$$\sum_{n\geq 0} \overline{A}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n$$

$$= (-q^\eta; q^\eta)_\infty \sum_{n\geq 0} A_{1-j}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n.$$

The relation between $\overline{B}_1$ and B_0

Theorem (He-Ji-Zhao, 2022)

For $k \geq r \geq \lambda \geq 0$ and $k > \lambda \geq 0$,

$$\sum_{n\geq 0} \overline{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$
$$= (-q^\eta;q^\eta)_\infty \sum_{n\geq 0} B_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n.$$

Thomas Y. He, Kathy Q. Ji and Alice X.H. Zhao, Overpartitions and Bressoud's conjecture, I, Adv. Math. 404 (2022), Paper No. 108449, 81 pp.

The sketch of the proof of Bressoud's conjecture

$$(-q^{\eta};q^{\eta})_{\infty}\sum_{n\geq 0}A_0(\alpha_1,\ldots,\alpha_{\lambda};\eta,k,r;n)q^n$$

$$=\sum_{n\geq 0}\overline{A}_1(\alpha_1,\ldots,\alpha_{\lambda};\eta,k,r;n)q^n$$

$$\overset{????}{=}\sum_{n\geq 0}\overline{B}_1(\alpha_1,\ldots,\alpha_{\lambda};\eta,k,r;n)q^n$$

$$\overset{M1}{=}(-q^{\eta};q^{\eta})_{\infty}\sum_{n\geq 0}B_0(\alpha_1,\ldots,\alpha_{\lambda};\eta,k,r;n)q^n$$

.

An overpartition analogue of Bressoud's conjecture for $j = 1$

By generalizing Kim's method, we main show that

$$\sum_{n\geq 0} \overline{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

$$= \frac{(-q^{\alpha_1},\ldots,-q^{\alpha_\lambda},-q^{\eta};q^{\eta})_\infty(q^{\eta(r-\frac{\lambda}{2})},q^{\eta(2k-r-\frac{\lambda}{2})},q^{\eta(2k-\lambda)};q^{\eta(2k-\lambda)}}{(q^\eta;q^\eta)_\infty}$$

$$= \sum_{n\geq 0} \overline{A}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n.$$

Theorem (He-Ji-Zhao)

Let λ, k, r and η be the integers such that $k \geq r \geq \lambda \geq 0$ and $k > \lambda$. Then for $n \geq 0$,

$$\overline{A}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n) = \overline{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n).$$

Thomas Y. He, Kathy Q. Ji and Alice X.H. Zhao, Overpartitions and Bressoud's conjecture, II, submitted. arXiv: 2001.00162.

The sketch of the proof of Bressoud's conjecture

$$(-q^\eta;q^\eta)_\infty \sum_{n\geq 0} A_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

$$= \sum_{n\geq 0} \overline{A}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

$$\overset{M2}{=} \sum_{n\geq 0} \overline{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

$$\overset{M1}{=} (-q^\eta;q^\eta)_\infty \sum_{n\geq 0} B_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

.

The relation between $\overline{B}_0$ and B_1

Theorem (He-Ji-Zhao, 2022)

For $k > r \geq \lambda \geq 0$ and $k - 1 > \lambda \geq 0$,

$$\sum_{n\geq 0} \overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n$$

$$= (-q^\eta; q^\eta)_\infty \sum_{n\geq 0} B_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r; n) q^n.$$

For $k - 1 > \lambda \geq 0$,

$$\sum_{n\geq 0} \overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, k; n) q^n$$

$$= (-q^\eta; q^\eta)_\infty \sum_{n\geq 0} B_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, k-1; n) q^n.$$

An application of this relation

$$\sum_{n\geq 0}\overline{A}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

$$=(-q^\eta;q^\eta)_\infty\sum_{n\geq 0}A_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

$$\overset{Kim's\_result}{=}(-q^\eta;q^\eta)_\infty\sum_{n\geq 0}B_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

$$\overset{M1}{=}\sum_{n\geq 0}\overline{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

.

An overpartition analogue of Bressoud's conjecture for $j = 0$

Theorem (He-Ji-Zhao)

Let λ, k and r be the integers such that $k \geq r \geq \lambda \geq 0$ and $k - 1 > \lambda$. Then for $n \geq 0$,

$$\overline{A}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) = \overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n).$$

Remark: Kim-Bressoud:
$A_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) = B_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$.

Thomas Y. He, Kathy Q. Ji and Alice X.H. Zhao, Overpartitions and Bressoud's conjecture, I, Adv. Math. 404 (2022), Paper No. 108449, 81 pp.

The corresponding q-identities

We also obtain the following analytic form with the aid of Bailey's pair. For $k \geq r > \lambda \geq 0$,

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}(1+q^{-\eta N_r})(-q^{\eta-\eta N_{\lambda+1}};q^\eta)_{N_{\lambda+1}}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^{(2-j)\eta};q^{(2-j)\eta})_{N_{k-1}}}$$

$$\times (-q^{\eta+\eta N_\lambda};q^\eta)_\infty \prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s} \prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty$$

$$= (-q^{\alpha_1},\ldots - q^{\alpha_\lambda}, -q^\eta;q^\eta)_\infty$$

$$\times \frac{(q^{(r-\frac{\lambda}{2})\eta}, q^{(2k-r-\frac{\lambda}{2}+j-1)\eta}, q^{(2k-\lambda+j-1)\eta}; q^{(2k-\lambda+j-1)\eta})_\infty}{(q^\eta;q^\eta)_\infty}.$$

Overpartition analogues of some classical partition theorems

By the relation of $\overline{B}_0$ and B_1 and the relation of $\overline{B}_1$ and B_0, we could obtain overpartition analogue of some classical partition theorems:

- Euler's partition theorem (new)
- Rogers-Ramanujan identities
- The Rogers-Ramanujan-Gordon identity
- The Rogers-Ramanujan-Bressoud identity
- The Göllnitz-Gordon identity (new)
- Andrews' generalization of Göllnitz-Gordon identity (new)
- Bressoud's generalization of Göllnitz-Gordon identity (new)

Bressoud's conjecture

Conjecture (Bressoud)

Let λ, k, r and $j = 0$ or 1 be the integers such that $(2k+j)/2 > r \geq \lambda \geq 0$. Then

$$\sum_{n\geq 0} B_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n$$

$$= \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^{(2-j)\eta};q^{(2-j)\eta})_{N_{k-1}}}$$

$$\times \prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s}\prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty.$$

Problem: How to give a direct proof of Bressoud's conjecture?

THANK YOU!