Ranks and Cranks of Partitions

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Overview

In this talk, I wish to report some work on ranks and cranks of partitions, include

(1) The definitions of ranks and cranks of partitions;

(2) Andrews-Dyson-Rhoades's conjecture on the unimodality of spt-cranks of spt-partitions;

(3) Bringmann and Mahlburg's conjectured inequalities on ranks and cranks of partitions;

(4) The moments of ranks and cranks of partitions;

(5) Nearly equal distributions of ranks and cranks of partitions;

(6) The distribution of cranks of partitions and some applications.

Definition

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Example: There are five partitions of 4, which are

$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).$$

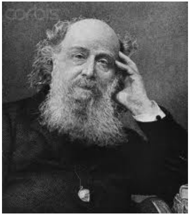
Let $p(n)$ denote the number of partitions of n, we see that $p(4) = 5$.

Background: The theory of partitions

- While G.W. Leibniz appears to have been the earliest to consider the partitioning of integers into sums, L. Euler was the first person to make truly deep discoveries. J.J. Sylvester was the next researcher to make major contributions.

G.W. Leibniz

L. Euler



J.J. Sylvester

Generating function

Theorem (Euler)

$$\sum_{n=0}^{+\infty} p(n)q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}.$$

proof: We first consider the following expansion

$$\begin{aligned}\prod_{j=1}^{M} \frac{1}{1-q^j} =& (1+q^1+q^{1+1}+q^{1+1+1}+\cdots) \\ & \times (1+q^2+q^{2+2}+q^{2+2+2}+\cdots) \\ & \times \cdots\cdots \\ & \times (1+q^M+q^{M+M}+q^{M+M+M}+\cdots) \\ & = \sum_{n=0} (\cdots\cdots) q^n,\end{aligned}$$

Obviously, the coefficient of q^n in the above expansion is equal to the number of solutions to Diophantine's equation $1 \times j_1 + 2 \times j_2 + 3 \times j_3 + \cdots = n$. On the other hand, it's easy to see that each

Euler's contributions

- The generating function of $p(n)$:

$$\sum_{n\geq 0} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

- Euler's pentagonal number theorem:

$$\prod_{i=1}^{\infty}(1-q^i) = \sum_{n=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}.$$

- The recurrence relation for $p(n)$:

$$(p(n)-p(n-1)) - (p(n-2)-p(n-5)) + \cdots$$
$$(-1)^m p\left(n-\frac{1}{2}m(3m-1)\right) + (-1)^m p\left(n-\frac{1}{2}m(3m+1)\right)$$
$$+\cdots = 0.$$

Assume that $p(M)=0$ for all negative M.

MacMahon took several months to calculate a table of values of $p(n)$ $(n \leq 200)$.

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42

n	11	12	13	14	15	16	17	18	19	20
$p(n)$	56	77	101	135	176	231	292	385	490	627

n	21	22	23	24	25	26	27	28
$p(n)$	792	1002	1255	1575	1958	2436	3010	3718

The partition function increases quite rapidly with n.

Percy Alexander MacMahon (1854–1929)

- MacMahon 16岁考入皇家陆军军官学校，19岁被分配到印度马德拉斯部队，开始了军旅生涯。24岁决定做一个数学家，于是在25岁那年，他参加了在伍尔维奇的炮兵军官高级课程。44岁，他退伍后当选了英国皇家学会会员。从1894年至1896年期间，他出任伦敦数学学会主席。
- MacMahon在不变量理论，对称函数理论，分拆理论做了开拓性的工作。他发展了一整套分析学技术，我们现在称之为组合分析学科。他的巨著《Combinatory Analysis》目前仍然在印刷发行。

Percy Alexander MacMahon (1854-1929)

- major index是以MacMahon的军衔命名的。他于1913年证明了major index和inversion number 是同分布的，即长为n的具有k个inversion的排列个数与长为n的major index为k的排列个数相等。这些数值被称为MacMahon number。

Srinivasa Ramanujan (1887-1920)

- Ramanujan was one of India's greatest mathematical geniuses.
- There is a colorful story between he and the English mathematician G. H. Hardy at the University of Cambridge.
- The Movie: The Man Who Knew Infinity. The mathematician Ken Ono served as a consultant for the film.

  The London Mathematical Society proclaimed that this film "outshines Good Will Hunting in almost every way".

- Sometime in the late 1920s, G.N. Watson （伦敦数学会会长）and B.M. Wilson began the task of editing Ramanujan's notebooks.
- G.E. Andrews and B.C. Berndt completed the project begun by Watson and Wilson.
- In the spring of 1976, G.E. Andrews visited Trinity College, Cambridge, to examine the papers left by Watson. Among Watson's papers, he found a manuscript containing 138 pages in the handwriting of Ramanujan. In view of the fame of Ramanujan's notebooks, it was natural for Andrews to call this newly found manuscript "Ramanujan's lost notebook."

Figure 1: *

G.E. Andrews and B.C. Berndt "Ramanujan's Lost Notebook" I–V

The discovery of Ramanujan

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42

n	11	12	13	14	15	16	17	18	19	20
$p(n)$	56	77	101	135	176	231	297	385	490	627

n	21	22	23	24	25	26	27	28
$p(n)$	792	1002	1255	1575	1958	2436	3010	3718

$$(1)\quad p(4),\quad p(9),\quad p(14),\quad p(19),\quad \cdots \equiv 0 \pmod 5,$$
$$(2)\quad p(5),\quad p(12),\quad p(19),\quad p(26),\quad \cdots \equiv 0 \pmod 7,$$
$$(3)\quad p(6),\quad p(17),\quad p(28),\quad p(39),\quad \cdots \equiv 0 \pmod{11}.$$

In 1919, Ramanujan found the the following striking congruences for ordinary partition function $p(n)$.

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}. \\ p(7n+5) &\equiv 0 \pmod{7}. \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

The prediction of Ramanujan

It appears that there are no equally simple properties for any moduli involving primes other than these three (i.e. $\ell = 5, 7, 11$).

S. Ramanujan, Some propertities of $p(n)$, the number of partitions of n, *Proc. Cambridge Philos. Soc.* 19 (1919) 207–210.

Some related results

(1) Ramanujan gave elementary proofs of congruences mod 5 and 7.

S. Ramanujan, Some propertities of $p(n)$, the number of partitions of n, *Proc. Cambridge Philos. Soc.* 19 (1919) 207–210.

(2) In 1969, Winquist first gave a proof of the congruence mod 11.

L. Winquist, An elementary proof of $p(11m+6) \equiv 0 \bmod 11$, J. Combin. Theory **6** (1969), 56–59.

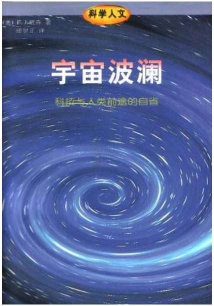
(3) In 2005, Ahlgren and Boylan present an affirmative answer to Ramanujan's prediction.

S. Ahlgren and M. Boylan, Arithmetic properties of the partition function, Invent. Math. 153 (2005) 487–502.

Freeman Dyson (1923–2020)

- 弗里曼·戴森，美籍英裔数学物理学家，美国科学院院士，普林斯顿高等研究院教授。 戴森早年在剑桥大学追随著名的数学家G.H. 哈代研究数学，二战后来到了美国康奈尔大学，跟随汉斯·贝特教授。
- 戴森获得许多殊荣：沃尔夫奖（the Wolf Prize）、伦敦皇家学会休斯奖（Hughes Medal）、德国物理学会普朗克奖（Max Planck Medal）、奥本海默纪念奖、哈维奖（Harvey Prize）等。

Freeman Dyson (1923–2020)



- 戴森还以在核武器政策和外星智能方面的工作而闻名。“戴森球”是弗里曼·戴森在1960年就提出的一种理论。

- 戴森也写了许多普及性读物，比如《全方位的无限》、《武器与希望》、《宇宙波澜》、《想象的未来》、《太阳、基因组与互联网：科学革命的工具》、《想象中的世界》等书，在科学界和大众中都激起极大的回响，先已有中文译本。

When he was in the second year of high-school in England, Dyson learnt Ramanujan's congruences from the book "An introduction to the theory of numbers" of Hardy and Wright. He tried hard to supply the missing proof of congruence mod 11, but did not succeed. However, he found a question which the proofs do not answer.

Dyson's problem

How to divide the partitions of $5n+4$ into five classes with the same number of partitions in each class?

Dyson wanted to find a concrete criterion, so that you could look at any particular partition of $5n+4$ and use the criterion to tell which of the five equal classes it belonged to.

Dyson's rank

Three years later, when he was a sophomore at Cambridge, he found it and called it "rank".

Definition (Dyson, 1944)

Let λ be a partition. The rank of λ is defined to be the largest part of λ minus the number of parts of λ.

For example, the rank of $(5, 4, 2, 1, 1, 1)$ is equal to $5 - 6 = -1$.

It is very simple to be happy, but it is very difficult to be simple.

— Rabindranath Tagore

F. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

Dyson's rank

Let $N(i, m, n)$ denote the number of partitions of n with rank $\equiv i \mod m$, Dyson first conjectured and later proved by Atkin and Swinnerton-Dyer.

Conjecture (Dyson, 1944)

$$\begin{aligned} N(k, 5, 5n+4) &= \frac{p(5n+4)}{5}, \quad 0 \leq k \leq 4, \\ N(k, 7, 7n+5) &= \frac{p(7n+5)}{7}, \quad 0 \leq k \leq 6, \end{aligned}$$

which imply $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$.

Example

	$n = 4$ $(p(4) = 5)$	rank	rank mod 5
$N(1,5,n)$	$(3,1)$	1	1
$N(2,5,n)$	$(1,1,1,1)$	-3	2
$N(3,5,n)$	(4)	3	3
$N(4,5,n)$	$(2,1,1)$	-1	4
$N(5,5,n)$	$(2,2)$	0	5

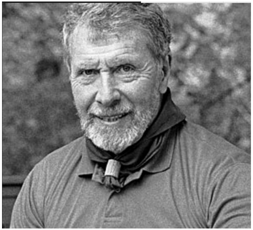
	$n = 5$ $(p(5) = 7)$	rank	rank mod 7
$N(1,7,n)$	$(3,2)$	1	1
$N(2,7,n)$	$(4,1)$	2	2
$N(3,7,n)$	(1^5)	-4	3
$N(4,7,n)$	(5)	4	4
$N(5,7,n)$	$(2,1^3)$	-2	5
$N(6,7,n)$	$(2,2,1)$	-1	6
$N(7,7,n)$	$(3,1,1)$	0	7

Dyson's observation

Dyson noted that ranks could not provide an explanation to $p(11n+6) \equiv 0 \pmod{11}$. He also conjectured that there should be a partition statistic (which he called crank) that would provide the missing combinatorial explanation of the above congruence.

	$n = 6$ ($p(6) = 11$)	rank	rank mod 11
$N(1, 11, n)$	$(4, 1^2)$ $(3, 3)$	1	1
$N(2, 11, n)$	$(4, 2)$	2	2
$N(3, 11, n)$	$(5, 1)$	3	3
$N(4, 11, n)$			
$N(5, 11, n)$	(6)	5	5
$N(6, 11, n)$	(1^6)	-5	6
$N(7, 11, n)$			
$N(8, 11, n)$	$(2, 1^4)$	-3	8
$N(9, 11, n)$	$(2, 2, 1^2)$	-2	9
$N(10, 11, n)$	$(3, 1^3)$ $(2, 2, 2)$	-1	10
$N(11, 11, n)$	$(3, 2, 1)$	0	11

Arthur Oliver Lonsdale Atkin (1925–2008)



- Arthur Oliver Lonsdale Atkin, who published under the name A. O. L. Atkin, was a British mathematician.
- He received his Ph.D. in 1952 from the University of Cambridge, where he was one of John Littlewood's research students.

Theorem (Atkin-Swinnerton-Dyer, Proc. London Math. Soc., 1954)

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \le k \le 4,$$

$$N(k,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \le k \le 6,$$

which imply $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$.

Andrews-Dyson-Garvan's crank

Forty four years later, Andrews and Garvan, building on the work of Garvan finally unveiled Dyson's crank of a partition λ:

Definition (Andrews and Garvan, 1988)

For a partition λ, its crank $c(\lambda)$ is defined as

$$c(\lambda) = \begin{cases} \lambda_1, & \text{if } n_1(\lambda) = 0, \\ \mu(\lambda) - n_1(\lambda), & \text{if } \ n_1(\lambda) > 0. \end{cases}$$

where $n_1(\lambda)$ denotes the number of parts equal to one in λ and $\mu(\lambda)$ denotes the number of parts in λ larger than $n_1(\lambda)$.

For example, $\lambda = (5, 3, 2, 1, 1, 1)$, its crank $c(\lambda) = \mu(\lambda) - n_1(\lambda) = 1 - 3 = -2$.

G. E. Andrews and F. G. Garvan, Dyson's crank of a partition. Bull. Amer. Math. Soc. **18** (1988),167–171.

Theorem (Garvan, 1988)

Let $M(j,m,n)$ denote the number of partitions of n with crank $\equiv j \bmod m$,

$$M(k,5,5n+4)=\frac{p(5n+4)}{5}, \quad \textit{for} \quad 0\le k\le 4,$$

$$M(k,7,7n+5)=\frac{p(7n+5)}{7}, \quad \textit{for} \quad 0\le k\le 6,$$

$$M(k,11,11n+6)=\frac{p(11n+6)}{11}, \quad \textit{for} \quad 0\le k\le 10,$$

which imply $p(5n+4)\equiv 0 \pmod 5$, $p(7n+5)\equiv 0 \pmod 7$ and $p(11n+6)\equiv 0 \pmod{11}$.

F. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7, 11. Trans. Am. Math. Soc. **305** (1988) 47–77.

Example

	$n = 4$ $(p(4) = 5)$	crank	crank mod 5
$M(1,5,n)$	(1^4)	-4	1
$M(2,5,n)$	$(2,2)$	2	2
$M(3,5,n)$	$(2,1^2)$	-2	3
$M(4,5,n)$	(4)	4	4
$M(5,5,n)$	$(3,1)$	0	5

	$n = 5$ $(p(5) = 7)$	crank	crank mod 7
$M(1,7,n)$	$(2^2,1)$	1	1
$M(2,7,n)$	(1^5)	-5	2
$M(3,7,n)$	$(3,2)$	3	3
$M(4,7,n)$	$(2,1^3)$	-3	4
$M(5,7,n)$	(5)	5	5
$M(6,7,n)$	$(3,1^2)$	-1	6
$M(7,7,n)$	$(4,1)$	0	7

Example

	$n = 6$ $(p(6) = 11)$	crank	crank mod 11
$M(1, 11, n)$	$(3, 2, 1)$	1	1
$M(2, 11, n)$	(2^3)	2	2
$M(3, 11, n)$	(3^2)	3	3
$M(4, 11, n)$	$(4, 2)$	4	4
$M(5, 11, n)$	(1^6)	-6	5
$M(6, 11, n)$	(6)	6	6
$M(7, 11, n)$	$(2, 1^4)$	-4	7
$M(8, 11, n)$	$(3, 1^3)$	-3	8
$M(9, 11, n)$	$(2^2, 1^2)$	-2	9
$M(10, 11, n)$	$(4, 1^2)$	-1	10
$M(11, 11, n)$	$(5, 1)$	0	11

George Eyre Andrews (1938-)

- an Evan Pugh Professor of Mathematics at Pennsylvania State University;
- Former president of the American Mathematical Society;
- He is considered to be the world's leading expert in the theory of integer partitions.

Freeman J. Dyson: A walk through Ramanujan's garden, the lecture given at Ramanujan Centenary Conference, University of Illinios, 1987.

> "George Andrews is now the chief gardener of Ramanujan's" garden and is doing a magnificent job. He does not stand like Proserpine, gathering all things mortal with cold immortal hands. He likes to have live human beings in his garden, trampling over the flower-beds. Andrews also enlarged the territory of the garden by finding the famous lost notebook of Ramanujan.

The generating function of rank

Let $N(m,n)$ denote the number of partitions of n with rank m.

Theorem (Atkin-Swinnerton-Dyer, Dyson)

The generating function for $N(m;n)$ is given by

$$\sum_{n=1}^{+\infty} N(m;n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{+\infty} (-1)^{n-1} q^{n(3n-1)/2+|m|n}(1-q^n), \quad |q|<1,$$

where $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ *and* $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$.

This result implies $N(m,n) = N(-m,n)$.

A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, Proc. Lond. Math. Soc. III. Ser. 4 (1954) 84–106.

F. J. Dyson, A new symmetry of partitions, J. Combin. Theory Ser. A 7 (1969) 56–61.

The generating function of crank

Theorem

The generating function for $M(m;n)$ is given by

$$\sum_{n=1}^{+\infty} M(m;n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{+\infty} (-1)^{n-1} q^{n(n-1)/2+|m|n}(1-q^n), \quad |q|<1,$$

where $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ *and* $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$.

This result implies $M(m,n) = M(-m,n)$.

F. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7, 11. Trans. Am. Math. Soc. **305** (1988) 47–77.

G. E. Andrews and F. G. Garvan, Dyson's crank of a partition. Bull. Amer. Math. Soc. **18** (1988),167–171.

Atkin's method

Hence the generating function of $N(m, 5, n)$ is

$$\sum_{n=0}^{\infty} N(m, 5, n) q^n = \frac{1}{(q)_\infty} \sum_{n \neq 0} (-1)^n q^{n(3n+1)/2} \frac{q^{mn} + q^{(5-m)n}}{1 - q^{5n}}.$$

The proof of $N(0, 5, 5n+4) = N(1, 5, 5n+4)$ is equivalent to show that the coefficients of q^{5n+4} in the series expansion of the following generating function is equal to 0.

$$\sum_{n=0}^{\infty} (N(0, 5, n) - N(1, 5, n)) q^n$$

$$= \frac{1}{(q)_\infty} \sum_{n \neq 0} (-1)^n q^{n(3n+1)/2} \left(\frac{1 + q^{5n}}{1 - q^{5n}} - \frac{q^n + q^{4n}}{1 - q^{5n}} \right).$$

The first problem

How to give a bijective proof of Dyson's conjecture?

- Ranks can be used to divide the set of partitions of $5n+4$ into five equivalent classes. How to build a bijection between these five equivalent classes?
- It should be noted that Garvan-Kim-Stanton (1990) gave combinatorial proofs of Ramanujan's congruences by using t-core. This is another beautiful story!

F.G. Garvan, D. Kim and D. Stanton, cranks and t-cores, Invent. Math. 101 (1990) 1–17.

Part II: Andrews-Dyson-Rhoades's conjecture on the unimodality of spt-cranks of spt-function

Definition (Andrews, 2008)

The spt-function $spt(n)$, called the smallest part function, is defined to be the total number of smallest parts in all partitions of n.

By the following table, we may see that $p(4) = 5$ and $spt(4) = 10$.

						Total
$\lambda \in \mathcal{P}(4)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$	5
$n_s(\lambda)$	1	1	2	2	4	10

Let $P(n)$ denote the set of ordinary partitions of n, we see that

$$spt(n) = \sum_{\lambda \in P(n)} n_s(\lambda),$$

where $n_s(\lambda)$ denotes the number of occurrences of the smallest part in λ.

Andrews showed that the spt-function satisfies the following remarkable congruences:

Theorem (Andrews, 2008)

$$
\begin{aligned}
spt(5n+4) &\equiv 0 \pmod{5}. \\
spt(7n+5) &\equiv 0 \pmod{7}. \\
spt(13n+6) &\equiv 0 \pmod{13}.
\end{aligned}
$$

In his paper, Andrews: "The appearance of 13 in this result is completely unexpected and thanks to Frank Garvan who not only pointed out O'Brien thesis to him but also supplied him with a copy."

O'Brien thesis, Durham University, 1965.

G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142.

Connection to the moments of ranks and cranks

Based on the generating function of the spt-function and Watson's q-analog of Whipple's theorem, Andrews showed that the spt-function can be expressed in terms of the second moment of ranks and cranks.

Theorem (Andrews, 2008)

$$spt(n) = \frac{1}{2}\sum_{m=-\infty}^{+\infty} m^2 M(m,n) - \frac{1}{2}\sum_{m=-\infty}^{+\infty} m^2 N(m,n).$$

G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142.

Kathy Q. Ji, A combinatorial proof of Andrews' smallest parts partition function, Electr. J. Combin. 15 (2008) N12.

Watson's q-analog of Whipple's theorem

Watson's q-analog of Whipple's theorem:

$$\sum_{k=0}^{\infty}(-1)^k q^{k(k-1)/2}\frac{1-aq^{2k}}{1-a}\frac{(a,b,c,d,e;q)_k}{(q,aq/b,aq/c,aq/d,aq/e;q)_k}\left(\frac{a^2q^2}{bcde}\right)^k$$

$$=\frac{(aq,aq/de;q)_\infty}{(aq/d,aq/e;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq/bc,d,e;q)_k}{(q,aq/b,aq/c;q)_k}\left(\frac{aq}{be}\right)^k.$$

To give a combinatorial interpretation of spt-congruences, Andrews, Garvan and Liang introduced the spt-crank which is defined on a restricted set of vector partitions.

Definition (S-partitions)

Let $\mathcal{D}$ denote the set of partitions into distinct parts and $\mathcal{P}$ denote the set of partitions. For $\pi \in \mathcal{P}$, we use $s(\pi)$ to denote the minimum part of π with the convention that $s(\emptyset) = +\infty$. Define

$$S = \{(\pi_1, \pi_2, \pi_3) \in \mathcal{D} \times \mathcal{P} \times \mathcal{P} : \pi_1 \neq \emptyset \text{ and } s(\pi_1) \leq \min\{s(\pi_2), s(\pi_3)\}\}.$$

The triplet $\pi = (\pi_1, \pi_2, \pi_3) \in S$ is called to be an S-partition of n with weight $\omega(\pi) = (-1)^{\ell(\pi_1)-1}$ if $|\pi| = |\pi_1| + |\pi_2| + |\pi_3| = n$.

For example, $((2,1),(2,2),(3,3,2))$ is an S-partition of 15 with weight -1.

The generating function for weighted S-partitions

From the definition of S-partitions, we see that

$$\sum_{\pi \in S} \omega(\pi) q^{|\pi|} = \sum_{n=1}^{+\infty} \frac{q^n (q^{n+1};q)_\infty}{(q^n;q)_\infty (q^n;q)_\infty} = \sum_{n=1}^{+\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty} = \sum_{n \geq 0} spt(n) q^n,$$

so the net number of S-partitions of n is equal to $spt(n)$.

S-partitions	$\omega(\pi)$	$spt - fuction$
$((1),(1),(1,1))$	$+1$	(4)
$((1),\emptyset,(3))$	$+1$	$(3,1)$
$((2,1),\emptyset,(1))$	-1	$(2,2)$
$((2),\emptyset,(2))$	$+1$	$(2,2)$
$((1),(1,1,1),\emptyset)$	$+1$	$(2,1,1)$
$((1),\emptyset,(2,1))$	$+1$	$(2,1,1)$
$((1),(2,1),\emptyset)$	$+1$	$(1,1,1,1)$
$((1),\emptyset,(1,1,1))$	$+1$	$(1,1,1,1)$
$((1),(1,1),(1))$	$+1$	$(1,1,1,1)$
$((1),(3),\emptyset)$	$+1$	$(1,1,1,1)$
$((2,1),(1),\emptyset)$	-1	
$((2),(2),\emptyset)$	$+1$	
$((1),(2),(1))$	$+1$	
$((1),(1),(2))$	$+1$	
$((3,1),\emptyset,\emptyset)$	-1	
$((4),\emptyset,\emptyset)$	$+1$	
16	10	10

The definition of spt-crank is the same as Garvan's crank for weighted vector partitions used to interpret Ramanujan's congruences for $p(n)$.

Definition (Andrews, Garvan, Liang, 2012)

Let π be an S-partition, the spt-crank of π, denoted $r(\pi)$, is defined to be the difference between the number of parts of π_2 and π_3, that is,

$$r(\pi) = \ell(\pi_2) - \ell(\pi_3).$$

G.E. Andrews, F.G. Garvan and J.L. Liang, Combinatorial interpretations of congruences for the spt-function, Ramanujan J. 29 (2012) 321–338.

Andrews, Garvan and Liang showed the following relations

Theorem (Andrews, Garvan and Liang, 2012)

Let $N_S(j,m,n)$ denote the net number of S-partitions of n with spt-crank $\equiv j \bmod m$,

$$\begin{aligned} N_S(k,5,5n+4) &= \frac{spt(5n+4)}{5}, \quad \text{for} \quad 0 \le k \le 4, \\ N_S(k,7,7n+5) &= \frac{spt(7n+5)}{7}, \quad \text{for} \quad 0 \le k \le 6, \end{aligned}$$

which imply $spt(5n+4) \equiv 0 \pmod 5$ and $spt(7n+5) \equiv 0 \pmod 7$.

G.E. Andrews, F.G. Garvan and J.L. Liang, Combinatorial interpretations of congruences for the spt-function, Ramanujan J. 29 (2012) 321–338.

S-partitions	$\omega(\pi)$	spt-crank mod 5	*spt*-function
$((1),(1),(1,1))$	$+1$	4	(4)
$((1),\emptyset,(3))$	$+1$	4	$(3,1)$
$((2,1),\emptyset,(1))$	-1	4	$(2,2)$
$((2),\emptyset,(2))$	$+1$	4	$(2,2)$
$((1),(1,1,1),\emptyset)$	$+1$	3	$(2,1,1)$
$((1),\emptyset,(2,1))$	$+1$	3	$(2,1,1)$
$((1),(2,1),\emptyset)$	$+1$	2	$(1,1,1,1)$
$((1),\emptyset,(1,1,1))$	$+1$	2	$(1,1,1,1)$
$((1),(1,1),(1))$	$+1$	1	$(1,1,1,1)$
$((1),(3),\emptyset)$	$+1$	1	$(1,1,1,1)$
$((2,1),(1),\emptyset)$	-1	1	
$((2),(2),\emptyset)$	$+1$	1	
$((1),(2),(1))$	$+1$	0	
$((1),(1),(2))$	$+1$	0	
$((3,1),\emptyset,\emptyset)$	-1	0	
$((4),\emptyset,\emptyset)$	$+1$	0	

Remarks

- Andrews, Garvan and Liang proposed a problem to find a definition of the spt-crank in terms of ordinary partitions.
- This problem was considered by Andrews, Dyson and Rhoades as an interesting and apparently challenging problem.
- Chen-Ji-Zang (2016) gave an answer to this problem, and showed that $N_S(m,n)$ also counts the number of doubly marked partitions of n with spt-crank m.

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (2013) 76–88.

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, The spt-crank for ordinary partitions, J. Reine Angew. Math. 711 (2016) 231–249.

The second problem

How to give a combinatorial interpretation of $spt(13n+6) \equiv 0 \pmod{13}$?

By the following table, we may see that $p(6) = 11$ and $spt(6) = 26$.

$\|\lambda\| = 6$	$n_s(\lambda)$
(6)	1
$(5,1)$	1
$(4,2)$	1
$(4,1^2)$	2
(3^2)	2
$(3,2,1)$	1
$(3,1^3)$	3
(2^3)	3
$(2,2,1^2)$	2
$(2,1^4)$	4
(1^6)	6
11	26

The Andrews-Dyson-Rhoades conjecture

Recall that $N_S(m,n)$ denotes the net number of S-partitions of n with spt-crank m, by our result, it is known that $N_S(m,n)$ also counts the number of doubly marked partitions of n with spt-crank m. It can be showed that

$$N_S(-m,n) = N_S(m,n).$$

Andrews, Dyson and Rhoades posed the following conjecture on the unimodality of the spt-crank:

Conjecture (Andrews, Dyson and Rhoades, 2013)

For all $n \geq 0$ and $m \geq 0$,

$$N_S(m,n) \geq N_S(m+1,n).$$

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank,MDPI-Mathematics 1 (3) (2013) 76–88.

Example

For example, the following table gives details to support this conjecture.

$n \setminus m$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1									
2	1	1								
3	1	1	1							
4	2	2	1	1						
5	2	2	2	1	1					
6	4	4	3	2	1	1				
7	5	4	4	3	2	1	1			
8	7	7	6	5	3	2	1	1		
9	10	9	8	6	5	3	2	1	1	
10	13	13	11	10	7	5	3	2	1	1

Andrews, Dyson and Rhoades showed this conjecture holds for fixed m and sufficiently large n.

Theorem (Andrews, Dyson and Rhoades, 2013)

For each $m \geq 0$, we have

$$N_S(m,n) - N_S(m+1,n) \sim \frac{(2m+1)\pi^2}{192\sqrt{3}n^2} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as} \quad n \to \infty.$$

Chen-Ji-Zang (2015) give a constructive proof of this conjecture.

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (3) (2013) 76–88.

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Proof of the Andrews-Dyson-Rhoades conjecture on the spt-crank, Adv. Math. 270 (2015) 60–96.

Part III: Bringmann and Mahlburg's conjectured inequalities on ranks and cranks of partitions

The rank and the crank functions

- Let $N(\leq m, n)$ denote the number of partitions of n with rank not greater than m;
- let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m.

In the concluding remarks of a paper in 2009, Bringmann and Mahlburg said that they found an interesting phenomenon by testing with Maple

Bringmann and Mahlburg's Observation

For $1 \leq n \leq 100$ and $1 \leq m \leq n-1$,

$$N(\leq m, n) \geq M(\leq m, n) \geq N(\leq m-1, n).$$

K. Bingmann and K. Mahlburg, Inequlaitities between ranks and cranks, Proc. Amer. Math. Soc. 137 (2009) 2567–2574.

Example

For example, let $n = 8$, we have the following table:

m	$N(\leq m, 8)$	$M(\leq m, 8)$	$N(\leq m-1, 8)$
0	12	12	10
1	15	14	12
2	17	16	15
3	19	17	17
4	20	19	19
5	21	20	20
6	21	21	21
7	22	21	21
8	22	22	22

Related to the spt-crank

Andrews, Dyson and Rhoades established the following relation:

Theorem (Andrews, Dyson and Rhoades, 2013)

For $m \geq 0$ and $n \geq 0$,

$$N(\leq m, n) - M(\leq m, n) = N_S(m, n) - N_S(m+1, n).$$

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (3) (2013) 76–88.

Proof of the Andrews-Dyson-Rhoades conjecture

We proved this conjecture holds for all nonnegative integers n and m by constructing an injection.

Theorem (Chen, Ji and Zang, 2015)

For all $n \geq 0$ and $m \geq 0$,

$$N(\leq m, n) \geq M(\leq m, n)$$

$$\Leftrightarrow$$

$$N_S(m, n) \geq N_S(m+1, n).$$

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Proof of the Andrews-Dyson-Rhoades conjecture on the spt-crank, Adv. Math. 270 (2015) 60–96.

Sketch of the proof

- Let $Q(m, n)$ denote the set of partitions of n such that m appears in the rank-set of λ. $\#Q(m, n) = M(\leq m, n)$ (Dyson, 1989);
- Let $P(m, n)$ denote the set of partitions of n with rank not less than $-m$; $\#P(m, n) = N(\leq m, n)$ (By the symmetry of ranks).

To prove the conjecture, we only need to build an injection from the set $Q(m, n)$ to the set $P(m, n)$.

m-Durfee rectangle symbol

In order to characterize the partitions in $Q(m,n)$ and $P(m,n)$, we define m-Durfee rectangle symbol, which is a generalization of the Durfee symbol introduced by Andrews in 2007.

Definition (Chen, Ji and Zang, 2015)

An m-Durfee rectangle symbol is defined as follows

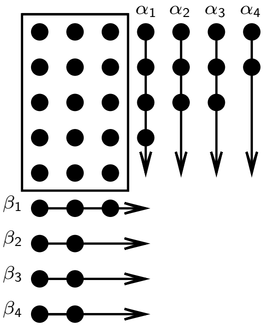
$$(\alpha,\beta)_{(m+j)\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j},$$

where $(m+j)\times j$ is the m-Durfee rectangle of the Ferrers diagram of λ and α consists of columns to the right of the m-Durfee rectangle and β consists of rows below the m-Durfee rectangle.

G.E. Andrews, Partitions, Durfee symbols, and the Atkin–Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.

For example, the 2-Durfee rectangle symbol of $(7,7,6,4,3,3,2,2,2)$ is

$$\begin{pmatrix} 4, & 3, & 3, & 2 \\ 3, & 2, & 2, & 2 \end{pmatrix}_{5\times 3}.$$



Propositions

The set of partitions in $Q(m,n)$ can be described in terms of m-Durfee rectangle symbols as follows.

Proposition (Chen, Ji and Zang, 2015)

Let λ be an ordinary partition and $(\alpha,\beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then m appears in the rank-set of λ if and only if either $j=0$ or $j\geq 1$ and $\beta_1=j$.

We describe the set of partitions in $P(m,n)$ in terms of m-Durfee rectangle symbols as follows.

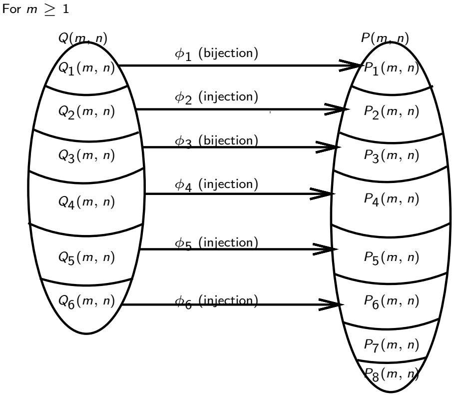
Proposition (Chen, Ji and Zang, 2015)

Let λ be an ordinary partition and $(\alpha,\beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then the rank of λ is not less than $-m$ if and only if either $j=0$ or $j\geq 1$ and $\ell(\beta)\leq\ell(\alpha)$.

There are two cases: $m \geq 1$ and $m = 0$.

For $m \geq 1$, By using the m-Durfee rectangle symbol, we divide the set $Q(m, n)$ into six disjoint subsets $Q_i(m, n)$, where $1 \leq i \leq 6$, as well as divide the set $P(m, n)$ into eight disjoint subsets $P_i(m, n)$, where $1 \leq i \leq 8$.

Then we construct six injections ϕ_i from $Q_i(m,n)$ to $P_i(m,n)$ for $m \geq 1$. So the conjecture for case $m \geq 1$ has been proved.



The case $m = 0$ is not simpler than $m \geq 1$, in this case, the injection ϕ consists of three more injections.

An email from F. Dyson

From: Freeman Dyson `dyson@ias.edu`
To: W.Y.C. Chen `chenyc@tju.edu.cn`
Subject: Re: cranks-s.pdf

Dear Bill Chen,
Congratulations for proving our conjecture. This is a formidable piece of work and will take me some time to understand. If you continue to work on the problem, y-ou might find a shorter proof. Anyhow, this is a big step forward.

Yours sincerely,
Freeman Dyson

Yet, there is neither analytic nor algebraic proof of this simple conjecture. The third problem is how to give a shorter proof of this conjecture.

Proof of the Bringmann-Mahlburg conjectured inequalities

Chen-Ji-Zang (2017) found the injections ϕ_1, ϕ_2 and ϕ_3 can also be used to show that the Bringmann and Mahlburg's conjectured inequality $M(\leq m, n) \geq N(\leq m-1, n)$ holds.

Theorem (Chen, Ji and Zang, 2017)

For $n \geq 1$ and $1 \leq m \leq n-1$,

$$N(\leq m, n) \geq M(\leq m, n) \geq N(\leq m-1, n).$$

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Nearly equivalent relation between the rank and crank of partitions, In: G.E. Andrews and F. Garvan (eds.), Analytic Number Theory, Modular Forms and q-Hypergeometric Series, 2017.

Part IV: The moments of ranks and cranks of partitions

The moments of ranks and cranks

Definition (Atkin and Garvan, 2003)

$$N_k(n) = \sum_{m=-\infty}^{+\infty} m^k N(m,n),$$

$$M_k(n) = \sum_{m=-\infty}^{+\infty} m^k M(m,n).$$

A.O.L. Atkin and F. Garvan, Relations between the ranks and cranks of partitions, Ramanujan J. 7 (2003) 343–366.

Just as we said before, Andrews found that

$$spt(n) = \frac{1}{2}(M_2(n) - N_2(n)) > 0.$$

A natural question is that whether for k, we have

$$M_k(n) - N_k(n) > 0$$

G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142.

$$
\begin{aligned}
N_k(n) &= \sum_{m=-\infty}^{+\infty} m^k N(m,n) \\
&= \sum_{m=-\infty}^{+\infty} (-m)^k N(-m,n) \\
&\qquad\qquad \Downarrow \text{ symmetry} \\
&= \sum_{m=-\infty}^{+\infty} (-1)^k m^k N(m,n) \\
&= (-1)^k N_k(n).
\end{aligned}
$$

This implies that

$$N_{2k+1}(n) = 0.$$

Using the same argument, we could show that

$$M_{2k+1}(n) = 0.$$

The positive moments of ranks and cranks

To study the odd moments of ranks and cranks, Andrews, Chan and Kim modified the definition of the moments of ranks and cranks.

Definition (Andrews, Chan and Kim, 2013)

The kth positive moment of the rank $\overline{N}_k(n)$ is defined by

$$\overline{N}_k(n) = \sum_{m=1}^{+\infty} m^k N(m, n).$$

The kth positive moment of the crank $\overline{M}_k(n)$ is defined by

$$\overline{M}_k(n) = \sum_{m=1}^{+\infty} m^k M(m, n).$$

G.E. Andrews, S.H. Chan and B. Kim, The odd moments of ranks and cranks, J. Combin Theory, Ser. A 120 (2013) 77–91.

Relations

- The new odd moments of ranks and cranks are now nontrivial.
- For even moments of ranks and cranks, we have

$$N_{2k}(n) = 2\overline{N}_{2k}(n), \quad M_{2k}(n) = 2\overline{M}_{2k}(n).$$

Theorem (Garvan, 2011)

For all $k \geq 1$ and $n \geq 1$, we have

$$M_{2k}(n) > N_{2k}(n).$$

Theorem (Andrews, Chan and Kim, 2013)

For $k \geq 1$ and $n \geq 1$, we have

$$\overline{M}_k(n) > \overline{N}_k(n).$$

Using

$$N_{2k}(n) = 2\overline{N}_{2k}(n), \quad M_{2k}(n) = 2\overline{M}_{2k}(n),$$

it is easy to see that Andrews-Chan-Kim's inequality is equivalent to Garvan's inequality when k is even.

F.G. Garvan, Higher order spt-functions, Adv. Math. 228 (2011) 241–265.

G.E. Andrews, S.H. Chan, and B. Kim, The odd moments of ranks and cranks, J. Combin Theory, Ser. A 120 (2013) 77–91.

Connection

We found that the Andrews-Dyson-Rhoades conjecture can easily lead to Andrews-Chan-Kim's inequality.

Theorem (Chen, Ji and Zang, 2015)

For $k \geq 1$ and $n \geq 1$, we have

$$\overline{N}_k(n) = \frac{1}{2}\sum_{m=1}^{+\infty}(m^k - (m-1)^k)\,(p(n) - N_{\leq m-1}(n)),$$

$$\overline{M}_k(n) = \frac{1}{2}\sum_{m=1}^{+\infty}(m^k - (m-1)^k)\,(p(n) - M_{\leq m-1}(n)).$$

The proof of these two relations is just based on Abel's lemma.

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Proof of the Andrews-Dyson-Rhoades conjecture on the spt-crank, Adv. Math. 270 (2015) 60–96.

Alternative proof of Andrews-Chan-Kim's inequality

From these two relations, we see that

$$\boxed{\overline{M}_k(n) - \overline{N}_k(n) = \frac{1}{2}\sum_{m=1}^{n-1}(m^k - (m-1)^k)(N_{\le m-1}(n) - M_{\le m-1}(n)) + n^k - (n-1)^k.}$$

By the Andrews-Dyson-Rhoades conjecture (the Bringmann-Mahlburg conjectured inequality), we have for $n \geq 1$ and $m \geq 1$

$$N_{\le m-1}(n) - M_{\le m-1}(n) \geq 0.$$

Since for $m \geq 1$ and $k \geq 1$,

$$m^k - (m-1)^k > 0,$$

we reach the assertion that for $n \geq 1$ and $k \geq 1$

$$\overline{M}_k(n) - \overline{N}_k(n) > 0.$$

Partition congruences

In addition, Andrews found the following congruences:

Theorem (Andrews, 2007)

$$N_3(5n+4) \equiv 0 \pmod 5,$$
$$N_3(7n+5) \equiv 0 \pmod 7,$$
$$N_5(7n+5) \equiv 0 \pmod 7.$$

To give combinatorial interpretations of these three congruences, Andrews introduced k-marked Durfee symbols.

George E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.

Kathy Q. Ji, The combinatorics of k-marked Durfee symbols, Trans. Amer. Math. Soc. 363 (2011) 987–1005.

More References

- William Y. C. Chen, Kathy Q. Ji, Erin Y. Y. Shen, On the positive moments of ranks of partitions, Electron. J. Combin. 21 (2014), no. 1, Paper 1.29, 10 pp.
- Kathy Q. Ji and Alice X. H. Zhao, The crank moments weighted by the parity of cranks, Ramanujan J. 44 (2017) 631–640.
- Liuquan Wang, Arithmetic properties of odd ranks and k-marked odd Durfee symbols, Adv. in Appl. Math. 121 (2020), 102098, 28 pp.

Part V: Nearly equal distributions of ranks and cranks of partitions

Bringmann-Mahlburg's observation

Bringmann and Mahlburg also speculated that " the observation on the inequalities may also be stated in terms of ordered lists of partitions. Specifically, for $1 \leq n \leq 100$, there must be some re-ordering τ_n of partitions λ of n such that

$$|\text{crank}(\lambda)| - |\text{rank}(\tau_n(\lambda))| = 0 \text{ or } 1,$$

although we do not have an explicit combinatorial description of τ_n."

K. Bingmann and K. Mahlburg, Inequlaitities between ranks and cranks, Proc. Amer. Math. Soc. 137 (2009) 2567–2574.

The bijection τ_n

- If we list the set of partitions of n in two ways, one by the ranks, and the other by the cranks, then we are led to a re-ordering τ_n of the partitions of n.
- Using Bringmann and Mahlburg's inequalities, we show that the rank and the crank are nearly equidistributed over the set of partitions of n.

Theorem (Chen, Ji and Zang, 2017)

Let τ_n be a reordering on the set of partitions of n as defined above. Then for $\lambda \in \mathcal{P}(n)$, we have

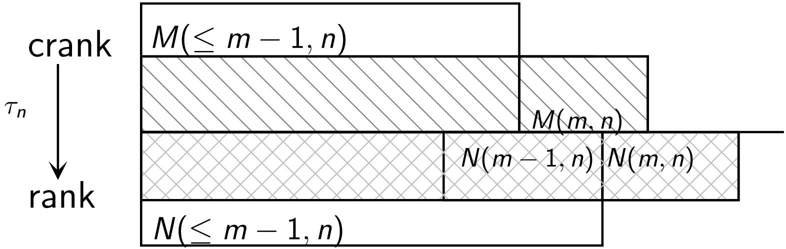
$$|\mathrm{crank}(\lambda)| - |\mathrm{rank}(\tau_n(\lambda))| = 0, \text{ or } 1.$$

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Nearly equivalent relation between the rank and crank of partitions, In: G.E. Andrews and F. Garvan (eds.), Analytic Number Theory, Modular Forms and q-Hypergeometric Series, 2017.

Example

$\lambda \in P(5)$	(crank)	$\xrightarrow{\tau_5}$	$\mu \in P(5)$	(rank)	$\lvert crank(\lambda)\rvert - \lvert rank(\mu)$
$(1,1,1,1,1)$	(-5)	$\rightarrow$	$(1,1,1,1,1)$	(-4)	1
$(2,1,1,1)$	(-3)	$\rightarrow$	$(2,1,1,1)$	(-2)	1
$(3,1,1)$	(-1)	$\rightarrow$	$(2,2,1)$	(-1)	0
$(4,1)$	(0)	$\rightarrow$	$(3,1,1)$	(0)	0
$(2,2,1)$	(1)	$\rightarrow$	$(3,2)$	(1)	0
$(3,2)$	(3)	$\rightarrow$	$(4,1)$	(2)	1
(5)	(5)	$\rightarrow$	(5)	(4)	1

Using the inequalities $N(\leq m, n) \geq M(\leq m, n) \geq N(\leq m-1, n)$, we have the following illustration for $m \geq 1$:



It is clear that for $crank(\lambda) = m$, we have $rank(\tau_n(\lambda)) = m$ or $m-1$. Thus we have $|crank(\lambda)| - |rank(\tau_n(\lambda))| = 0$ or 1. The case $m \leq 0$ can be proved in a similar way.

Problem

It should be noted that the above description of τ_n relies on the two orderings of partitions of n.

The fourth problem

It would be interesting to find a direct definition of τ_n which is defined explicitly on a partition λ of n.

An application of the bijection τ_n

Using the bijection τ_n and the Cauchy-Schwartz inequality, we show that the following inequality between $spt(n)$ and $p(n)$ holds.

Theorem (Chen, Ji and Zang, 2017)

For $n \geq 1$, we have

$$spt(n) \leq \sqrt{2n}p(n).$$

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Nearly equivalent relation between the rank and crank of partitions, In: G.E. Andrews and F. Garvan (eds.), Analytic Number Theory, Modular Forms and q-Hypergeometric Series, 2017.

Proof of the upper bound

Since τ_n is a bijection, we have

$$2spt(n) = \sum_{\lambda \in \mathcal{P}(n)} crank^2(\lambda) - \sum_{\lambda \in \mathcal{P}(n)} rank^2(\tau_n(\lambda)).$$

From the difference of two squares, we have

$$2spt(n) = \sum_{\lambda \in \mathcal{P}(n)} (|crank(\lambda)| - |rank(\tau_n(\lambda))|)(|crank(\lambda)| + |rank(\tau_n(\lambda))|).$$

From the inequality

$$|crank(\lambda)| - |rank(\tau_n(\lambda))| = 0 \text{ or } 1,$$

we find that

$$2spt(n) \leq 2 \sum_{\lambda \in \mathcal{P}(n)} |crank(\lambda)|.$$

Proof of the upper bound

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} spt(n) &\leq \sum_{\lambda\in\mathcal{P}(n)} |crank(\lambda)| \\ &\leq \sqrt{\sum_{\lambda\in\mathcal{P}(n)} 1^2 \sum_{\lambda\in\mathcal{P}(n)} crank(\lambda)^2} \\ &= \sqrt{p(n) \sum_{\lambda\in\mathcal{P}(n)} crank(\lambda)^2}. \end{aligned}$$

Thanks to the following equation given by Dyson,

$$\sum_{\lambda\in\mathcal{P}(n)} crank(\lambda)^2 = 2np(n).$$

Thus we show that $spt(n) \leq \sqrt{2n}p(n)$.

The conjecture of $spt(n)$'s upper bound

Chan and Mao raised the following conjecture:

Conjecture (Chan and Mao, 2014)

$$spt(n) \leq \sqrt{n}p(n).$$

F.J. Dyson, A new symmetry of partitions, J. Combin. Theory A 7 (1969) 56–61.

S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258(2014) 414–437.

The definition of $ospt(n)$

In 2013, Andrews, Chan and Kim define $ospt(n)$ as follows:

Definition (Andrews, Chan and Kim, 2013)

$$ospt(n) = \overline{M}_1(n) - \overline{N}_1(n).$$

Using the generating function, Andrews, Chan and Kim proved the positivity of $ospt(n)$ and found a combinatorial interpretation of $ospt(n)$ in terms of even strings and odd strings of a partition.

G.E. Andrews, S.H. Chan, B. Kim, The odd moments of ranks and cranks, J. Combin. Theory Ser. A 120 (2013) 77–91.

The combinatorial interpretation of $ospt(n)$

We showed that $ospt(n)$ can also be interpreted in terms of $\tau(n)$.

Theorem (Chen, Ji and Zang, 2017)

For $n > 1$, $ospt(n)$ equals the number of partitions λ of n such that $crank(\lambda) - rank(\tau_n(\lambda)) = 1$.

- It can be seen that $\tau_n((n)) = (n)$ for $n > 1$ since the partition (n) has the largest rank and the largest crank among all partitions of n.
- It follows that $\text{crank}((n)) - \text{rank}(\tau_n((n))) = 1$ when $n > 1$.
- Thus the above result implies that $ospt(n) > 0$ for $n > 1$.
- Chan and Mao, 2014

$$ospt(n) < \frac{p(n)}{2} \quad \text{for } n \geq 3.$$

S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258 (2014) 414–437.

Part VI: The distribution of the cranks of partitions

Andrew-Dyson-Rhoades's claim

Andrews, Dyson and Rhoades (2013) pointed out that

This table suggests that the sequence $\{N_S(m,n)\}_m$ is (weakly) unimodal.

Conjecture 1.1. *For each $m \geq 0$ and $n \geq 0$ we have*

$$N_S(m,n) \geq N_S(m+1,n).$$

This property is not true for the ordinary rank or crank statistic. For example,

$$N(n-1,n) = N(n-3,n) = 1 \text{ and } N(n-2,n) = 0$$

0 for all $n > 2$ and a similar statement holds for the crank. Our first statement reinterprets

$$M(n,n)=1, \quad M(n-1,n)=0, \quad M(n-2,n)=1,$$

for $n > 2$.

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (2013) 76–88.

The unimodality of the crank

Theorem (Ji and Zang, 2021)

For $n \geq 44$ and $1 \leq m \leq n-1$,

$$M(m-1,n) \geq M(m,n),$$

By the symmetry $M(m,n) = M(-m,n)$, the above inequality implies the following corollary.

Corollary (Ji and Zang, 2021)

For $n \geq 44$, the sequence $\{M(m,n)\}_{2-n \leq m \leq n-2}$ is unimodal.

Kathy Q. Ji and Wenston J.T. Zang, Unimodality of the Andrews-Garvan-Dyson cranks of partitions, Adv. Math, 2021, pp.52.

The conjecture on the unimodality of $N(m, n)$

Conjecture

When $n \geq 39$ *and* $1 \leq m \leq n-2$,

$$N(m-1, n) \geq N(m, n).$$

By the symmetry $N(m, n) = N(-m, n)$, we see that the above conjecture implies the sequence $\{N(m, n)\}_{|m| \leq n-2}$ is unimodal for $n \geq 39$.

In 2014, Chan and Mao gave the following two inequalities.

Theorem (Chan and Mao, 2014)

For $n \geq 2$ and $2 \leq m \leq n-2$,

$$N(m-2, n) \geq N(m, n).$$

S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258 (2014) 414–437.

Definition

A sequence $\{a_i\}_{1\leq i\leq n}$ of real numbers is said to be unimodal if for some $0 \leq j \leq n$, we have $a_0 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$.

For example, the binomial coefficients $\{\binom{n}{k}\}_{0\leq k\leq n}$: the number of k-subsets of the set $[n] = \{1, 2, \ldots, n\}$.

nk	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

Log-concave sequences

Definition

A sequence $\{a_i\}_{1\leq i\leq n}$ of real numbers is said to be log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n-1$.

It is well known that

Theorem

If a sequence $\{a_i\}_{1\leq i\leq n}$ of positive integers is log-concave, then $\{a_i\}_{1\leq i\leq n}$ is unimodal.

$$a_i^2 \geq a_{i-1}a_{i+1} \Longrightarrow \frac{a_i}{a_{i+1}} \geq \frac{a_{i-1}}{a_i}.$$

More precisely,

$$\frac{a_{n-1}}{a_n} \geq \frac{a_{n-2}}{a_{n-1}} \geq \cdots \frac{a_2}{a_3} \geq \frac{a_1}{a_2}$$

The conjecture on the log-concavity of $M(m, n)$

Conjecture

For $n \geq 72$ and $72 - n \leq m \leq n - 72$,

$$M(m, n)^2 \geq M(m-1, n)M(m+1, n).$$

In other words, for $n \geq 72$, the sequence $\{M(m, n)\}_{|m| \leq n-71}$ is log-concave.

K. Bringmann, C. Jennings-Shaffer and K. Mahlburg, The asymptotic distribution of the rank for unimodal sequences, J. Number Theory (2021), https://doi.org/10.1016/j.jnt.2020.11.016

The conjecture on the log-concavity of $N(m,n)$

Conjecture

For $n \geq 73$ and $73 - n \leq m \leq n - 73$,

$$N(m,n)^2 \geq N(m-1,n)N(m+1,n).$$

In other words, for $n \geq 73$, the sequence $\{N(m,n)\}_{|m|\leq n-72}$ is log-concave.

It can be shown that this two conjectures implies that $\{M(m,n)\}_{|m|\leq n-2}$ is unimodal for $n \geq 44$ and $\{N(m,n)\}_{|m|\leq n-2}$ is unimodal for $n \geq 39$.

Kathy Q. Ji and Wenston J.T. Zang, Unimodality of the Andrews-Garvan-Dyson cranks of partitions, Adv. Math, 2021, pp.52.

More inequalities on the crank

Theorem (Ji and Zang, 2021)

For $n \geq 14$ and $0 \leq m \leq n-2$,

$$M(m, n) \geq M(m, n-1).$$

Kathy Q. Ji and Wenston J.T. Zang, Unimodality of the Andrews-Garvan-Dyson cranks of partitions, Adv. Math, 2021, pp.52.

The second generating function of crank

Theorem (Garvan)

For $m \geq 0$,

$$\sum_{n=0}^{\infty} M(m,n)q^n = \frac{(1-q)q^m}{(q;q)_m} + \sum_{k=1}^{\infty} \frac{q^{k(k+m)+2k+m}}{(q;q)_k (q^2;q)_{k+m-1}}.$$

F.G. Garvan, Combinatorial interpretations of Ramanujan's partition congruences, Ramanujan revisited, 29–45, Academic Press, Boston, MA, 1988.

Outline of the proof of $M(m,n) \geq M(m,n-1)$

By Garvan's generating function, it's easy to see that

$$\sum_{n=0}^{\infty} (M(m,n) - M(m,n-1))\, q^n$$

$$= \frac{(1-q)^2 q^m}{(q;q)_m} + \frac{q^{2m+3}}{(q^2;q)_m} + \sum_{k=2}^{\infty} \frac{q^{k(k+m)+2k+m}}{(q^2;q)_{k-1}(q^2;q)_{k+m-1}}.$$

To prove $M(m,n) - M(m,n-1) \geq 0$, it suffices to show that the coefficients of q^n ($n \geq 14$) in

$$\frac{(1-q)^2 q^m}{(q;q)_m}$$

are nonnegative when $0 \leq m \leq n-2$.

The definition of $p_m(n)$

Theorem

Let $p_m(n)$ be the number of partitions of n with parts taken from $2, 3, \ldots, m$. Then

$$\sum_{n\geq 0} p_m(n)q^n = \frac{1-q}{(q;q)_m}.$$

By the definition, we see that

$$\begin{aligned}\sum_{n\geq 0} p_m(n)q^n &= (1+q^2+q^{2+2}+\cdots)\\ &\quad \times(1+q^3+q^{3+3}+\cdots)\cdots(1+q^m+q^{m+m}+\cdots)\\ &= \frac{1}{1-q^2}\times\frac{1}{1-q^3}\times\cdots\times\frac{1}{1-q^m}.\end{aligned}$$

Small values of $p_m(n)$

For $m = 3$,

$$p_3(n) = \begin{cases} \left\lfloor \frac{n}{6} \right\rfloor + 1, & \textit{if } n \not\equiv 1 \pmod 6; \\ \left\lfloor \frac{n}{6} \right\rfloor, & \textit{if } n \equiv 1 \pmod 6. \end{cases}$$

For $m = 4$,

$$p_4(n) = \begin{cases} 3a^2 + 3a + 1, & \text{if } n = 12a \text{ or } n = 12a + 3; \\ 3a^2 + 4a + 1, & \text{if } n = 12a + 2 \text{ or } n = 12a + 5; \\ 3a^2 + 5a + 2, & \text{if } n = 12a + 4 \text{ or } n = 12a + 7; \\ 3a^2 + 6a + 3, & \text{if } n = 12a + 6 \text{ or } n = 12a + 9; \\ 3a^2 + 7a + 4, & \text{if } n = 12a + 8 \text{ or } n = 12a + 11; \\ 3a^2 + 8a + 5, & \text{if } n = 12a + 10 \text{ or } n = 12a + 13. \end{cases}$$

The monotonicity property of $p_m(n)$

Theorem (Ji and Zang)

For $r \geq 2$, define

$$d_m(n) = p_m(n) - p_m(n-1).$$

Then

(1) $d_r(0) = 1$ *and* $d_r(1) = -1$ *for all* $r \geq 2$.

(2) $d_2(n) = 1$ *when* n *is even and* $d_2(n) = -1$ *when* n *is odd.*

(3) $d_3(n) = 1$ *when* $n \equiv 0, 2 \pmod 6$, $d_3(n) = -1$ *when* $n \equiv 1 \pmod 6$ *and* $d_3(n) = 0$ *when* $n \equiv 3, 4, 5 \pmod 6$.

(4) $d_4(n) > 0$ *when* n *is even,* $d_4(n) = -\lfloor (n+11)/12 \rfloor$ *when* $n \equiv 1 \pmod 2$ *and* $n \not\equiv 3 \pmod{12}$ *and* $d_4(n) = -\lfloor n/12 \rfloor$ *when* $n \equiv 3 \pmod{12}$.

(5) $d_5(n) \geq 0$ *for* $n \geq 2$. *Moreover,* $d_5(n) \geq 1$ *for* $n \geq 14$.

(6) $d_6(n) \geq 0$ *for* $n \geq 0$ *except for* $d_6(1) = d_6(7) = d_6(13) = -1$.

(7) *When* $r \geq 7$, $d_r(n) \geq 0$ *for* $n \geq 2$. *Moreover,* $d_r(r+2) \geq 1$ *and* $d_r(2r+7) \geq 1$.

Outline of the proof of $M(m,n) \geq M(m,n-1)$

Recall that

$$\sum_{n=0}^{\infty} (M(m,n) - M(m,n-1))\, q^n$$

$$= \frac{(1-q)^2 q^m}{(q;q)_m} + \frac{q^{2m+3}}{(q^2;q)_m} + \sum_{k=2}^{\infty} \frac{q^{k(k+m)+2k+m}}{(q^2;q)_{k-1}(q^2;q)_{k+m-1}}.$$

and

$$\frac{(1-q)^2 q^m}{(q;q)_m} = \sum_{n \geq m} d_m(n-m) q^n$$

It is easy to see that for $n \geq m$

$$M(m,n) - M(m,n-1) \geq d_m(n-m).$$

Hence it could follow from the nonnegativity of $d_m(n)$ that $M(m,n) - M(m,n-1) \geq 0$ when $n \geq 14$ and $0 \leq m \leq n-2$.

F.G. Garvan, Combinatorial interpretations of Ramanujan's partition congruences, Ramanujan revisited, 29–45, Academic Press, Boston, MA,

Outline of the proof of $M(m-1,n) \geq M(m,n)$

By Garvan's generating function, it's easy to see that

$$\sum_{n=0}^{\infty} (M(m-1,n) - M(m,n))\, q^n$$

$$= \sum_{k=1}^{\infty} \frac{q^{k(k+m-1)+2k+m-1}}{(q;q)_k (q^2;q)_{k+m-2}} - \sum_{k=1}^{\infty} \frac{q^{k(k+m)+2k+m}}{(q;q)_k (q^2;q)_{k+m-1}}$$

$$+ \frac{(1-q)q^{m-1}}{(q;q)_{m-1}} - \frac{q^m}{(q^2;q)_{m-1}}.$$

we aim to show that the coefficients of q^n in the above summation are nonnegative when $n \geq 44$ and $1 \leq m \leq n-1$. It turns out that this will be more difficult and it's required to transform the above summation into several summations which have nonnegative power series coefficients.

The inequalities on $ospt(n)$

- Andrews, Chan and Kim, 2013

$$ospt(n) = \overline{M}_1(n) - \overline{N}_1(n).$$

- Chen, Ji and Zang, 2017

  $ospt(n) = \#$ of partitions λ of n such that crank$(\lambda) -$ rank$(\tau_n(\lambda)) = 1$.

- Chan and Mao, 2014

$$ospt(n) < \frac{p(n)}{2} \quad \text{for } n \geq 3.$$

- Chan and Mao raised the following conjecture:

$$ospt(n) < p(n)/3 \quad \text{for} \quad n \geq 10.$$

S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258 (2014) 414–437.

In order to prove the conjecture of Chan and Mao on $ospt(n)$, we will use the following four inequalities.

- Chan and Mao, 2014

$$ospt(n) < \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} + \frac{N(1,n)}{2} \quad \text{for} \quad n \geq 7.$$

- Chen-Ji-Zang, 2017

$$N(\leq m-1, n) \leq M(\leq m, n) \leq N(\leq m, n) \quad \text{for} n \geq 1 \quad \text{and} \quad m \geq 0.$$

- Ji-Zang, 2021

$$M(m,n) \geq M(m+1,n) \quad \text{for} \quad n \geq 44 \quad \text{and} \quad 0 \leq m \leq n-2.$$

- Ji-Zang, 2021

$$p(n) \geq 21M(0,n) \quad \text{for} \quad n \geq 39.$$

We confirmed Chan and Mao's conjecture.

Theorem (Ji-Zang, 2021)

For $n \geq 10$,

$$ospt(n) < p(n)/3.$$

Kathy Q. Ji and Wenston J.T. Zang, Unimodality of the Andrews-Garvan-Dyson cranks of partitions, Adv. Math, 2021, pp.52.

More references

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Concluding remark

Ramanujan discovered so much, and yet he left so much more in his garden for other people to discover.

—Freeman J. Dyson,
A walk through Ramanujan's garden.

THANK YOU!