Inequalities on Ranks and Cranks of Partitions

Kathy Qing Ji（季青）

Center for Applied Mathematics, Tianjin University, P.R. China

kathyji@tju.edu.cn

Joint work with William Y.C. Chen （陈永川）and Wenston J.T. Zang（臧经涛）

Overview

In this talk, I wish to report some recent work on ranks and cranks of partitions:

(1) The definitions of ranks and cranks of partitions;

(2) Proof of two Bringmann and Mahlburg's conjectured inequalities on ranks and cranks of partitions;

(3) Nearly equal distributions of ranks and cranks of partitions;

(4) The distribution of cranks of partitions and some applications.

Part I: The definitions of ranks and cranks of partitions

Integer partitions

Definition

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_l = n.$$

Example: There are five partitions of 4, which are

$$(4), (3,1), (2,2), (2,1,1), (1,1,1,1).$$

Let $p(n)$ denote the number of partitions of n, we see that $p(4) = 5$.

Ramanujan's congruences for $p(n)$

In 1919, Ramanujan found the following striking congruences for $p(n)$.

$$p(5n+4) \equiv 0 \pmod 5.$$
$$p(7n+5) \equiv 0 \pmod 7.$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

The prediction of Ramanujan

It appears that there are no equally simple properties with the following form for any moduli involving primes other than these three (i.e. $\ell = 5, 7, 11$):

$$p(\ell n + \beta) \equiv 0 \mod \ell, \quad \text{for all } n$$

with a prime ℓ and some fixed integer β.

S. Ramanujan, Some propertities of $p(n)$, the number of partitions of n, *Proc. Cambridge Philos. Soc.* 19 (1919) 207–210.

Some related results

(1) Ramanujan gave elementary proofs of congruences mod 5 and 7.

S. Ramanujan, Some propertities of $p(n)$, the number of partitions of n, *Proc. Cambridge Philos. Soc.* 19 (1919) 207–210.

(2) In 1969, Winquist first gave a proof of the congruence mod 11.

L. Winquist, An elementary proof of $p(11m+6) \equiv 0 \bmod 11$, J. Combin. Theory **6** (1969), 56–59.

(3) In 2005, Ahlgren and Boylan present an affirmative answer to Ramanujan's prediction.

S. Ahlgren and M. Boylan, Arithmetic properties of the partition function, Invent. Math. 153 (2005) 487–502.

When he was in the second year of high-school in England (1941), Dyson learnt Ramanujan's congruences from the book "An introduction to the theory of numbers" of Hardy and Wright. He tried hard to supply the missing proof of congruence mod 11, but did not succeed. However, he found the question on the congruences mod 5 and 7 which Ramanujan's proofs do not answer.

Dyson raised such a question on the congruence $p(5n+4) \equiv 0 \pmod 5$:

Dyson's question

How to divide the partitions of $5n+4$ into five classes such that each class has the same number of partitions?

Dyson wanted to find a concrete criterion, so that you could look at any particular partition of $5n+4$ and use the criterion to tell which of the five equal classes it belonged to.

Three years later, when he was a sophomore at Cambridge, he found it and called it "rank".

Dyson's rank

Definition

For a partition λ, its rank is defined as its largest part minus the number of parts.

For example, $\lambda = (5, 4, 2, 1, 1, 1)$, the $\text{rank}(\lambda) = 5 - 6 = -1$.

F. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

Let $N(i, m, n)$ denote the number of partitions of n with rank $\equiv i \mod m$, Dyson first conjectured and later proved by Atkin and Swinnerton-Dyer.

Theorem (Dyson, 1944, Atkin and Swinnerton-Dyer, 1954)

$$\begin{aligned} N(k,5,5n+4) &= \frac{p(5n+4)}{5}, \quad 0 \le k \le 4, \\ N(k,7,7n+5) &= \frac{p(7n+5)}{7}, \quad 0 \le k \le 6, \end{aligned}$$

which imply $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$.

A.O.L. Atking and P. Swinnerton-Dyer, Some properties of partitions, Proc. London Math. Soc. (3) 4 (1954) 84–106.

Example

	$n = 4$ $(p(4) = 5)$	rank mod 5
$N(1,5,n)$	$(3,1)$	1
$N(2,5,n)$	$(1,1,1,1)$	2
$N(3,5,n)$	(4)	3
$N(4,5,n)$	$(2,1,1)$	4
$N(5,5,n)$	$(2,2)$	5

	$n = 5$ $(p(5) = 7)$	rank mod 7
$N(1,7,n)$	$(3,2)$	1
$N(2,7,n)$	$(4,1)$	2
$N(3,7,n)$	(1^5)	3
$N(4,7,n)$	(5)	4
$N(5,7,n)$	$(2,1^3)$	5
$N(6,7,n)$	$(2,2,1)$	6
$N(7,7,n)$	$(3,1,1)$	7

Dyson's observation

Dyson noted that ranks could not provide an interpretation to $p(11n+6) \equiv 0 \pmod{11}$. He also conjectured that there should be a partition statistic (which he called crank) that would provide the missing combinatorial interpretation of the above congruence.
$p(6) = 11$

$\lvert\lambda\rvert = 6$	$r(\lambda)$	$r(\lambda) \pmod{11}$
(6)	5	5
$(5,1)$	4	4
$(4,2)$	2	2
$(4,1^2)$	1	1
$(3,3)$	1	1
$(3,2,1)$	0	0
$(3,1^3)$	-1	10
$(2,2,2)$	-1	10
$(2,2,1^2)$	-2	9
$(2,1^4)$	-3	8
(1^6)	-5	6

Andrews-Garvan-Dyson's crank

Forty-four years later, Garvan first found the definition of the crank defined on vector partitions which became the forerunners of the crank. In the same year, Andrews and Garvan finally found the definition of crank in terms of ordinary partitions based on Garvan's cranks of vector partitions.

G.E. Andrews and F.G. Garvan, Dyson's crank of a partition, Bull. Amer. Math. Soc. 18 (1988) 167–171.

F.G. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7, 11, Trans. Amer. Math. Soc. 305 (1988) 47–77.

Andrews-Garvan-Dyson's crank

Definition

For a partition λ, the crank of λ is defined as follows:

$$crank(\lambda) = \begin{cases} \lambda_1, & \text{if } n_1(\lambda) = 0; \\ \mu(\lambda) - n_1(\lambda), & \text{if } n_1(\lambda) > 0, \end{cases}$$

where $n_1(\lambda)$ denotes the number of 1's in λ and $\mu(\lambda)$ denotes the number of parts of λ larger than $n_1(\lambda)$.

For example, let $\lambda = (7, 7, 6, 4, 3, 1, 1, 1, 1)$, then $n_1(\lambda) = 4$ and $\mu(\lambda) = 3$. This implies $crank(\lambda) = 3 - 4 = -1$.

Let $M(i,m,n)$ denote the number of partitions of n with crank $\equiv i \mod m$, Andrews and Garvan showed that

Theorem (Andrews and Garvan, 1988; Garvan, 1988)

$$\begin{aligned} M(k,5,5n+4) &= \frac{p(5n+4)}{5}, \quad 0 \le k \le 4, \\ M(k,7,7n+5) &= \frac{p(7n+5)}{7}, \quad 0 \le k \le 6, \\ M(k,11,11n+6) &= \frac{p(11n+6)}{11}, \quad 0 \le k \le 10. \end{aligned}$$

which imply $p(5n+4) \equiv 0 \pmod 5$, $p(7n+5) \equiv 0 \pmod 7$ and $p(11n+6) \equiv 0 \pmod{11}$.

Identities of ranks and cranks modulo other numbers

- G.E. Andrews and R. Lewis, The ranks and cranks of partitions moduli 2, 3 and 4, J. Number Theory 85 (2000) 74–84.
- F.G. Garvan, The crank of partitions mod 8, 9 and 10, Trans. Amer. Math. Soc. 322 (1990) 79–94.
- R. Lewis, On the rank and the crank modulo 4, Proc. Amer. Math. Soc. 112 (1991) 925–933.
- R. Lewis, On the ranks of partitions modulo 9, Bull. London Math. Soc. 23 (1991) 417–421.
- R. Lewis, Relations between the rank and the crank modulo 9, J. London. Math. Soc.(2) 45 (1992) 222–231.
- R. Lewis and N. Santa-Gadea, On the rank and the crank modulo 4 and 8, Trans. Amer. Math. Soc. 341 (1994) 449–465.

Connected to mock theta functions

- G.E. Andrews and F.G. Garvan, Ramanujan's "lost" notebook VI: The mock theta conjectures, Adv. Math. 73 (1989) 242–255.
- D. Hickerson, A proof of the mock theta conjectures, Invent. Math. 94 (1988) 639–660.
- D. Hickerson, On the seventh order mock theta functions, Invent. Math. 94 (1988) 661–677.
- K. Bringmann and K. Ono, The $f(q)$ mock theta function conjecture and partition ranks, Invent. Math. 165 (2) (2006) 243–266.
- K. Bringmann and K. Ono, Dyson's ranks and Maass forms, Ann. Math. (2) 171 (1) (2010) 419–449.

Part II: Bringmann and Mahlburg's conjectured inequalities

The rank and the crank functions

- Let $N(\leq m, n)$ denote the number of partitions of n with rank not greater than m;
- let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m.

In the concluding remarks of a paper in 2009, Bringmann and Mahlburg said that they found an interesting phenomenon by testing with Maple

Bringmann and Mahlburg's Observation

For $m \geq 0$ and $1 \leq n \leq 100$,

$$N(\leq m, n) \geq M(\leq m, n) \geq N(\leq m-1, n).$$

K. Bingmann and K. Mahlburg, Inequlaitities between ranks and cranks, Proc. Amer. Math. Soc. 137 (2009) 2567–2574.

Example

For example, let $n = 8$, we have the following table:

m	$N(\leq m, 8)$	$M(\leq m, 8)$	$N(\leq m-1, 8)$
0	12	12	10
1	15	14	12
2	17	16	15
3	19	17	17
4	20	19	19
5	21	20	20
6	21	21	21
7	22	21	21
8	22	22	22

Related to the spt-Function

In 2013, Andrews, Dyson and Rhoades noted that one of Bringmann and Mahlburg's conjectured inequalities is equivalent to their conjecture on the spt-crank of the spt-function.

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (2013) 76–88.

Definition (Andrews, 2008)

The spt-function $spt(n)$, called the smallest part function, is defined to be the total number of smallest parts in all partitions of n.

By the following table, we may see that $p(4) = 5$ and $spt(4) = 10$.

						Total
$\lambda \in \mathcal{P}(4)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$	5
$n_s(\lambda)$	1	1	2	2	4	10

Let $P(n)$ denote the set of ordinary partitions of n, we see that

$$spt(n) = \sum_{\lambda \in P(n)} n_s(\lambda),$$

where $n_s(\lambda)$ denotes the number of occurrences of the smallest part in λ.

The generating function for $spt(n)$

From the definition of the spt-function, we see that

$$\sum_{n\geq 1} spt(n)q^n$$

$$= \sum_{n=1}^{\infty}(q^n + 2q^{2n} + 3q^{3n} + \cdots) \prod_{j=n+1}^{\infty} (1 + q^j + q^{2j} + \cdots)$$

$$= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty}.$$

Connection to the moments of ranks and cranks

Based on the generating function of the spt-function and Watson's q-analog of Whipple's theorem, Andrews found the following connection between the spt-function and the moments of ranks and cranks.

Theorem (Andrews, 2008)

$$2spt(n) = \sum_{\lambda \in \mathcal{P}(n)} crank^2(\lambda) - \sum_{\lambda \in \mathcal{P}(n)} rank^2(\lambda),$$

G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142.

Kathy Q. Ji, A combinatorial proof of Andrews' smallest parts partition function, Electr. J. Combin. 15 (2008) N12.

Based on such connection and using the results for the ranks of partitions, Andrews showed that the spt-function satisfies the following remarkable congruences:

Theorem (Andrews, 2008)

$$\begin{aligned} spt(5n+4) &\equiv 0 \pmod{5}. \\ spt(7n+5) &\equiv 0 \pmod{7}. \\ spt(13n+6) &\equiv 0 \pmod{13}. \end{aligned}$$

G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142.

The definition of the spt-crank

Andrews, Garvan and Liang followed Dyson's way and tried to give combinatorial interpretations of the spt-congruences.

They introduced the spt-crank which is defined on a restricted set of vector partitions.

G.E. Andrews, F.G. Garvan and J.L. Liang, Combinatorial interpretations of congruences for the spt-function, Ramanujan J. 29 (2012) 321–338.

The definition of the spt-crank

Let $N_S(m;n)$ denote the net number of S-partitions of n with spt-crank m. Andrews, Garvan and Liang obtained the following generating function.

$$\sum_m N_S(m;n)z^m q^n = \sum_{n=1}^{\infty} \frac{q^n(q^{n+1};q)_\infty}{(zq^n;q)_\infty(z^{-1}q^n;q)_\infty}.$$

Setting $z=1$ in the above identity, we find that

$$\sum_{m,n} N_S(m;n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty} = \sum_{n\geq 0} spt(n)q^n.$$

G.E. Andrews, F.G. Garvan and J.L. Liang, Combinatorial interpretations of congruences for the spt-function, Ramanujan J. 29 (2012) 321–338.

Combinatorial interpretations

Let $N_S(s,t;n)$ denote the net number of S-partitions of n with spt-crank $\equiv s \mod t$. Andrews, Garvan and Liang showed the following relations:

Theorem (Andrews, Garvan and Liang, 2013)

$$\begin{aligned} N_S(k,5,5n+4) &= \frac{spt(5n+4)}{5}, \quad \text{for} \quad 0 \le k \le 4, \\ N_S(k,7,7n+5) &= \frac{spt(7n+5)}{7}, \quad \text{for} \quad 0 \le k \le 6, \end{aligned}$$

which imply $spt(5n+4) \equiv 0 \pmod 5$ and $spt(7n+5) \equiv 0 \pmod 7$.

Remarks

- Recall that the spt-crank was defined on the set of vector partitions.
- Andrews, Garvan and Liang proposed a problem to find a definition of the spt-crank in terms of ordinary partitions.
- This problem was considered by Andrews, Dyson and Rhoades as an interesting and apparently challenging problem.
- In our recent paper, we gave an answer to this problem, and showed that $N_S(m,n)$ also counts the number of doubly marked partitions of n with spt-crank m.

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (2013) 76–88.

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, The spt-crank for ordinary partitions, J. Reine Angew. Math. 711 (2016) 231–249.

Recall that $N_S(m; n)$ denotes the net number of S-partitions of n with spt-crank m, which also counts the number of doubly marked partitions of n with spt-crank m.
Andrews, Dyson and Rhoades established the following relation:

Theorem (Andrews, Dyson and Rhoades, 2013)

For $m \geq 0$ and $n \geq 0$,

$$N(\leq m, n) - M(\leq m, n) = N_S(m, n) - N_S(m+1, n).$$

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (3) (2013) 76–88.

Therefore the Bringmann and Mahlburg's conjectured inequality $N(\leq m, n) \geq M(\leq m, n)$ is equivalent to the following conjecture on the spt-crank:

Conjecture (Andrews, Dyson and Rhoades, 2013)

For $m \geq 0$ and $n \geq 0$,

$$N_S(m, n) \geq N_S(m+1, n).$$

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (3) (2013) 76–88.

Andrews, Dyson and Rhoades showed this conjecture holds for fixed m and sufficiently large n.

Theorem (Andrews, Dyson and Rhoades, 2013)

For each $m \geq 0$, we have

$$N(\leq m, n) - M(\leq m, n) \sim \frac{(2m+1)\pi^2}{384\sqrt{3}n^2} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as} \quad n \to \infty.$$

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the *spt*-crank, MDPI-Mathematics 1 (3) (2013) 76–88.

Proof of the Andrews-Dyson-Rhoades conjecture

We proved this conjecture holds for all nonnegative integers n and m by constructing an injection.

Theorem (Chen, Ji and Zang, 2017)

For all $n \geq 0$ and $m \geq 0$,

$$N(\leq m, n) \geq M(\leq m, n)$$

$$\Leftrightarrow$$

$$N_S(m, n) \geq N_S(m+1, n).$$

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Proof of the Andrews-Dyson-Rhoades conjecture on the spt-crank, Adv. Math. 270 (2015) 60–96.

Definition (Dyson, 1989)

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an ordinary partition. The rank-set of λ is an infinite sequence

$$[-\lambda_1, 1-\lambda_2, \ldots, j-\lambda_{j+1}, \ldots, \ell-1-\lambda_\ell, \ell, \ell+1, \ldots].$$

For example, the rank-set of $\lambda = (5, 5, 4, 3, 1)$ is

$$[-5, -4, -2, 0, 3, 5, 6, 7, 8, \ldots].$$

F.J. Dyson, Mappings and symmetries of partitions, J. Combin. Theory A 51 (1989) 169–180.

Rank-set

Let $q(m, n)$ be the number of partitions λ of n such that m appears in the rank-set of λ. Dyson established the the following connection: For $n > 1$,

$$M(\leq m, n) = q(m, n).$$

Based on this identity, Dyson also showed that

$$\sum_{\lambda \in \mathcal{P}(n)} crank(\lambda)^2 = 2np(n),$$

which will be used in the proof of an inequality of $spt(n)$.

The sketch of proof

- Let $Q(m,n)$ denote the set of partitions of n such that m appears in the rank-set of λ. $\#Q(m,n) = M(\leq m,n)$.
- Let $P(m,n)$ denote the set of partitions of n with rank not less than $-m$; $\#P(m,n) = N(\leq m,n)$;

To prove the conjecture, we only need to build an injection from the set $Q(m,n)$ to the set $P(m,n)$.

m-Durfee rectangle symbol

In order to characterize the partitions in $Q(m,n)$ and $P(m,n)$, we define m-Durfee rectangle symbol, which is a generalization of the Durfee symbol introduced by Andrews in 2007.

Definition (Chen, Ji and Zang, 2015)

An m-Durfee rectangle symbol is defined as follows

$$(\alpha,\beta)_{(m+j)\times j}=\begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j},$$

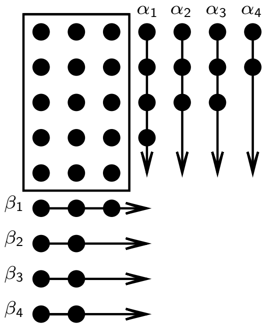
where $(m+j)\times j$ is the m-Durfee rectangle of the Ferrers diagram of λ and α consists of columns to the right of the m-Durfee rectangle and β consists of rows below the m-Durfee rectangle.

G.E. Andrews, Partitions, Durfee symbols, and the Atkin–Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.

Example

For example, the 2-Durfee rectangle symbol of $(7, 7, 6, 4, 3, 3, 2, 2, 2)$ is

$$\begin{pmatrix} 4, & 3, & 3, & 2 \\ 3, & 2, & 2, & 2 \end{pmatrix}_{5\times 3}.$$



Propositions

The set of partitions in $Q(m, n)$ can be described in terms of m-Durfee rectangle symbols as follows.

Proposition (Chen, Ji and Zang, 2015)

Let λ be an ordinary partition and $(\alpha, \beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then m appears in the rank-set of λ if and only if either $j = 0$ or $j \geq 1$ and $\beta_1 = j$.

We describe the set of partitions in $P(m, n)$ in terms of m-Durfee rectangle symbols as follows.

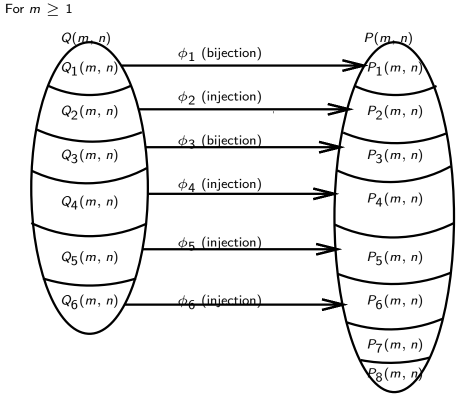
Proposition (Chen, Ji and Zang, 2015)

Let λ be an ordinary partition and $(\alpha, \beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then the rank of λ is not less than $-m$ if and only if either $j = 0$ or $j \geq 1$ and $\ell(\beta) \leq \ell(\alpha)$.

There are two cases: $m \geq 1$ and $m = 0$.

For $m \geq 1$, By using the m-Durfee rectangle symbol, we divide the set $Q(m, n)$ into six disjoint subsets $Q_i(m, n)$, where $1 \leq i \leq 6$, as well as divide the set $P(m, n)$ into eight disjoint subsets $P_i(m, n)$, where $1 \leq i \leq 8$.

Then we construct six injections ϕ_i from $Q_i(m,n)$ to $P_i(m,n)$ for $m \geq 1$. So the conjecture for case $m \geq 1$ has been proved.



The case $m = 0$ is not simpler than $m \geq 1$, in this case, the injection ϕ consists of three more injections.

Proof of Bringmann-Mahlburg's conjectured inequalities

Recently, we found the injections ϕ_1, ϕ_2 and ϕ_3 can also be used to show that the Bringmann and Mahlburg's conjectured inequality $M(\leq m, n) \geq N(\leq m-1, n)$ holds.

Theorem (Chen, Ji and Zang, 2017)

For $m \geq 0$ and $n \geq 0$,

$$N(\leq m, n) \geq M(\leq m, n) \geq N(\leq m-1, n).$$

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Nearly equivalent relation between the rank and crank of partitions, In: G.E. Andrews and F. Garvan (eds.), Analytic Number Theory, Modular Forms and q-Hypergeometric Series, 2017.

Part III: Nearly equal distributions of the rank and the crank of partitions

Bringmann-Mahlburg's observation

Bringmann and Mahlburg also speculated that " the observation on the inequalities may also be stated in terms of ordered lists of partitions. Specifically, for $1 \leq n \leq 100$, there must be some re-ordering τ_n of partitions λ of n such that

$$|\text{crank}(\lambda)| - |\text{rank}(\tau_n(\lambda))| = 0 \text{ or } 1,$$

although we do not have an explicit combinatorial description of τ_n."

K. Bingmann and K. Mahlburg, Inequlaitities between ranks and cranks, Proc. Amer. Math. Soc. 137 (2009) 2567–2574.

The bijection τ_n

- If we list the set of partitions of n in two ways, one by the ranks, and the other by the cranks, then we are led to a re-ordering τ_n of the partitions of n.
- Using Bringmann and Mahlburg's inequalities, we show that the rank and the crank are nearly equidistributed over the set of partitions of n.

Theorem (Chen, Ji and Zang, 2017)

Let τ_n be a reordering on the set of partitions of n as defined above. Then for $\lambda \in \mathcal{P}(n)$, we have

$$|\mathrm{crank}(\lambda)| - |\mathrm{rank}(\tau_n(\lambda))| = 0, \text{ or } 1.$$

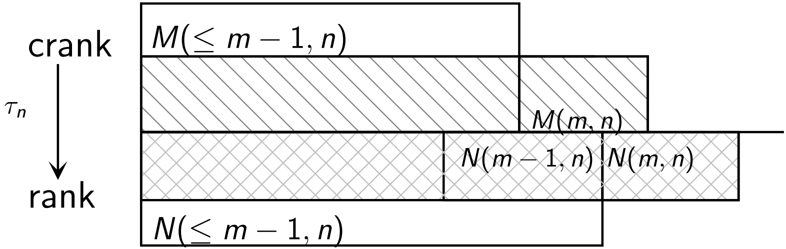
W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Nearly equivalent relation between the rank and crank of partitions, In: G.E. Andrews and F. Garvan (eds.), Analytic Number Theory, Modular Forms and q-Hypergeometric Series, 2017.

Example

$\lambda \in P(5)$	(crank)	$\overset{\tau_5}{\rightarrow}$	$\mu \in P(5)$	(rank)	$\lvert crank(\lambda)\rvert - \lvert rank(\mu$
$(1,1,1,1,1)$	(-5)	$\rightarrow$	$(1,1,1,1,1)$	(-4)	1
$(2,1,1,1)$	(-3)	$\rightarrow$	$(2,1,1,1)$	(-2)	1
$(3,1,1)$	(-1)	$\rightarrow$	$(2,2,1)$	(-1)	0
$(4,1)$	(0)	$\rightarrow$	$(3,1,1)$	(0)	0
$(2,2,1)$	(1)	$\rightarrow$	$(3,2)$	(1)	0
$(3,2)$	(3)	$\rightarrow$	$(4,1)$	(2)	1
(5)	(5)	$\rightarrow$	(5)	(4)	1

The sketch of proof

Using the inequalities $N(\leq m, n) \geq M(\leq m, n) \geq N(\leq m-1, n)$, we have the following illustration for $m \geq 1$:



It is clear that for $crank(\lambda) = m$, we have $rank(\tau_n(\lambda)) = m$ or $m-1$. Thus we have $|crank(\lambda)| - |rank(\tau_n(\lambda))| = 0$ or 1. The case $m \leq 0$ can be proved in a similar way.

Remark

It should be noted that the above description of τ_n relies on the two orderings of partitions of n, it would be interesting to find a direct definition of τ_n which is defined explicitly on a partition λ of n.

An application of the bijection τ_n

Using the bijection τ_n and the Cauchy-Schwartz inequality, we show that the following inequality between $spt(n)$ and $p(n)$ holds.

Theorem (Chen, Ji and Zang, 2017)

For $n \geq 1$, we have

$$spt(n) \leq \sqrt{2n}p(n).$$

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Nearly equivalent relation between the rank and crank of partitions, In: G.E. Andrews and F. Garvan (eds.), Analytic Number Theory, Modular Forms and q-Hypergeometric Series, 2017.

The conjecture on $spt(n)$

In 2014, Chan and Mao raised the following conjecture:

Conjecture (Chan and Mao, 2014)

$$\frac{\sqrt{6n}}{\pi}p(n) \leq spt(n) \leq \sqrt{n}p(n).$$

Note that

$$\frac{\sqrt{6}}{\pi} \approx 0.7796968012$$

S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258 (2014) 414–437.

Bringmann and Mahlburg showed

Theorem (Bringmann and Mahlburg, 2014)

$$spt(n) \sim \frac{\sqrt{6n}}{\pi} p(n) \quad \text{as} \quad n \to \infty.$$

K. Bringmann and K. Mahlburg, Asymptotic inequalities for positive crank and rank moments, Trans. Amer. Math. Soc. 366 (2014) 1073–1094.

Part V: The distribution of the crank of partitions

The rank and crank functions

- Let $N(m, n)$ denote the number of partitions of n with rank m;
- Let $M(m, n)$ denote the number of partitions of n with crank m.

The unimodal-type inequalities on the rank

In 2014, Chan and Mao gave the following two inequalities.

Theorem (Chan and Mao, 2014)

For $m \geq 0$ and $n \geq 0$,

$$N(m, n) \geq N(m+2, n),$$

and for $n \geq 12$, $n \neq m+2$,

$$N(m, n) \geq N(m, n-1).$$

S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258 (2014) 414–437.

The unimodality on the crank

Theorem (Ji and Zang, preprint)

For $n \geq 44$ and $1 \leq m \leq n-1$,

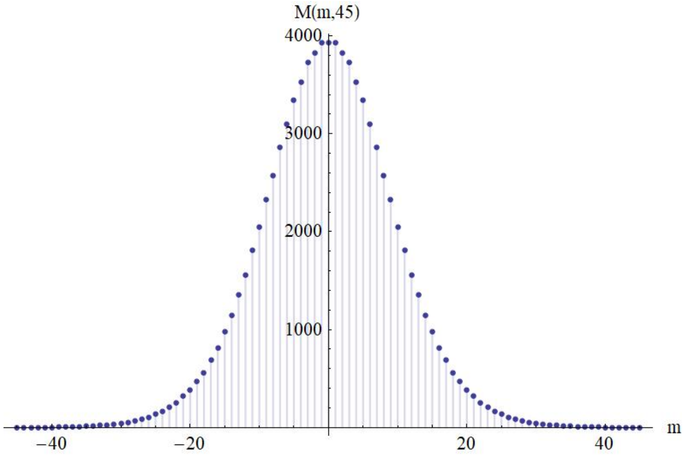
$$M(m-1, n) \geq M(m, n),$$

By the symmetry $M(m, n) = M(-m, n)$, the above inequality implies the following corollary.

Corollary (Ji and Zang, preprint)

For $n \geq 44$, the sequence $\{M(m, n)\}_{2-n \leq m \leq n-2}$ is unimodal.

Kathy Q. Ji and Wenston J.T. Zang, On the distribution of the Andrews-Garvan-Dyson cranks of partitions, preprint.


M(m,45)
4000
3000
2000
1000
−40
−20
20
40
m

More inequalities on the crank

By the above inequality, we also obtain the following inequalities on $M(m,n)$ similar to the inequalities on $N(m,n)$ due to Chan and Mao.

Theorem (Ji and Zang, preprint)

For $n \geq 4$ *and* $0 \leq m \leq n-2$,

$$M(m,n) \geq M(m+2,n),$$

and for $n \geq 14$ *and* $0 \leq m \leq n-2$,

$$M(m,n) \geq M(m,n-1).$$

Kathy Q. Ji and Wenston J.T. Zang, On the distribution of the Andrews-Garvan-Dyson cranks of partitions, preprint.

The definition of $p_k(n)$

To prove these inequalities on $M(m, n)$, we investigated the distribution of the partition function $p_k(n)$,

Definition

Let $p_k(n)$ be the number of partitions of n into at most k parts such that the largest part appears at least two times.

When $k = 3$ and $n = 6$, there are six partitions of 6 with at most 3 parts, which are

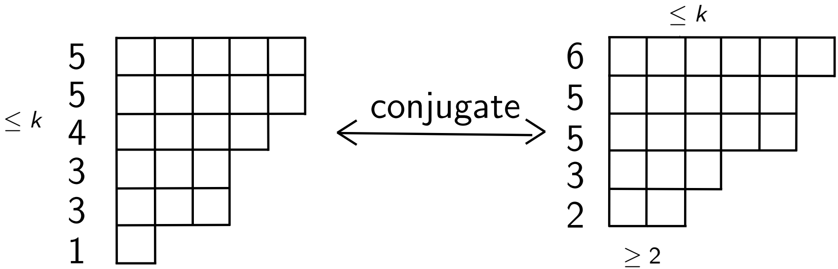
$$(6), (5, 1), (4, 2), (3, 3), (4, 1, 1), (3, 2, 1),$$

and there are only one partition of 6 with at most 3 parts such that the largest part appears at least two times, that is $(3, 3)$. So

$$p_3(6) = 1.$$

The definition of $p_k(n)$

By the conjugate of partitions, we see that $p_k(n)$ also counts the number of partitions of n such that each part is larger than 1 and not exceeding k.



The generating function of $p_k(n)$

Hence the generating function of $p_k(n)$ can be expressed as follows:

$$\begin{aligned}\sum_{n\geq 0} p_k(n)q^n &= (1+q^2+q^{2+2}+\cdots)\\ &\times(1+q^3+q^{3+3}+\cdots)\cdots(1+q^k+q^{k+k}+\cdots)\\ &= \frac{1}{1-q^2}\times\frac{1}{1-q^3}\times\cdots\times\frac{1}{1-q^k}.\end{aligned}$$

For $k = 3$,

$$p_3(n) = \begin{cases} \left\lfloor \frac{n}{6} \right\rfloor + 1, & \textit{if} \ \ n \not\equiv 1 \pmod 6; \\ \left\lfloor \frac{n}{6} \right\rfloor, & \textit{if} \ \ n \equiv 1 \pmod 6. \end{cases}$$

For $k = 4$,

$$p_4(n) = \begin{cases} 3a^2 + 3a + 1, & \text{if } n = 12a \text{ or } n = 12a + 3; \\ 3a^2 + 4a + 1, & \text{if } n = 12a + 2 \text{ or } n = 12a + 5; \\ 3a^2 + 5a + 2, & \text{if } n = 12a + 4 \text{ or } n = 12a + 7; \\ 3a^2 + 6a + 3, & \text{if } n = 12a + 6 \text{ or } n = 12a + 9; \\ 3a^2 + 7a + 4, & \text{if } n = 12a + 8 \text{ or } n = 12a + 11; \\ 3a^2 + 8a + 5, & \text{if } n = 12a + 10 \text{ or } n = 12a + 13. \end{cases}$$

The monotonicity of $p_k(n)$

We also showed that $p_k(n)$ has the following monotonicity property.

Theorem (Ji and Zang, preprint)

For $k = 4$ or $k \geq 6$ and $n \geq 2$, we have

$$p_k(n) \geq p_k(n-1).$$

Using the above inequality on $p_k(n)$, we give proofs of these inequalities on $M(m,n)$.

Kathy Q. Ji and Wenston J.T. Zang, On the distribution of the Andrews-Garvan-Dyson cranks of partitions, preprint.

The definition of $ospt(n)$

In 2013, Andrews, Chan and Kim define $ospt(n)$ as follows:

Definition (Andrews, Chan and Kim, 2013)

$$ospt(n) = \sum_{m \geq 0} mM(m,n) - \sum_{m \geq 0} mN(m,n).$$

Andrews, Chan and Kim showed that $ospt(n) > 0$ and gave a combinatorial interpretation of $ospt(n)$ in terms of even strings and odd strings of a partition.

G.E. Andrews, S.H. Chan, B. Kim, The odd moments of ranks and cranks, J. Combin. Theory Ser. A 120 (2013) 77–91.

The combinatorial interpretation of $ospt(n)$

We showed that $ospt(n)$ can also be interpreted in terms of $\tau(n)$.

Theorem (Chen, Ji and Zang, 2017)

For $n > 1$, $ospt(n)$ equals the number of partitions λ of n such that $crank(\lambda) - rank(\tau_n(\lambda)) = 1$.

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Nearly equivalent relation between the rank and crank of partitions, In: G.E. Andrews and F. Garvan (eds.), Analytic Number Theory, Modular Forms and q-Hypergeometric Series, 2017.

The inequalities on $ospt(n)$

Chan and Mao gave the following inequalities on $ospt(n)$ by using the inequalities on the ranks of partitions:

Theorem (Chan and Mao, 2014)

$$ospt(n) > \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4}, \qquad \text{for } n \geq 8,$$

$$ospt(n) < \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} + \frac{N(1,n)}{2}, \qquad \text{for } n \geq 7,$$

$$ospt(n) < \frac{p(n)}{2}, \qquad \text{for } n \geq 3.$$

S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258 (2014) 414–437.

These inequalities on $ospt(n)$ imply the following asymptotic estimate:

Theorem (Bringmann and Mahlburg, 2014)

$$ospt(n) \sim \frac{1}{4} p(n) \quad \text{as} \quad n \to \infty.$$

Chan and Mao raised the following conjecture:

$$ospt(n) < p(n)/3 \quad \text{for} \quad n \geq 10.$$

K. Bringmann and K. Mahlburg, Asymptotic inequalities for positive crank and rank moments, Trans. Amer. Math. Soc. 366 (2014) 1073–1094.

In order to prove the conjecture of Chan and Mao on $ospt(n)$, we will use the following four inequalities.

- Chan and Mao, 2014

$$ospt(n) < \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} + \frac{N(1,n)}{2} \quad \text{for} \quad n \geq 7.$$

- Chen-Ji-Zang, 2017

$$N(\leq m-1, n) \leq M(\leq m, n) \leq N(\leq m, n) \quad \text{for} n \geq 1 \quad \text{and} \quad m \geq 0.$$

- Ji-Zang, 2018

$$M(m,n) \geq M(m+1,n) \quad \text{for} \quad n \geq 44 \quad \text{and} \quad 0 \leq m \leq n-2.$$

- Ji-Zang, 2018

$$p(n) \geq 21M(0,n) \quad \text{for} \quad n \geq 76.$$

We confirmed Chan and Mao's conjecture.

Theorem

For $n \geq 10$,

$$ospt(n) < p(n)/3.$$

Kathy Q. Ji and Wenston J.T. Zang, On the distribution of the Andrews-Garvan-Dyson cranks of partitions, preprint.

THANK YOU!