On Stanley's Partition Function

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Abstract

Stanley defined a partition function $t(n)$ as the number of partitions λ of n such that the number of odd parts of λ is congruent to the number of odd parts of the conjugate partition λ' modulo 4. We show that $t(n)$ equals the number of partitions of n with an even number of hooks of even length. We derive a closed-form formula for the generating function for the numbers $p(n) - t(n)$. As a consequence, we see that $t(n)$ has the same parity as the ordinary partition function $p(n)$. A simple combinatorial explanation of this fact is also provided.

1 Introduction

This note is concerned with the partition function $t(n)$ introduced by Stanley [8, 9]. We shall give a combinatorial interpretation of $t(n)$ in terms of hook lengths and shall prove that $t(n)$ and the partition function $p(n)$ have the same parity. Moreover, we compute the generating function for $p(n) - t(n)$.

We shall adopt the common notation on partitions in Andrews [1] or Andrews and Eriksson [3]. A partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r)$ of a nonnegative integer n is a nonincreasing sequence of nonnegative integers such that the sum of the components λ_i equals n. A part is meant to be a positive component, and the number of parts of λ is called the length, denoted $l(\lambda)$. The conjugate partition of λ is defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$, where λ'_i $(1 \leqslant i \leqslant t,\ t = l(\lambda))$ is the number of parts in $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ which are greater than or equal to i. The number of odd parts in $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is denoted by $\mathcal{O}(\lambda)$.

For $|q| < 1$, the q-shifted factorial is defined by

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \geqslant 1,$$

and

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots,$$

see Gasper and Rahman [5].

Stanley [8, 9] introduced the partition function $t(n)$ as the number of partitions λ of n such that $\mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda')$ (mod 4), and obtained the following formula

$$t(n) = \frac{1}{2}\left(p(n) + f(n)\right), \tag{1.1}$$

where $p(n)$ is the number of partitions of n and $f(n)$ is determined by the generating function

$$\sum_{n=0}^{\infty} f(n)q^n = \prod_{i\geqslant 1} \frac{(1+q^{2i-1})}{(1-q^{4i})(1+q^{4i-2})^2}. \tag{1.2}$$

Andrews [2] obtained the following closed-form formula for the generating function of $t(n)$

$$\sum_{n=0}^{\infty} t(n)q^n = \frac{(q^2;q^2)_\infty^2 (q^{16};q^{16})_\infty^5}{(q;q)_\infty (q^4;q^4)_\infty^5 (q^{32};q^{32})_\infty^2}. \tag{1.3}$$

He also derived the congruence relation

$$t(5n+4) \equiv 0 \ (\mathrm{mod}\ 5). \tag{1.4}$$

In this note, we shall consider the complementary partition function of $t(n)$, namely, the partition function $u(n) = p(n) - t(n)$, which is the number of partitions λ of n such that $\mathcal{O}(\lambda) \not\equiv \mathcal{O}(\lambda')$ (mod 4). We obtain a closed-form formula for the generating function of $u(n)$ which implies that Stanley's partition function $t(n)$ and ordinary partition function $p(n)$ have the same parity for any n. We also present a simple combinatorial explanation of this fact. Furthermore, we derive formulas for the generating functions for the numbers $u(4n), u(4n+1), u(4n+2)$ and $u(4n+3)$, which are analogous to the generating function formulas for the partition functions $t(4n)$, $t(4n+1)$, $t(4n+2)$ and $t(4n+3)$ due to Andrews [2]. In the last section, we find combinatorial interpretations for $t(n)$ and $u(n)$ in terms of hooks of even length.

2 The generating function formula

In this section, we shall derive a generating function formula for the partition function $u(n) = p(n) - t(n)$. The proof is similar to Andrews' proof of (1.3) for $t(n)$. As a consequence, one sees that $t(n)$ and $p(n)$ have the same parity for any nonnegative integer n. This fact also has a simple combinatorial interpretation. We shall also compute the generating functions for the numbers $u(4n), u(4n+1), u(4n+2)$ and $u(4n+3)$.

Theorem 2.1 *We have*

$$\sum_{n=0}^{\infty} u(n)q^n = \frac{2q^2(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2 (q^{32};q^{32})_\infty^2}{(q;q)_\infty (q^4;q^4)_\infty^5 (q^{16};q^{16})_\infty}. \tag{2.5}$$

Proof. We notice that the definition of $t(n)$ implies

$$u(n) = p(n) - t(n) = \frac{p(n) - f(n)}{2}. \tag{2.6}$$

Hence we have

$$\begin{aligned}
\sum_{n=0}^{\infty} u(n)q^n &= \frac{1}{2}\left(\frac{1}{(q;q)_\infty} - \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)^2_\infty}\right) \\
&= \frac{1}{2}\left(\frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(q^2;q^4)^2_\infty} - \frac{(-q;q^2)_\infty}{(q^4;q^4)_\infty(-q^2;q^4)^2_\infty}\right) \\
&= \frac{(-q;q^2)_\infty}{2(q^4;q^4)^2_\infty(q^2;q^4)^2_\infty(-q^2;q^4)^2_\infty}\left((q^4;q^4)_\infty(-q^2;q^4)^2_\infty - (q^4;q^4)_\infty(q^2;q^4)^2_\infty\right).
\end{aligned}$$

Using Jacobi's triple product identity [4, p.10]

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty, \tag{2.7}$$

we see that

$$(q^4;q^4)_\infty(-q^2;q^4)^2_\infty = \sum_{n=-\infty}^{\infty} q^{2n^2} \tag{2.8}$$

and

$$(q^4;q^4)_\infty(q^2;q^4)^2_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}. \tag{2.9}$$

Clearly,

$$\sum_{n=-\infty}^{\infty} q^{2n^2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} = 2\sum_{n=-\infty}^{\infty} q^{2(2n+1)^2}. \tag{2.10}$$

Thus we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} u(q)q^n &= \frac{(-q;q^2)_\infty}{(q^4;q^4)^2_\infty(q^2;q^4)^2_\infty(-q^2;q^4)^2_\infty}\sum_{n=-\infty}^{\infty} q^{2(2n+1)^2} \\
&= \frac{q^2(-q;q^2)_\infty}{(q^4;q^4)^2_\infty(q^4;q^8)^2_\infty}\sum_{n=-\infty}^{\infty} q^{8n^2+8n}. \qquad (2.11)
\end{aligned}$$

Using Jacobi's triple product identity, we find

$$\sum_{n=-\infty}^{\infty} q^{8n^2+8n} = (-q^{16};q^{16})_\infty(-1;q^{16})_\infty(q^{16};q^{16})_\infty. \tag{2.12}$$

Observe that

$$(-1;q^{16})_\infty = 2(-q^{16};q^{16})_\infty. \tag{2.13}$$

In view of (2.11), we get

$$\sum_{n=0}^{\infty} u(q)q^n = \frac{2q^2(-q^{16};q^{16})_\infty(-q^{16};q^{16})_\infty(-q;q^2)_\infty(q^{16};q^{16})_\infty}{(q^4;q^4)^2_\infty(q^4;q^8)^2_\infty}$$
$$= \frac{2q^2(q^{32};q^{32})_\infty(-q;q^2)_\infty(-q^{16};q^{16})_\infty}{(q^4;q^4)^2_\infty(q^4;q^8)^2_\infty}.$$

Now,

$$(-q;q^2)_\infty = \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty(q^4;q^4)_\infty}, \tag{2.14}$$

$$(q^4;q^8)_\infty = \frac{(q^4;q^4)_\infty}{(q^8;q^8)_\infty} \tag{2.15}$$

and

$$(-q^{16};q^{16})_\infty = \frac{(q^{32};q^{32})_\infty}{(q^{16};q^{16})_\infty}. \tag{2.16}$$

Consequently,

$$\sum_{n=0}^{\infty} u(q)q^n = \frac{2q^2(q^{32};q^{32})_\infty(q^8;q^8)^2_\infty(q^2;q^2)^2_\infty(q^{32};q^{32})_\infty}{(q^4;q^4)^2_\infty(q^4;q^4)^2_\infty(q;q)_\infty(q^4;q^4)_\infty(q^{16};q^{16})_\infty}$$
$$= \frac{2q^2(q^2;q^2)^2_\infty(q^8;q^8)^2_\infty(q^{32};q^{32})^2_\infty}{(q;q)_\infty(q^4;q^4)^5_\infty(q^{16};q^{16})_\infty}.$$

This completes the proof. ∎

Corollary 2.2 *For $n \geqslant 0$,*

$$t(n) \equiv p(n) \pmod 2.$$

We remark that there is a simple combinatorial explanation of the above parity property. We observe that for any partition λ of n,

$$\mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda') \pmod 2 \tag{2.17}$$

because we have both $\mathcal{O}(\lambda) \equiv n \pmod 2$ and $\mathcal{O}(\lambda') \equiv n \pmod 2$. By the definition of $u(n)$ and relation (2.17), we deduce that $u(n)$ equals the number of partitions of n such that

$$\mathcal{O}(\lambda) - \mathcal{O}(\lambda') \equiv 2 \pmod 4. \tag{2.18}$$

Suppose λ is a partition counted by $u(n)$. From (2.18) it is evident that its conjugation λ' is also counted by $u(n)$. Once more, from (2.18) we deduce that $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda')$ are not equal, so that λ is different from λ'. Thus we reach the conclusion that $u(n)$ must be even, and so $t(n)$ has the same parity as $p(n)$ since $p(n) = t(n) + u(n)$.

In view of (2.6), we have the following congruence relation.

Corollary 2.3 *For $n \geqslant 0$,*

$$f(n) \equiv p(n) \pmod 4.$$

Theorem 2.1 enables us to derive the generating functions for $u(4n+i)$, where $i = 0, 1, 2, 3$. Andrews [2] has obtained formulas for the generating functions of $t(4n+i)$ for $i = 0, 1, 2, 3$.

Theorem 2.4 *We have*

$$\sum_{n=0}^{\infty} u(4n)q^n = 2q^2(q^{16};q^{16})_\infty(-q;q^{16})_\infty(-q^{15};q^{16})_\infty V(q),$$

$$\sum_{n=0}^{\infty} u(4n+1)q^n = 2q(q^{16};q^{16})_\infty(-q^3;q^{16})_\infty(-q^{13};q^{16})_\infty V(q),$$

$$\sum_{n=0}^{\infty} u(4n+2)q^n = 2(q^{16};q^{16})_\infty(-q^7;q^{16})_\infty(-q^9;q^{16})_\infty V(q),$$

$$\sum_{n=0}^{\infty} u(4n+3)q^n = 2(q^{16};q^{16})_\infty(-q^5;q^{16})_\infty(-q^{11};q^{16})_\infty V(q),$$

where

$$V(q) = \frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty}.$$

Proof. By Theorem 2.1, we find

$$\begin{aligned}\sum_{n=0}^{\infty} u(n)q^n &= \frac{2q^2(q^2;q^2)_\infty^2}{(q;q)_\infty}V(q^4)\\ &= \frac{2q^2(q^2;q^2)_\infty}{(q;q^2)_\infty}V(q^4)\end{aligned}$$

Since

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty \tag{2.19}$$

and

$$(q^2;q^2)_\infty = (q;q)_\infty(-q;q)_\infty, \tag{2.20}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} u(n)q^n &= 2q^2(q;q)_\infty(-q;q)_\infty(-q;q)_\infty V(q^4)\\ &= q^2(q;q)_\infty(-1;q)_\infty(-q;q)_\infty V(q^4).\end{aligned}$$

Using Jacobi's triple product identity, we get

$$(q;q)_\infty(-1;q)_\infty(-q;q)_\infty = \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}}. \tag{2.21}$$

Thus we have

$$\sum_{n=0}^{\infty} u(n)q^n = q^2 \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} V(q^4) = 2q^2 \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} V(q^4). \tag{2.22}$$

It is easy to check that

$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \sum_{n=-\infty}^{\infty} q^{2n^2-n}. \tag{2.23}$$

In virtue of (2.22), we get

$$\begin{aligned}\sum_{n=0}^{\infty} u(n)q^n &= 2q^2 \sum_{n=-\infty}^{\infty} q^{2n^2-n} V(q^4)\\ &= 2q^2 \sum_{i=0}^{3} \sum_{k=-\infty}^{\infty} q^{2(4k+i)^2-(4k+i)} V(q^4). \end{aligned}\tag{2.24}$$

For $i=0$, extracting the terms of the form q^{4j+2} in (2.24) for any integer j, we obtain

$$\sum_{n=0}^{\infty} u(4n+2)q^{4n+2} = 2q^2 \sum_{j=-\infty}^{\infty} q^{32j^2-4j} V(q^4).$$

Again, Jacobi's triple product identity gives

$$\sum_{j=-\infty}^{\infty} q^{32j^2-4j} = (q^{64};q^{64})_\infty(-q^{28};q^{64})_\infty(-q^{36};q^{64})_\infty. \tag{2.25}$$

Hence we get

$$\sum_{n=0}^{\infty} u(4n+2)q^{4n+2} = 2q^2(q^{64};q^{64})_\infty(-q^{28};q^{64})_\infty(-q^{36};q^{64})_\infty V(q^4),$$

which simplifies to

$$\sum_{n=0}^{\infty} u(4n+2)q^{n} = 2(q^{16};q^{16})_\infty(-q^{7};q^{16})_\infty(-q^{9};q^{16})_\infty V(q).$$

The remaining cases can be verified using similar arguments. This completes the proof. ∎

3 Combinatorial interpretations for $t(n)$ and $u(n)$

In [8, Proposition 3.1], Stanley found three partition statistics that have the same parity as $(\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2$, and gave several combinatorial interpretations for $t(n)$. We shall present combinatorial interpretations of partition functions $t(n)$ and $u(n)$ in terms of the number of hooks of even length. For the definition of hook lengths, see Stanley [7, p. 373]. A hook of even length is called an even hook. The following theorem shows that the number of even hooks has the same parity as $(\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2$.

Theorem 3.1 *For any partition λ of n, $\mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda') \pmod 4$ if and only if λ has an even number of even hooks.*

Proof. We use induction on n. It is clear that Theorem 3.1 holds for $n = 1$. Suppose that it is true for all partitions of n. We aim to show that the conclusion also holds for all partitions of $n + 1$. Let λ be a partition of $n + 1$ and $v = (i, j)$ be any inner corner of the Young diagram of λ, that is, the removal of the square v gives a Young diagram of a partition of n. Let λ^- denote the partition obtained by removing the square v from the Young diagram of λ. We use $H_e(\lambda)$ to denote the number of squares with even hooks in the Young diagram of λ. We claim that

$$H_e(\lambda) \equiv H_e(\lambda^-) \pmod 2 \quad \text{if and only if} \quad \lambda_i \equiv \lambda'_j \pmod 2. \tag{3.26}$$

Let $\mathcal{T}(\lambda, v)$ denote the set of all squares in the Young diagram of λ which are in the same row as v or in the same column as v. After removing the square v from the Young diagram of λ, the hook lengths of the squares in $\mathcal{T}(\lambda, v)$ decrease by one. Meanwhile, the hook lengths of other squares remain the same. Furthermore, if λ_i and λ'_j have the same parity, then the number of squares in $\mathcal{T}(\lambda, v)$ is even. This implies that the parity of the number of squares in $\mathcal{T}(\lambda, v)$ of even hook length coincides with the parity of the number of squares in $\mathcal{T}(\lambda, v)$ of odd hook length. Similarly, for the case when λ_i and λ'_j have different parities, it can be shown that the number of squares in $\mathcal{T}(\lambda, v)$ of even hook length is of opposite parity to the number of squares in $\mathcal{T}(\lambda, v)$ of odd hook length. Hence we arrive at (3.26).

By the inductive hypothesis, we see that $\mathcal{O}(\lambda^-) \equiv \mathcal{O}((\lambda^-)') \pmod 4$ if and only if $H_e(\lambda^-)$ is even. For any inner corner $v = (i, j)$ of λ, if $\lambda_i \equiv \lambda'_j \pmod 2$, then $\mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda') \pmod 4$ if and only if $\mathcal{O}(\lambda^-) \equiv \mathcal{O}((\lambda^-)') \pmod 4$. By (3.26), we find that in this case, $H_e(\lambda)$ and $H_e(\lambda^-)$ have the same parity. Thus the assertion holds for any partition λ of $n+1$. The case that $\lambda_i \not\equiv \lambda'_j \pmod 2$ can be justified in the same manner. This completes the proof. ∎

From Theorem 3.1, we obtain a combinatorial interpretation for Stanley's partition function $t(n)$, which can be recast as a combinatorial interpretation for $u(n)$.

Theorem 3.2 *The partition function $t(n)$ is equal to the number of partitions of n with an even number of even hooks, and the partition function $u(n)$ is equal to the number of partitions of n with an odd number of even hooks.*

Combining Theorem 2.1 and Theorem 3.2, we have the following parity property.

Corollary 3.3 *For any positive integer n, the number of partitions of n with an odd number of even hooks is always even.*

Since $f(n) = t(n) - u(n)$, from Theorem 3.2 we see that $f(n)$ can be interpreted as a signed counting of partitions of n with respect to the number of even hooks, as stated below.

Corollary 3.4 *The function $f(n)$ equals the number of partitions of n with an even number of even hooks minus the number of partitions of n with an odd number of even hooks.*

To conclude, we remark that Corollary 3.4 can also be deduced from an identity of Han [6, Corollary 5.2] by setting $t = 2$.

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