A Combinatorial Proof of Andrews' Smallest Parts Partition Function

Kathy Qing Ji
Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
ji@nankai.edu.cn

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Abstract

We give a combinatorial proof of Andrews' smallest parts partition function with the aid of rooted partitions introduced by Chen and the author.

1 Introduction

We adopt the common notation on partitions as used in [1]. *A partition* λ of a positive integer n is a finite nonincreasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$$

such that $\sum_{i=1}^{r} \lambda_i = n$. Then λ_i are called the parts of λ. The number of parts of λ is called the length of λ, denoted by $l(\lambda)$. The weight of λ is the sum of parts, denoted by $|\lambda|$. We let $\mathcal{P}(n)$ denote the set of partitions of n.

Let $spt(n)$ denote the number of smallest parts in all partitions of n and $n_s(\lambda)$ denote the number of the smallest parts in λ, we then have

$$spt(n) = \sum_{\lambda \in \mathcal{P}(n)} n_s(\lambda). \tag{1.1}$$

Below is a list of the partitions of 4 with their corresponding number of smallest parts. We see that $spt(4) = 10$.

$\lambda \in \mathcal{P}(4)$	$n_s(\lambda)$
(4)	1
$(3,1)$	1
$(2,2)$	2
$(2,1,1)$	2
$(1,1,1,1)$	4

The rank of a partition λ introduced by Dyson [6] is defined as the largest part minus the number of parts, which is usually denoted by $r(\lambda) = \lambda_1 - l(\lambda)$. Let $N(m,n)$ denote the number of partitions of n with rank m. Atkin and Garvan [4] define the kth moment of the rank by

$$N_k(n) = \sum_{m=-\infty}^{+\infty} m^k N(m,n). \tag{1.2}$$

In [2], Andrews shows the following partition function on $spt(n)$ analytically:

Theorem 1.1 (Andrews)

$$spt(n) = np(n) - \frac{1}{2}N_2(n), \tag{1.3}$$

where $p(n)$ *is the number of partitions of* n*.*

At the end of the paper, Andrews states that "In addition the connection of $N_2(n)/2$ to the enumeration of 2-marked Durfee symbols in [3] suggests the fact that there are also serious problems concerning combinatorial mappings that should be investigated." In this paper, we give a combinatorial proof of (1.3) with the aid of rooted partitions introduced by Chen and the author [5], instead of using a 2-marked Durfee symbols.

A rooted partition of n can be formally defined as a pair of partitions (α, β), where $|\alpha| + |\beta| = n$ and β is a nonempty partition with equal parts. The union of the parts of α and β are regarded as the parts of the rooted partition (α, β).

Example 1.2 *There are twelve rooted partitions of* 4*:*

$$\begin{array}{cccc}
(\emptyset,\ (4)) & ((1),\ (3)) & ((3),\ (1)) & ((2),\ (2)) \\
(\emptyset,\ (2,2)) & ((1,1),\ (2)) & ((2,1),(1)) & ((2),(1,1)) \\
((1,1,1),\ (1)) & ((1,1),\ (1,1)) & ((1),\ (1,1,1)) & (\emptyset,\ (1,1,1,1))
\end{array}$$

Let $\mathcal{RP}(n)$ denote the set of rooted partitions of n.

2 Combinatorial proof

In this section, we will first build the connection between rooted partitions and ordinary partitions, then interpret $np(n)$, $\frac{1}{2}N_2(n)$ in terms of rooted partitions (see Theorems 2.2 and 2.5). In this framework, a combinatorial justification of (1.3) reduces to build a bijection between the set of ordinary partitions of n and the set of the rooted partitions (α, β) of n with $\beta_1 > \alpha_1$.

We now make a connection between rooted partitions and ordinary partitions by extending the construction in [5, Theorems 3.5, 3.6].

Lemma 2.1 *The number of rooted partitions of n is equal to the sum of lengths over partitions of n, namely*

$$\sum_{(\alpha,\beta)\in\mathcal{RP}(n)} 1 = \sum_{\lambda\in\mathcal{P}(n)} l(\lambda). \tag{2.4}$$

Proof. For a given partition $\lambda = (\lambda_1,\, \lambda_2, \ldots,\, \lambda_l) \in \mathcal{P}(n)$, we could get $l(\lambda)$ distinct rooted partitions $(\alpha,\, \beta)$ of n by designating any part of λ as the part of β and keep the remaining parts of λ as parts of α. Assume that d is a part that appears m_d times ($m_d \geq 2$) in λ, we then choose β as the partition with d repeated i times, where $i = 1, 2, \ldots, m_d$. Conversely, for a rooted partition (α, β), we could get an ordinary partition λ by uniting the parts of α and β. It's clear to see that there are *exactly* $l(\lambda)$ distinct rooted partitions corresponding to λ in $\mathcal{RP}(n)$. ∎

For example, there are five partitions of 4: (4), $(3,1)$, $(2,2)$, $(2,1,1)$, $(1,1,1,1)$, and the sum of lengths is twelve. From Example 1.2, we see that there are also twelve rooted partitions of 4.

We are ready to interpret $np(n)$ in terms of rooted partitions using the construction in Lemma 2.1.

Theorem 2.2 *$np(n)$ is equal to the sum of β_1 over all rooted partitions (α, β) of n, namely*

$$np(n) = \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} \beta_1. \tag{2.5}$$

Proof. As $np(n) = \sum_{\lambda\in\mathcal{P}(n)} |\lambda|$, it suffices to prove

$$\sum_{\lambda\in\mathcal{P}(n)} |\lambda| = \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} \beta_1. \tag{2.6}$$

From Lemma 2.1, one sees that for $\lambda \in \mathcal{P}(n)$, there exists exactly $l(\lambda)$ distinct rooted partitions (α, β) corresponding to it in $\mathcal{RP}(n)$. Furthermore, the sum of β_1 over these $l(\lambda)$ distinct rooted partitions equals to $|\lambda|$, this is because that β is obtained by designating some equal parts of λ as its parts. Thus we get the identity (2.6). ∎

For example, for the case of $n = 4$, there are five partitions of 4, so we have $4p(4) = 20$. While the sum of β_1 over all twelve rooted partitions (α, β) of 4 is also twenty (see Example 1.2).

For the combinatorial explanation of $\frac{1}{2}N_2(n)$ in terms of rooted partitions, we first reinterpret $\frac{1}{2}N_2(n)$ in terms of ordinary partitions. Here we need to define the conjugate of the partition. For a partition $\lambda = (\lambda_1, \ldots,\, \lambda_r)$, the conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots,\, \lambda'_t)$ of λ by setting λ'_i to be the number of parts of λ that are greater than or equal to i. Clearly, $l(\lambda) = \lambda'_1$ and $\lambda_1 = l(\lambda')$. Therefore, it's straightforward to verify the following partition identity:

$$\sum_{\lambda\in\mathcal{P}(n)} \lambda_1^2 = \sum_{\lambda\in\mathcal{P}(n)} l(\lambda)^2. \tag{2.7}$$

We have the following lemma:

Lemma 2.3

$$\frac{1}{2}N_2(n) = \sum_{\lambda\in\mathcal{P}(n)} l(\lambda)^2 - \sum_{\lambda\in\mathcal{P}(n)} [\lambda_1 \cdot l(\lambda)]. \tag{2.8}$$

Proof. From the definition of rank and the moment of rank, we know that

$$\frac{1}{2}N_2(n) = \sum_{\lambda\in\mathcal{P}(n)} \frac{(\lambda_1 - l(\lambda))^2}{2}. \tag{2.9}$$

Clearly,

$$\sum_{\lambda\in\mathcal{P}(n)} \frac{(\lambda_1 - l(\lambda))^2}{2} = \frac{1}{2}\sum_{\lambda\in\mathcal{P}(n)} \lambda_1^2 + \frac{1}{2}\sum_{\lambda\in\mathcal{P}(n)} l(\lambda)^2 - \sum_{\lambda\in\mathcal{P}(n)} [\lambda_1 \cdot l(\lambda)]. \tag{2.10}$$

Thus we obtain the combinatorial explanation (2.8) for $\frac{1}{2}N_2(n)$ when substitute (2.7) into (2.10). ∎

We next transform Lemma 2.3 on ordinary partitions to the following statement on rooted partitions by the construction in Lemma 2.1.

Lemma 2.4

$$\frac{1}{2}N_2(n) = \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} [l(\alpha) + l(\beta)] - \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} h(\alpha, \beta), \tag{2.11}$$

where $h(\alpha, \beta)$ denote the largest part of the rooted partition (α, β), that is $h(\alpha, \beta) = \beta_1$ if $\alpha_1 \le \beta_1$; otherwise $h(\alpha, \beta) = \alpha_1$.

Proof. From Lemma 2.1, it's known that for $\lambda \in \mathcal{P}(n)$, we will get exactly $l(\lambda)$ distinct rooted partitions (α, β) corresponding to it in $\mathcal{RP}(n)$. Furthermore for each of these $l(\lambda)$ distinct rooted partitions (α, β), we have $l(\alpha) + l(\beta) = l(\lambda)$ and $h(\alpha, \beta) = \lambda_1$.

Therefore, the sum of $l(\alpha)+l(\beta)$ over all $l(\lambda)$ rooted partitions (α, β) is equal to $l(\lambda)^2$, and we deduce that

$$\sum_{\lambda\in\mathcal{P}(n)} l(\lambda)^2 = \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} [l(\alpha) + l(\beta)]. \tag{2.12}$$

Furthermore, the sum of $h(\alpha, \beta)$ over all $l(\lambda)$ rooted partitions (α, β) is equal to $\lambda_1 \cdot l(\lambda)$, so we have

$$\sum_{\lambda\in\mathcal{P}(n)} [\lambda_1 \cdot l(\lambda)] = \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} h(\alpha, \beta). \tag{2.13}$$

According to Lemma 2.3, (2.12), and (2.13), we deduce (2.11). ∎

When applying the conjugation into α in the rooted partition (α,β), we see that each rooted partition (α,β) with $l(\alpha)$ corresponds to a rooted partition (α',β') with α'_1 such that $l(\alpha)=\alpha'_1$. Thus we obtain the following partition identity:

$$\sum_{(\alpha,\beta)\in\mathcal{RP}(n)} l(\alpha) = \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} \alpha_1. \tag{2.14}$$

Similarly, when employing the conjugation to β in (α,β), we find that each rooted partition (α,β) with $l(\beta)$ corresponds to a rooted partition (α',β') with β'_1 such that $l(\beta)=\beta'_1$. So we have:

$$\sum_{(\alpha,\beta)\in\mathcal{RP}(n)} l(\beta) = \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} \beta_1. \tag{2.15}$$

When subscribe (2.14) and (2.15) into (2.11), we obtain the following combinatorial interpretation for $\frac{1}{2}N_2(n)$ in terms of rooted partitions.

Theorem 2.5 *$\frac{1}{2}N_2(n)$ is equal to the sum of α_1 over all rooted partitions (α,β) of n with $\alpha_1<\beta_1$, add the sum of β_1 over all rooted partitions (α,β) of n with $\alpha_1\geq\beta_1$, namely*

$$\frac{1}{2}N_2(n) = \sum_{\substack{(\alpha,\beta)\in\mathcal{RP}(n)\\ \alpha_1<\beta_1}} \alpha_1 + \sum_{\substack{(\alpha,\beta)\in\mathcal{RP}(n)\\ \alpha_1\geq\beta_1}} \beta_1. \tag{2.16}$$

Based on Theorems 2.2 and 2.5, we may reformulate Andrews' smallest parts partition function (1.3) as the following theorem:

Theorem 2.6

$$\sum_{\lambda\in\mathcal{P}(n)} n_s(\lambda) = \sum_{(\alpha,\beta)\in\mathcal{RP}(n)} \beta_1 - \left[\sum_{\substack{(\alpha,\beta)\in\mathcal{RP}(n)\\ \alpha_1<\beta_1}} \alpha_1 + \sum_{\substack{(\alpha,\beta)\in\mathcal{RP}(n)\\ \alpha_1\geq\beta_1}} \beta_1\right]. \tag{2.17}$$

Proof. Evidently, the proof of this theorem is equivalent to the proof of the following partition identity:

$$\sum_{\lambda\in\mathcal{P}(n)} n_s(\lambda) = \sum_{\substack{(\alpha,\beta)\in\mathcal{RP}(n)\\ \alpha_1<\beta_1}} [\beta_1-\alpha_1]. \tag{2.18}$$

We now build a bijection ψ between the set of ordinary partitions of n and the set of rooted partitions (α,β) of n with $\alpha_1<\beta_1$. Furthermore, for $\lambda\in\mathcal{P}(n)$ and $(\alpha,\beta)=\psi(\lambda)$, we have $n_s(\lambda)=\beta_1-\alpha_1$.

The map ψ: For $\lambda\in\mathcal{P}(n)$, we will construct a rooted partition (α,β) where $\beta_1>\alpha_1$. Assume that $l(\lambda)=l$ and $\lambda_1=a$, consider its conjugate $\lambda'=(\lambda'_1,\ \lambda'_2,\ldots,\lambda'_a)$ where $\lambda'_1=l$. Supposed that the largest part of λ' repeats m_l times, that is, there are m_l parts of size l in λ'. We then have $n_s(\lambda)=\lambda'_1-\lambda'_{m_l+1}$. Define β as the partitions with parts of size l repeated m_l times, and keep the remaining parts of λ' as parts of α.

From the above construction, one could see that $\alpha_1 = \lambda'_{m_l+1}$ and $\beta_1 = \lambda'_1$, that is $\beta_1 > \alpha_1$. Furthermore, $n_s(\lambda) = \lambda'_1 - \lambda'_{m_l+1} = \beta_1 - \alpha_1$. Hence the map ψ satisfies the conditions and one can easily see that this process is reversible. Thus we complete the proof of Theorem 2.6. ∎

For example, there are five partitions of 4: (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1). We also have five rooted partitions (α, β) with $\alpha_1 < \beta_1$.

$$(\emptyset,\ (4))\ ((1),\ (3))\ (\emptyset,\ (2,2))\ ((1,1),\ (2))\ (\emptyset,\ (1,1,1,1)).$$

Applying the above bijection, we get the following correspondence:

$$\begin{array}{lll} (4) \leftrightarrows (\emptyset,\ (1,1,1,1)) & (3,1) \leftrightarrows ((1,1),\ (2)) & (2,2) \leftrightarrows (\emptyset,\ (2,2)) \\ (2,1,1) \leftrightarrows ((1),\ (3)) & (1,1,1,1) \leftrightarrows (\emptyset,\ (4)). & \end{array}$$

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