THE BINOMIAL-STIRLING-EULERIAN POLYNOMIALS

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ABSTRACT. We introduce the binomial-Stirling-Eulerian polynomials, denoted $\tilde{A}_n(x, y|\alpha)$, which encompass binomial coefficients, Eulerian numbers and two Stirling statistics: the left-to-right minima and the right-to-left minima. When $\alpha = 1$, these polynomials reduce to the binomial-Eulerian polynomials $\tilde{A}_n(x, y)$, originally named by Shareshian and Wachs and explored by Chung-Graham-Knuth and Postnikov-Reiner-Williams. We investigate the γ-positivity of $\tilde{A}_n(x, y|\alpha)$ from two aspects:

- firstly by employing the grammatical calculus introduced by Chen;
- and secondly by constructing a new group action on permutations.

These results extend the symmetric Eulerian identity found by Chung, Graham and Knuth, and the γ-positivity of $\tilde{A}_n(x, y)$ first demonstrated by Postnikov, Reiner and Williams.

1. INTRODUCTION

The binomial-Eulerian polynomials, named by Shareshian and Wachs [30], incorporate both binomial coefficients and Eulerian numbers. They originated from Postnikov, Reiner and Williams's work on generalized permutohedra [29] and a symmetric Eulerian identity found by Chung, Graham and Knuth [12].

Let $\mathfrak{S}_n$ denote the set of permutations on $[n] := \{1, 2, \ldots, n\}$. An index $i \in [n-1]$ is called a *descent* (resp., *ascent*) of $\pi \in \mathfrak{S}_n$ if $\pi_i > \pi_{i+1}$ (resp., $\pi_i < \pi_{i+1}$). The number of permutations in $\mathfrak{S}_n$ with k descents (or k ascents) is known (see [28, 31]) as the classical *Eulerian number* $\left\langle {n \atop k} \right\rangle$. For fixed positive integers a and b, Chung, Graham and Knuth [12] found the following symmetric Eulerian identity

$$\sum_{k\geq 0} \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle = \sum_{k\geq 0} \binom{a+b}{k} \left\langle {k \atop b-1} \right\rangle, \tag{1.1}$$

where by convention $\left\langle {0 \atop 0} \right\rangle = 1$. Several generalizations of this identity were subsequently obtained by Chung and Graham [13], Han, Lin and Zeng [17] and Lin [19].

As observed by Shareshian and Wachs [30], the identity (1.1) is equivalent to the symmetry of the *binomial-Eulerian polynomials*

$$\tilde{A}_n(x) := 1 + x \sum_{m=1}^{n} \binom{n}{m} A_m(x), \tag{1.2}$$

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where $A_m(x)$ are the classical *Eulerian polynomials* given by

$$A_m(x)=\sum_{k=0}^{m-1}\left\langle {m \atop k} \right\rangle x^k.$$

In other words, if we write $\tilde{A}_n(x)=\sum_{k=0}^{n}\tilde{A}(n,k)x^k$, then identity (1.1) is equivalent to $\tilde{A}(n,k)=\tilde{A}(n,n-k)$.

It is known [28, Sec. 8.8] that the h-polynomials of dual permutohedra are the Eulerian polynomials. Foata and Schützenberger [14] first proved the following γ-positivity expansion of Eulerian polynomials:

Theorem 1.1 (Foata-Schützenberger). *For $n\geq 1$,*

$$A_n(x)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\gamma_{n,k}x^k(1+x)^{n-1-2k}, \tag{1.3}$$

where $\gamma_{n,k}$ counts the number of permutations in $\mathfrak{S}_n$ with k descents and without double descents (see Definition 1.4 for double descents).

An elegant combinatorial proof of (1.3) via a group action was later constructed by Foata and Strehl [15] (see also [2–4,23]). Since then various refinements of this γ-positivity expansion, with or without combinatorial proofs, have been found [3,20,23,24,27].

Postnikov, Reiner and Williams [29, Section 10.4] proved that the h-polynomials of dual stellohedra equal the binomial-Eulerian polynomials and provided the combinatorial interpretation

$$\tilde{A}_n(x)=\sum_{\pi\in\mathrm{PRW}_{n+1}}x^{\mathrm{asc}(\pi)}, \tag{1.4}$$

where PRW_n is the set of permutations $\pi\in\mathfrak{S}_n$ such that the first ascent (if any) of π appears at the letter 1. For instance,

$$\mathrm{PRW}_1=\{1\},\ \mathrm{PRW}_2=\{12,21\},\ \mathrm{PRW}_3=\{123,132,213,312,321\},$$

and

$$\begin{aligned}\mathrm{PRW}_4=\{&1234,1243,1324,1342,2134,2143,3124,3142,3214,4123,4132,4213,\\ &4312,4321,1423,1432\}.\end{aligned}$$

They also showed the following γ-positivity expansion of $\tilde{A}_n(x)$, which implies the symmetry and unimodality of $\tilde{A}_n(x)$.

Theorem 1.2 (Postnikov-Reiner-Williams). *For $n\geq 1$,*

$$\tilde{A}_n(x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\tilde{\gamma}_{n,k}x^k(1+x)^{n-2k}, \tag{1.5}$$

where $\tilde{\gamma}_{n,k}$ counts the number of permutations in PRW_{n+1} with k ascents and without double ascents (see Definition 1.4 for double ascents).

It's worth noting that this result bears an analogy to Theorem 1.1. Concerning more γ-positive polynomials arising in enumerative and geometric combinatorics, the interested reader is referred to the two surveys by Brändén [4] and Athanasiadis [2], and the book by Petersen [28].

Recently, some related generalizations of Theorem 1.2 have been found. Two different q-analogs of $\tilde{A}_n(t)$ that possess refined γ-positivity were investigated by Shareshian and Wachs [30] and Lin, Wang and Zeng [22]. As pointed out by Brändén [4, Remark 7.3.1], the real-rootedness of a polynomial with symmetric coefficients implies the γ-positivity of such polynomial. Haglund and Zhang [16] proved that the binomial Eulerian polynomials $\tilde{A}_n(t)$ are real-rooted, affirming a conjecture by Ma, Ma and Yeh [25]. Brändén and Jochemko [5] refines and strengthens the aforementioned results by Haglund and Zhang [16]. In particular, by employing symmetric functions, Shareshian and Wachs provided a new interpretation of $\tilde{\gamma}_{n,k}$ in (1.5). A combinatorial proof via the Foata–Strehl action was later provided by Lin, Wang and Zeng [22].

Theorem 1.3 (Shareshian-Wachs)**.** *Suppose that $\tilde{\gamma}_{n,k}$ is the coefficients in the γ-expansion (1.5) of $\tilde{A}_n(x)$. Then $\tilde{\gamma}_{n,k}$ also counts the number of permutations $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ with k descents and without interior double descents (see Definition 1.4 for interior double descents).*

In this paper, we introduce the generalized binomial-Eulerian polynomials $\tilde{A}_n(t)$, called the binomial-Stirling-Eulerian polynomials, using two Stirling statistics. Recall that a *left-to-right minimum* (resp., *right-to-left minimum*) of $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$ is a letter σ_i such that $\sigma_j > \sigma_i$ for every $j < i$ (resp., $j > i$). Let $\mathrm{LRmin}(\sigma)$ (resp., $\mathrm{RLmin}(\sigma)$) be the number of left-to-right minima (resp. right-to-left minima) of σ. It is well known [31] that

$$\sum_{\sigma\in\mathfrak{S}_n} \alpha^{\mathrm{LRmin}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{\mathrm{RLmin}(\sigma)} = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1),$$

which is the generating function for the unsigned *Stirling numbers of the first kind*. Thus, the two statistics 'LRmin' and 'RLmin' are usually referred to as Stirling statistics on permutations. The *binomial-Stirling Eulerian polynomials*, denoted $\tilde{A}_n(x, y|\alpha)$, are defined by

$$\tilde{A}_n(x, y|\alpha) = \sum_{\sigma\in\mathrm{PRW}_{n+1}} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}.$$

It should be noted that the *Stirling-Eulerian polynomials*

$$A_n(x, y|\alpha) := \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}$$

were introduced by Carlitz and Scoville [6] and investigated in further details in the recent work [18] of the first named author.

The main objective of this paper is to investigate the refined γ-positivity of $\tilde{A}_n(x, y|\alpha)$ from both the algebraic and the combinatorial aspects. The first few γ-positivity expansions of $\tilde{A}_n(x,y|\alpha)$ read as

$$\begin{aligned}
\tilde{A}_1(x,y|\alpha) &= \alpha(x+y),\\
\tilde{A}_2(x,y|\alpha) &= \alpha^2(x+y)^2+\alpha xy,\\
\tilde{A}_3(x,y|\alpha) &= \alpha^3(x+y)^3+(\alpha+3\alpha^2)xy(x+y),\\
\tilde{A}_4(x,y|\alpha) &= \alpha^4(x+y)^4+(6\alpha^3+4\alpha^2+\alpha)xy(x+y)^2+(3\alpha^2+2\alpha)x^2y^2.
\end{aligned}$$

The following four classical permutation statistics are crucial in our investigation.

Definition 1.4. *For $\sigma=\sigma_1\sigma_2\cdots\sigma_n\in\mathfrak{S}_n$, a value σ_i $(1\le i\le n)$ is called a* **double ascent** *(resp.,* **double descent, peak, valley***) of σ if $\sigma_{i-1}<\sigma_i<\sigma_{i+1}$ (resp., $\sigma_{i-1}>\sigma_i>\sigma_{i+1}$, $\sigma_{i-1}<\sigma_i>\sigma_{i+1}$, $\sigma_{i-1}>\sigma_i<\sigma_{i+1}$), where we use the convention $\sigma_0=\sigma_{n+1}=+\infty$. In particular, a double descent σ_i with $1<i<n$ will be called an* **interior double descent** *of σ. Let $\mathrm{da}(\sigma)$ (resp., $\mathrm{dd}(\sigma)$, $\mathrm{M}(\sigma)$, $\mathrm{V}(\sigma)$) denote the number of double ascents (resp., double descents, peaks, valleys) of σ.*

From the algebraic side, we employ the grammatical calculus introduced by Chen [7] and prove that $\tilde{A}_n(x,y|\alpha)$ can be expressed as the following refinement of the binomial-Stirling-Eulerian polynomials.

Theorem 1.5. *For $n\ge1$,*

$$\tilde{A}_n(x,y|\alpha)=\sum_{\sigma\in\mathrm{PRW}_{n+1}}(u_1u_2)^{\mathrm{M}(\sigma)}u_3^{\mathrm{da}(\sigma)}u_4^{\mathrm{dd}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2},\tag{1.6}$$

where $x+y=u_3+u_4$ and $xy=u_1u_2$.

This result draws a parallel with the following refinement obtained by the first named author [18] with the aid of the grammar calculus method.

Theorem 1.6. (Ji [18, Thm. 6.1 ($\alpha=\beta$)]) *For $n\ge1$,*

$$A_n(x,y|\alpha)=\sum_{\sigma\in\mathfrak{S}_n}(u_1u_2)^{\mathrm{M}(\sigma)}u_3^{\mathrm{da}(\sigma)}u_4^{\mathrm{dd}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2},$$

where $u_3+u_4=x+y$ and $u_1u_2=xy$.

It turns out that Theorem 1.5 not only provides three combinatorial interpretations of coefficients in the γ-expansion of $\tilde{A}_n(x,y|\alpha)$ (see Theorem 2.1) but also can be used to deduce a relation (see Theorem 2.2) that parallels the relation between the enumerator of permutations by $(\mathrm{M},\mathrm{des})$ and the Eulerian polynomials found by Zhuang [33].

From the combinatorial side, we introduce a new group action on permutations in the spirit of the so-called *Foata–Strehl action* [15] on permutations. This new group action enables us to prove combinatorially Theorem 1.5. Furthermore, we found that this new group action on permutations can also be applied to prove Theorem 1.6. It should be noted that the Foata–Strehl action proof of a q-analog of Theorem 1.3 provided by Lin, Wang and Zeng [22] cannot be adopted to prove Theorem 1.5.

The rest of this paper is organized as follows. In Section 2, we present some relevant consequences of Theorem 1.5. In Section 3, we provide the context-free grammar proof of Theorem 1.5. In Section 4, we construct an involution to prove the symmetry of $\tilde{A}_n(x,y|\alpha)$. This involution enables us to provide bijective proofs for an α-extension of (1.1) and the equivalence of two combinatorial interpretations given by (2.1) and (2.3) in Theorem 2.1 for the coefficients in the γ-expansion of $\tilde{A}_n(x,y|\alpha)$. In Section 5, we introduce a new group action on permutations to prove combinatorially Theorem 1.5 and Theorem 1.6.

2. Relevant consequences of Theorem 1.5

Based on Theorem 1.5, we obtain the following combinatorial interpretations of the coefficients in the γ-expansion of $\tilde{A}_n(x,y|\alpha)$. Note that when $\alpha=1$ and $y=1$ in Theorem 2.1, we recover Theorem 1.2 from (2.1), and similarly, we obtain Theorem 1.3 from (2.3). In the case of $\alpha=1$, the relation (2.2) bears a resemblance to the one between peak polynomials and Eulerian polynomials, as established by Stembridge [32].

Theorem 2.1. *For $n\geq 1$, let*

$$\tilde{A}_n(x,y|\alpha)=\sum_{k=0}^{\lfloor n/2\rfloor}\tilde{\gamma}_{n,k}(\alpha)(xy)^k(x+y)^{n-2k}.$$

Then

$$\tilde{\gamma}_{n,k}(\alpha)=\sum_{\sigma\in\Gamma_{n,k}^{(1)}}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2} \tag{2.1}$$

$$=2^{-n+2k}\sum_{\sigma\in\Gamma_{n,k}^{(2)}}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2} \tag{2.2}$$

$$=\sum_{\sigma\in\Gamma_{n,k}^{(3)}}\alpha^{\mathrm{RLmin}(\sigma)}, \tag{2.3}$$

where

- $\Gamma_{n,k}^{(1)}$ *denotes the set of permutations in* PRW_{n+1} *with k ascents and without double ascents;*
- $\Gamma_{n,k}^{(2)}$ *denotes the set of permutations in* PRW_{n+1} *with k peaks;*
- $\Gamma_{n,k}^{(3)}$ *denotes the set of permutations $\sigma=\sigma_1\cdots\sigma_n\in\mathfrak{S}_n$ with k descents and without interior double descents.*

Proof. (1) By definition, it is not hard to see that for $\sigma\in\mathfrak{S}_n$,

$$\mathrm{M}(\sigma)+\mathrm{dd}(\sigma)=\mathrm{des}(\sigma)\quad\text{and}\quad\mathrm{M}(\sigma)+\mathrm{da}(\sigma)=\mathrm{asc}(\sigma). \tag{2.4}$$

Setting $u_3=0$ in Theorem 1.5, and using (2.4), we obtain (2.1).

(2) When $u_3=u_4=v$ and $u_1=u_2=u$ in Theorem 1.5, we find that

$$u_3=u_4=\frac{x+y}{2}\quad\text{and}\quad u_1=u_2=\sqrt{xy},$$

and in this case, Theorem 1.5 can be reformulated as (2.2).

(3) From (2.4), we find that

$$\mathrm{dd}(\sigma)+\mathrm{da}(\sigma)=n-1-2\mathrm{M}(\sigma). \tag{2.5}$$

Notice that if $\mathrm{dd}(\sigma)=0$, then

$$\mathrm{M}(\sigma)=\mathrm{des}(\sigma). \tag{2.6}$$

Setting $u_4=0$ in Theorem 1.5 and using (2.5) yields

$$\tilde{A}_n(x,y|\alpha)=\sum_{\substack{\sigma\in\mathrm{PRW}_{n+1}\\ \mathrm{dd}(\sigma)=0}}(xy)^{\mathrm{des}(\sigma)}(x+y)^{n-2\mathrm{des}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}.$$

Let NDD_n denote the set of permutations $\sigma=\sigma_1\cdots\sigma_n\in\mathfrak{S}_n$, in which there does not exist any index $1<i<n$ such that $\sigma_{i-1}>\sigma_i>\sigma_{i+1}$. As the first letter of each $\sigma\in\mathrm{PRW}_{n+1}$ with $\mathrm{dd}(\sigma)=0$ must be 1, the expansion (2.3) then follows from the simple one-to-one correspondence

$$\sigma_1\sigma_2\cdots\sigma_{n+1}\mapsto(\sigma_2-1)\cdots(\sigma_n-1)$$

between $\{\sigma\in\mathrm{PRW}_{n+1}:\mathrm{dd}(\sigma)=0\}$ and NDD_n. □

When set $u_1=uv$, $u_2=u_3=w$, and $u_4=v$ in Theorem 1.5, we arrive at the following relation, which resembles the relationship between joint polynomials involving peaks and descents and the Eulerian polynomials established by Zhuang [33].

Theorem 2.2. *For $n\geq 1$,*

$$\sum_{\sigma\in\mathrm{PRW}_{n+1}}u^{\mathrm{M}(\sigma)}v^{\mathrm{des}(\sigma)}w^{\mathrm{asc}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}$$
$$=\sum_{\sigma\in\mathrm{PRW}_{n+1}}x^{\mathrm{des}(\sigma)}y^{\mathrm{asc}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2},$$

where

$$x=\frac{(w+v)-\sqrt{(w+v)^2-4uvw}}{2}\quad\text{and}\quad y=\frac{(w+v)+\sqrt{(w+v)^2-4uvw}}{2}.$$

We conclude this section with an immediate consequence of Theorem 2.1, which provides an interpretation of the α-extension of secant numbers. Recall that a permutation $\sigma\in\mathfrak{S}_n$ is *alternating* (or *down-up*) if

$$\sigma_1>\sigma_2<\sigma_3>\sigma_4<\sigma_5>\cdots.$$

Let $\mathcal{A}_n$ be the set of all alternating permutations in $\mathfrak{S}_n$. Recall that the *Euler numbers* $\{E_n\}_{n\geq 0}$ are defined by

$$\sec(x)+\tan(x)=\sum_{n\geq 0}E_n\frac{x^n}{n!}.$$

The numbers E_n with even indices and odd indices are known as *secant numbers* and *tangent numbers*, respectively. A classical result due to André [1] asserts that $|\mathcal{A}_n|=E_n$.

Setting $x=-1$ and $y=1$ in Theorem 2.1, we obtain the following interpretation of the α-extension of secant numbers.

Theorem 2.3. *For $n \geq 1$,*

$$\tilde{A}_n(-1,1|\alpha) = \begin{cases} (-1)^{n/2} \sum_{\sigma \in \mathcal{A}_n} \alpha^{\mathrm{RLmin}(\sigma)}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

It should be noted that the first named author [18, Thm. 1.12] also found an interpretation of the α-extension of secant numbers.

Theorem 2.4. ([18, Thm. 1.12]) *For $n \geq 1$,*

$$A_{n+1}(-1,1|\alpha/2) = \begin{cases} (-1)^{n/2} \sum_{\sigma \in \mathcal{A}_n} \alpha^{\mathrm{RLmin}(\sigma)}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Combining Theorem 2.3 with Theorem 2.4 results in the following connection between these two kinds of Stirling-Eulerian polynomials.

Theorem 2.5. *For $n \geq 1$,*

$$\tilde{A}_n(-1,1|\alpha) = A_{n+1}(-1,1|\alpha/2).$$

Remark 2.6. *It would be interesting to construct a direct bijective proof of the above relationship.*

3. Context-free grammar proof of Theorem 1.5 and relevant results

In this section, we employ the context-free grammar approach to present the proof of Theorem 1.5. It turns out that the grammar approach played an important role in the study of the γ-positivity of the Eulerian polynomials or other combinatorial polynomials, see Chen and Fu [9,10], Chen, Fu and Yan [11], Lin, Ma and Zhang [21], and Ma, Ma and Yeh [26].

A context-free grammar G over a set $V = \{x, y, z, \ldots\}$ of variables is a set of substitution rules replacing a variable in V by a Laurent polynomial of variables in V. For a context-free grammar G over V, the formal derivative D with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V such that each substitution rule is treated as the common differential rule that satisfies the following relations:

$$D(u+v) = D(u) + D(v) \quad \text{and} \quad D(uv) = D(u)v + uD(v).$$

Let

$$\widetilde{P}_n(u_1,u_2,u_3,u_4|\alpha) = \sum_{\sigma \in \mathrm{PRW}_{n+1}} (u_1u_2)^{\mathrm{M}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}.$$

To prove Theorem 1.5 by employing the grammatical calculus, we are required to establish the grammars for $\tilde{A}_n(x,y|\alpha)$ and $\widetilde{P}_n(u_1,u_2,u_3,u_4|\alpha)$. Rather than focusing on these two classes of polynomials, we will explore two more general polynomials $\tilde{A}_n(x,y,z|\alpha)$ and $\widetilde{P}_n(u_1,u_2,u_3,u_4,u_5|\alpha)$. The primary reason for this choice is that the grammars for both $\tilde{A}_n(x,y,z|\alpha)$ and $\widetilde{P}_n(u_1,u_2,u_3,u_4,u_5|\alpha)$ can be readily determined. Let

$$\tilde{A}_n(x,y,z|\alpha) = \sum_{\sigma \in \mathrm{PRW}_{n+1}} x^{\mathrm{des}(\sigma)-\mathrm{LRmin}(\sigma)+1} y^{\mathrm{asc}(\sigma)} z^{\mathrm{LRmin}(\sigma)-1} \alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}$$

and

$$\begin{aligned}&\widetilde{P}_n(u_1,u_2,u_3,u_4,u_5|\alpha)\\&\quad=\sum_{\sigma\in\mathrm{PRW}_{n+1}}(u_1u_2)^{\mathrm{M}(\sigma)}u_3^{\mathrm{da}(\sigma)}u_4^{\mathrm{dd}(\sigma)-\mathrm{LRmin}(\sigma)+1}u_5^{\mathrm{LRmin}(\sigma)-1}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}.\end{aligned}$$

We have the following grammatical interpretations for $\tilde{A}_n(x,y,z|\alpha)$ and $\widetilde{P}_n(u_1,u_2,u_3,u_4,u_5|\alpha)$.

Lemma 3.1. *Let D_G be the formal derivative with respect to the following grammar*

$$G=\{a\to a\alpha(z+y),\ x\to xy,\ y\to xy\}. \tag{3.1}$$

Then for $n\geq 0$,

$$D_G^n(a)=a\tilde{A}_n(x,y,z|\alpha).$$

Lemma 3.2. *Let $D_{\widetilde{G}}$ be the formal derivative with respect to the following grammar*

$$\widetilde{G}=\{a\to a\alpha(u_3+u_5),\ u_4\to u_1u_2,\ u_3\to u_1u_2,\ u_1\to u_1u_3, u_2\to u_2u_4\}. \tag{3.2}$$

Then for $n\geq 0$,

$$D_{\widetilde{G}}^n(a)=a\widetilde{P}_n(u_1,u_2,u_3,u_4,u_5|\alpha).$$

Using the grammatical calculus, we ultimately demonstrate that

Theorem 3.3. *For $n\geq 1$,*

$$\widetilde{P}_n(u_1,u_2,u_3,u_4,u_5|\alpha)=\tilde{A}_n(x,y,z|\alpha),$$

where $x+y=u_3+u_4$, $y+z=u_3+u_5$ and $xy=u_1u_2$.

Evidently, Theorem 1.5 follows from Theorem 3.3 by setting $z=x$ and $u_5=u_4$. The rest of this section is devoted to the grammatical proof of Theorem 3.3.

3.1. **The grammatical labelings.** In this subsection, we give proofs of Lemma 3.1 and Lemma 3.2 by using the grammatical labeling. The notion of a grammatical labeling was introduced by Chen and Fu [8].

Proof of Lemma 3.1. Let $\sigma=\sigma_1\cdots\sigma_n\in\mathrm{PRW}_n$ such that $\sigma_k=1$. For $1\leq i\leq n+1$, we refer to the position i as the one immediately before σ_i (i.e., the space between σ_{i-1} and σ_i), whereas the position $n+1$ is meant to be the position after σ_n. To define the labeling for $\tilde{A}_{n-1}(x,y,z|\alpha)$, we patch $+\infty$ to $\sigma=\sigma_1\sigma_2\cdots\sigma_n$ at the end of σ, or equivalently, set $\sigma_{n+1}=+\infty$. Then the label of the position i $(2\leq i\leq n+1)$ is given by the following procedure:

- if σ_{i-1} is a ascent, then label the position i by y;
- if σ_{i-1} is a descent and $i>k$, then label the position i by x;
- if σ_{i-1} is a descent and $i\leq k$, then label the position i by z;
- If $i=n+1$, then label it by a;
- if σ_i is a left-to-right minimum or a right-to-left minimum and $i\neq k$, then label α below σ_i.

Below is an example,

$$\begin{array}{l} 7\quad 5\quad 4\quad 1\quad 2\quad 3\quad 9\quad 8\quad 6 \\ \longrightarrow \begin{array}{ccccccccccccccccccc} 7 & z & 5 & z & 4 & z & 1 & y & 2 & y & 3 & y & 9 & x & 8 & x & 6 & a & +\infty \\ \alpha & & \alpha & & \alpha & & & & \alpha & & \alpha & & & & & & \alpha & & \end{array}. \end{array}$$

Evidently, the weight of σ is given by

$$\omega(\sigma)=ax^{\mathrm{des}(\sigma)-\mathrm{LRmin}(\sigma)+1}y^{\mathrm{asc}(\sigma)}z^{\mathrm{LRmin}(\sigma)-1}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}.$$

From the definition of the above labeling, we see that

$$a\tilde{A}_n(x,y,z|\alpha)=\sum_{\sigma\in\mathrm{PRW}_{n+1}}\omega(\sigma). \tag{3.3}$$

We proceed to show Lemma 3.1 by induction on n. For $n=0$, the statement is obvious. Assume that this assertion holds for $n-1$. To show that it is valid for n, we represent a permutation $\sigma=\sigma_1\cdots\sigma_n$ in PRW_n by patching $+\infty$ at the end of σ. Assume that $\sigma_k=1$ and π is the permutation obtained from σ by inserting the element $n+1$ at any position i. To ensure that $\pi\in\mathrm{PRW}_{n+1}$, the element $n+1$ could not be inserted at any position with the label z. We consider the following three cases:

Case 1: If i is labeled by x in the labeling of σ, then the position i of π is labeled by y and the position $i+1$ of π is labeled by x in the labeling of π, so the change of weights of π is consistent with the substitution rule $x\to xy$.

$$\cdots\ \sigma_{i-1}\ x\quad \sigma_i\ \cdots\Rightarrow\cdots\sigma_{i-1}\ y\quad n+1\quad x\quad \sigma_i\ \cdots.$$

Case 2: If i is labeled by y in the labeling of σ, then it can be checked that the change of weights is in accordance with the rule $y\to xy$.

$$\cdots\ \sigma_{i-1}\quad y\quad \sigma_i\ \cdots\Rightarrow\ \cdots\ \sigma_{i-1}\quad y\quad n+1\quad x\qquad \sigma_i\ \cdots.$$

Case 3: If i is labeled by a in the labeling of σ, that means $i=n+1$, then the changes of weights caused by the insertion are coded by the rule $a\to a\alpha(z+y)$.

$$\begin{array}{cccc} \cdots & \sigma_n & a & +\infty \\ \cdots & \alpha & & \end{array} \Rightarrow \begin{array}{cccccc} \cdots & \sigma_n & y & n+1 & a & +\infty \\ \cdots & \alpha & & \alpha & & \end{array}$$

or

$$\begin{array}{ccccccccc} \sigma_1 & z & \cdots & \sigma_{k-1} & z & 1 & \cdots & \sigma_n & a & +\infty \\ \alpha & & \cdots & \alpha & & & \cdots & \alpha & & \end{array}$$

$$\Rightarrow \begin{array}{cccccccccccc} n+1 & z & \sigma_1 & z & \cdots & \sigma_{k-1} & z & 1 & \cdots & \sigma_n & a & +\infty \\ \alpha & & \alpha & & \cdots & \alpha & & & \cdots & \alpha & & \end{array}.$$

Summing up all the cases shows that this assertion is valid for n. This completes the proof of Lemma 3.1. □

We proceed to show that the grammar (3.2) generates $\widetilde{P}_n(u_1,u_2,u_3,u_4,u_5|\alpha)$.

Proof of Lemma 3.2. The labeling for $\widetilde{P}_{n-1}(u_1,u_2,u_3,u_4,u_5|\alpha)$ can be described as follows. Let $\sigma=\sigma_1\cdots\sigma_n\in\mathrm{PRW}_n$ such that $\sigma_k=1$. We patch $+\infty$ to σ at both ends so that there are $n+1$ positions between two adjacent elements. For $1\le i\le n+1$, recall that the position i is said to be the position immediately before σ_i, whereas the position $n+1$ is meant to be the position after σ_n. For $2 \le i \le n+1$, we label the position i as follows:

- If σ_i is a peak, then label the position i by u_2 and label the position $i+1$ by u_1;
- If σ_i is a double ascent, then label the position i by u_3;
- If σ_{i-1} is a double descent and $i \le k$, then label the position i by u_5;
- If σ_{i-1} is a double descent and $i > k$, then label the position i by u_4;
- If $i = n+1$, then label it by a;
- if σ_i is a left-to-right minimum or a right-to-left minimum and $i \neq k$, then label α below σ_i.

The weight of σ is defined to be the product of all the labels. For the permutation $7\,5\,4\,1\,2\,3\,9\,8\,6$, we have the following labeling:

$$
\begin{array}{ccccccccc}
7 & 5 & 4 & 1 & 2 & 3 & 9 & 8 & 6
\end{array}
$$

$$
\longrightarrow \begin{array}{ccccccccccccccccccc}
+\infty & 7 & u_5 & 5 & u_5 & 4 & u_5 & 1 & u_3 & 2 & u_3 & 3 & u_2 & 9 & u_1 & 8 & u_4 & 6 & a & +\infty \\
 & \alpha & & \alpha & & \alpha & & & & \alpha & & \alpha & & & & & & \alpha & &
\end{array}.
$$

It is clear to see that the weight of σ is given by

$$
\omega(\sigma) = a(u_1u_2)^{\mathrm{M}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)-\mathrm{LRmin}(\sigma)+1} u_5^{\mathrm{LRmin}(\sigma)-1} \alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}.
$$

From the definition of the above labeling, we see that

$$
aP_n(u_1,u_2,u_3,u_4,u_5|\alpha,\beta) = \sum_{\sigma \in \mathrm{PRW}_{n+1}} \omega(\sigma). \tag{3.4}
$$

We proceed to show Lemma 3.2 by induction on n. For $n=0$, the statement is obvious. Assume that this assertion holds for $n-1$. To show that it is valid for n, we represent a permutation $\sigma = \sigma_1 \cdots \sigma_n$ in PRW_n by patching $+\infty$ at both ends so that there are $n+1$ positions between two adjacent elements. Assume that $\sigma_k = 1$ and π is the permutation in PRW_{n+1} obtained from σ by inserting the element $n+1$ at the position i. Then the label of the position i could not be u_5 since $\pi \in \mathrm{PRW}_{n+1}$. We consider the following six cases:

Case 1: If i is labeled by u_3 in the labeling of σ, then the position i of π is labeled by u_2 and the position $i+1$ of π is labeled by u_1 in the labeling of π, so the change of weights of π is consistent with the substitution rule $u_3 \to u_1u_2$.

$$
\cdots\ \sigma_{i-1}\ u_3\quad \sigma_i\ \cdots\ \Rightarrow \cdots \sigma_{i-1}\ u_2\quad n+1\quad u_1\quad \sigma_i\ \cdots.
$$

Case 2: If i is labeled by u_4 in the labeling of σ, then it can be checked that the change of weights is in accordance with the rule $u_4 \to u_1u_2$.

$$
\cdots\ \sigma_{i-1}\quad u_4\quad \sigma_i\ \cdots\ \Rightarrow\ \cdots\ \sigma_{i-1}\quad u_2\quad n+1\quad u_1\qquad \sigma_i\ \cdots.
$$

Case 3: If i is labeled by u_1 in the labeling of σ, then it can be checked that the change of weights is in accordance with the rule $u_1 \to u_1u_3$.

$$
\cdots\ \sigma_{i-2}\quad u_2\quad \sigma_{i-1}\quad u_1\quad \sigma_i\ \cdots\ \Rightarrow\ \cdots \sigma_{i-2}\quad u_3\quad \sigma_{i-1}\quad u_2\quad n+1\quad u_1\quad \sigma_i\ \cdots.
$$

Case 4: If i is labeled by u_2 in the labeling of σ, then it can be checked that the change of weights is in accordance with the rule $u_2 \to u_2u_4$.

$$
\cdots\ \sigma_{i-1}\quad u_2\quad \sigma_i\quad u_1\quad \sigma_{i+1}\ \cdots\ \Rightarrow\ \cdots\ \sigma_{i-1}\quad u_2\quad n+1\quad u_1\quad \sigma_i\quad u_4\quad \sigma_{i+1}\ \cdots.
$$

Case 5: If i is labeled by a in the labeling of σ, that means $i = n+1$, then the changes of weights caused by the insertion are coded by the rule $a \to a\alpha(u_3 + u_5)$.

$$\begin{array}{cccc} \cdots & \sigma_n & a & +\infty \\ \cdots & \alpha & & \end{array} \Rightarrow \begin{array}{cccccc} \cdots & \sigma_n & u_3 & n+1 & a & +\infty \\ \cdots & \alpha & & \alpha & & \end{array}$$

or

$$\begin{array}{cccccccccc} \sigma_1 & u_5 & \cdots & \sigma_{k-1} & u_5 & 1 & \cdots & \sigma_n & a & +\infty \\ \alpha & & \cdots & \alpha & & & \cdots & \alpha & & \end{array}$$

$$\Rightarrow \begin{array}{cccccccccccc} n+1 & u_5 & \sigma_1 & u_5 & \cdots & \sigma_{k-1} & u_5 & 1 & \cdots & \sigma_n & a & +\infty \\ \alpha & & \alpha & & \cdots & \alpha & & & \cdots & \alpha & & \end{array}.$$

Summing up all the cases shows that this assertion is valid for n. This completes the proof. ☐

3.2. **The grammatical derivation for Theorem 3.3.** With Lemma 3.1 and Lemma 3.2 in hand, we are now in a position to give a proof of Theorem 3.3 by using the transformations between the grammar (3.1) and the grammar (3.2). It should be noted that the idea of transformations of grammars was proposed by Ma, Ma and Yeh [26] and further developed by Chen and Fu [10].

Proof of Theorem 3.3. By Lemma 3.1 and Lemma 3.2, it suffices to show that for $n \geq 1$,

$$D_G^n(a) = D_{\widetilde{G}}^n(a) \tag{3.5}$$

under the assumption that

$$x + y = u_3 + u_4, \quad y + z = u_3 + u_5 \quad \text{and} \quad xy = u_1u_2. \tag{3.6}$$

To begin with, by employing induction, it is easily established that

$$D_G^n(a) = a \sum_{2i+j+k=n} C_{i,j,k}(\alpha)(xy)^i(y+z)^j(x+y)^k, \tag{3.7}$$

and

$$D_{\widetilde{G}}^n(a) = a \sum_{2i+j+k=n} \widetilde{C}_{i,j,k}(\alpha)(u_1u_2)^i(u_3+u_5)^j(u_3+u_4)^k, \tag{3.8}$$

where $C_{i,j,k}(\alpha)$ and $\widetilde{C}_{i,j,k}(\alpha)$ are polynomials in α with integer coefficients.

Under the assumption (3.6), it is easy to check that

$$D_G(a) = a\alpha(z+y) = a\alpha(u_3+u_5) = D_{\widetilde{G}}(a), \tag{3.9}$$

$$D_G(x+y) = 2xy = 2u_1u_2 = D_{\widetilde{G}}(u_3+u_4), \tag{3.10}$$

$$D_G(y+z) = xy = u_1u_2 = D_{\widetilde{G}}(u_5+u_4), \tag{3.11}$$

$$D_G(xy) = xy(x+y) = u_1u_2(u_3+u_4) = D_{\widetilde{G}}(u_1u_2). \tag{3.12}$$

We proceed to show (3.5) is valid under the assumption (3.6) by induction on n. By (3.9), we see that (3.5) is valid when $n = 1$. Assume that (3.5) holds for n under the assumption (3.6), that is, $D_G^n(a) = D_{\widetilde{G}}^n(a)$. It implies that for $0 \leq i, j, k \leq n$, where $2i + j + k = n$,

$$C_{i,j,k}(\alpha) = \widetilde{C}_{i,j,k}(\alpha). \tag{3.13}$$

To obtain $D_G^{n+1}(a)$, we apply D_G to (3.7) to derive that

$$
\begin{aligned}
D_G^{n+1}(a) = & \sum_{2i+j+k=n} C_{i,j,k}(\alpha) D_G(a)(xy)^i(y+z)^j(x+y)^k \\
& + \sum_{2i+j+k=n} iC_{i,j,k}(\alpha) a(xy)^{i-1} D_G(xy)(y+z)^j(x+y)^k \\
& + \sum_{2i+j+k=n} jC_{i,j,k}(\alpha) a(xy)^i(y+z)^{j-1} D_G(y+z)(x+y)^k \\
& + \sum_{2i+j+k=n} kC_{i,j,k}(\alpha) a(xy)^i(y+z)^j(x+y)^{k-1} D_G(x+y).
\end{aligned} \tag{3.14}
$$

Under the assumption (3.6), and using (3.9), (3.10), (3.11), (3.12) and (3.13), we find that

$$
\begin{aligned}
D_G^{n+1}(a) = & \sum_{2i+j+k=n} \widetilde{C}_{i,j,k}(\alpha) D_{\widetilde{G}}(a)(u_1u_2)^i(u_3+u_5)^j(u_3+u_4)^k \\
& + \sum_{2i+j+k=n} i\widetilde{C}_{i,j,k}(\alpha) a(u_1u_2)^{i-1} D_{\widetilde{G}}(u_1u_2)(u_3+u_5)^j(u_3+u_4)^k \\
& + \sum_{2i+j+k=n} j\widetilde{C}_{i,j,k}(\alpha) a(u_1u_2)^i(u_3+u_5)^{j-1} D_{\widetilde{G}}(u_3+u_5)(u_3+u_4)^k \\
& + \sum_{2i+j+k=n} k\widetilde{C}_{i,j,k}(\alpha) a(u_1u_2)^i(u_3+u_5)^j(u_3+u_4)^{k-1} D_{\widetilde{G}}(u_3+u_4),
\end{aligned} \tag{3.15}
$$

which is also equal to $D_{\widetilde{G}}^{n+1}(a)$ when we applying $D_{\widetilde{G}}$ to (3.8). Hence (3.5) is also valid for $n+1$. This completes the proof. □

4. An involution for the symmetry of $\tilde{A}_n(x, y|\alpha)$

This section aims to define an involution ϕ on the set PRW_n such that for each permutation $\sigma \in \mathrm{PRW}_n$ and $\pi = \phi(\sigma)$, we have

$$
\mathrm{des}(\sigma) = \mathrm{asc}(\pi), \quad \mathrm{asc}(\sigma) = \mathrm{des}(\pi), \quad \mathrm{dd}(\sigma) = \mathrm{da}(\pi), \quad \mathrm{da}(\sigma) = \mathrm{dd}(\pi) \tag{4.1}
$$

and

$$
\mathrm{LRmin}(\sigma) + \mathrm{RLmin}(\sigma) = \mathrm{LRmin}(\pi) + \mathrm{RLmin}(\pi). \tag{4.2}
$$

Clearly, this involution not only supplies a bijective proof for the symmetry of $\tilde{A}_n(x, y|\alpha)$ in x and y but also establishes a bijective proof for the equivalence between (2.1) and (2.3) as presented in Theorem 2.1.

It should be noted that the symmetry of $\tilde{A}_n(x, y|\alpha)$ is equivalent to a symmetric Stirling-Eulerian identity. Define

$$
\left\langle {m \atop k} \right\rangle_\alpha := \sum_{\substack{\sigma \in \mathfrak{S}_m \\ \mathrm{asc}(\sigma)=k}} \alpha^{\mathrm{RLmin}(\sigma)},
$$

where $\left\langle {m \atop k} \right\rangle_\alpha$ is called the *Stirling-Eulerian number*. It is not hard to see that

$$\tilde{A}_n(1, y|\alpha) = \alpha^n + y \sum_{m=1}^{n} \binom{n}{m} \alpha^{n-m} \sum_{k=0}^{m-1} \left\langle {m \atop k} \right\rangle_\alpha y^k.$$

Hence, the symmetry of $\tilde{A}_n(x, y|\alpha)$ in x and y is equivalent to the following symmetric Stirling-Eulerian identity, from which we recover Chung-Graham-Knuth's identity (1.1) by taking $\alpha = 1$.

Theorem 4.1. *For fixed integers $a, b \geq 1$,*

$$\sum_{k\geq 0} \alpha^{a+b-k} \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle_\alpha = \sum_{k\geq 0} \alpha^{a+b-k} \binom{a+b}{k} \left\langle {k \atop b-1} \right\rangle_\alpha, \tag{4.3}$$

where by convention $\left\langle {0 \atop 0} \right\rangle_\alpha = 1$.

Remark 4.2. *Note that Chung, Graham and Knuth's bijection [12] does not work for* (4.3).

The construction of ϕ. Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathrm{PRW}_n$ such that $\sigma_k = 1$. We first put a bar after each right-to-left minimum of σ. For the permutation

$$\sigma = 5\,4\,1\,2\,7\,3\,6\,10\,9\,8, \tag{4.4}$$

we have

$$\sigma = 5\,4\,1\,|\,2\,|\,7\,3\,|6\,|\,10\,9\,8\,|.$$

If there is only one element σ_j between two bars, we call σ_j isolated. For σ given by (4.4), we see that it has two isolated right-to-left minima, that is, $\sigma_4 = 2$ and $\sigma_7 = 6$. For each block of entries between two bars of σ, assume that it looks like

$$|\sigma_{i_1}\, \sigma_{i_1+1}\, \ldots\, \sigma_{i_2-1}\, \sigma_{i_2}|.$$

We then transform them to

$$|\sigma_{i_2-1}\, \sigma_{i_2-2}\, \ldots\, \sigma_{i_1}\, \sigma_{i_2}|.$$

We denote the resulting permutation by τ. For σ given by (4.4), the permutation τ corresponding to σ will be

$$\tau = 5\,4\,1\,|\,2\,|\,7\,3\,|\,6\,|\,9\,10\,8\,|.$$

Assume that there are r isolated right-to-left minima of τ, which are $\tau_{i_1} < \ldots < \tau_{i_r}$. Observe that $\tau_k = 1$, so there are k left-to-right minima, which are $\tau_1 > \cdots > \tau_{k-1} > \tau_k = 1$. We next remove isolated right-to-left minima and left-to-right minima except $\tau_k = 1$ from τ and put $\tau_{i_r} > \tau_{i_{r-1}} > \ldots > \tau_{i_1}$ before $\tau_k = 1$ and then insert $\tau_1 > \cdots > \tau_{k-1}$ into the resulting permutation so that τ_i becomes an isolated right-to-left minimum for $1 \leq i \leq k-1$. We denote the resulting permutation by π. For the permutation τ above, we obtain

$$\pi = 6\,2\,1\,|\,7\,3\,|\,4\,|\,5\,|\,9\,10\,8\,|.$$

It is not hard to check that π satisfies (4.1) and (4.2) and this process is reversible. Therefore, ϕ is an involution on PRW_n, as desired.

Applying the above involution ϕ, we get the following correspondence defined on the set PRW_4 interchanging "des" and "asc".

$$1\,2\,3\,4 \leftrightarrows 4\,3\,2\,1, \quad 1\,2\,4\,3 \leftrightarrows 2\,1\,4\,3, \quad 1\,3\,2\,4 \leftrightarrows 4\,1\,3\,2, \quad 1\,3\,4\,2 \leftrightarrows 1\,4\,3\,2$$

$$1\,4\,2\,3 \leftrightarrows 3\,1\,4\,2, \quad 2\,1\,3\,4 \leftrightarrows 4\,3\,1\,2, \quad 3\,1\,2\,4 \leftrightarrows 4\,2\,1\,3, \quad 4\,1\,2\,3 \leftrightarrows 3\,4\,1\,4.$$

5. A new group action on permutations for Stirling-Eulerian polynomials

In this section, we introduce a new group action on permutations which enables us to prove combinatorially a unified generalization of Theorems 1.5 and 1.6.

5.1. **Introducing a new group action on permutations.** In this subsection, we aim to introduce a new group action on permutations.

For any $x \in [n]$ and $\sigma \in \mathfrak{S}_n$, the *$x$-factorization* of σ is the partition of σ into the form $\sigma = w_1 w_2 x w_3 w_4$, where w_i's are intervals of σ and w_2 (resp. w_3) is the maximal contiguous interval (possibly empty) immediately to the left (resp. right) of x whose letters are all greater than x. Following Foata and Strehl [15], the action φ_x is defined by

$$\varphi_x(\sigma) = w_1 w_3 x w_2 w_4.$$

For example, if $\sigma = 217685439 \in \mathfrak{S}_9$ and $x = 5$, then $w_1 = 21$, $w_2 = 768$, $w_3 = \emptyset$, $w_4 = 439$ and we get $\varphi_x(\sigma) = 215768439$. The map φ_x is an involution acting on $\mathfrak{S}_n$, and for all $x, y \in [n]$, φ_x and φ_y commute.

For a double ascent (resp., double desent) σ_i of σ, if it happens to be also a right-to-left (resp., left-to-right) minimum, then it is called a *rlmin-double ascent* (resp., *lrmin-double descent*); otherwise, it is called an *internal double ascent (resp., internal double descent)*. If $x = \sigma_i$ is a rlmin-double ascent (resp., lrmin-double descent) of σ, then let $\psi_x(\sigma)$ be obtained from σ by removing the letter σ_i and inserting it immediately before (resp., after) the greatest left-to-right (resp., right-to-left) minimum that is smaller than σ_i. For instance, if $\sigma = 6\,10\,8\,3\,1\,4\,9\,2\,\mathbf{5}\,11\,7 \in \mathfrak{S}_{11}$ and $x = 5$ is a rlmin-double ascent, then 3 is the greatest left-to-right minimum smaller than 5 and so $\psi_x(\sigma) = 6\,10\,8\,\mathbf{5}\,3\,1\,4\,9\,2\,11\,7$. Moreover, if x is neither a rlmin-double ascent nor a lrmin-double descent of σ, then set $\psi_x(\sigma) = \sigma$. It is clear that the map ψ_x is another involution acting on $\mathfrak{S}_n$.

Now, for any $x \in [n]$ and $\sigma \in \mathfrak{S}_n$, introduce the new action Φ_x by

$$\Phi_x(\sigma) := \begin{cases} \sigma, & \text{if } x \text{ is a peak or a valley of } \sigma; \\ \varphi_x(\sigma), & \text{if } x \text{ is an internal double ascent or an internal double descent of } \sigma; \\ \psi_x(\sigma), & \text{if } x \text{ is a rlmin-double ascent or a lrmin-double descent of } \sigma. \end{cases}$$

It is routine to check that all Φ_x's are involutions and commute. Therefore, for any $S \subseteq [n]$ we can define the function $\Phi_S : \mathfrak{S}_n \to \mathfrak{S}_n$ by $\Phi_S = \prod_{x \in S} \Phi_x$, where multiplication is the composition of functions. Hence the group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the function Φ_S. See Fig. 1 for a nice visualization of the group action Φ_S on permutations.

5.2. **A unified generalization of Theorems 1.5 and 1.6.** The main features of the action Φ_S lies in the following two lemmas, whose proofs are straightforward and will be omitted.

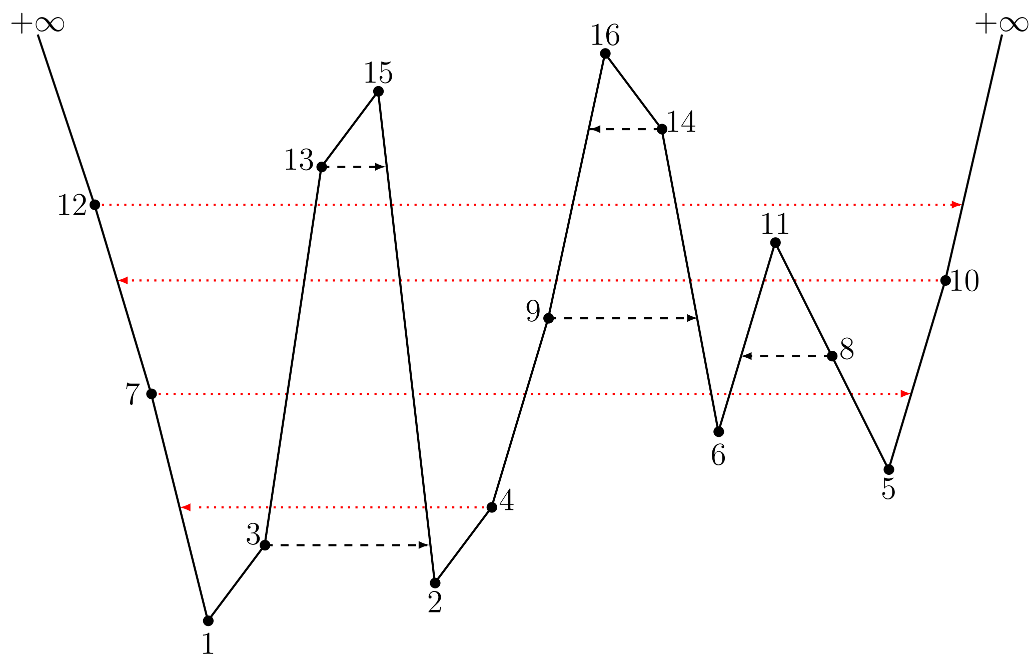

FIGURE 1. The group action Φ_S on $\sigma = 12\,7\,1\,3\,13\,15\,2\,4\,9\,16\,14\,6\,11\,8\,5\,10$ (the dotted lines in red indicate the movements of rlmin-double ascents and lrmin-double descents).

Lemma 5.1. *If x is an internal double ascent (resp., internal double descent) of σ, then x becomes a internal double descent (resp., internal double ascent) of $\Phi_x(\sigma)$.*

Lemma 5.2. *If x is a rlmin-double ascent (resp., lrmin-double descent) of σ, then x becomes a lrmin-double descent (resp., rlmin-double ascent) of $\Phi_x(\sigma)$. Consequently,*

$$\mathrm{LRmin}(\sigma) + \mathrm{RLmin}(\sigma) = \mathrm{LRmin}(\Phi_x(\sigma)) + \mathrm{RLmin}(\Phi_x(\sigma)).$$

For $\Pi \subseteq \mathfrak{S}_n$, we introduce the *generalized Stirling-Eulerian polynomial*

$$A(\Pi; x, y|\alpha) = \sum_{\sigma\in\Pi} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}.$$

We aim to prove the following common generalization of Theorems 1.5 and 1.6.

Theorem 5.3. *Suppose that $\Pi \subseteq \mathfrak{S}_n$ for $n \geq 1$ is invariant under the action Φ_S for any $S \subseteq [n]$. Then,*

$$A(\Pi; x, y|\alpha) = \sum_{\sigma\in\Pi} (u_1u_2)^{\mathrm{M}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}, \tag{5.1}$$

where $u_3 + u_4 = x + y$ and $u_1u_2 = xy$.

Proof. For the sake of convenience, we denote $\mathrm{st}(\sigma) = \mathrm{LRmin}(\sigma) + \mathrm{RLmin}(\sigma) - 2$ for any $\sigma \in \mathfrak{S}_n$. Consider the group $\mathbb{Z}_2^n$ acts on $\Pi \subseteq \mathfrak{S}_n$ via the function Φ_S and let $\mathrm{Orb}(\sigma) =$

$\{g(\sigma) : g \in \mathbb{Z}_2^n\}$ be the orbit of $\sigma \in \Pi$ under this action. By Lemmas 5.1 and 5.2, we have

$$\sum_{\pi \in \mathrm{Orb}(\sigma)} (u_1u_2)^{\mathrm{M}(\pi)} u_3^{\mathrm{da}(\pi)} u_4^{\mathrm{dd}(\pi)} \alpha^{\mathrm{st}(\pi)} = (u_1u_2)^{\mathrm{M}(\bar\sigma)} (u_3+u_4)^{\mathrm{da}(\bar\sigma)} \alpha^{\mathrm{st}(\bar\sigma)}, \tag{5.2}$$

where $\bar\sigma$ denotes the unique element in $\mathrm{Orb}(\sigma)$ that has no double descents. Applying the same argument and utilizing (2.4), we deduce that

$$\sum_{\pi \in \mathrm{Orb}(\sigma)} x^{\mathrm{des}(\pi)} y^{\mathrm{asc}(\pi)} \alpha^{\mathrm{st}(\pi)} = (xy)^{\mathrm{M}(\bar\sigma)} (x+y)^{\mathrm{da}(\bar\sigma)} \alpha^{\mathrm{st}(\bar\sigma)}.$$

Comparing with (5.2) yields

$$\sum_{\pi \in \mathrm{Orb}(\sigma)} x^{\mathrm{des}(\pi)} y^{\mathrm{asc}(\pi)} \alpha^{\mathrm{st}(\pi)} = \sum_{\pi \in \mathrm{Orb}(\sigma)} (u_1u_2)^{\mathrm{M}(\pi)} u_3^{\mathrm{da}(\pi)} u_4^{\mathrm{dd}(\pi)} \alpha^{\mathrm{st}(\pi)}$$

when $u_3 + u_4 = x + y$ and $u_1u_2 = xy$. Summing over all orbits of Π then gives (5.1). □

Taking $\Pi = \mathfrak{S}_n$ in Theorem 5.3 gives Theorem 1.6. As PRW_{n+1} is invariant under the action Φ_S for any $S \subseteq [n+1]$, taking $\Pi = \mathrm{PRW}_{n+1}$ in Theorem 5.3 then gives Theorem 1.5.

Theorem 5.4. *Suppose that $\Pi \subseteq \mathfrak{S}_n$ for $n \geq 1$ is invariant under the action Φ_S for any $S \subseteq [n]$. Then, $A(\Pi; x, y|\alpha)$ admits the γ-positivity expansion*

$$A(\Pi; x, y|\alpha) = \sum_{\substack{\sigma \in \Pi \\ \mathrm{dd}(\sigma)=0}} (xy)^{\mathrm{des}(\sigma)} (x+y)^{n-1-\mathrm{des}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}.$$

Proof. Setting $u_4 = 0$ in Theorem 5.3 and using relationships (2.5) and (2.6) yields the desired result. □

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