The (α, β)-Eulerian Polynomials and Descent-Stirling Statistics on Permutations

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Abstract. Carlitz and Scoville introduced the polynomials $A_n(x, y|\alpha, \beta)$, which we refer to as the (α, β)-Eulerian polynomials. These polynomials count permutations based on Eulerian-Stirling statistics, including descents, ascents, left-to-right maxima, and right-to-left maxima. Carlitz and Scoville obtained the generating function of $A_n(x, y|\alpha, \beta)$. In this paper, we introduce a new family of polynomials, $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$, defined on permutations, incorporating descent-Stirling statistics including valleys, exterior peaks, right double descents, left double ascents, left-to-right maxima, and right-to-left maxima. By employing the grammatical calculus introduced by Chen, we establish the connection between the generating function of $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ and the generating function of the (α, β)-Eulerian polynomials $A_n(x, y|\alpha, \beta)$ introduced by Carlitz and Scoville. Using this connection, we derive the generating function of $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$, which can be specialized to obtain the (α, β)-extensions of generating functions for peaks, left peaks, double ascents, right double ascents and left-right double ascents given by David-Barton, Elizalde and Noy, Entringer, Gessel, Kitaev and Zhuang. Moreover, we establish two relations between $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ and $A_n(x, y|\alpha, \beta)$, which enable us to derive (α, β)-extensions of results of Stembridge, Petersen, Brändén, and Zhuang. We also obtain the left peak version of Stembridge's formula and the peak version of Petersen's formula, along with their respective (α, β)-extensions, by utilizing these two relations. Specializing (α, β)-extensions of Stembridge's formula and the left peak version of Stembridge's formula allows us to derive the (α, β)-extensions of the tangent and secant numbers.

Keywords: Permutations, descents, ascents, peaks, double descents, double ascents, left-to-right maxima, generating functions, alternating permutations, the tangent and the secant numbers, context-free grammars

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1 Introduction

The objective of this paper is to investigate the polynomials involving descent-Stirling statistics. Let us first recall Eulerian, Stirling and descent statistics on permutations. Let $\mathfrak{S}_n$ denote the set of permutations on $[n] := \{1, 2, \ldots, n\}$. We say that i is a descent of

$\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ if $1 \le i < n$ and $\sigma_i > \sigma_{i+1}$. The Eulerian polynomials are defined by

$$A_n(x) := \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)+1} \tag{1.1}$$

with the convention that $A_0(x) = 1$, where $\mathrm{des}(\sigma)$ counts the number of descents of σ. The generating function of $A_n(t)$ is well known:

$$\sum_{n\ge 0} A_n(x)\frac{t^n}{n!} = \frac{1-x}{1-xe^{(1-x)t}}. \tag{1.2}$$

Eulerian polynomials carry a profound historical legacy and play a pivotal role across diverse combinatorial landscapes. For an extensive analysis, please refer to Petersen [29].

A permutation statistic whose generating function is given by (1.1) is called Eulerian. Let $\sigma = \sigma_1\cdots\sigma_n \in \mathfrak{S}_n$. A number $1 \le i < n$ for which $\sigma_i < \sigma_{i+1}$ is called an ascent of σ and a number $1 \le i < n$ for which $\sigma_i > i$ is called an excedance of σ. Let $\mathrm{asc}(\sigma)$ denote the number of ascents of σ and let $\mathrm{exc}(\sigma)$ denote the number of excedances of σ. It is known that $\mathrm{asc}(\sigma)$ and $\mathrm{exc}(\sigma)$ are Eulerian statistics, see MacMahon [25, p.186] and Stanley [25, p.186]. To wit,

$$A_n(x) := \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)+1} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{asc}(\sigma)+1} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{exc}(\sigma)+1}.$$

A permutation statistic is called a Stirling statistic if

$$\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{sst}(\sigma)} = x(x+1)(x+2)\cdots(x+n-1). \tag{1.3}$$

That is, sst has the same generating function as the unsigned Stirling number of the first kind. See [34, Proposition 1.3.7]. Here we describe five Stirling statistics. The first is the number of cycles in a decomposition of σ into disjoint cycles, including those of length 1, which is denoted $\mathrm{cyc}(\sigma)$. For the permutation $\sigma = 2\,7\,1\,8\,3\,6\,5\,4$, its cycle decomposition is $(1\,2\,7\,5)\,(4\,8)\,(6)$, and so $\mathrm{cyc}(\sigma) = 3$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$. A left-to-right maximum (resp. a left-to-right minimum) of σ is an element σ_i such that $\sigma_j < \sigma_i$ (resp. $\sigma_j > \sigma_i$) for every $j < i$ and a right-to-left maximum (resp. a right-to-left minimum) of σ is an element σ_i such that $\sigma_j < \sigma_i$ (reps. $\sigma_j > \sigma_i$) for every $j > i$. Let $\mathrm{LRmax}(\sigma)$, $\mathrm{LRmin}(\sigma)$, $\mathrm{RLmax}(\sigma)$ and $\mathrm{RLmin}(\sigma)$ denote the number of left-to-right maxima, left-to-right minima, right-to-left maxima and right-to-left minima of σ, respectively. For the permutation $\sigma = 2\,7\,1\,8\,3\,6\,5\,4$, we see that

$$\mathrm{LRmax}(\sigma) = 3,\ \mathrm{LRmin}(\sigma) = 2,\ \mathrm{RLmax}(\sigma) = 4,\ \mathrm{RLmin}(\sigma) = 3.$$

It is well known that

$$\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{cyc}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{LRmax}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{RLmax}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{LRmin}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{RLmin}(\sigma)} \tag{1.4}$$

$$= x(x+1)(x+2)\cdots(x+n-1). \tag{1.5}$$

Carlitz and Scoville [4] considered the following polynomials involving Eulerian-Stirling statistics, which we refer to as the (α, β)-Eulerian polynomials:

$$A_n(x, y|\alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} \alpha^{\mathrm{LRmax}(\sigma)-1} \beta^{\mathrm{RLmax}(\sigma)-1}. \tag{1.6}$$

They obtained the following generating function of $A_n(x, y|\alpha, \beta)$:

Theorem 1.1. (Carlitz and Scoville [4, Theorem 9])

$$\sum_{n \geq 0} A_n(x, y|\alpha, \beta) \frac{t^n}{n!} = (1 + xF(x, y; t))^{\alpha} (1 + yF(x, y; t))^{\beta}, \tag{1.7}$$

where $F(x, y; t)$ is given by

$$F(x, y; t) = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}}. \tag{1.8}$$

It's worth mentioning that Carlitz and Scoville [4] used the terms fall for descent and rise for ascent. They referred to a left-to-right maximum and a right-to-left maximum of a permutation as a left upper record and a right upper record of a permutation.

Note that when $\alpha = 0$, $\beta = 1$ and $x = 1$, the polynomials $A_n(x, y|\alpha, \beta)$ reduce to the classical Eulerian polynomials $A_n(y)$ given by (1.1). Accordingly, we recover the generating function (1.2) of $A_n(y)$ by setting $\alpha = 0$, $\beta = 1$ and $x = 1$ in (1.7).

When $x = y = 1$ and $\beta = 0$, it is not difficult to find that

$$A_n(1, 1|\alpha, 0) = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{LRmax}(\sigma)} \tag{1.9}$$

and by (1.7), we see that

$$\sum_{n \geq 0} A_n(1, 1|\alpha, 0) \frac{t^n}{n!} = (1 + F(1, 1; t))^{\alpha} = \left(\frac{1}{1-t} \right)^{\alpha}. \tag{1.10}$$

Comparing the coefficients of $t^n/n!$ yields the generating function (1.4).

We would also like to mention that Foata and Schützenberger [16] introduced the following polynomial, which incorporates Eulerian-Stirling statistics and has been referred to as the q-analogue of Eulerian polynomials by Brenti [3]:

$$A_n(x, q) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} q^{\mathrm{cyc}(\sigma)}. \tag{1.11}$$

The grammar of $A_n(x, q)$ was found by Ma-Ma-Yeh-Zhu [24]. Brenti [3] obtained the following generating function of $A_n(x, q)$ and showed that $A_n(x, q)$ is log-concave and unimodal.

$$1 + \sum_{n \geq 1} A_n(x, q) \frac{t^n}{n!} = \left(\frac{e^{t(x-1)} - x}{1 - x} \right)^{-q}. \tag{1.12}$$

Thanks to the first fundamental transformation of Foata and Schützenberger [16], we see that q-Eulerian polynomial (1.11) is a special case of (α,β)-Eulerian polynomials $A_n(x,y|\alpha,\beta)$. More precisely, we have

$$A_n(x,q)=\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{exc}(\sigma)}q^{\mathrm{cyc}(\sigma)}=\sum_{\overline{\sigma}\in\mathfrak{S}_n} x^{\mathrm{des}(\overline{\sigma})}q^{\mathrm{LRmax}(\overline{\sigma})}=A_n(1,x|q,0).$$

Consequently, we can retrieve the generating function (1.12) for $A_n(x,q)$ by setting $x=1$ and $\beta=0$ and replacing y and α with x and q respectively in (1.7). It should be noted that the polynomials $A_n(x,q)$ has been shown to be related to the $1/k$-Eulerian polynomials, see Savage and Viswanathan [31] and Ma and Mansour [23] for example.

The descent statistics are related to the Eulerian statistics, which are permutation s-tatistics that depend only on the descent and length of a permutation, see Gessel and Zhuang [19], Zhuang [36, 37]. The classical descent statistics include variations of peaks and valleys, double ascents and double descents. In this paper, we adopt the terminology for these descent statistics provided by Gessel and Zhuang [19] and Zhuang [36, 37]. For the detailed definitions of these descent statistics, see Section 2.

Carlitz and Scoville [4] considered the generating function on the joint distribution of the number of exterior peaks, the number of descents and the number of ascents. It should be noted that an exterior peak is referred to as a maximum by Carlitz and Scoville [4].

Theorem 1.2. (Carlitz and Scoville [4, Theorem 2]) *Let $W(\sigma)$ denote the number of exterior peaks of σ (see Definition 2.1). Then*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{\mathrm{W}(\sigma)-1}v^{\mathrm{des}(\sigma)}w^{\mathrm{asc}(\sigma)}\right)\frac{t^n}{n!}=\left(1+yF(x,y;t)\right)\left(1+xF(x,y;t)\right), \quad (1.13)$$

where $F(x,y;t)$ was given by (1.8) *and*

$$x=\frac{(w+v)+\sqrt{(w+v)^2-4uvw}}{2},\quad y=\frac{(w+v)-\sqrt{(w+v)^2-4uvw}}{2}. \quad (1.14)$$

Goulden and Jackson [20, Exercise 3.3.46] and Stanley [34, Exercise 1.61] reformulat-ed Carlitz and Scoville's result in terms of left double ascents and right double descents, see Definition 2.8 and Definition 2.9. It should be noted that Goulden and Jackson [20, Exercise 3.3.46] and Stanley [34, Exercise 1.61] refer to a left double ascent as a double rise, a right double descent as a double fall. A valley is called a modified minimum by Goulden and Jackson [20, Exercise 3.3.46].

Let $V(\sigma)$, $\mathrm{lda}(\sigma)$ and $\mathrm{rdd}(\sigma)$ denote the number of valleys of σ, left double ascents of σ, right double descents of σ respectively (see Section 2). Goulden and Jackson [20, Exercise 3.3.46] and Stanley [34, Exercise 1.61] reformulated Theorem 1.2 as

$$\sum_{n\geq 1}\left(\sum_{\sigma\in\mathfrak{S}_n} u_1^{\mathrm{V}(\sigma)}u_2^{\mathrm{W}(\sigma)-1}u_3^{\mathrm{lda}(\sigma)}u_4^{\mathrm{rdd}(\sigma)}\right)\frac{t^n}{n!}=F(x,y;t), \quad (1.15)$$

where $x+y=u_3+u_4$ and $xy=u_1u_2$.

Fu [18] provided a grammatical proof of (1.15). Pan and Zeng [27] provided *inv* q-analogue of (1.15). Differentiating both sides of (1.15) with respect to t, we have

Theorem 1.3. (Carlitz and Scoville II [4, Theorem 2])

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_{n+1}} u_1^{\mathrm{V}(\sigma)}u_2^{\mathrm{W}(\sigma)-1}u_3^{\mathrm{lda}(\sigma)}u_4^{\mathrm{rdd}(\sigma)}\right)\frac{t^n}{n!}=\left(1+yF(x,y;t)\right)\left(1+xF(x,y;t)\right), \tag{1.16}$$

where $x+y=u_3+u_4$ *and* $xy=u_1u_2$*.*

Setting $u_1=u_3=w$, $u_2=uv$ and $u_4=v$ in Theorem 1.3, and using (2.2) and (2.11), one could recover Theorem 1.2.

The main objective of this paper is to investigate the following polynomial involving descent-Stirling statistics.

$$P_n(u_1,u_2,u_3,u_4|\alpha,\beta)=\sum_{\sigma\in\mathfrak{S}_{n+1}} u_1^{\mathrm{V}(\sigma)}u_2^{\mathrm{W}(\sigma)-1}u_3^{\mathrm{rdd}(\sigma)}u_4^{\mathrm{lda}(\sigma)}\alpha^{\mathrm{LRmax}(\sigma)-1}\beta^{\mathrm{RLmax}(\sigma)-1}. \tag{1.17}$$

By using the grammatical calculus introduced by Chen [5], we establish the connection between the generating function of $P_n(u_1,u_2,u_3,u_4|\alpha,\beta)$ and the generating function (1.6) of $A_n(x,y|\alpha,\beta)$.

Theorem 1.4. *We have*

$$\sum_{n\geq 0}P_n(u_1,u_2,u_3,u_4|\alpha,\beta)\frac{t^n}{n!}=\left(1+yF(x,y;t)\right)^{\frac{\alpha+\beta}{2}}\left(1+xF(x,y;t)\right)^{\frac{\alpha+\beta}{2}}e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}, \tag{1.18}$$

where $x+y=u_3+u_4$*,* $xy=u_1u_2$ *and* $F(x,y;t)$ *is given by* (1.8)*.*

When $\alpha=\beta=1$ in Theorem 1.4, we recover Theorem 1.3.

Using Theorem 1.4, we derive the following generating function of $P_n(u_1,u_2,u_3,u_4|\alpha,\beta)$.

Theorem 1.5. *We have*

$$\sum_{n\geq 0}P_n(u_1,u_2,u_3,u_4|\alpha,\beta)\frac{t^n}{n!}=e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}\times\left(\cosh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right.$$
$$\left.-\frac{u_3+u_4}{\sqrt{(u_3+u_4)^2-4u_1u_2}}\sinh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right)^{-(\alpha+\beta)}.$$

As applications of Theorem 1.5, we obtain (α,β)-extensions of the generating functions of peaks, left peaks, double ascents, right double ascents and left-right double ascents (see Theorems 4.2, 4.3, 4.4, 4.5 and 4.6). For more detailed explanations on the generating functions of these statistics, please see Section 2.

Based on Theorem 1.5, we obtain the following explicit expression of $P_n(u_1, u_2, u_3, u_4; \alpha, \beta)$ when $\alpha + \beta = -1$. In particular, we obtain the following interesting enumerative consequences.

Theorem 1.6. *Let $M(\sigma)$ denote the number of interior peaks of σ. When $\alpha + \beta = -1$ and for $n \geq 1$,*

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} u^{\mathrm{M}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)-1} \beta^{\mathrm{RLmin}(\sigma)-1} = \begin{cases} (1-u)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is even}, \\ -(1-u)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd}. \end{cases} \tag{1.19}$$

Theorem 1.7. *Let $L(\sigma)$ denote the number of left peaks of σ. For $n \geq 1$,*

$$\sum_{\sigma \in \mathfrak{S}_n} u^{\mathrm{L}(\sigma)} (-1)^{\mathrm{RLmin}(\sigma)} = \begin{cases} (1-u)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is even}, \\ -(1-u)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 1.8. *Let $\mathrm{rda}(\sigma)$ denote the number of right double ascents of σ. For $n \geq 1$,*

$$2^n \sum_{\sigma \in \mathfrak{S}_{n+1}} u^{\mathrm{rda}(\sigma)} \left(-\frac{1}{2}\right)^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}$$

$$= \begin{cases} ((1+u)^2-4)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is even}, \\ -(1+u)((1+u)^2-4)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd}. \end{cases} \tag{1.20}$$

Combining Theorem 1.1 and Theorem 1.4, we derive two relations between $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ and $A_n(x, y|\alpha, \beta)$ (see Theorem 6.1 and Theorem 6.2). These two relations enable us to establish (α, β)-extensions of the relations related to the Eulerian polynomial due to Stembridge, Petersen, Brändén and Zhuang, see Theorems 1.9, 6.4, 6.5 and 6.9. We also obtain the left peak version of Stembridge's formula and peak version of Petersen's formula (see Theorem 6.3 and Theorem 6.7) and their (α, β)-extensions, see Theorem 1.10 and Theorem 6.6. The following two consequences can be viewed as the (α, β)-extensions of Stembridge's formula and the left peak version of Stembridge's formula

Theorem 1.9. *For $n \geq 1$,*

$$\sum_{\sigma \in \mathfrak{S}_n} (xy)^{\mathrm{M}(\sigma)} \left(\frac{x+y}{2}\right)^{n-2\mathrm{M}(\sigma)-1} \alpha^{\mathrm{LRmin}(\sigma)-1} \beta^{\mathrm{RLmin}(\sigma)-1}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)} y^{n-\mathrm{des}(\sigma)-1} \left(\frac{\alpha+\beta}{2}\right)^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}, \tag{1.21}$$

where $M(\sigma)$ counts the number of interior peaks of σ.

Theorem 1.10. *For $n \geq 0$,*

$$\sum_{\sigma \in \mathfrak{S}_n} (xy)^{\mathrm{L}(\sigma)} \left(\frac{x+y}{2}\right)^{n-2\mathrm{L}(\sigma)} \beta^{\mathrm{RLmin}(\sigma)}$$

$$= \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{des}(\sigma)} y^{n-\mathrm{des}(\sigma)} \left(\frac{\beta}{2}\right)^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}, \tag{1.22}$$

where $L(\sigma)$ counts the number of left peaks of σ.

Specializing Theorem 1.9 and Theorem 1.10 allows us to derive the (α, β)-extensions of the tangent and secant numbers. Recall that the tangent number E_{2n+1} and the secant number E_{2n} are defined by

$$\sum_{n\geq 0} E_{2n+1}\frac{t^{2n+1}}{(2n+1)!} = \tan(t) \quad \text{and} \quad \sum_{n\geq 0} E_{2n}\frac{t^{2n}}{(2n)!} = \sec(t).$$

A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ is down-up (or alternating) if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \ldots$ and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ is up-down (or reverse alternating) if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \ldots$. The down-up permutations in $\mathfrak{S}_4$ are $2\,1\,4\,3$, $3\,1\,4\,2$, $3\,2\,4\,1$, $4\,1\,3\,2$, $4\,2\,3\,1$ and the up-down permutations in $\mathfrak{S}_4$ are $3\,4\,1\,2$, $2\,4\,1\,3$, $2\,3\,1\,4$, $1\,4\,2\,3$, $1\,3\,2\,4$. It is easy to show that the number of down-up permutations of $[n]$ equals the number of up-down permutations of $[n]$. André [1] showed that E_n counts the number of down-up (or up-down) permutations of $[n]$.

Euler [14] found the following interesting relation: For $n \geq 1$,

$$\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathrm{exc}(\sigma)} = \begin{cases} (-1)^{\frac{n-1}{2}} E_n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \tag{1.23}$$

Roselle [30] obtained the following parallel result to Euler involving secant numbers: For $n \geq 1$,

$$\sum_{\sigma\in\mathfrak{D}_n} (-1)^{\mathrm{exc}(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \tag{1.24}$$

where $\mathfrak{D}_n$ counts the number of permutations in $\mathfrak{S}_n$ without fixed points. It should be noted many different q-analogues of (1.23) and (1.24) have been established by [15, 21, 32, 33].

Setting $x = -1$ and $y = 1$ in Theorem 1.9, we have

Theorem 1.11. *Let $\mathfrak{S}_n^a$ denote the set of up-down permutations of $[n]$. For $n \geq 1$,*

$$\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathrm{des}(\sigma)} \left(\frac{\alpha+\beta}{2}\right)^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}$$
$$= \begin{cases} (-1)^{\frac{n-1}{2}} \sum_{\sigma\in\mathfrak{S}_n^a} \alpha^{\mathrm{LRmin}(\sigma)-1}\beta^{\mathrm{RLmin}(\sigma)-1}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \tag{1.25}$$

Setting $x = -1$ and $y = 1$ in Theorem 1.10, we have

Theorem 1.12. *Let $\mathfrak{S}_n^a$ denote the set of down-up permutations of $[n]$. For $n \geq 1$,*

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{\mathrm{des}(\sigma)} \left(\frac{\beta}{2}\right)^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}$$
$$= \begin{cases} (-1)^{\frac{n}{2}} \sum_{\sigma\in\mathfrak{S}_n^a} \beta^{\mathrm{RLmin}(\sigma)}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.26}$$

Setting $\alpha = \beta = 1$ in Theorem 1.11, we could recover Euler's relation (1.23) with the aid of the first fundamental transformation of Foata and Schützenberger [16]. Setting $\beta = 1$ in Theorem 1.12, we obtain the following identity, which seems to be new.

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{\mathrm{des}(\sigma)} \left(\frac{1}{2}\right)^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2} = \begin{cases} (-1)^{\frac{n}{2}} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.27}$$

Combining (1.24) and (1.27), we obtain the following identity:

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{\mathrm{des}(\sigma)} \left(\frac{1}{2}\right)^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2} = \sum_{\sigma\in\mathfrak{D}_n} (-1)^{\mathrm{exc}(\sigma)}. \tag{1.28}$$

It would be interesting to give a combinatorial proof of the above identity.

This paper is organized as follows. Section 2 provides a review of some classical descent statistics, including left peaks, interior peaks, exterior peaks, valleys, left double ascents, double ascents, right double ascents, left-right ascents, left double descents, double descents, right double descents, left-right descents. We then collect the generating functions associated with these statistics and their relations with the Eulerian polynomials. Section 3 is dedicated to proving the main result of this paper (Theorem 1.4) using the grammatical calculus introduced by Chen [5]. In Section 4, we first derive the generating function of $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ using Theorem 1.4. We then present (α, β)-extensions of some known generating functions related to descent statistics by specializing the generating function of $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$. Section 5 aims to establish an explicit expression of $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ when $\alpha + \beta = -1$. This result can be specialized to obtain Theorems 1.6, 1.7, and 1.8. In Section 6, we first establish two relations between $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ and $A_n(x, y|\alpha, \beta)$ using Theorem 1.4. We then derive (α, β)-extensions of some known relations between descent statistics and the Eulerian polynomials given by Stembridge, Petersen, Brändén and Zhuang by specializing these two relations.

2 Descent statistics

In this section, we begin by revisiting classical descent statistics, which encompass variations related to peaks, valleys, double ascents, and double descents. We then collect the generating functions associated with these statistics and their relationships with Eulerian polynomials. Here we follow Stanley's terminology for peaks and its variations, as described in [34, Exercise 1.61]. For double ascents, double descents and their variations, we adhere to the definitions provided by Zhuang in [36].

Definition 2.1 (Variations in peaks)**.** *Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$,*

(1) *we say that i is a* **left peak** *of σ if $1 \le i < n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ under the assumption that $\sigma_0 = \sigma_{n+1} = 0$. Let $L(\sigma)$ denote the number of left peaks of σ.*

(2) *we say that i is an* **interior peak** *(or a peak for short) of σ if $1 < i < n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$. Let $M(\sigma)$ denote the number of peaks of σ.*

(3) *we say that i is an* **exterior peak** *of σ if $1 \le i \le n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ under the assumption that $\sigma_0 = \sigma_{n+1} = 0$. Let $W(\sigma)$ denote the number of exterior peaks of σ.*

(4) *we say that i is a* **valley** *of σ if $1 \le i \le n$ and $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$ under the assumption that $\sigma_0 = \sigma_{n+1} = 0$. Let $\mathrm{V}(\sigma)$ denote the number of valleys of σ.*

For the permutation $\sigma = 7\,1\,3\,8\,5\,9\,6\,2\,4 \in \mathfrak{S}_9$, we see that

$$L(\sigma) = 3, \quad M(\sigma) = 2, \quad W(\sigma) = 4, \quad \mathrm{V}(\sigma) = 3.$$

Note that the symbols $L(\sigma)$, $M(\sigma)$, and $W(\sigma)$ used in this context were introduced by Chen and Fu [7], which are meaningful and easy to remember. The letter L looks like having a peak on the left. It should also be noted that Carlitz and Scoville [4] refer to an exterior peak as a maximum, while Goulden and Jackson [20, Exercise 3.3.46] describe it as a modified maximum. Similarly, Goulden and Jackson [20, Exercise 3.3.46] term a valley as a modified minimum.

It is evident that for $\sigma \in \mathfrak{S}_n$,

$$\mathrm{V}(\sigma) = \mathrm{W}(\sigma) - 1 \tag{2.1}$$

and

$$\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{M}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{V}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{W}(\sigma)-1} \tag{2.2}$$

Hence it suffices to consider left peak and interior peak. The generating function for the left peak polynomials is attribute to Gessel [26, Sequence A008971], see Zhuang [36, Theorem 10].

Theorem 2.2 (Gessel)**.** *We have*

$$\sum_{n\ge 0} \left(\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{L}(\sigma)} \right) \frac{t^n}{n!} = \frac{\sqrt{1-u}}{\sqrt{1-u}\cosh\left(t\sqrt{1-u}\right) - \sinh\left(t\sqrt{1-u}\right)}. \tag{2.3}$$

The following presents the generating function for the peak polynomials. As brought up by Stanley [34], the generating function of the peak polynomials can be deduced from an equation of David-Barton [10], see Chen and Fu [8] for more information. The equivalent formulae have been found by Entringer [13], Kitaev [22] and Zhuang [36, Theorem 9].

Theorem 2.3. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{M}(\sigma)}\right)\frac{t^n}{n!} = \frac{\sqrt{1-u}\cosh\left(t\sqrt{1-u}\right)}{\sqrt{1-u}\cosh\left(t\sqrt{1-u}\right) - v\sinh\left(t\sqrt{1-u}\right)}. \tag{2.4}$$

Chen and Fu [9] provide a grammatical proof of (2.3) and (2.4).

Stembridge [35] first considered the relation between the peak polynomials and the Eulerian polynomials. He obtained the following relation between the peak polynomials and the Eulerian polynomials in the study of his theory of enriched P-partitions, which was rediscovered by Brändén [2] with the aid of the "modified Foata-Strehl action", a variant of a group action on permutations originally defined by Foata and Strehl [17].

Theorem 2.4 (Stembridge)**.** *For $n \geq 1$,*

$$\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)} = \left(\frac{1+x}{2}\right)^{n-1}\sum_{\sigma\in\mathfrak{S}_n}\left(\frac{4x}{(1+x)^2}\right)^{\mathrm{M}(\sigma)}. \tag{2.5}$$

Petersen [28, Observation 3.1.2] established a relation between the left peak polynomials and the Eulerian polynomials, stated as follows.

Theorem 2.5 (Petersen)**.** *For $n \geq 1$,*

$$(1+x)^n\sum_{\sigma\in\mathfrak{S}_n}\left(\frac{4x}{(1+x)^2}\right)^{\mathrm{L}(\sigma)} = \sum_{k=1}^{n}\binom{n}{k}2^k(1-x)^{n-k}\sum_{\sigma\in\mathfrak{S}_k}x^{\mathrm{des}(\sigma)+1} + (1-x)^n. \tag{2.6}$$

Chen and Fu [8] provided grammatical proofs of (2.5) and (2.6).

Recently, Zhuang [37] established two relations between the joint polynomials of peaks (or left peaks) and descents and the Eulerian polynomials.

Theorem 2.6. (Zhuang [37, Theorem 4.2]) *For $n \geq 1$,*

$$\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{M}(\sigma)+1}v^{\mathrm{des}(\sigma)+1} = \left(\frac{1+b}{1+ab}\right)^{n+1}\sum_{\sigma\in\mathfrak{S}_n} a^{\mathrm{des}(\sigma)+1}, \tag{2.7}$$

where

$$a = \frac{(1+v)^2 - 2uv - (1+v)\sqrt{(1+v)^2-4uv}}{2uv} \tag{2.8}$$

and

$$b = \frac{1+v^2-2uv-(1-v)\sqrt{(1+v)^2-4uv}}{2(1-u)v}. \tag{2.9}$$

Theorem 2.7. (Zhuang [37, Theorem 4.7]) *For* $n \geq 1$*,*

$$\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{L}(\sigma)} v^{\mathrm{des}(\sigma)} = \frac{1}{(1+ab)^n}\left(\sum_{k=1}^{n}\binom{n}{k}(1+b)^k(1-a)^{n-k}\sum_{\sigma\in\mathfrak{S}_k} a^{\mathrm{des}(\sigma)+1} + (1-a)^n\right), \tag{2.10}$$

where a *and* b *are defined by* (2.8) *and* (2.9).

Definition 2.8 (Variations in double ascents)**.** *Given a permutation* $\sigma = \sigma_1 \cdots \sigma_n$*,*

(1) *we say that* i *is a* **left double ascent** *of* σ *if* $1 \leq i < n$ *and* $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$ *under the assumption that* $\sigma_0 = 0$*. Let* $\mathrm{lda}(\sigma)$ *denote the number of left double ascents of* σ*.*

(2) *we say that* i *is a* **double ascent** *of* σ *if* $1 < i < n$ *and* $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$*. Let* $\mathrm{da}(\sigma)$ *denote the number of double ascents of* σ*.*

(3) *we say that* i *is a* **right double ascent** *of* σ *if* $1 < i \leq n$ *and* $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$ *under the assumption that* $\sigma_{n+1} = +\infty$*. Let* $\mathrm{rda}(\sigma)$ *denote the number of right double ascents of* σ*.*

(4) *we say that* i *is a* **left-right double ascent** *of* σ *if* $1 \leq i \leq n$ *and* $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$ *under the assumption that* $\sigma_0 = 0$ *and* $\sigma_{n+1} = +\infty$*. Let* $\mathrm{lrda}(\sigma)$ *denote the number of left-right double ascents of* σ*.*

Definition 2.9 (Variations in double descents)**.** *Given a permutation* $\sigma = \sigma_1 \cdots \sigma_n$*,*

(1) *we say that* i *is a* **left double descent** *of* σ *if* $1 \leq i < n$ *and* $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$ *under the assumption that* $\sigma_0 = +\infty$*. Let* $\mathrm{ldd}(\sigma)$ *denote the number of left double descents of* σ*.*

(2) *we say that* i *is a* **double descent** *of* σ *if* $1 < i < n$ *and* $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$*. Let* $\mathrm{dd}(\sigma)$ *denote the number of double descents of* σ*.*

(3) *we say that* i *is a* **right double descent** *of* σ *if* $1 < i \leq n$ *and* $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$ *under the assumption that* $\sigma_{n+1} = 0$*. Let* $\mathrm{rdd}(\sigma)$ *denote the number of right double descents of* σ*.*

(4) *we say that* i *is a* **left-right double ascent** *of* σ *if* $1 \leq i \leq n$ *and* $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$ *under the assumption that* $\sigma_0 = +\infty$ *and* $\sigma_{n+1} = 0$*. Let* $\mathrm{lrdd}(\sigma)$ *denote the number of left-right double ascents of* σ*.*

For the permutation $\sigma = 7\,1\,3\,8\,5\,9\,6\,2\,4 \in \mathfrak{S}_9$, we see that

$$\mathrm{da}(\sigma) = 1, \quad \mathrm{lda}(\sigma) = 1, \quad \mathrm{rda}(\sigma) = 2, \quad \mathrm{lrda}(\sigma) = 2$$

and

$$\mathrm{dd}(\sigma) = 1, \quad \mathrm{ldd}(\sigma) = 2, \quad \mathrm{rdd}(\sigma) = 1, \quad \mathrm{lrdd}(\sigma) = 2.$$

By definition, we see that for $\sigma \in \mathfrak{S}_n$,

$$\operatorname{des}(\sigma) = \mathrm{W}(\sigma) + \operatorname{rdd}(\sigma) - 1, \qquad \operatorname{asc}(\sigma) = \mathrm{W}(\sigma) + \operatorname{lda}(\sigma) - 1 \tag{2.11}$$

and

$$\operatorname{des}(\sigma) = \mathrm{L}(\sigma) + \operatorname{dd}(\sigma), \qquad \operatorname{asc}(\sigma) = \mathrm{L}(\sigma) + \operatorname{lrda}(\sigma) - 1. \tag{2.12}$$

It is evident from taking reverses and complements that we only need to consider double ascents, right double ascents and left-right double ascents. By generalizing Gessel's reciprocity formula for noncommutative symmetric functions, Zhuang gave a systematic method for obtaining the generating functions for double ascents, right double ascents and left-right double ascents. It should be noted that the equivalent form of the generating function for double ascents was established by Elizalde and Noy [12]. Elizalde and Noy [12] referred to a double descent as a proper double descent.

Theorem 2.10. (Elizalde and Noy [12], Zhuang [36, Theorem 12])

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\operatorname{da}(\sigma)}\right)\frac{t^n}{n!} = \frac{ve^{\frac{(1-u)}{2}t}}{v\cosh\left(\frac{1}{2}vt\right) - (1+u)\sinh\left(\frac{1}{2}vt\right)}, \tag{2.13}$$

where $v = \sqrt{(u+1)^2-4}$.

Theorem 2.11. (Zhuang [36, Theorem 13])

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\operatorname{rda}(\sigma)}\right)\frac{t^n}{n!} = \frac{v\cosh\left(\frac{1}{2}vt\right) + (1-u)\sinh\left(\frac{1}{2}vt\right)}{v\cosh\left(\frac{1}{2}vt\right) - (1+u)\sinh\left(\frac{1}{2}vt\right)} \tag{2.14}$$

and

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\operatorname{lrda}(\sigma)}\right)\frac{t^n}{n!} = \frac{ve^{\frac{(u-1)}{2}t}}{v\cosh\left(\frac{1}{2}vt\right) - (1+u)\sinh\left(\frac{1}{2}vt\right)}, \tag{2.15}$$

where $v = \sqrt{(u+1)^2-4}$.

3 The grammatical derivation for Theorem 1.4

The main objective of this section is to give a proof of Theorem 1.4 by using the grammatical calculus introduced by Chen [5]. A context-free grammar G over a set $V = \{x, y, z, \ldots\}$ of variables is a set substitution rules replacing a variable in V by a Laurent polynomial of variables in V. For a context-free grammar G over V, the formal derivative D with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V such that each substitution rule is treated as the common differential rule that satisfies the following relations:

$$D(u+v) = D(u) + D(v) \tag{3.1}$$

$$D(uv) = D(u)v + uD(v). \tag{3.2}$$

Hence, it obeys the Leibniz's rule

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v).$$

For a constant c, we have $D(c) = 0$.

A formal derivative D with respect to G is also associated with an exponential generating function. For a Laurent polynomial w of variables in V, let

$$\mathrm{Gen}^{(G)}(w,t) = \sum_{n\geq 0} D^n(w) \frac{t^n}{n!}. \tag{3.3}$$

Then, by (3.1) and (3.2), we derive that

$$\mathrm{Gen}^{(G)}(u+v,t) = \mathrm{Gen}^{(G)}(u,t) + \mathrm{Gen}^{(G)}(v,t). \tag{3.4}$$

$$\mathrm{Gen}^{(G)}(uv,t) = \mathrm{Gen}^{(G)}(u,t)\mathrm{Gen}^{(G)}(v,t). \tag{3.5}$$

For more information on the grammatical calculus, we refer to Chen [5] and Chen and Fu [6, 8].

Dumont [11] showed the following grammar

$$G_1 = \{x \to xy,\ y \to xy\}. \tag{3.6}$$

generates the Eulerian polynomials $A_n(x)$. More precisely, let D_{G_1} be the formal derivative with respect to the grammar G_1 given by (3.6), then for $n \geq 1$,

$$D^n_{G_1}(y) = xA_n(x, y|0, 1),$$

Here we adopt the notion $A_n(x, y|\alpha, \beta)$ given by (1.6) to represent the bivariate Eulerian polynomials, where

$$xA_n(x, y|0, 1) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{asc}(\sigma)+1} y^{\mathrm{des}(\sigma)+1}$$

Chen and Fu [7] showed that

$$\mathrm{Gen}^{(G_1)}(y,t) := \sum_{n\geq 0} D^n_{G_1}(y) \frac{t^n}{n!} = x(1 + yF(x, y; t)), \tag{3.7}$$

where $F(x, y; t)$ is given in (1.8). Together with Dumont's result, they provided a grammatical proof of the generating function (1.2) of $A_n(x)$.

Similarly, it can be shown that

$$\mathrm{Gen}^{(G_1)}(x,t) := \sum_{n\geq 0} D^n_{G_1}(x) \frac{t^n}{n!} = y(1 + xF(x, y; t)). \tag{3.8}$$

In this section, we first show that the following grammar

$$\widetilde{G} = \{a \to a\alpha u_4,\ b \to b\beta u_3,\ u_4 \to u_1u_2,\ u_3 \to u_1u_2,\ u_1 \to u_1u_3, u_2 \to u_2u_4\}. \tag{3.9}$$

can be used to generate the polynomial $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$. More precisely,

Theorem 3.1. *Let $D_{\widetilde{G}}$ be the formal derivative with respect to the grammar defined in* (3.9)*, we have*

$$D_{\widetilde{G}}^n(ab) = abP_n(u_1, u_2, u_3, u_4|\alpha, \beta). \tag{3.10}$$

Based on Theorem 3.1, we give a proof of Theorem 1.4 using the grammatical calculus. More precisely, it suffices to demonstrate the following theorem.

Theorem 3.2. *Let $D_{\widetilde{G}}$ be the formal derivative with respect to the grammar defined in* (3.9)*, we have*

$$\mathrm{Gen}^{(\widetilde{G})}(ab, t) = ab\,(1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}}\,(1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}}\,e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}, \tag{3.11}$$

where $x + y = u_3 + u_4$ and $xy = u_1u_2$ and $F(x, y; t)$ is given by (1.8).

3.1 A grammatical labeling of $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$

To prove Theorem 3.1, we are required to give the combinatorial definition of $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ involving the left-to-right minima and the right-to-left minima. Recall that the complement of $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ is given by

$$\sigma^c = (n + 1 - \sigma_1)(n + 1 - \sigma_2) \cdots (n + 1 - \sigma_n).$$

For example, if $\sigma = 7\,1\,3\,8\,5\,6\,2\,4$, then the complement of σ is given by $\sigma^c = 2\,8\,6\,1\,4\,3\,7\,5$.

Evidently, the complement provides a bijection between $\mathfrak{S}_n$ and $\mathfrak{S}_n$. Moreover, for $\sigma \in \mathfrak{S}_n$ and σ^c is the complement of σ, we have

$$\mathrm{asc}(\sigma) = \mathrm{des}(\sigma^c), \quad \mathrm{des}(\sigma) = \mathrm{asc}(\sigma^c), \tag{3.12}$$

$$\mathrm{lda}(\sigma) = \mathrm{ldd}(\sigma^c), \quad \mathrm{rdd}(\sigma) = \mathrm{rda}(\sigma^c), \tag{3.13}$$

$$\mathrm{LRmax}(\sigma) = \mathrm{LRmin}(\sigma^c), \quad \mathrm{RLmax}(\sigma) = \mathrm{RLmin}(\sigma^c) \tag{3.14}$$

and

$$\mathrm{W}(\sigma) - 1 = \mathrm{V}(\sigma) = \mathrm{M}(\sigma^c). \tag{3.15}$$

Hence we find that $A_n(x, y|\alpha, \beta)$ and $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ can also be interpreted as follows:

$$A_n(x, y|\alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)-1} \beta^{\mathrm{RLmin}(\sigma)-1} \tag{3.16}$$

and

$$P_n(u_1, u_2, u_3, u_4|\alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1u_2)^{\mathrm{M}(\sigma)} u_3^{\mathrm{rda}(\sigma)} u_4^{\mathrm{ldd}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)-1} \beta^{\mathrm{RLmin}(\sigma)-1}. \tag{3.17}$$

Please take note that the polynomial $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ encompasses interior peaks, right double ascents, and left double descents. However, it's worth mentioning that left peaks, left-right double ascents, and double descents can also be characterized by specializing the polynomial $P_n(u_1,u_2,u_3,u_4|\alpha,\beta)$. More precisely, setting $\alpha=0$ in (3.17), we find that

$$\begin{aligned} P_n(u_1,u_2,u_3,u_4|0,\beta) &= \sum_{\sigma\in\mathfrak{S}_{n+1}} (u_1u_2)^{\mathrm{M}(\sigma)}u_3^{\mathrm{rda}(\sigma)}u_4^{\mathrm{ldd}(\sigma)}0^{\mathrm{LRmin}(\sigma)-1}\beta^{\mathrm{RLmin}(\sigma)-1} \\ &= \sum_{\substack{\sigma\in\mathfrak{S}_{n+1}\\ \sigma_1=1}} (u_1u_2)^{\mathrm{M}(\sigma)}u_3^{\mathrm{rda}(\sigma)}u_4^{\mathrm{ldd}(\sigma)}\beta^{\mathrm{RLmin}(\sigma)-1} \end{aligned}$$

Given $\sigma=(1,\sigma_2,\ldots,\sigma_{n+1})\in\mathfrak{S}_{n+1}$. Define $\overline{\sigma}=(\sigma_2-1,\sigma_3-1,\ldots,\sigma_{n+1}-1)$. It is easy to check that $\overline{\sigma}\in\mathfrak{S}_n$ and

$$\mathrm{M}(\sigma)=\mathrm{L}(\overline{\sigma}),\ \mathrm{rda}(\sigma)=\mathrm{lrda}(\overline{\sigma}),\ \mathrm{ldd}(\sigma)=\mathrm{dd}(\overline{\sigma}),\ \mathrm{RLmin}(\sigma)-1=\mathrm{RLmin}(\overline{\sigma}).$$

Moreover, this process is reversible. Hence we derive that

$$P_n(u_1,u_2,u_3,u_4|0,\beta)=\sum_{\overline{\sigma}\in\mathfrak{S}_n}(u_1u_2)^{\mathrm{L}(\overline{\sigma})}u_3^{\mathrm{lrda}(\overline{\sigma})}u_4^{\mathrm{dd}(\overline{\sigma})}\beta^{\mathrm{RLmin}(\overline{\sigma})}. \tag{3.18}$$

Using the same argument, we have

$$A_n(x,y|\alpha,0)=\sum_{\sigma\in\mathfrak{S}_n}x^{\mathrm{des}(\sigma)+1}y^{\mathrm{asc}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)} \tag{3.19}$$

and

$$A_n(x,y|0,\beta)=\sum_{\sigma\in\mathfrak{S}_n}x^{\mathrm{des}(\sigma)}y^{\mathrm{asc}(\sigma)+1}\beta^{\mathrm{RLmin}(\sigma)} \tag{3.20}$$

We are now in a position to show Theorem 3.1 by using the grammatical labeling. The notion of a grammatical labeling was introduced by Chen and Fu [6].

Let $\sigma=\sigma_1\sigma_2\cdots\sigma_n\in\mathfrak{S}_n$. For $1\leq i\leq n+1$, recall that the position i is said to be the position immediately before σ_i, whereas the position $n+1$ is meant to be the position after σ_n. The labeling for $P_n(u_1,u_2,u_3,u_4|\alpha,\beta)$ can be described as follows. We patch $+\infty$ to σ at both ends so that there are $n+1$ positions between two adjacent elements. For $1\leq i\leq n+1$, we label the position i as follows:

- If $i=1$, then label it by a;
- If $i=n+1$, then label it by b;
- If i is a right double ascent, then label the position i by u_3;
- If $i-1$ is a left double descent, then label the position i by u_4;
- If i is a peak, then label the position i by u_2 and label the position $i+1$ by u_1;

- If σ_j is a left-to-right minimum and $\sigma_j \neq 1$, then label α below σ_j;
- If σ_j is a right-to-left minimum and $\sigma_j \neq 1$, then label β below σ_j.

The weight ω of σ is defined to be the product of all the labels. Below is an example,

$$
\begin{array}{l}
7 \quad 1 \quad 2 \quad 8 \quad 3 \quad 6 \quad 5 \quad 4 \\
\longrightarrow \begin{array}{ccccccccccccccccccc}
+\infty & a & 7 & u_4 & 1 & u_3 & 2 & u_2 & 8 & u_1 & 3 & u_2 & 6 & u_1 & 5 & u_4 & 4 & b & +\infty \\
 & & \alpha & & & & \beta & & & & \beta & & & & & & \beta & &
\end{array}
\end{array}
$$

If σ has k peaks, d_1 right double ascents, d_2 left double descents, and m_1 left-to-right minima and m_2 right-to-left minima, then its weight is given by

$$\omega(\sigma) = ab(u_1u_2)^k u_3^{d_1} u_4^{d_2} \alpha^{m_1-1} \beta^{m_2-1}.$$

From the definition of the above labeling, we see that

$$abP_n(u_1, u_2, u_3, u_4|\alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} \omega(\sigma). \tag{3.21}$$

Proof of Theorem 3.1. We proceed by induction on n. For $n = 0$, the statement is obvious. Assume that this assertion holds for $n-1$. To show that it is valid for n, we represent a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ in $\mathfrak{S}_n$ by patching $+\infty$ at both ends so that there are $n + 1$ positions between two adjacent elements, and these are $n + 1$ positions to insert the element $n + 1$ into σ to generate a permutation in $\mathfrak{S}_{n+1}$. Suppose that π is permutation in $\mathfrak{S}_{n+1}$ obtained from σ by inserting the element $n + 1$ at the position i, where $1 \leq i \leq n + 1$. We consider the following six cases:

Case 1: If i is labeled by a in the labeling of σ, that means that $i = 1$, then the first position and the second position of π are labeled by a and u_4 in the labeling of π respectively. Moreover, $n+1$ is the left-to-right minimum of π, so label α below $n+1$. In this case, we find that the change of weights of π is consistent with the substitution rule $a \to a\alpha u_4$.

$$
\begin{array}{cccc}
+\infty & a & \sigma_1 & \cdots \\
 & & \alpha &
\end{array}
\Rightarrow
\begin{array}{cccccc}
+\infty & a & n+1 & u_4 & \sigma_1 & \cdots \\
 & & \alpha & & \alpha & \cdots
\end{array}.
$$

Case 2: If i is labeled by b in the labeling of σ, that means $i = n+1$, then the changes of weights caused by the insertion are coded by the rule $b \to b\beta u_3$.

$$
\begin{array}{cccc}
\cdots & \sigma_n & b & +\infty \\
\cdots & \beta & &
\end{array}
\Rightarrow
\begin{array}{cccccc}
\cdots & \sigma_n & u_3 & n+1 & b & +\infty \\
\cdots & \beta & & \beta & &
\end{array}.
$$

Case 3: If i is labeled by u_3 in the labeling of σ, then the position i of π is labeled by u_2 and the position $i + 1$ of π is labeled by u_1 in the labeling of π, so the change of weights of π is consistent with the substitution rule $u_3 \to u_1u_2$.

$$\cdots\ \sigma_{i-1}\ u_3 \quad \sigma_i\ \cdots \Rightarrow \cdots \sigma_{i-1}\ u_2 \quad n+1 \quad u_1 \quad \sigma_i\ \cdots.$$

Case 4: If i is labeled by u_4 in the labeling of σ, then it can be checked that the change of weights is in accordance with the rule $u_4 \to u_1u_2$, see the figure below.

$$\cdots\ \sigma_{i-1} \quad u_4 \quad \sigma_i\ \cdots\ \Rightarrow\ \cdots\ \sigma_{i-1} \quad u_2 \quad n+1 \quad u_1 \qquad \sigma_i\ \cdots .$$

Case 5: If i is labeled by u_1 in the labeling of σ, then it can be checked that the change of weights is in accordance with the rule $u_1 \to u_1u_3$, see the figure below.

$$\cdots\ \sigma_{i-2} \quad u_2 \quad \sigma_{i-1} \quad u_1 \quad \sigma_i\ \cdots\ \Rightarrow\ \cdots\sigma_{i-2} \quad u_3 \quad \sigma_{i-1} \quad u_2 \quad n+1 \quad u_1 \quad \sigma_i\ \cdots .$$

Case 6: If i is labeled by u_2 in the labeling of σ, then it can be checked that the change of weights is in accordance with the rule $u_2 \to u_2u_4$, see the figure below.

$$\cdots\ \sigma_{i-1} \quad u_2 \quad \sigma_i \quad u_1 \quad \sigma_{i+1}\ \cdots\ \Rightarrow\ \cdots\ \sigma_{i-1} \quad u_2 \quad n+1 \quad u_1 \quad \sigma_i \quad u_4 \quad \sigma_{i+1}\ \cdots .$$

Summing up all the cases shows that this assertion is valid for n. This completes the proof. ∎

3.2 Proof of Theorem 3.2

We are now in a position to give a grammatical derivation of Theorem 1.4. By employing Theorem 3.1, it is sufficient to demonstrate Theorem 3.2.

Proof of Theorem 3.2. Let $D_{\widetilde{G}_1}$ is the formal derivative with respect to the grammar

$$\widetilde{G}_1 = \{u_4 \to u_1u_2,\ u_3 \to u_1u_2,\ u_1 \to u_1u_3, u_2 \to u_2u_4\}. \tag{3.22}$$

We first show that

$$\begin{aligned}\mathrm{Gen}^{(\widetilde{G}_1)}(u_1u_2, t) &:= \sum_{n\geq 0} D^n_{\widetilde{G}_1}(u_1u_2)\frac{t^n}{n!} \\ &= xy(1+yF(x,y;t))(1+xF(x,y;t))\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\mathrm{Gen}^{(\widetilde{G}_1)}(u_1, t) &:= \sum_{n\geq 0} D^n_{\widetilde{G}_1}(u_1)\frac{t^n}{n!} \\ &= u_1\sqrt{(1+yF(x,y;t))(1+xF(x,y;t))}e^{\frac{(u_3-u_4)t}{2}},\end{aligned} \tag{3.24}$$

where $x+y = u_3+u_4$ and $xy = u_1u_2$ and $F(x,y;t)$ is given by (1.8).

Recall that D_{G_1} is the formal derivative with respect to the grammar (3.6) and $D_{\widetilde{G}_1}$ is the formal derivative with respect to the grammar (3.22). If we set $x+y = u_3+u_4$ and $xy = u_1u_2$, we find that

$$D_{G_1}(xy) = xy(x+y) = u_1u_2(u_3+u_4) = D_{\widetilde{G}_1}(u_1u_2) \tag{3.25}$$

and

$$D_{G_1}(x+y) = 2xy = 2u_1u_2 = D_{\widetilde{G}_1}(u_3+u_4). \tag{3.26}$$

We claim that for $n \geq 1$,

$$D^n_{G_1}(xy) = D^n_{\widetilde{G}_1}(u_1u_2). \tag{3.27}$$

By (3.25), we see that (3.27) is valid when $n = 1$. Assume that (3.27) holds for n. Observe that $D^n_{G_1}(xy)$ is symmetric in x, y, so we may write $D^n_{G_1}(xy)$ in the following form:

$$D^n_{G_1}(xy) = \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j (xy)^j (x+y)^{n+2-2j}, \tag{3.28}$$

where a_j are positive integers. By the induction hypothesis, we see that

$$D^n_{\widetilde{G}_1}(u_1u_2) = D^n_{G_1}(xy).$$

and so

$$D^n_{\widetilde{G}_1}(u_1u_2) = \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j (u_1u_2)^j (u_3+u_4)^{n+2-2j} \tag{3.29}$$

Applying D_{G_1} to (3.28), we obtain that

$$\begin{aligned} D^{n+1}_{G_1}(xy) = &\sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j j (xy)^{j-1} (x+y)^{n-2j+2} D_{G_1}(xy) \\ &+ \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j (n-2j+2)(xy)^j (x+y)^{n-2j+1} D_{G_1}(x+y) \end{aligned}$$

Since $x+y = u_3+u_4$ and $xy = u_1u_2$, we find that

$$\begin{aligned} D^{n+1}_{G_1}(xy) = &\sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j j (u_1u_2)^{j-1} (u_3+u_4)^{n-2j+2} D_{G_1}(xy) \\ &+ \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j (n-2j+2)(u_1u_2)^j (u_3+u_4)^{n-2j+1} D_{G_1}(x+y) \end{aligned} \tag{3.30}$$

Applying (3.25) and (3.26) into (3.30), and by (3.29), we conclude that

$$D^{n+1}_{G_1}(xy) = D^{n+1}_{\widetilde{G}_1}(u_1u_2),$$

and so (3.27) also is valid for $n+1$, and hence the claim is verified. Therefore, we obtain

$$\mathrm{Gen}^{(\widetilde{G}_1)}(u_1u_2, t) = \mathrm{Gen}^{(G_1)}(xy, t). \tag{3.31}$$

By applying the multiplicative property (3.5), we can deduce from (3.7) and (3.8) that

$$\mathrm{Gen}^{(G_1)}(xy,t) = \sum_{n\geq 0} D^n_{G_1}(xy)\frac{t^n}{n!} = xy(1+yF(x,y;t))(1+xF(x,y;t)). \tag{3.32}$$

Substituting (3.32) into (3.31), we obtain (3.23).

To prove (3.24), we first observe that

$$D_{\widetilde{G}_1}(u_1u_2^{-1}) = u_1u_2^{-1}(u_3-u_4). \tag{3.33}$$

Since

$$D_{\widetilde{G}_1}(u_3-u_4) = 0,$$

it follows that for $n \geq 0$,

$$D^n_{\widetilde{G}_1}(u_1u_2^{-1}) = u_1u_2^{-1}(u_3-u_4)^n.$$

Hence

$$\mathrm{Gen}^{(\widetilde{G}_1)}(u_1u_2^{-1},t) := \sum_{n\geq 0} D^n_{\widetilde{G}_1}(u_1u_2^{-1})\frac{t^n}{n!} = u_1u_2^{-1}e^{(u_3-u_4)t}. \tag{3.34}$$

By the multiplicative property (3.5), we deduce from (3.23) and (3.34) that

$$\begin{aligned}(\mathrm{Gen}^{(\widetilde{G}_1)}(u_1,t))^2 &= \mathrm{Gen}^{(\widetilde{G}_1)}(u_1u_2^{-1},t)\mathrm{Gen}^{(\widetilde{G}_1)}(u_1u_2,t)\\ &= u_1^2((1+yF(x,y;t))(1+xF(x,y;t)))e^{(u_3-u_4)t},\end{aligned}$$

from which, we obtain (3.24).

Let α, β be two fixed numbers. We see that

$$D_{\widetilde{G}_1}(u_2^{\alpha}) = \alpha u_2^{\alpha-1}D_{\widetilde{G}_1}(u_2) = \alpha u_2^{\alpha}u_4, \tag{3.35}$$

$$D_{\widetilde{G}_1}(u_1^{\beta}) = \beta u_1^{\beta-1}D_{\widetilde{G}_1}(u_1) = \beta u_1^{\beta}u_3. \tag{3.36}$$

Let $D_{\widetilde{G}}$ is the formal derivative with respect to the grammar (3.9). Setting $a = u_2^{\alpha}$ and $b = u_1^{\beta}$, then by (3.35) and (3.36), we find that

$$D_{\widetilde{G}_1}(u_2^{\alpha}) = D_{\widetilde{G}}(a) \quad \text{and} \quad D_{\widetilde{G}_1}(u_1^{\beta}) = D_{\widetilde{G}}(b).$$

Moreover, it is easy to check that

$$D_{\widetilde{G}_1}(u_1) = D_{\widetilde{G}}(u_1), \quad D_{\widetilde{G}_1}(u_2) = D_{\widetilde{G}}(u_2), \quad D_{\widetilde{G}_1}(u_3) = D_{\widetilde{G}}(u_3), \quad \text{and} \quad D_{\widetilde{G}_1}(u_4) = D_{\widetilde{G}}(u_4).$$

Hence we can use the induction on n to deduce that for $n \geq 0$,

$$D^n_{\widetilde{G}_1}(u_2^{\alpha}) = D^n_{\widetilde{G}}(a) \quad \text{and} \quad D^n_{\widetilde{G}_1}(u_1^{\beta}) = D^n_{\widetilde{G}}(b).$$

Consequently, for $n \geq 0$,

$$D^n_{\widetilde{G}}(ab) = D^n_{\widetilde{G}_1}(u_2^{\alpha}u_1^{\beta}). \tag{3.37}$$

It follows that

$$\begin{aligned}
\mathrm{Gen}^{(\widetilde{G})}(ab,t) &= \mathrm{Gen}^{(\widetilde{G}_1)}(u_2^{\alpha}u_1^{\beta},t)\\
&= (\mathrm{Gen}^{(\widetilde{G}_1)}(u_2u_1,t))^{\alpha}(\mathrm{Gen}^{(\widetilde{G}_1)}(u_1,t))^{\beta-\alpha}\\
&= u_1^{\beta}u_2^{\alpha}\,(1+yF(x,y;t))^{\frac{\alpha+\beta}{2}}\,(1+xF(x,y;t))^{\frac{\alpha+\beta}{2}}\,e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}\\
&= ab\,(1+yF(x,y;t))^{\frac{\alpha+\beta}{2}}\,(1+xF(x,y;t))^{\frac{\alpha+\beta}{2}}\,e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}
\end{aligned}$$

as desired. This completes the proof of Theorem 3.2. ∎

It should be noted that the generating functions (3.23) and (3.24) in the proof of Theorem 1.4 could also be derived by using the results of Fu [18].

4 The generating functions

In this section, we give a proof of Theorem 1.5 with the aid of Theorem 1.4. We then derive many consequences of Theorem 1.5, which provide (α,β)-extensions of the generating functions of peaks, left peaks, double ascents, right double ascents and left-right double ascents.

Proof of Theorem 1.5. From Theorem 1.4, we see that

$$\begin{aligned}
&\sum_{n\geq 0} P_n(u_1,u_2,u_3,u_4|\alpha,\beta)\frac{t^n}{n!}\\
&\quad = ((1+xF(x,y;t))(1+yF(x,y;t)))^{\frac{\alpha+\beta}{2}}\,e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}, \qquad (4.1)
\end{aligned}$$

where $x+y=u_3+u_4$, $xy=u_1u_2$.

Recall that

$$F(x,y;t)=\frac{e^{xt}-e^{yt}}{xe^{yt}-ye^{xt}},$$

we find that

$$\begin{aligned}
1+xF(x,y;t) &= \frac{(x-y)e^{xt}}{xe^{yt}-ye^{xt}}\\
&= \left(-\frac{x}{y-x}e^{(y-x)t}+\frac{y}{y-x}\right)^{-1}. \qquad (4.2)
\end{aligned}$$

Similarly, we have

$$1+yF(x,y;t)=\left(\frac{y}{y-x}e^{(x-y)t}-\frac{x}{y-x}\right)^{-1}. \qquad (4.3)$$

Since $x+y=u_3+u_4$ and $xy=u_1u_2$, it follows that

$$x=\frac{(u_3+u_4)-\sqrt{(u_3+u_4)^2-4u_1u_2}}{2}, \tag{4.4}$$

$$y=\frac{(u_3+u_4)+\sqrt{(u_3+u_4)^2-4u_1u_2}}{2}. \tag{4.5}$$

Hence, we have

$$y-x=\sqrt{(u_3+u_4)^2-4u_1u_2}, \tag{4.6}$$

$$\frac{x}{y-x}=-\frac{1}{2}+\frac{1}{2}\frac{(u_3+u_4)}{\sqrt{(u_3+u_4)^2-4u_1u_2}}, \tag{4.7}$$

$$\frac{y}{y-x}=\frac{1}{2}+\frac{1}{2}\frac{(u_3+u_4)}{\sqrt{(u_3+u_4)^2-4u_1u_2}}. \tag{4.8}$$

Putting (4.6), (4.7) and (4.8) into (4.2), we obtain

$$\begin{aligned}
&-\frac{x}{y-x}e^{(y-x)t}+\frac{y}{y-x}\\
&=\frac{e^{t\sqrt{(u_3+u_4)^2-4u_1u_2}}+1}{2}-\frac{(u_3+u_4)}{\sqrt{(u_3+u_4)^2-4u_1u_2}}\frac{e^{t\sqrt{(u_3+u_4)^2-4u_1u_2}}-1}{2}\\
&=\frac{1}{e^{-\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}}}\left(\cosh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right.\\
&\qquad\left.-\frac{(u_3+u_4)}{\sqrt{(u_3+u_4)^2-4u_1u_2}}\sinh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right).
\end{aligned}$$

Similarly, plugging (4.6), (4.7) and (4.8) into (4.3), we derive that

$$\begin{aligned}
&-\frac{x}{y-x}+\frac{y}{y-x}e^{(x-y)t}\\
&=\frac{1}{e^{\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}}}\left(\cosh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right.\\
&\qquad\left.-\frac{(u_3+u_4)}{\sqrt{(u_3+u_4)^2-4u_1u_2}}\sinh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
(1+xF(x,y;t))(1+yF(x,y;t))=&\left(\cosh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right.\\
&\left.-\frac{(u_3+u_4)}{\sqrt{(u_3+u_4)^2-4u_1u_2}}\sinh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right)^{-2}.
\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.1) yields the generating function of $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ as stated in Theorem 1.5. This completes the proof. ∎

Setting $\alpha = 0$ in Theorem 1.5, and by (3.18) , we have

Theorem 4.1. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n}(u_1u_2)^{\mathrm{L}(\sigma)}u_3^{\mathrm{lrda}(\sigma)}u_4^{\mathrm{dd}(\sigma)}\beta^{\mathrm{RLmin}(\sigma)}\right)\frac{t^n}{n!} = e^{\frac{\beta}{2}(u_3-u_4)t}\times\left(\cosh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right.$$
$$\left.-\frac{u_3+u_4}{\sqrt{(u_3+u_4)^2-4u_1u_2}}\sinh\left(\frac{t}{2}\sqrt{(u_3+u_4)^2-4u_1u_2}\right)\right)^{-\beta}.$$

Setting $u_1 = u_2 = u$ and $u_3 = u_4 = v$ in Theorem 4.1, and using (2.12), we derive the following β-extension of the generating function for the left peak polynomials.

Theorem 4.2. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n}u^{2\mathrm{L}(\sigma)}v^{n-2\mathrm{L}(\sigma)}\beta^{\mathrm{RLmin}(\sigma)}\right)\frac{t^n}{n!}$$
$$=\left(\frac{\sqrt{v^2-u^2}}{\sqrt{v^2-u^2}\cosh\left(t\sqrt{v^2-u^2}\right)-v\sinh\left(t\sqrt{v^2-u^2}\right)}\right)^{\beta}.$$

Setting $\beta = 1$ in Theorem 4.2, we recover the generating function (2.3) for the left peak polynomials established by Gessel [26, Sequence A008971].

Setting $u_1 = u_2 = u_4 = 1$ and $u_3 = u$ in Theorem 4.1, we acquire the β-extension of the generating function for left-right double ascents, from which we recover the generating function (2.15) for the left-right double ascents by setting $\beta = 1$.

Theorem 4.3. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n}u^{\mathrm{lrda}(\sigma)}\beta^{\mathrm{RLmin}(\sigma)}\right)\frac{t^n}{n!}$$
$$=e^{\frac{\beta(u-1)}{2}t}\left(\frac{\sqrt{(u+1)^2-4}}{\sqrt{(u+1)^2-4}\cosh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)-(1+u)\sinh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)}\right)^{\beta}.$$

Setting $u_1 = u_2 = u_3 = 1$ and $u_4 = u$ in Theorem 4.1, and by taking reverse of a permutation, we obtain the α-extension of the generating function for double ascents. This allows us to retrieve the generating function (2.13) for double ascents when $\alpha = 1$.

Theorem 4.4. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{da}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)}\right)\frac{t^n}{n!}$$
$$= e^{\frac{\alpha(1-u)}{2}t}\left(\frac{\sqrt{(u+1)^2-4}}{\sqrt{(u+1)^2-4}\cosh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)-(1+u)\sinh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)}\right)^{\alpha}.$$

Setting $u_1=u_2=u$ and $u_3=u_4=v$ in Theorem 1.5 and using (3.17), we arrive at

Theorem 4.5. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{2\mathrm{M}(\sigma)}v^{n-2\mathrm{M}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)-1}\beta^{\mathrm{RLmin}(\sigma)-1}\right)\frac{t^n}{n!}$$
$$=\left(\frac{\sqrt{v^2-u^2}}{\sqrt{v^2-u^2}\cosh\left(t\sqrt{v^2-u^2}\right)-v\sinh\left(t\sqrt{v^2-u^2}\right)}\right)^{\alpha+\beta}.$$

By setting $v=\alpha=\beta=1$ in Theorem 4.5, replacing u with $\sqrt{u}$, and performing integration on both sides with respect to t, we retrieve the generating function (2.4) for the peak polynomials.

Setting $u_1=u_2=u_4=1$ and $u_3=u$ in Theorem 1.5 and using (3.17) gives that

Theorem 4.6. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{\mathrm{rda}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)-1}\beta^{\mathrm{RLmin}(\sigma)-1}\right)\frac{t^n}{n!}=e^{\frac{(\beta-\alpha)(u-1)}{2}t}$$
$$\times\left(\frac{\sqrt{(u+1)^2-4}}{\sqrt{(u+1)^2-4}\cosh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)-(1+u)\sinh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)}\right)^{\alpha+\beta}.$$

Setting $\alpha=\beta=1$ in Theorem 4.6 and then integrating both sides of the mentioned identity with respect to t, we bring back the generating function (2.14) for the right double ascents.

5 The expression of $P_n(u_1,u_2,u_3,u_4|\alpha,\beta)$ with $\alpha+\beta=-1$

This section is dedicated to deriving an explicit expression of $P_n(u_1,u_2,u_3,u_4;\alpha,\beta)$ when $\alpha+\beta=-1$ by utilizing Theorem 1.5.

Theorem 5.1. *When $\alpha+\beta=-1$ and for $n\geq 1$,*

$$
\begin{aligned}
&2^n P_n(u_1,u_2,u_3,u_4|\alpha,\beta) \qquad (5.1)\\
&\quad = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k}((u_3+u_4)^2-4u_1u_2)^k(\beta-\alpha)^{n-2k}(u_3-u_4)^{n-2k}\\
&\qquad -(u_3+u_4)\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k+1}((u_3+u_4)^2-4u_1u_2)^k(\beta-\alpha)^{n-2k-1}(u_3-u_4)^{n-2k-1}.
\end{aligned}
$$

Proof. When $\alpha+\beta=-1$, and if we let $v=\sqrt{(u_3+u_4)^2-4u_1u_2}$, then Theorem 1.5 becomes

$$
\begin{aligned}
&\sum_{n\geq 0}P_n(u_1,u_2,u_3,u_4|\alpha,\beta)\frac{t^n}{n!}\\
&= e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}\times\left(\cosh\left(\frac{1}{2}vt\right)-\frac{u_3+u_4}{v}\sinh\left(\frac{1}{2}vt\right)\right)\\
&= e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}\times\left(\sum_{n\geq 0}\frac{v^{2n}}{2^{2n}}\frac{t^{2n}}{(2n)!}-(u_3+u_4)\left(\sum_{n\geq 0}\frac{v^{2n}}{2^{2n+1}}\frac{t^{2n+1}}{(2n+1)!}\right)\right)\\
&= \left(\sum_{n\geq 0}\frac{(\beta-\alpha)^n(u_3-u_4)^n}{2^n}\frac{t^n}{n!}\right)\times\left(\sum_{n\geq 0}\frac{v^{2n}}{2^{2n}}\frac{t^{2n}}{(2n)!}-(u_3+u_4)\left(\sum_{n\geq 0}\frac{v^{2n}}{2^{2n+1}}\frac{t^{2n+1}}{(2n+1)!}\right)\right)\\
&= \sum_{n\geq 0}\frac{t^n}{n!}\left(\sum_{k\geq 0}\binom{n}{2k}(\beta-\alpha)^{n-2k}(u_3-u_4)^{n-2k}\frac{v^{2k}}{2^n}\right.\\
&\qquad \left.-(u_3+u_4)\sum_{k\geq 0}\binom{n}{2k+1}(\beta-\alpha)^{n-2k-1}(u_3-u_4)^{n-2k-1}\frac{v^{2k}}{2^n}\right).
\end{aligned}
$$

Comparing the coefficients of $t^n/n!$ on the both sides yields (5.1). This completes the proof. ∎

Setting $u_3=u_4=u_1=1$ and $u_2=u$ in Theorem 5.1 and using (3.17) yields Theorem 1.6. Theorem 1.7 follows from Theorem 1.6 by setting $\alpha=0$.

By choosing $\alpha=\beta=-1/2$ in Theorem 5.1, we derive that

Theorem 5.2. *For $n\geq 1$,*

$$
\begin{aligned}
&2^nP_n\left(u_1,u_2,u_3,u_4|-\frac{1}{2},-\frac{1}{2}\right)\\
&\quad=\begin{cases}((u_3+u_4)^2-4u_1u_2)^{\lfloor \frac{n}{2}\rfloor}, & \text{if } n \text{ is even},\\ -(u_3+u_4)((u_3+u_4)^2-4u_1u_2)^{\lfloor \frac{n}{2}\rfloor}, & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}
$$

Setting $u_1 = u_2 = u_4 = 1$, $u_3 = u$ in Theorem 5.2 and employing (3.17) yields Theorem 1.8.

6 Relations between $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ and $A_n(x, y|\alpha, \beta)$

In this section, we begin by presenting two relations between $P_n(u_1, u_2, u_3, u_4|\alpha, \beta)$ and $A_n(x, y|\alpha, \beta)$ and give their proofs. Subsequently, we derive several consequences from these connections. These specific derivations will not only yield the (α, β)-extensions of the related relations associated with the Eulerian polynomial due to Stembridge, Petersen, Brändén and Zhuang, but will also provide the left peak version of Stembridge's formula, the peak version of Petersen's formula and their (α, β)-extensions.

6.1 Two relations

Theorem 6.1. *For $n \geq 0$,*

$$P_n(u_1, u_2, u_3, u_4|\alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k} A_k\left(x, y|\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right) \frac{(\beta-\alpha)^{n-k}(u_3-u_4)^{n-k}}{2^{n-k}}, \tag{6.1}$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$.

Proof. Combining Theorem 1.1 and Theorem 1.4, we derive that

$$\begin{aligned}
&\sum_{n\geq 0} P_n(u_1, u_2, u_3, u_4|\alpha, \beta)\frac{t^n}{n!} \\
&= e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t} \sum_{k\geq 0} A_k\left(x, y|\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right)\frac{t^k}{k!} \\
&= \left(\sum_{m\geq 0} \frac{(\beta-\alpha)^m(u_3-u_4)^m}{2^m}\frac{t^m}{m!}\right)\left(\sum_{k\geq 0} A_k\left(x, y|\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right)\frac{t^k}{k!}\right) \\
&= \sum_{n\geq 0} \frac{t^n}{n!}\left(\sum_{k=0}^{n}\binom{n}{k} A_k\left(x, y|\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right)\frac{(\beta-\alpha)^{n-k}(u_3-u_4)^{n-k}}{2^{n-k}}\right).
\end{aligned} \tag{6.2}$$

Equating the coefficients of $t^n/n!$ yields the result. ∎

Theorem 6.2. *For $n \geq 0$,*

$$
\begin{aligned}
&P_n(u_1, u_2, u_3, u_4|\alpha, \beta) \\
&= \sum_{k=0}^{n} \binom{n}{k} A_k\left(x, y|0, \alpha+\beta\right)(\alpha x - \beta y + (\beta-\alpha)u_3)^{n-k} \qquad (6.3) \\
&= \sum_{k=0}^{n} \binom{n}{k} A_k\left(x, y|\alpha+\beta, 0\right)(\alpha y - \beta x + (\beta-\alpha)u_3)^{n-k} \qquad (6.4)
\end{aligned}
$$

where $x + y = u_3 + u_4$ and $xy = u_1u_2$.

Proof. Since, by (1.8), we see that

$$1 + xF(x, y; t) = \frac{(x-y)e^{xt}}{xe^{yt} - ye^{xt}}$$

and

$$1 + yF(x, y; t) = \frac{(x-y)e^{yt}}{xe^{yt} - ye^{xt}}.$$

Hence

$$(1 + xF(x, y; t))(1 + yF(x, y; t)) = \frac{(x-y)^2 e^{(x+y)t}}{(xe^{yt} - ye^{xt})^2},$$

so that

$$
\begin{aligned}
&((1 + xF(x, y; t))(1 + yF(x, y; t)))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t} \\
&= \left(\frac{x-y}{xe^{yt} - ye^{xt}}\right)^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(u_3-u_4)}{2} + \frac{(\alpha+\beta)(x+y)}{2}\right)t} \\
&= (1 + yF(x, y; t))^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(u_3-u_4)}{2} + \frac{(\alpha+\beta)(x-y)}{2}\right)t} \qquad (6.5) \\
&= (1 + xF(x, y; t))^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(u_3-u_4)}{2} + \frac{(\alpha+\beta)(y-x)}{2}\right)t} \qquad (6.6)
\end{aligned}
$$

Applying (1.7) into (6.5), and applying Theorem 1.4, we find that

$$
\begin{aligned}
&\sum_{n\geq 0} P_n(u_1, u_2, u_3, u_4|\alpha, \beta)\frac{t^n}{n!} \\
&= (1 + yF(x, y; t))^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(u_3-u_4)}{2} + \frac{(\alpha+\beta)(x-y)}{2}\right)t} \\
&= \left(\sum_{k\geq 0} A_k\left(x, y|0, \alpha+\beta\right)\frac{t^k}{k!}\right)\left(\sum_{m\geq 0} (\alpha x - \beta y + (\beta-\alpha)u_3)^m \frac{t^m}{m!}\right),
\end{aligned}
$$

where the last line follows from the relation $x + y = u_3 + u_4$. Equating the coefficients of $t^n/n!$ yields the relation (6.3).

If we plug (1.7) into (6.5), and by Theorem 1.4, we derive that

$$\sum_{n\geq 0} P_n(u_1,u_2,u_3,u_4|\alpha,\beta)\frac{t^n}{n!}$$
$$= (1+xF(x,y;t))^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(u_3-u_4)}{2}+\frac{(\alpha+\beta)(y-x)}{2}\right)t}$$
$$= \left(\sum_{k\geq 0} A_k\left(x,y|\alpha+\beta,0\right)\frac{t^k}{k!}\right)\left(\sum_{m\geq 0}(\alpha y-\beta x+(\beta-\alpha)u_3)^m\frac{t^m}{m!}\right).$$

Equating the coefficients of $t^n/n!$ yields the relation (6.4). This completes the proof of Theorem 6.2. ∎

6.2 Some consequences

Setting $u_3=u_4$ and $u_1=u_2$ in Theorem 6.1, we find that

$$u_3=u_4=\frac{x+y}{2} \quad \text{and} \quad u_1=u_2=\sqrt{xy}, \tag{6.7}$$

and using the combinatorial definition (3.16) of $A_n(x,y|\alpha,\beta)$ and the combinatorial definition (3.17) of $P_n(u_1,u_2,u_3,u_4|\alpha,\beta)$, we obtain Theorem 1.9. Setting $\alpha=\beta=1$ and $y=1$ in Theorem 1.9, we reacquire the relation (2.5) due to Stembridge.

Choosing $\alpha=0$ in Theorem 1.9 yields Theorem 1.10. Setting $\beta=1$ and $y=1$ in Theorem 1.10, we obtain the following consequence, which can be viewed as the left peak version of Stembridge's formula.

Theorem 6.3. *For* $n\geq 1$,

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} x^{\mathrm{des}(\sigma)}\left(\frac{1}{2}\right)^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}=\left(\frac{1+x}{2}\right)^n\sum_{\sigma\in\mathfrak{S}_n}\left(\frac{4x}{(1+x)^2}\right)^{\mathrm{L}(\sigma)}. \tag{6.8}$$

Setting $\alpha=\beta$, $u_1=uv$, $u_2=u_3=w$ and $u_4=v$ in Theorem 6.1, and by (3.16) and (3.17), and invoking (2.11), (3.12) (3.13) and (3.15), we deduce the following consequence, which can be viewed as the α-extension of Zhuang's relation (2.7).

Theorem 6.4. *For* $n\geq 1$,

$$\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{M}(\sigma)}v^{\mathrm{des}(\sigma)}w^{\mathrm{asc}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}$$
$$=\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)}y^{\mathrm{asc}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)+\mathrm{RLmin}(\sigma)-2}, \tag{6.9}$$

where

$$x=\frac{(w+v)-\sqrt{(w+v)^2-4uvw}}{2}, \tag{6.10}$$

and

$$y=\frac{(w+v)+\sqrt{(w+v)^2-4uvw}}{2}. \tag{6.11}$$

Note that (6.10) and (6.11) follows from (4.4) and (4.5) upon setting $u_1=uv$, $u_2=u_3=w$ and $u_4=v$. Setting $\alpha=1$, $w=1$, $a=x/y$ and $b=(y-1)/(1-x)$ in Theorem 6.4, we regain the relation (2.7) established by Zhuang [37, Theorem 4.2].

By choosing $\alpha=0$, $u_3=u_4$ and $u_1=u_2$ in (6.4) of Theorem 6.2, and using (3.18) and (3.19), we derive that the β-extension of Petersen's relation, from which we recover the relation (2.6) by setting $y=1$ and $\beta=1$.

Theorem 6.5. *For* $n\geq 1$*,*

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n}(xy)^{\mathrm{L}(\sigma)}\left(\frac{x+y}{2}\right)^{n-2\mathrm{L}(\sigma)}\beta^{\mathrm{RLmin}(\sigma)}\\=\sum_{k=1}^{n}\binom{n}{k}\frac{(\beta(y-x))^{n-k}}{2^{n-k}}\left(\sum_{\sigma\in\mathfrak{S}_k}x^{\mathrm{des}(\sigma)+1}y^{\mathrm{asc}(\sigma)}\beta^{\mathrm{LRmin}(\sigma)}\right)\\+\left(\frac{(\beta(y-x))}{2}\right)^n.\end{aligned} \tag{6.12}$$

Setting $u_3=u_4$ and $u_1=u_2$ in (6.4) of Theorem 6.2, and using (3.17) and (3.19), we get

Theorem 6.6. *For* $n\geq 1$*,*

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_{n+1}}(xy)^{\mathrm{M}(\sigma)}\left(\frac{x+y}{2}\right)^{n-2\mathrm{M}(\sigma)}\alpha^{\mathrm{LRmin}(\sigma)-1}\beta^{\mathrm{RLmin}(\sigma)-1}\\=\sum_{k=1}^{n}\binom{n}{k}\frac{((\alpha+\beta)(y-x))^{n-k}}{2^{n-k}}\left(\sum_{\sigma\in\mathfrak{S}_k}x^{\mathrm{des}(\sigma)+1}y^{\mathrm{asc}(\sigma)}(\alpha+\beta)^{\mathrm{LRmin}(\sigma)}\right)\\+\left(\frac{((\alpha+\beta)(y-x))}{2}\right)^n.\end{aligned} \tag{6.13}$$

By setting $y=1$ and $\alpha=\beta=1$ in Theorem 6.6, we derive the following relation, which can be viewed as the peak version of Petersen's formula (2.6).

Theorem 6.7. *For* $n\geq 1$*,*

$$\begin{aligned}\left(\frac{1+x}{2}\right)^n\sum_{\sigma\in\mathfrak{S}_{n+1}}\left(\frac{4x}{(1+x)^2}\right)^{\mathrm{M}(\sigma)}\\=\sum_{k=1}^{n}\binom{n}{k}(1-x)^{n-k}\sum_{\sigma\in\mathfrak{S}_k}x^{\mathrm{des}(\sigma)+1}2^{\mathrm{LRmin}(\sigma)}+(1-x)^n.\end{aligned}$$

Setting $u_1 = uv$, $u_2 = u_3 = w$ and $u_4 = v$ in (6.4) of Theorem 6.2, and using (3.17) and (3.19), we obtain the following consequence. As we will see, this result can be viewed as the (α, β)-extensions of the peak version of Zhuang's relation (2.10).

Theorem 6.8. *For $n \geq 1$,*

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} u^{\mathrm{M}(\sigma)} v^{\mathrm{des}(\sigma)} w^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{LRmin}(\sigma)-1} \beta^{\mathrm{RLmin}(\sigma)-1} \tag{6.14}$$

$$= \sum_{k=1}^{n} \binom{n}{k} (\alpha y - \beta x + (\beta - \alpha) w)^{n-k} \left(\sum_{\sigma \in \mathfrak{S}_k} x^{\mathrm{des}(\sigma)+1} y^{\mathrm{asc}(\sigma)} (\alpha + \beta)^{\mathrm{LRmin}(\sigma)} \right)$$

$$+ (\alpha y - \beta x + (\beta - \alpha) w)^{n}, \tag{6.15}$$

where x and y are given by (6.10) *and* (6.11) *respectively.*

The following consequence follows from Theorem 6.8 by setting $\alpha = 0$.

Theorem 6.9. *For $n \geq 0$,*

$$\sum_{\sigma \in \mathfrak{S}_n} u^{\mathrm{L}(\sigma)} v^{\mathrm{des}(\sigma)} w^{n-\mathrm{des}(\sigma)} \beta^{\mathrm{RLmin}(\sigma)}$$

$$= \sum_{k=1}^{n} \binom{n}{k} (\beta(w - x))^{n-k} \left(\sum_{\sigma \in \mathfrak{S}_k} x^{\mathrm{des}(\sigma)+1} y^{\mathrm{asc}(\sigma)} \beta^{\mathrm{LRmin}(\sigma)} \right) + (\beta(w - x))^{n}, \tag{6.16}$$

where x and y are given by (6.10) *and* (6.11) *respectively.*

Setting $\beta = 1$, $w = 1$, $a = x/y$ and $b = (y-1)/(1-x)$ in Theorem 6.9, we recover the relation (2.10) due to Zhuang [37, Theorem 4.7].

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