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UNIMODALITY OF k-REGULAR PARTITIONS INTO DISTINCT PARTS WITH BOUNDED LARGEST PART

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ABSTRACT. A k-regular partition into distinct parts is a partition into distinct parts with no part divisible by k. In this paper, we provide a general method to establish the unimodality of k-regular partitions into distinct parts where the largest part is at most $km+k-1$. Let $d_{k,m}(n)$ denote the number of k-regular partitions of n into distinct parts where the largest part is at most $km+k-1$. In line with this method, we show that $d_{4,m}(n) \geq d_{4,m}(n-1)$ for $m \geq 0$, $1 \leq n \leq 3(m+1)^2$ and $n \neq 4$, and $d_{8,m}(n) \geq d_{8,m}(n-1)$ for $m \geq 2$ and $1 \leq n \leq 14(m+1)^2$. When $5 \leq k \leq 10$ and $k \neq 8$, we show that $d_{k,m}(n) \geq d_{k,m}(n-1)$ for $m \geq 0$ and $1 \leq n \leq \left\lfloor \frac{k(k-1)(m+1)^2}{4} \right\rfloor$.

1. Introduction

The main theme of this paper is to investigate the unimodality of k-regular partitions into distinct parts where the largest part is at most $km+k-1$. A k-regular partition into distinct parts is a partition into distinct parts with no part divisible by k. For example, below are the 4-regular partitions of 10 into distinct parts,

$$(10),\ (9,1),\ (7,3),\ (7,2,1),\ (6,3,1),\ (5,3,2).$$

Let $d_{k,m}(n)$ denote the number of k-regular partitions into distinct parts where the largest part is at most $km+k-1$. From the example above, we see that $d_{4,1}(10)=4$ and $d_{4,2}(10)=6$. By definition, it is easy to see that the generating function of $d_{k,m}(n)$ is given by

$$D_{k,m}(q) := \sum_{n=0}^{N(k,m)} d_{k,m}(n)q^n = \prod_{j=0}^{m} \left(1+q^{jk+1}\right)\left(1+q^{jk+2}\right)\cdots\left(1+q^{jk+k-1}\right), \tag{1.1}$$

where

$$N(k,m) = \frac{k(k-1)(m+1)^2}{2}.$$

Recall that a polynomial $a_0+a_1q+\cdots+a_Nq^N$ with integer coefficients is called unimodal if for some $0 \leq j \leq N$,

$$a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_N,$$

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and is called symmetric if for all $0 \leq j \leq N$, $a_j = a_{N-j}$, see [16, p. 124, Ex. 50]. It is well-known that the Gaussian polynomials

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$$

are symmetric and unimodal, as conjectured by Cayley [4] in 1856 and confirmed by Sylvester [18] in 1878 based on semi-invariants of binary forms. For more information, we refer to [3, 9, 11, 13]. Since then, the unimodality of polynomials (or combinatorial sequences) has drawn great attention in recent decades. In particular, the unimodality of several special k-regular partitions have been investigated by several authors. For example, the polynomials

$$(1+q)(1+q^2)\cdots(1+q^m) \tag{1.2}$$

are proved to be symmetric and unimodal for $m \geq 1$. The first proof of the unimodality of the polynomials (1.2) was given by Hughes [8] resorting to Lie algebra results. Stanley [15] provided an alternative proof by using the hard Lefschetz theorem. Stanley [14] also established the general result of this type based on a result of Dynkin [6]. An analytic proof of the unimodality of the polynomials (1.2) was attributed to Odlyzko and Richmond [10] by extending the argument of van Lint [19] and Entringer [7].

Stanley [15] conjectured the polynomials

$$(1+q)(1+q^3)\cdots(1+q^{2m+1}) \tag{1.3}$$

are symmetric and unimodal for $m \geq 26$, except at the coefficients of q^2 and $q^{(m+1)^2-2}$. More precisely, let

$$\sum_{n=0}^{(m+1)^2} d_{2,m}(n)q^n = (1+q)(1+q^3)\cdots(1+q^{2m+1}).$$

Stanley conjectured that $d_{2,m}(n) \geq d_{2,m}(n-1)$ for $m \geq 26$, $1 \leq n \leq \left\lfloor \frac{(m+1)^2}{2} \right\rfloor$ and $n \neq 2$. This conjecture has been proved by Almkvist [1] via refining the method of Odlyzko and Richmond [10]. Pak and Panova [12] showed that the polynomials (1.3) are strict unimodal by interpreting the differences between numbers of certain partitions as Kronecker coefficients of representations of S_n. By refining the method of Odlyzko and Richmond [10], we show that the polynomials

$$\prod_{j=0}^{m}(1+q^{3j+1})(1+q^{3j+2}) \tag{1.4}$$

are symmetric and unimodal for $m \geq 0$, see [5].

In this paper, we aim to establish the symmetry and unimodality of $D_{k,m}(q)$ for $k \geq 4$. It should be noted that the polynomial (1.2) is associated with $D_{1,m}(q)$, while the polynomial (1.3) is associated with $D_{2,m}(q)$. When $k = 3$, $D_{k,m}(q)$ reduces to the polynomial (1.4).

One main result of this paper is to show that $D_{4,m}(q)$ is almost unimodal.

Theorem 1.1. *The polynomials*

$$\prod_{j=0}^{m}(1+q^{4j+1})(1+q^{4j+2})(1+q^{4j+3}) \tag{1.5}$$

are symmetric and unimodal for $m \geq 0$, except at the coefficients of q^4 and $q^{6(m+1)^2-4}$. More precisely, let

$$\sum_{n=0}^{6(m+1)^2} d_{4,m}(n)q^n = \prod_{j=0}^{m}(1+q^{4j+1})(1+q^{4j+2})(1+q^{4j+3}).$$

Then for $m \geq 0$, $d_{4,m}(n) \geq d_{4,m}(n-1)$ for $1 \leq n \leq 3(m+1)^2$and $n \neq 4$.

We also provide an effective way to establish the unimodality of $D_{k,m}(q)$ for $k \geq 5$.

Theorem 1.2. *For $k \geq 5$, if there exists $m_0 \geq 0$ such that $D_{k,m_0}(q)$ is unimodal and for $m_0 < m < 8k^{\frac{3}{2}}$ and $\left\lceil \frac{k(k-1)m^2}{4} \right\rceil \leq n \leq \left\lfloor \frac{k(k-1)(m+1)^2}{4} \right\rfloor$,*

$$d_{k,m}(n) \geq d_{k,m}(n-1), \tag{1.6}$$

then $D_{k,m}(q)$ is unimodal for $m \geq m_0$.

By utilizing Theorem 1.2 and conducting tests with Maple, we obtain the following two consequences.

Corollary 1.3. *When $5 \leq k \leq 10$ and $k \neq 8$, the polynomials*

$$\prod_{j=0}^{m}\left(1+q^{jk+1}\right)\left(1+q^{jk+2}\right)\cdots\left(1+q^{jk+k-1}\right)$$

are symmetric and unimodal for $m \geq 0$.

Corollary 1.4. *The polynomials*

$$\prod_{j=0}^{m}\left(1+q^{8j+1}\right)\left(1+q^{8j+2}\right)\cdots\left(1+q^{8j+7}\right)$$

are symmetric and unimodal for $m \geq 2$.

It should be noted that Zhan and Zhu [20] explored the unimodality of k-regular partitions into distinct parts where the largest is at most $kn+j$, with $0 \leq j \leq k-1$, by extending the methodology presented in this paper.

2. A key lemma

This section is devoted to the proof of the following lemma. It turns out that this lemma figures prominently in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 2.1. *If $k \geq 4$, $m \geq 8k^{\frac{3}{2}}$ and $\frac{k(k-1)m^2}{4} \leq n \leq \frac{k(k-1)(m+1)^2}{4}$, then*

$$d_{k,m}(n) > d_{k,m}(n-1). \tag{2.1}$$

Before demonstrating Lemma 2.1, we collect several identities and inequalities which will be useful in its proof.

$$e^{ix} = \cos(x) + i\sin(x), \tag{2.2}$$

$$\cos(2x) = 2\cos^2(x) - 1, \tag{2.3}$$

$$\sin(2x) = 2\sin(x)\cos(x), \tag{2.4}$$

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta), \tag{2.5}$$

$$\sin(x) \geq xe^{-x^2/3} \quad \text{for } 0 \leq x \leq 2, \tag{2.6}$$

$$\cos(x) \geq e^{-\gamma x^2} \quad \text{for } |x| \leq 1, \ (\gamma = -\ln\cos(1) = 0.615626\ldots), \tag{2.7}$$

$$x - \frac{x^3}{6} \leq \sin(x) \leq x \quad \text{for } x \geq 0, \tag{2.8}$$

$$\cos(x) \leq e^{-x^2/2} \quad \text{for } |x| \leq \frac{\pi}{2}, \tag{2.9}$$

$$|\cos(x)| \leq \exp\left(-\frac{1}{2}\sin^2(x) - \frac{1}{4}\sin^4(x)\right), \tag{2.10}$$

$$\left|\frac{\sin(nx)}{\sin(x)}\right| \leq n \quad \text{for } x \neq i\pi, \ i = 0, 1, 2, \ldots, \tag{2.11}$$

$$\sum_{k=1}^{n} \sin^2(kx) = \frac{n}{2} - \frac{\sin((2n+1)x)}{4\sin(x)} + \frac{1}{4} \quad \text{for } x \neq i\pi, \ i = 0, 1, 2, \ldots, \tag{2.12}$$

$$\sum_{k=1}^{n} \sin^4(kx) = \frac{3n}{8} - \frac{\sin((2n+1)x)}{4\sin(x)} + \frac{\sin((2n+1)2x)}{16\sin(2x)} + \frac{3}{16} \quad \text{for } x \neq \frac{i\pi}{2}, \ i = 0, 1, 2, \ldots. \tag{2.13}$$

The identity (2.2) is Euler's identity, see [17, p. 4]. The formulas (2.3)–(2.5) of trigonometric functions can be found in [2, Chap. 8]. The inequalities (2.6)–(2.11) are due to Odlyzko and Richmond [10, p. 81]. The identities (2.12) and (2.13) have been proved in [5].

We are now in a position to prove Lemma 2.1 by considering $d_{k,m}(n)$ as the Fourier coefficients of $D_{k,m}(q)$ and proceeding to estimate its integral.

Proof of Lemma 2.1: Putting $q = e^{2i\theta}$ in (1.1), we get

$$D_{k,m}(e^{2i\theta}) = \prod_{j=0}^{m}(1+(e^{2i\theta})^{jk+1})(1+(e^{2i\theta})^{jk+2})\cdots(1+(e^{2i\theta})^{jk+k-1})$$

$$\overset{(2.2)}{=} \prod_{j=0}^{m}\prod_{l=1}^{k-1}(1+\cos(2(jk+l)\theta)+i\sin(2(jk+l)\theta))$$

$$\overset{(2.3)\&(2.4)}{=} \prod_{j=0}^{m}\prod_{l=1}^{k-1}\left(2\cos^2((jk+l)\theta)+2i\sin((jk+l)\theta)\cos((jk+l)\theta)\right)$$

$$\overset{(2.2)}{=} \prod_{j=0}^{m}\prod_{l=1}^{k-1}2\cos((jk+l)\theta)\exp(i(jk+l)\theta)$$

$$= 2^{(k-1)(m+1)}\exp(iN(k,m)\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta). \tag{2.14}$$

Using Taylor's theorem [17, pp. 47–49], we derive that

$$\begin{aligned} d_{k,m}(n) &= \frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{D_{k,m}\left(e^{2i\theta}\right)}{\left(e^{2i\theta}\right)^{n+1}}\mathrm{d}\left(e^{2i\theta}\right) \\ &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}D_{k,m}\left(e^{2i\theta}\right)e^{-2in\theta}\mathrm{d}\theta \\ &\overset{(2.14)}{=} \frac{2^{(k-1)(m+1)}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\exp(i(N(k,m)-2n)\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta \\ &\overset{(2.2)}{=} \frac{2^{(k-1)(m+1)}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos((N(k,m)-2n)\theta)+i\sin((N(k,m)-2n)\theta)) \\ &\qquad\qquad \times\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta. \end{aligned}$$

Observe that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin((N(k,m)-2n)\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta = 0,$$

so we conclude that

$$d_{k,m}(n) = \frac{2^{(k-1)(m+1)+1}}{\pi}\int_{0}^{\frac{\pi}{2}}\cos((N(k,m)-2n)\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta.$$

To show that $d_{k,m}(n)$ increases with n, we take the derivative with respect to n,

$$\frac{\partial}{\partial n}d_{k,m}(n) = \frac{2^{(k-1)(m+1)+2}}{\pi}\int_{0}^{\frac{\pi}{2}}\theta\sin((N(k,m)-2n)\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta.$$

Let $N(k,m)-2n=\mu$, and let

$$I_{k,m}(\mu) = \int_{0}^{\frac{\pi}{2}}\theta\sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta.$$

Thus it suffices to show that

$$I_{k,m}(\mu) > 0 \text{ for } k\geq 4,\ m\geq 8k^{\frac{3}{2}} \text{ and } 0<\mu\leq\frac{k(k-1)(2m+1)}{2}. \tag{2.15}$$

We will separate the integral $I_{k,m}(\mu)$ into three parts,

$$I_{k,m}(\mu)=\left\{\int_0^{\frac{2\pi}{k(k-1)(2m+1)}}+\int_{\frac{2\pi}{k(k-1)(2m+1)}}^{\frac{\pi}{2km+2(k-1)}}+\int_{\frac{\pi}{2km+2(k-1)}}^{\frac{\pi}{2}}\right\}\theta\sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta$$

$$=I_{k,m}^{(1)}(\mu)+I_{k,m}^{(2)}(\mu)+I_{k,m}^{(3)}(\mu),$$

and aim to show that when $k\geq 4,\ m\geq 8k^{\frac{3}{2}}$ and $0<\mu\leq\frac{k(k-1)(2m+1)}{2}$,

$$I_{k,m}^{(1)}(\mu)>\left|I_{k,m}^{(2)}(\mu)\right|+\left|I_{k,m}^{(3)}(\mu)\right|, \tag{2.16}$$

from which, it is immediate that (2.15) is valid.

We first estimate the value of $I_{k,m}^{(1)}(\mu)$. Recall that

$$I_{k,m}^{(1)}(\mu)=\int_0^{\frac{2\pi}{k(k-1)(2m+1)}}\theta\sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta. \tag{2.17}$$

When $0\leq\theta\leq\frac{4}{k(k-1)(2m+1)}$, we see that $0\leq\mu\theta\leq 2$ and $0\leq(jk+l)\theta\leq 1$ for $0\leq j\leq m$ and $1\leq l\leq k-1$. Using (2.6) and (2.7), we deduce that

$$\sin(\mu\theta)\geq\mu\theta\exp\left(-\frac{\mu^2\theta^2}{3}\right)\ \text{and}\ \cos((jk+l)\theta)\geq\exp\left(-\gamma(jk+l)^2\theta^2\right).$$

Hence

$$\begin{aligned}
&\theta\sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\\
&\geq\mu\theta^2\exp\left(-\frac{\mu^2\theta^2}{3}\right)\exp\left(-\gamma\theta^2\sum_{j=0}^{m}\sum_{l=1}^{k-1}(jk+l)^2\right)\\
&\geq\mu\theta^2\exp\left(-\frac{k^2(k-1)^2(m+\frac{1}{2})^2\theta^2}{3}\right)\\
&\quad\times\exp\left(-\gamma\theta^2k(k-1)\left(\frac{km^3}{3}+km^2+\frac{(6k-1)m}{6}+\frac{2k-1}{6}\right)\right).
\end{aligned}$$

Put

$$c_k(m)=k^2(k-1)^2\left(\frac{1}{3m}+\frac{1}{3m^2}+\frac{1}{12m^3}\right)+\gamma k(k-1)\left(\frac{k}{3}+\frac{k}{m}+\frac{6k-1}{6m^2}+\frac{2k-1}{6m^3}\right).$$

When $k\geq 4$ and $m\geq 8k^{\frac{3}{2}}$, we find that

$$\begin{aligned}
c_k(m)&\leq c_k\left(8k^{\frac{3}{2}}\right)\\
&=k^{\frac{1}{2}}(k-1)^2\left(\frac{1}{3\cdot 8}+\frac{1}{3\cdot 8^2k^{\frac{3}{2}}}+\frac{1}{12\cdot 8^3k^3}\right)\\
&\quad+\gamma k^2(k-1)\left(\frac{1}{3}+\frac{1}{8k^{\frac{3}{2}}}+\frac{6-k^{-1}}{6\cdot 8^2k^3}+\frac{2-k^{-1}}{6\cdot 8^3k^{\frac{9}{2}}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq k^3\left(\frac{1}{24}+\frac{1}{192k^{\frac{3}{2}}}+\frac{1}{6144k^3}+\gamma\left(\frac{1}{3}+\frac{1}{8k^{\frac{3}{2}}}+\frac{1}{64r^3}+\frac{1}{1536k^{\frac{9}{2}}}\right)\right)\\
&\leq k^3\left(\frac{1}{24}+\frac{1}{192\cdot 4^{\frac{3}{2}}}+\frac{1}{6144\cdot 4^3}\right.\\
&\quad\left.+0.616\cdot\left(\frac{1}{3}+\frac{1}{8\cdot 4^{\frac{3}{2}}}+\frac{1}{64\cdot 4^3}+\frac{1}{1536\cdot 4^{\frac{9}{2}}}\right)\right) \quad (\text{by } k\geq 4)\\
&< 0.26k^3 := c_k,
\end{aligned}
$$

and so

$$
\theta\sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\geq \mu\theta^2\exp\left(-c_k m^3\theta^2\right). \tag{2.18}
$$

Applying (2.18) to (2.17), we deduce that when $k\geq 4$, $m\geq 8k^{\frac{3}{2}}$ and $0<\mu\leq\frac{k(k-1)(2m+1)}{2}$,

$$
\begin{aligned}
I_{k,m}^{(1)}(\mu) &= \int_0^{\frac{2\pi}{k(k-1)(2m+1)}} \theta\sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta\\
&\geq \int_0^{\frac{4}{k(k-1)(2m+1)}} \theta\sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta\\
&\geq \int_0^{\frac{4}{k(k-1)(2m+1)}} \mu\theta^2\exp\left(-c_k m^3\theta^2\right)\mathrm{d}\theta\\
&= \left\{\int_0^{\infty}-\int_{\frac{4}{k(k-1)(2m+1)}}^{\infty}\right\}\mu\theta^2\exp\left(-c_k m^3\theta^2\right)\mathrm{d}\theta\\
&= \frac{\mu}{2c_k^{\frac{3}{2}}m^{\frac{9}{2}}}\left(\int_0^{\infty}v^{\frac{1}{2}}e^{-v}\mathrm{d}v-\int_{\frac{16c_k m^3}{k^2(k-1)^2(2m+1)^2}}^{\infty}v^{\frac{1}{2}}e^{-v}\mathrm{d}v\right)\\
&= \frac{\mu}{2c_k^{\frac{3}{2}}m^{\frac{9}{2}}}\left(\frac{\sqrt{\pi}}{2}-\int_{\frac{16c_k m^3}{k^2(k-1)^2(2m+1)^2}}^{\infty}v^{\frac{1}{2}}e^{-v}\mathrm{d}v\right).
\end{aligned}
$$

When $m\geq 8k^{\frac{3}{2}}$, we see that

$$
\begin{aligned}
\frac{16c_k m^3}{k^2(k-1)^2(2m+1)^2} &\geq \frac{16\cdot 0.26k^3\cdot 8^3k^{\frac{9}{2}}}{k^2(k-1)^2(2\cdot 8k^{\frac{3}{2}}+1)^2}\\
&\geq \frac{16\cdot 0.26k^3\cdot 8^3k^{\frac{9}{2}}}{k^2k^2(17k^{\frac{3}{2}})^2}\\
&= \frac{2129.92\sqrt{k}}{289}
\end{aligned}
$$

$$\geq \frac{2129\sqrt{4}}{289} > 14.7 \quad (\text{by } k \geq 4), \tag{2.19}$$

so

$$\int_{\frac{16c_k m^3}{k^2(k-1)^2(2m+1)^2}}^{\infty} v^{\frac{1}{2}} e^{-v} \mathrm{d}v < \int_{14.7}^{\infty} v^{\frac{1}{2}} e^{-v} \mathrm{d}v < 1.64 \times 10^{-6}.$$

As a result, we can assert that when $k \geq 4$, $m \geq 8k^{\frac{3}{2}}$ and $0 < \mu \leq \frac{k(k-1)(2m+1)}{2}$,

$$I_{k,m}^{(1)}(\mu) > \frac{\mu}{2c_k^{\frac{3}{2}} m^{\frac{9}{2}}} \left(\frac{\sqrt{\pi}}{2} - 1.64 \times 10^{-6} \right) > \frac{3.34\mu}{k^{\frac{9}{2}} m^{\frac{9}{2}}}. \tag{2.20}$$

We now turn to estimate the value of $I_{k,m}^{(2)}(\mu)$ given by

$$I_{k,m}^{(2)}(\mu) = \int_{\frac{2\pi}{k(k-1)(2m+1)}}^{\frac{\pi}{2km+2(k-1)}} \theta \sin(\mu\theta) \prod_{j=0}^{m} \prod_{l=1}^{k-1} \cos((jk+l)\theta) \mathrm{d}\theta. \tag{2.21}$$

When $\frac{2\pi}{k(k-1)(2m+1)} \leq \theta \leq \frac{\pi}{2km+2(k-1)}$, we have $0 \leq (jk+l)\theta \leq \frac{\pi}{2}$ for $0 \leq j \leq m$ and $1 \leq l \leq k-1$. In light of (2.9), we derive that

$$\cos((jk+l)\theta) \leq \exp\left(-\frac{(jk+l)^2\theta^2}{2} \right).$$

Hence

$$\begin{aligned}
&\left| \prod_{j=0}^{m} \prod_{l=1}^{k-1} \cos((jk+l)\theta) \right| \\
&\leq \exp\left(-\frac{1}{2}\theta^2 \sum_{j=0}^{m} \sum_{l=1}^{k-1} (jk+l)^2 \right) \\
&= \exp\left(-\frac{1}{2} k(k-1)\theta^2 \left(\frac{km^3}{3} + km^2 + \frac{(6k-1)m}{6} + \frac{2k-1}{6} \right) \right) \\
&\leq \exp\left(-\frac{\pi^2}{2k(k-1)\left(m+\frac{1}{2}\right)^2} \left(\frac{km^3}{3} + km^2 + \frac{(6k-1)m}{6} + \frac{2k-1}{6} \right) \right) \\
&\qquad \left(\text{by } \frac{2\pi}{k(k-1)(2m+1)} \leq \theta \leq \frac{\pi}{2km+2(k-1)} \right) \\
&= \exp\left(-\frac{\pi^2}{2k(k-1)} \cdot \frac{km}{3} \cdot \frac{m^2+3m+\frac{6k-1}{2k}+\frac{2k-1}{2km}}{m^2+m+\frac{1}{4}} \right) \\
&\leq \exp\left(-\frac{\pi^2}{2k(k-1)} \cdot \frac{km}{3} \right) = \exp\left(-\frac{\pi^2 m}{6(k-1)} \right) < \exp\left(-\frac{\pi^2 m}{6k} \right).
\end{aligned} \tag{2.22}$$

Applying (2.22) to (2.21), and in view of (2.8) and (2.20), we derive that when $k \geq 4$, $m \geq 8k^{\frac{3}{2}}$ and $0 < \mu \leq \frac{k(k-1)(2m+1)}{2}$,

$$|I_{k,m}^{(2)}(\mu)| \overset{(2.8)}{\leq} \mu \exp\left(-\frac{\pi^2 m}{6k} \right) \int_{\frac{2\pi}{k(k-1)(2m+1)}}^{\frac{\pi}{2km+2(k-1)}} \theta^2 \mathrm{d}\theta$$

$$\le \frac{\mu\pi^3}{3}\left(\frac{1}{(2km+2(k-1))^3}-\frac{8}{(k(k-1)(2m+1))^3}\right)\exp\left(-\frac{\pi^2 m}{6k}\right)$$

$$\le \frac{\mu\pi^3}{3(2km+2(k-1))^3}\exp\left(-\frac{\pi^2 m}{6k}\right)$$

$$\le \frac{\mu\pi^3}{3(8m)^3}\exp\left(-\frac{\pi^2 m}{6k}\right) \quad (\text{by } k\ge 4)$$

$$\stackrel{(2.20)}{\le} \frac{\pi^3 k^{\frac{9}{2}} m^{\frac{3}{2}}}{5130}\exp\left(-\frac{\pi^2 m}{6k}\right) I^{(1)}_{k,m}(\mu). \tag{2.23}$$

Define

$$f_k(m) := \frac{\pi^3 k^{\frac{9}{2}} m^{\frac{3}{2}}}{5130}\exp\left(-\frac{\pi^2 m}{6k}\right).$$

We claim that $f_k'(m)<0$ for $k\ge 4$ and $m\ge 8k^{\frac{3}{2}}$. Since $f_k(m)>0$ for $k\ge 4$ and $m\ge 8k^{\frac{3}{2}}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}m} f_k(m) = \frac{\mathrm{d}}{\mathrm{d}m} e^{\ln f_k(m)} = f_k(m)\frac{\mathrm{d}}{\mathrm{d}m}\ln f_k(m). \tag{2.24}$$

Observe that when $k\ge 4$ and $m\ge 8k^{\frac{3}{2}}$,

$$\frac{\mathrm{d}}{\mathrm{d}m}\ln f_k(m) = \frac{3}{2m}-\frac{\pi^2}{6k} \le \frac{3}{2\cdot 8k^{\frac{3}{2}}}-\frac{\pi^2}{6k} = \frac{\pi^2}{6k}\left(\frac{9}{8\pi^2 k^{\frac{1}{2}}}-1\right)<0,$$

and this yields that $f_k'(m)<0$ for $k\ge 4$ and $m\ge 8k^{\frac{3}{2}}$ as claimed. Consequently,

$$f_k(m)\le f_k(8k^{\frac{3}{2}}) = \frac{8^{\frac{3}{2}}\pi^3}{5130} k^{\frac{27}{4}}\exp\left(-\frac{4\pi^2 k^{\frac{1}{2}}}{3}\right). \tag{2.25}$$

Applying (2.25) to (2.23), we obtain

$$|I^{(2)}_{k,m}(\mu)| \le \frac{8^{\frac{3}{2}}\pi^3}{5130} k^{\frac{27}{4}}\exp\left(-\frac{4\pi^2 k^{\frac{1}{2}}}{3}\right) I^{(1)}_{k,m}(\mu). \tag{2.26}$$

Define

$$h_1(k) := \exp\left(-\frac{4\pi^2 k^{\frac{1}{2}}}{3}\right) k^{\frac{27}{4}}.$$

Since $h_1(k)>0$ for $k\ge 4$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}k} h_1(k) = \frac{\mathrm{d}}{\mathrm{d}k} e^{\ln h_1(k)} = h_1(k)\frac{\mathrm{d}}{\mathrm{d}k}\ln h_1(k), \tag{2.27}$$

and since

$$\frac{\mathrm{d}}{\mathrm{d}k}\ln h_1(k) = \frac{27}{4k}-\frac{2\pi^2}{3k^{\frac{1}{2}}}$$

$$= \frac{1}{k}\left(\frac{27}{4}-\frac{2\pi^2 k^{\frac{1}{2}}}{3}\right)$$

$$\le \frac{1}{k}\left(\frac{27}{4}-\frac{4\pi^2}{3}\right) \quad (\text{by } k\ge 4)$$

$$< -\frac{6}{k} < 0,$$

it follows that $h_1'(k) < 0$ for $k \geq 4$. Hence $h_1(k) \leq h_1(4)$ for $k \geq 4$. Therefore,

$$\begin{aligned}|I^{(2)}_{k,m}(\mu)| &\overset{(2.26)}{\leq} \frac{8^{\frac{3}{2}}\pi^3}{5130}\exp\left(-\frac{8\pi^2}{3}\right)\cdot 4^{\frac{27}{4}} I^{(1)}_{k,m}(\mu)\\ &< 5.89\times 10^{-9} I^{(1)}_{k,m}(\mu). \end{aligned} \tag{2.28}$$

Finally, we turn to estimate the value of $I^{(3)}_{k,m}(\mu)$ defined by

$$I^{(3)}_{k,m}(\mu) = \int_{\frac{\pi}{2km+2(k-1)}}^{\frac{\pi}{2}} \theta \sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta. \tag{2.29}$$

Let $C = \left\{\frac{i\pi}{2k} | i = 1,2,\ldots,k\right\}$, it is easy to see that

$$\int_C \theta \sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta = 0,$$

so

$$I^{(3)}_{k,m}(\mu) = \int_{\left[\frac{\pi}{2km+2(k-1)},\frac{\pi}{2}\right]\setminus C} \theta \sin(\mu\theta)\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\mathrm{d}\theta. \tag{2.30}$$

When $\frac{\pi}{2km+2(k-1)} \leq \theta \leq \frac{\pi}{2}$ and $\theta \neq \frac{i\pi}{2k}$ $(i = 1,2,\ldots,k)$, by (2.10), (2.12) and (2.13), we deduce that

$$\begin{aligned}&\left|\prod_{j=0}^{m}\prod_{l=1}^{k-1}\cos((jk+l)\theta)\right|\\ &\overset{(2.10)}{\leq} \exp\left(-\frac{1}{2}\sum_{j=0}^{m}\sum_{l=1}^{k-1}\sin^2((jk+l)\theta) - \frac{1}{4}\sum_{j=0}^{m}\sum_{l=1}^{k-1}\sin^4((jk+l)\theta)\right)\\ &= \exp\left(-\frac{1}{2}\left(\sum_{j=1}^{km+k-1}\sin^2(j\theta) - \sum_{j=1}^{m}\sin^2(jk\theta)\right)\right.\\ &\quad \left. -\frac{1}{4}\left(\sum_{j=1}^{km+k-1}\sin^4(j\theta) - \sum_{j=1}^{m}\sin^4(jk\theta)\right)\right)\\ &\overset{(2.12)\&(2.13)}{=} \exp\left(-\frac{11(k-1)(m+1)}{32} + \frac{3\sin((2km+2k-1)\theta)}{16\sin(\theta)}\right.\\ &\quad \left. -\frac{\sin((2km+2k-1)2\theta)}{64\sin(2\theta)} - \frac{3\sin((2m+1)k\theta)}{16\sin(k\theta)} + \frac{\sin((2m+1)2k\theta)}{64\sin(2k\theta)}\right)\\ &:= E_{k,m}(\theta). \end{aligned} \tag{2.31}$$

We claim that for $k \geq 4$, $m \geq 8k^{\frac{3}{2}}$ and $\frac{\pi}{2km+2(k-1)} \leq \theta \leq \frac{\pi}{2}$ (where $\theta \neq \frac{i\pi}{2k}$, $i = 1,2,\ldots,k$),

$$E_{k,m}(\theta) < \exp(-0.381m - 0.224). \tag{2.32}$$

We approach the proof of (2.32) through a two-step process. First, we consider the interval $\frac{\pi}{2km+2(k-1)} \le \theta < \frac{\pi}{2k}$. Since $\frac{\pi}{2km+2(k-1)} \le \theta < 2\theta < k\theta < \frac{\pi}{2}$, by (2.8), we get

$$\begin{aligned} \sin(i\theta) &\ge \sin\left(\frac{i\pi}{2km+2(k-1)}\right) \\ &\ge \frac{i\pi}{2km+2(k-1)} - \frac{\left(\frac{i\pi}{2km+2(k-1)}\right)^3}{6} \\ &\ge \frac{i\pi}{2km+2(k-1)}\left(1-\frac{\left(\frac{k\pi}{2km+2(k-1)}\right)^2}{6}\right), \end{aligned} \tag{2.33}$$

where $i = 1, 2, k$. Applying (2.11) and (2.33) in (2.31), we obtain

$$\begin{aligned} E_{k,m}(\theta) &\le \exp\left(-\frac{11(k-1)(m+1)}{32} + \frac{3}{16\sin(\theta)} + \frac{1}{64\sin(2\theta)} + \frac{3}{16\sin(k\theta)} \right. \\ &\qquad \left. + \left|\frac{\sin((2m+1)2k\theta)}{64\sin(2k\theta)}\right|\right) \\ &\overset{(2.33)\&(2.11)}{\le} \exp\left(-\frac{11(k-1)(m+1)}{32} + \frac{2m+1}{64} + \frac{3}{16\left(\frac{\pi}{2km+2(k-1)}\left(1-\frac{\left(\frac{k\pi}{2km+2(k-1)}\right)^2}{6}\right)\right)} \right. \\ &\qquad \left. + \frac{1}{64\left(\frac{2\pi}{2km+2(k-1)}\left(1-\frac{\left(\frac{k\pi}{2km+2(k-1)}\right)^2}{6}\right)\right)} + \frac{3}{16\left(\frac{k\pi}{2km+2(k-1)}\left(1-\frac{\left(\frac{k\pi}{2km+2(k-1)}\right)^2}{6}\right)\right)}\right) \\ &= \exp\left(\frac{(12-11k)m}{32} + \frac{23-22k}{64} + \frac{24+25k}{128k\left(\frac{\pi}{2km+2(k-1)}\left(1-\frac{\left(\frac{k\pi}{2km+2(k-1)}\right)^2}{6}\right)\right)}\right) \\ &= \exp\left(\frac{(12-11k)m}{32} + \frac{23-22k}{64} + \frac{(24+25k)(2km+2(k-1))}{128\pi k\left(1-\frac{\pi^2k^2}{6(2km+2(k-1))^2}\right)}\right). \end{aligned}$$

When $k \ge 4$ and $m \ge 8k^{\frac{3}{2}}$, we have

$$1-\frac{\pi^2k^2}{6(2km+2(k-1))^2} \ge 1-\frac{\pi^2k^2}{6\left(16k^{\frac{5}{2}}+2(k-1)\right)^2} \quad (\text{by } m \ge 8k^{\frac{3}{2}})$$

$$
\begin{aligned}
&= 1-\frac{\pi^2}{6\left(16k^{\frac{3}{2}}+2-2k^{-1}\right)^2}\\
&\geq 1-\frac{\pi^2}{6\left(16\cdot 4^{\frac{3}{2}}+2-\frac{1}{2}\right)^2} \quad (\text{by } k\geq 4)\\
&= 1-\frac{\pi^2}{100621.5} > 0.9999.
\end{aligned}
$$

It follows that for $k\geq 4$, $m\geq 8k^{\frac{3}{2}}$ and $\frac{\pi}{2km+2(k-1)}\leq\theta<\frac{\pi}{2k}$,

$$
\begin{aligned}
E_{k,m}(\theta) &\leq \exp\left(\frac{(12-11k)m}{32}+\frac{23-22k}{64}+\frac{(24+25k)(2km+2(k-1))}{0.9999\cdot 128\pi k}\right)\\
&= \exp\left(\left(\frac{12-11k}{32}+\frac{24+25k}{0.9999\cdot 64\pi}\right)m+\frac{23-22k}{64}+\frac{(24+25k)(1-k^{-1})}{0.9999\cdot 64\pi}\right)\\
&\leq \exp\left((0.495-0.219k)m+0.479-0.219k\right)\\
&\leq \exp\left(-0.381m-0.397\right) \quad (\text{by } k\geq 4). \qquad (2.34)
\end{aligned}
$$

Next we consider the interval $\frac{\pi}{2k}\leq\theta\leq\frac{\pi}{2}$ and $\theta\neq\frac{i\pi}{2k}$ $(i=1,2,\ldots,k)$. Employing (2.8) and (2.11), we deduce that

$$
\begin{aligned}
E_{k,m}(\theta) \leq \exp\Bigg(&-\frac{11(k-1)(m+1)}{32}+\frac{3}{16\sin(\theta)}\\
&+\left|\frac{\sin((2km+2k-1)2\theta)}{64\sin(2\theta)}\right|+\left|\frac{3\sin((2m+1)k\theta)}{16\sin(k\theta)}\right|+\left|\frac{\sin((2m+1)2k\theta)}{64\sin(2k\theta)}\right|\Bigg)\\
\overset{(2.11)}{\leq} \exp\Bigg(&-\frac{11(k-1)(m+1)}{32}+\frac{3}{16\sin\left(\frac{\pi}{2k}\right)}\\
&+\frac{2km+2k-1}{64}+\frac{3(2m+1)}{16}+\frac{2m+1}{64}\Bigg)\\
\overset{(2.8)}{\leq} \exp\Bigg(&-\frac{11(k-1)(m+1)}{32}+\frac{3}{16\left(\frac{\pi}{2k}\left(1-\frac{\left(\frac{\pi}{2k}\right)^2}{6}\right)\right)}\\
&+\frac{2km+2k-1}{64}+\frac{3(2m+1)}{16}+\frac{2m+1}{64}\Bigg)\\
= \exp&\left(\left(\frac{3}{4}-\frac{5k}{16}\right)m-\frac{5k}{16}+\frac{17}{32}+\frac{3k}{8\pi\left(1-\frac{\pi^2}{24k^2}\right)}\right)\\
\leq \exp&\left(\left(\frac{3}{4}-\frac{5k}{16}\right)m-\frac{5k}{16}+\frac{17}{32}+\frac{3k}{0.9742\cdot 8\pi}\right) \quad (\text{by } k\geq 4)
\end{aligned}
$$

$$\le \exp\left(\left(\frac{3}{4}-\frac{5k}{16}\right)m+\frac{17}{32}-0.189k\right)$$

$$\le \exp(-0.5m-0.224) \quad (\text{by } k\ge 4). \tag{2.35}$$

Combining (2.34) and (2.35) yields (2.32), so the claim is verified. Substituting (2.32) to (2.30), and in view of (2.8) and (2.20), we derive that

$$|I_{k,m}^{(3)}(\mu)| \overset{(2.8)}{\le} \mu\exp(-0.381m-0.224)\int_{\frac{\pi}{2km+2(k-1)}}^{\frac{\pi}{2}}\theta^2 d\theta$$

$$\le \frac{\mu\pi^3}{3}\left(\frac{1}{8}-\frac{1}{(2km+2(k-1))^3}\right)\exp(-0.381m-0.224)$$

$$\le \frac{\mu\pi^3}{24}\exp(-0.381m-0.224)$$

$$\overset{(2.20)}{\le} \frac{\pi^3k^{\frac{9}{2}}m^{\frac{9}{2}}}{3.34\cdot 24}\exp(-0.381m-0.224)\,I_{k,m}^{(1)}(\mu). \tag{2.36}$$

Define

$$g_k(m):=\frac{\pi^3k^{\frac{9}{2}}m^{\frac{9}{2}}}{3.34\cdot 24}\exp(-0.381m-0.224).$$

Since when $k\ge 4$ and $m\ge 8k^{\frac{3}{2}}$, we have $g_k(m)>0$ and

$$\frac{d}{dm}g_k(m)=\frac{d}{dm}e^{\ln g_k(m)}$$

$$=g_k(m)\frac{d}{dm}\ln g_k(m)$$

$$=g_k(m)\left(\frac{9}{2m}-0.381\right)$$

$$\le g_k(m)\left(\frac{9}{2\cdot 8\cdot 4^{\frac{3}{2}}}-0.381\right)$$

$$< -0.31g_k(m)<0,$$

it follows that $g_k'(m)<0$ when $k\ge 4$ and $m\ge 8k^{\frac{3}{2}}$, and so for $k\ge 4$ and $m\ge 8k^{\frac{3}{2}}$,

$$g_k(m)\le g_k(8k^{\frac{3}{2}})=\frac{8^{\frac{9}{2}}\pi^3k^{\frac{45}{4}}}{3.34\cdot 24}\exp\left(-3.048k^{\frac{3}{2}}-0.224\right). \tag{2.37}$$

Define

$$h_2(k):=\exp\left(-3.048k^{\frac{3}{2}}-0.224\right)k^{\frac{45}{4}}.$$

When $k\ge 4$, we have $h_2(k)>0$ and

$$\frac{d}{dk}h_2(k)=\frac{d}{dk}e^{\ln h_2(k)}$$

$$=h_2(k)\frac{d}{dk}\ln h_2(k)$$

$$= h_2(k)\left(\frac{45}{4k} - 3.048\cdot\frac{3k^{\frac{1}{2}}}{2}\right)$$

$$\leq h_2(k)\left(\frac{45}{4\cdot 4} - 3.048\cdot\frac{3\sqrt{4}}{2}\right) \quad (\text{by } k\geq 4)$$

$$< -6.3h_2(k) < 0,$$

so $h_2'(k)<0$ for $k\geq 4$, and hence for $k\geq 4$,

$$g_k(m) \leq \frac{8^{\frac{9}{2}}\pi^3}{3.34\cdot 24}\exp\left(-3.048\cdot 4^{\frac{3}{2}} - 0.224\right)\cdot 4^{\frac{45}{4}} < 0.55. \tag{2.38}$$

Substituting (2.38) into (2.36), we have

$$|I^{(3)}_{k,m}(\mu)| < 0.55 I^{(1)}_{k,m}(\mu). \tag{2.39}$$

Combining (2.28) and (2.39) yields (2.16), and so (2.15) is valid. This leads to (2.1) holds for $k\geq 4$, $m\geq 8k^{\frac{3}{2}}$ and $\frac{k(k-1)m^2}{4}\leq n\leq \frac{k(k-1)(m+1)^2}{4}$, and so Lemma 2.1 is verified. ∎

3. Proofs of Theorem 1.1 and Theorem 1.2

This section is devoted to the proofs of Theorem 1.1 and Theorem 1.2. Prior to that, we demonstrate the symmetry of $D_{k,m}(q)$.

Theorem 3.1. *For $k\geq 0$, the polynomials $D_{k,m}(q)$ are symmetric.*

Proof. Replacing q by q^{-1} in (1.1), we find that

$$\begin{aligned} D_{k,m}(q^{-1}) &= \prod_{j=0}^{m}\left(1+q^{-(jk+1)}\right)\left(1+q^{-(jk+2)}\right)\cdots\left(1+q^{-(jk+k-1)}\right) \\ &= q^{-N(k,m)}\prod_{j=0}^{m}\left(1+q^{jk+1}\right)\left(1+q^{jk+2}\right)\cdots\left(1+q^{jk+k-1}\right) \\ &= q^{-N(k,m)}D_{k,m}(q). \end{aligned}$$

To wit,

$$D_{k,m}(q) = q^{N(k,m)}D_{k,m}(q^{-1}),$$

from which, it follows that $D_{k,m}(q)$ is symmetric. This completes the proof. ∎

We give an inductive proof of Theorem 1.1 with the aid of Lemma 2.1.

Proof of Theorem 1.1: From Theorem 3.1, we see that $D_{4,m}(q)$ is symmetric. Hence in order to prove Theorem 1.1, it suffices to show that

$$d_{4,m}(n)\geq d_{4,m}(n-1) \quad \text{for } m\geq 0,\ 1\leq n\leq 3(m+1)^2 \text{ and } n\neq 4. \tag{3.1}$$

Recall that $d_{4,m}(n)$ counts the number of 4-regular partitions into distinct parts where the largest part is at most $4m+3$, it is easy to check that for $m \geq 0$,

$$d_{4,m}(0) = d_{4,m}(1) = d_{4,m}(2) = 1,\ d_{4,m}(3) = 2,\ d_{4,m}(4) = 1. \tag{3.2}$$

Here we assume that $d_{4,m}(n) = 0$ when $n < 0$. It can be checked that (3.1) holds when $0 \leq m \leq 63$. In the following, we will demonstrate its validity for the case when $m \geq 64$. However, our main objective is to show that when $m \geq 64$,

$$d_{4,m}(n) \geq d_{4,m}(n-1), \quad \text{for } 5 \leq n \leq 12m+20 \tag{3.3}$$

and

$$d_{4,m}(n) \geq d_{4,m}(n-1)+1, \quad \text{for } 12m+21 \leq n \leq 3(m+1)^2, \tag{3.4}$$

which immediately led to (3.1). It can be checked that (3.3) and (3.4) are valid when $m = 64$. It remains to show that (3.3) and (3.4) hold when $m > 64$. We proceed by induction on m. Assume that (3.3) and (3.4) are valid for $m-1$, namely

$$d_{4,m-1}(n) \geq d_{4,m-1}(n-1), \quad \text{for } 5 \leq n \leq 12m+8 \tag{3.5}$$

and

$$d_{4,m-1}(n) \geq d_{4,m-1}(n-1)+1, \quad \text{for } 12m+9 \leq n \leq 3m^2. \tag{3.6}$$

We aim to show that (3.3) and (3.4) hold.

Comparing coefficients of q^n in

$$D_{4,m}(q) = \left(1+q^{4m+1}\right)\left(1+q^{4m+2}\right)\left(1+q^{4m+3}\right)D_{4,m-1}(q),$$

we obtain the following recurrence relation:

$$\begin{aligned} d_{4,m}(n) = {} & d_{4,m-1}(n) + d_{4,m-1}(n-4m-1) + d_{4,m-1}(n-4m-2) \\ & + d_{4,m-1}(n-4m-3) + d_{4,m-1}(n-8m-3) + d_{4,m-1}(n-8m-4) \\ & + d_{4,m-1}(n-8m-5) + d_{4,m-1}(n-12m-6), \end{aligned} \tag{3.7}$$

thereby leading to

$$\begin{aligned} d_{4,m}(n) - d_{4,m}(n-1) = {} & d_{4,m-1}(n) - d_{4,m-1}(n-1) \\ & + d_{4,m-1}(n-4m-1) - d_{4,m-1}(n-4m-4) \\ & + d_{4,m-1}(n-8m-3) - d_{4,m-1}(n-8m-6) \\ & + d_{4,m-1}(n-12m-6) - d_{4,m-1}(n-12m-7). \end{aligned} \tag{3.8}$$

When $5 \leq n \leq 12m+20$ and $n \neq 12m+10$, applying (3.5) and (3.6) to (3.8), we see that

$$d_{4,m}(n) - d_{4,m}(n-1) \geq 0.$$

When $n=12m+10$, we observe that

$$d_{4,m-1}(n-12m-6)-d_{4,m-1}(n-12m-7)=d_{4,m-1}(4)-d_{4,m-1}(3)=-1.$$

But by (3.6), we have

$$d_{4,m-1}(n)-d_{4,m-1}(n-1)=d_{4,m-1}(12m+10)-d_{4,m-1}(12m+9)\geq 1,$$

which leads to $d_{4,m}(n)-d_{4,m}(n-1)\geq 0$ when $n=12m+10$. To sum up, we get

$$d_{4,m}(n)-d_{4,m}(n-1)\geq 0, \quad \text{for } 5\leq n\leq 12m+20,$$

and so (3.3) is valid. Applying (3.5) and (3.6) to (3.8) again, we infer that

$$d_{4,m}(n)-d_{4,m}(n-1)\geq 1, \quad \text{for } 12m+21\leq n\leq 3m^2. \tag{3.9}$$

In view of Lemma 2.1, we see that

$$d_{4,m}(n)-d_{4,m}(n-1)\geq 1, \quad \text{for } 3m^2<n\leq 3(m+1)^2. \tag{3.10}$$

Combining (3.9) and (3.10), we confirm that (3.4) holds. Together with (3.3), we deduce (3.1) holds, and so $D_{4,m}(q)$ is unimodal, except at the coefficients of q^4 and $q^{N(4,m)-4}$. This completes the proof of Theorem 1.1. ∎

We conclude this paper with the proof of Theorem 1.2 by the utilization of Lemma 2.1.

Proof of Theorem 1.2: Given $k\geq 5$ and $m_0\geq 0$, assume that $D_{k,m_0}(q)$ is unimodal. We proceed to show that the polynomial $D_{k,m}(q)$ is unimodal for $m\geq m_0$ by induction on m. Considering the symmetry of $D_{k,m}(q)$, it suffices to show that for $m>m_0$ and $1\leq n\leq \left\lfloor \frac{k(k-1)(m+1)^2}{4}\right\rfloor$,

$$d_{k,m}(n)\geq d_{k,m}(n-1). \tag{3.11}$$

Assume that (3.11) is valid for $m-1$, that is, for $m>m_0$ and $1\leq n\leq \left\lfloor \frac{k(k-1)m^2}{4}\right\rfloor$,

$$d_{k,m-1}(n)\geq d_{k,m-1}(n-1). \tag{3.12}$$

We intend to show that (3.11) holds for $m>m_0$ and $1\leq n\leq \left\lfloor \frac{k(k-1)(m+1)^2}{4}\right\rfloor$. By comparing the coefficients of q^n in the polynomial

$$D_{k,m}(q)=\left(1+q^{km+1}\right)\left(1+q^{km+2}\right)\cdots\left(1+q^{km+k-1}\right)D_{k,m-1}(q),$$

it can be determined that

$$d_{k,m}(n)=\sum_{\substack{i_j=0 \text{ or } km+j\\ 1\leq j\leq k-1}} d_{k,m-1}(n-i_1-\cdots-i_{k-1}),$$

which leads to

$$\begin{aligned} &d_{k,m}(n)-d_{k,m}(n-1)\\ &=\sum_{\substack{i_j=0 \text{ or } km+j\\ 1\leq j\leq k-1}}\left(d_{k,m-1}(n-i_1-\cdots-i_{k-1})-d_{k,m-1}(n-i_1-\cdots-i_{k-1}-1)\right). \end{aligned} \tag{3.13}$$

Utilizing (3.12) in (3.13) yields that the validity of (3.11) for $m > m_0$ and $1 \le n \le \left\lfloor \frac{k(k-1)m^2}{4} \right\rfloor$. In view of Lemma 2.1, we see that (3.11) holds for $m \ge 8k^{\frac{3}{2}}$ and $\left\lceil \frac{k(k-1)m^2}{4} \right\rceil \le n \le \left\lfloor \frac{k(k-1)(m+1)^2}{4} \right\rfloor$. Given the condition that (3.11) holds for $m_0 < m < 8k^{\frac{3}{2}}$ and $\left\lceil \frac{k(k-1)m^2}{4} \right\rceil \le n \le \left\lfloor \frac{k(k-1)(m+1)^2}{4} \right\rfloor$, we reach the conclusion that (3.11) is valid for $m > m_0$ and $1 \le n \le \left\lfloor \frac{k(k-1)(m+1)^2}{4} \right\rfloor$. Therefore, $D_{k,m}(q)$ is unimodal for $m \ge m_0$. Thus, we complete the proof of Theorem 1.2. ∎

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