Convexity and log-concavity of the partition function weighted by the parity of the crank

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Abstract. Let $M_0(n)$ (resp. $M_1(n)$) denote the number of partitions of n with even (resp. odd) crank. Choi, Kang and Lovejoy established an asymptotic formula for $M_0(n) - M_1(n)$. By utilizing this formula with the explicit bound, we show that $M_k(n-1) + M_k(n+1) > 2M_k(n)$ for $k = 0$ or 1 and $n \geq 39$. This result can be seen as the refinement of the classical result regarding the convexity of the partition function $p(n)$, which counts the number of partitions of n. We also show that $M_0(n)$ (resp. $M_1(n)$) is log-concave for $n \geq 94$ and satisfies the higher order Turán inequalities for $n \geq 207$ with the aid of the upper bound and the lower bound for $M_0(n)$ and $M_1(n)$.

Keywords: partition, crank, equidistribution, convexity, log-concavity, the higher order Turán inequalities

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1 Introduction

This paper carries out a study of the partitions with even (resp. odd) crank from an analytic view. The crank of a partition was defined by Andrews and Garvan [2] as the largest part if the partition contains no ones, and otherwise as the number of parts larger than the number of ones minus the number of ones. Let $p(n)$ denote the number of partitions of n. It is known that the crank provides combinatorial explanations for Ramanujan's famous congruences $p(5n+4) \equiv 0 \pmod 5$, $p(7n+5) \equiv 0 \pmod 7$ and $p(11n+6) \equiv 0 \pmod{11}$, see Andrews and Garvan [2], Dyson [10] and Garvan [11]. More precisely, Let $M(r, Q; n)$ be the number of partitions of n with crank congruent to r modulo Q. Andrews and Garvan [2], building on the work of Garvan [11], established the following results:

$$M(r,5;5n+4)=\frac{p(5n+4)}{5}\quad \text{for each}\quad 0\le r\le 4,$$

$$M(r,7;7n+5)=\frac{p(7n+5)}{7}\quad \text{for each}\quad 0\le r\le 6,$$

$$M(r,11;11n+6)=\frac{p(11n+6)}{11}\quad \text{for each}\quad 0\le r\le 10.$$

Recently, Hamakiotes, Kriegman and Tsai [15] demonstrated that $M(r,Q;n)$ is asymptotically equidistributed modulo odd number Q. More precisely, let $0\le r<Q$ with Q an odd integer, they established the following asymptotic result:

$$\lim_{n\to\infty}\frac{M(r,Q;n)}{p(n)}=\frac{1}{Q}.$$

Their proof relies on the asymptotic formula for $M(r,Q;n)$ derived by Zapata Rolón [23].

The study of partitions with even and odd cranks was initially undertaken by Andrews and Lewis [3]. For convenience, let $M_0(n)$ and $M_1(n)$ denote the number of partitions of n with even and odd cranks, respectively. Andrews and Lewis [3] showed that

Theorem 1.1 (Andrews-Lewis)**.** *For $n\ge 0$,*

$$(-1)^n(M_0(n)-M_1(n))>0.$$

Following Andrews and Lewis' footsteps, Choi, Kang and Lovejoy [8] conducted a comprehensive study of $M_0(n)$ and $M_1(n)$. Among their main results, they derived a family of Ramanujan type congruences satisfied by $M_0(n)-M_1(n)$ and obtained the following asymptotic formula for $M_0(n)-M_1(n)$.

Theorem 1.2 (Choi-Kang-Lovejoy)**.**

$$M_0(n)-M_1(n)=E(n)+\frac{\sqrt{6}\pi}{3\mu(n)}\sum_{0<j<\frac{\sqrt{3}\mu(n)}{2\sqrt{\pi}}}\cosh\left(\frac{\mu(n)}{2j}\right)\frac{\hat{A}_j(n)}{\sqrt{j}},\tag{1.1}$$

where

$$|E(n)|\le\frac{95\cdot 6^{1/4}}{\sqrt{2\pi}}\mu(n)^{\frac{1}{2}}\tag{1.2}$$

and

$$\hat{A}_j(n)=\sum_{\substack{0\le h<2j\\ \gcd(h,2j)=1}}\exp\left(-\frac{\pi inh}{j}-\pi i\left(3s(h,2j)-2s\left(h,j\right)\right)\right),\tag{1.3}$$

where $s(h,j)$ is the Dedekind sum defined by

$$s(h,j)=\begin{cases}\displaystyle\sum_{r=1}^{j-1}\left(\frac{r}{j}-\left[\frac{r}{j}\right]-\frac{1}{2}\right)\left(\frac{hr}{j}-\left[\frac{hr}{j}\right]-\frac{1}{2}\right) & \text{for}\quad j\ge 2,\\ 0 & \text{for}\quad j=1.\end{cases}$$

Throughout this paper, we adopt the following notation as used in [18]

$$\mu(n)=\frac{\pi\sqrt{24n-1}}{6}. \tag{1.4}$$

The main objective of this paper is to employ the asymptotic formula (1.1) for $M_0(n)-M_1(n)$ to investigate the convexity and log-concavity of $M_0(n)$ and $M_1(n)$. We first show that $M_0(n)$ and $M_1(n)$ are convex when $n\geq 39$.

Theorem 1.3. *For $k=0$ or 1 and $n\geq 39$,*

$$M_k(n-1)+M_k(n+1)>2M_k(n).$$

It should be noted that Theorem 1.3 can be viewed as the refinements of the classical result involving the convexity of $p(n)$, that is, $p(n-1)+p(n+1)\geq 2p(n)$ for $n\geq 2$, see Gupta [14] or Honsberger [16, pp. 237–239].

Furthermore, we establish the log-concavity of $M_0(n)$ and $M_1(n)$.

Theorem 1.4. *For $k=0$ or 1 and $n\geq 94$,*

$$M_k(n)^2\geq M_k(n-1)M_k(n+1).$$

To wit, the sequences $\{M_0(n)\}_{n\geq 94}$ and $\{M_1(n)\}_{n\geq 93}$ are log-concave or satisfy the Turán inequalities.

To prove Theorem 1.4 , we establish the upper bound and the lower bound of $M_k(n-1)M_k(n+1)/M_k(n)^2$.

Theorem 1.5. *Let*

$$Y_k(n):=\frac{M_k(n-1)M_k(n+1)}{M_k(n)^2}.$$

For $k=0$ or 1 and $\mu(n)\geq 115$,

$$Y_k(n)<1-\frac{\pi^4}{9\mu(n)^3}+\frac{4\pi^4}{9\mu(n)^4}-\frac{\pi^4}{3\mu(n)^5}+\frac{\frac{\pi^8}{81}+5}{\mu(n)^6} \tag{1.5}$$

and

$$Y_k(n)>1-\frac{\pi^4}{9\mu(n)^3}+\frac{4\pi^4}{9\mu(n)^4}-\frac{\pi^4}{3\mu(n)^5}-\frac{60}{\mu(n)^6}. \tag{1.6}$$

These bounds further allow us to show that $M_0(n)$ and $M_1(n)$ satisfy the higher order Turán inequalities.

Theorem 1.6. *For $k=0$ or 1 and $n\geq 207$,*

$$\begin{aligned}&4(M_k(n)^2-M_k(n-1)M_k(n+1))(M_k(n+1)^2-M_k(n)M_k(n+2))\\&\quad\geq(M_k(n)M_k(n+1)-M_k(n-1)M_k(n+2))^2.\end{aligned}$$

That is, the sequences $\{M_0(n)\}_{n\geq 207}$ and $\{M_1(n)\}_{n\geq 206}$ satisfy the higher order Turán inequalities.

It should be noted that the Turán inequalities and the higher order Turán inequalities for the partition function $p(n)$ and its variations have been extensively investigated recently, see, for example, Bringmann, Kane, Rolen and Tripp [5], Chen [6], Chen, Jia and Wang [7], DeSalvo and Pak [9], Griffin, Ono, Rolen and Zagier [13] and Ono, Pujahari and Rolen [22]. In particular, Griffin, Ono, Rolen and Zagier [13] showed that $p(n)$ satisfies the order d Turán inequalities for sufficiently large n, confirming a conjectured of Chen, Jia and Wang [7]. For the definition of the order d Turán inequalities, please see Chen, Jia and Wang [7] or Griffin, Ono, Rolen and Zagier [13].

To conclude the introduction, let us say a few words about the order d Turán inequalities for $M_k(n)$. In fact, based on (2.10) and in view of Theorem 3 and Corollary 4 in [13], we could derive that

Theorem 1.7. *When $k = 0$ or 1 and $d \geq 1$, $M_k(n)$ satisfies the order d Turán inequalities for sufficiently large n.*

It would be interesting to establish the minimal number $O_{M_k}(d)$ such that $M_k(n)$ satisfies the order d Turán inequalities for $n \geq O_{M_k}(d)$. By Theorem 1.4, we see that

$$O_{M_0}(2) = 94 \quad \text{and} \quad O_{M_1}(2) = 93.$$

From Theorem 1.6, we find that

$$O_{M_0}(3) = 207 \quad \text{and} \quad O_{M_1}(3) = 206.$$

The paper is organized as follows. In Section 2, we first establish the error bound for the asymptotic formula for $M_0(n) - M_1(n)$ due to Choi, Kang and Lovejoy (that is, Theorem 2.1). We then give two additional application of Theorem 2.1, see Theorem 2.2 and Theorem 2.3. In Section 3, we employ Theorem 2.1 to establish the upper bound and the lower bound for $M_0(n)$ and $M_1(n)$ (that is, Theorem 3.1), which is useful in the study of the convexity and the log-concavity of $M_0(n)$ and $M_1(n)$. Section 4 focuses on exploring the convexity of $M_0(n)$ and $M_1(n)$ by utilizing Theorem 3.1. Section 5 is devoted to employing Theorem 3.1 to establish the upper bound and the lower bound of $M_k(n-1)M_k(n+1)/M_k(n)^2$ (that is, Theorem 1.5). These bounds play a crucial role in the proofs that $M_0(n)$ and $M_1(n)$ are log-concave for $n \geq 94$ and satisfy the higher order Turán inequalities for $n \geq 207$, which are established in Section 6.

2 The error bound

In this section, we state the error bound for the asymptotic formula of $M_0(n) - M_1(n)$ due to Choi, Kang and Lovejoy, which is required in our study of the convexity and log-concavity of $M_0(n)$ and $M_1(n)$. We also provide two additional applications of this error bound. First application is to provide a direct analytic proof of Andrews and Lewis' result (that is, Theorem 1.1) and the second one is to demonstrate that the cranks are asymptotically equidistributed modulo 2.

Using Theorem 1.2, we obtain the following asymptotic formula for $M_0(n) - M_1(n)$ with an effective bound on the error term.

Theorem 2.1. *For $\mu(n) \geq 4$, or equivalently, $n \geq 3$,*

$$M_0(n) - M_1(n) = \frac{(-1)^n \pi}{\sqrt{6}\mu(n)} e^{\frac{\mu(n)}{2}} + E_\beta(n),$$

where

$$|E_\beta(n)| \leq 63\mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}}. \tag{2.1}$$

Proof. Observing that $\hat{A}_1(n) = (-1)^n$, and using the formula of $\cosh(s)$ from [1, p. 459]

$$\cosh(s) = \frac{e^s + e^{-s}}{2},$$

we deduce from (1.1) that for $\mu(n) \geq 4$,

$$M_0(n) - M_1(n) = \frac{(-1)^n \pi}{\sqrt{6}\mu(n)} e^{\frac{\mu(n)}{2}} + E_\beta(n),$$

where

$$E_\beta(n) = E(n) + \frac{(-1)^n \pi}{\sqrt{6}\mu(n)} e^{-\frac{\mu(n)}{2}} + \frac{\sqrt{6}\pi}{3\mu(n)} \sum_{2 \leq j < \frac{\sqrt{3}\mu(n)}{2\sqrt{\pi}}} \cosh\left(\frac{\mu(n)}{2j}\right) \frac{\hat{A}_j(n)}{\sqrt{j}}. \tag{2.2}$$

We next establish the bound for $|E_\beta(n)|$. By the definition (1.3) of $\hat{A}_j(n)$, we derive that for any $n \geq 0$ and $j \geq 1$,

$$|\hat{A}_j(n)| \leq 2j.$$

Hence, we have

$$\begin{aligned}
\frac{\sqrt{6}\pi}{3\mu(n)} \sum_{2 \leq j < \frac{\sqrt{3}\mu(n)}{2\sqrt{\pi}}} \cosh\left(\frac{\mu(n)}{2j}\right) \frac{\hat{A}_j(n)}{\sqrt{j}} &\leq \frac{2\sqrt{6}\pi}{3\mu(n)} \sum_{2 \leq j \leq \frac{\mu(n)}{2}} \sqrt{j} \cosh\left(\frac{\mu(n)}{2j}\right) \\
&\leq \frac{2\pi}{\sqrt{3}\mu(n)^{\frac{1}{2}}} \sum_{2 \leq j \leq \frac{\mu(n)}{2}} \cosh\left(\frac{\mu(n)}{2j}\right) \\
&\leq \frac{\pi\mu(n)^{\frac{1}{2}}}{\sqrt{3}} \cosh\left(\frac{\mu(n)}{4}\right) \\
&\leq \frac{\pi\mu(n)^{\frac{1}{2}}}{2\sqrt{3}} \left(e^{\frac{\mu(n)}{4}} + e^{-\frac{\mu(n)}{4}}\right).
\end{aligned} \tag{2.3}$$

Applying (1.2) and (2.3) to (2.2), we derive that for $\mu(n) \geq 4$,

$$\begin{aligned}
|E_\beta(n)| &\leq \frac{\pi}{\sqrt{6}\mu(n)} e^{-\frac{\mu(n)}{2}} + \frac{\pi \mu(n)^{\frac{1}{2}}}{2\sqrt{3}} \left(e^{\frac{\mu(n)}{4}} + e^{-\frac{\mu(n)}{4}} \right) + \frac{95 \cdot 6^{1/4}}{\sqrt{2\pi}} \mu(n)^{\frac{1}{2}} \\
&\leq \frac{\pi}{2\sqrt{3}} \mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}} + \left(\frac{\pi}{4\sqrt{6}} + \frac{\pi}{2\sqrt{3}} + 60 \right) \mu(n)^{\frac{1}{2}} \\
&\leq \left(\frac{\pi}{4\sqrt{6}} + \frac{\pi}{\sqrt{3}} + 60 \right) \mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}} \\
&\leq 63 \mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}}.
\end{aligned}$$

This completes the proof. ∎

Using Theorem 2.1, we give a direct analytic proof of Theorem 1.1. In particular, we obtain the following more general result.

Theorem 2.2. *Given a positive integer* $d \geq 1$. *For* $n \geq \left\lceil \frac{24}{\pi^2} \left(\ln\left(\frac{7d}{2} \right) \right)^2 + \frac{1}{24} \right\rceil$,

$$(-1)^n (M_0(n) - M_1(n)) > d. \tag{2.4}$$

Proof. Using Theorem 2.1, we find that for $n \geq 3$,

$$(-1)^n (M_0(n) - M_1(n)) = \frac{\pi}{\sqrt{6}\mu(n)} e^{\frac{\mu(n)}{2}} + (-1)^n E_\beta(n), \tag{2.5}$$

where

$$|E_\beta(n)| \leq 63 \mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}}.$$

We first show that for $n \geq 1$,

$$(-1)^n (M_0(n) - M_1(n)) > \frac{1}{7} \mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}}. \tag{2.6}$$

By (2.5), it suffices to show that for $n \geq 1$,

$$\frac{\pi}{\sqrt{6}\mu(n)} e^{\frac{\mu(n)}{2}} - 63 \mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}} > \frac{1}{7} \mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}}. \tag{2.7}$$

We claim that for $\mu(n) \geq 38$, or equivalently, $n \geq 220$,

$$\frac{7\pi}{\sqrt{6}\mu(n)} e^{\frac{\mu(n)}{4}} > 442 \mu(n)^{\frac{1}{2}}. \tag{2.8}$$

Define

$$f(s) = \frac{7\pi}{442\sqrt{6} s^{\frac{3}{2}}} e^{\frac{s}{4}}.$$

It is easy to derive that for $s \geq 6$,

$$f'(s) = \frac{7\pi (s-6) e^{\frac{s}{4}}}{1768\sqrt{6} s^{\frac{5}{2}}} \geq 0.$$

Hence for $\mu(n) \geq 38$,

$$f(\mu(n)) = \frac{7\pi}{442\sqrt{6}\mu(n)^{\frac{3}{2}}} e^{\frac{\mu(n)}{4}} \geq f(38) > 1,$$

and so (2.8) is proved. Using (2.8), we derive that (2.7) is valid for $n \geq 220$. It follows that (2.6) is valid for $n \geq 220$. It can be checked that (2.6) is valid for $1 \leq n \leq 220$. Hence (2.6) is proved. On the other hand, when $d \geq 1$ and for $\mu(n) \geq 4 \ln\left(\frac{7d}{2}\right)$, or equivalently, $n \geq \left\lceil \frac{24}{\pi^2} \left(\ln\left(\frac{7d}{2}\right)\right)^2 + \frac{1}{24} \right\rceil$, we see that

$$\frac{1}{7}\mu(n)^{\frac{1}{2}} e^{\frac{\mu(n)}{4}} \geq \frac{1}{7}\sqrt{4\ln\left(\frac{7d}{2}\right)} e^{\frac{4\ln\left(\frac{7d}{2}\right)}{4}} > \frac{2}{7} \cdot \frac{7d}{2} = d. \tag{2.9}$$

Combining (2.6) and (2.9), we obtain (2.4). This completes the proof of Theorem 2.2. ∎

We conclude this section by showing that the cranks are asymptotically equidistributed modulo 2.

Theorem 2.3. *For $k = 0$ or 1,*

$$\lim_{n\to\infty} \frac{M_k(n)}{p(n)} = \frac{1}{2}. \tag{2.10}$$

It should be noted that asymptotically equidistribution of partition ranks modulo 2 has been recently proved by Gomez and Zhu [12] and Males [20] and Masri [21].

It turns out the proof of Theorem 2.3 also requires the following lower bound for $p(n)$ given by Bessenrodt and Ono [4]: For $n \geq 1$,

$$p(n) > \frac{\sqrt{3}}{12n}\left(1 - \frac{1}{\sqrt{n}}\right) e^{\mu(n)}. \tag{2.11}$$

We aim to prove the following consequence, which leads to the immediate proof of Theorem 2.3.

Theorem 2.4. *For $k = 0$ or 1 and for $n \geq 4$,*

$$\frac{M_k(n)}{p(n)} = \frac{1}{2} + (-1)^k E^c(n),$$

where

$$|E^c(n)| \leq 11578 e^{-\frac{\mu(n)}{4}}. \tag{2.12}$$

Proof. By definition, we see that

$$p(n) = M_0(n) + M_1(n). \tag{2.13}$$

It follows that for $k = 0$ or 1,

$$\frac{M_k(n)}{p(n)} = \frac{1}{2} + (-1)^k \frac{M_0(n) - M_1(n)}{2p(n)}.$$

Assume that

$$E^c(n) := \frac{M_0(n) - M_1(n)}{2p(n)}.$$

In light of Theorem 2.1 and (2.11), we derive that for $n \geq 4$,

$$\begin{aligned}
|E^c(n)| &\leq \frac{12\mu(n)^2}{\sqrt{3}} e^{-\mu(n)} \left(\frac{\pi}{\sqrt{6}\mu(n)} e^{\frac{\mu(n)}{2}} + |E_\beta(n)| \right) \\
&\leq 2\sqrt{2}\pi\mu(n) e^{-\frac{\mu(n)}{2}} + \frac{756}{\sqrt{3}} \mu(n)^{\frac{5}{2}} e^{-\frac{3\mu(n)}{4}} \\
&\leq \left(2\sqrt{2}\pi + \frac{756}{\sqrt{3}} \right) \mu(n)^{\frac{5}{2}} e^{-\frac{\mu(n)}{2}} \\
&\leq 446\mu(n)^{\frac{5}{2}} e^{-\frac{\mu(n)}{2}}.
\end{aligned}$$

We claim that for $\mu(n) > 0$,

$$446\mu(n)^{\frac{5}{2}} e^{-\frac{\mu(n)}{4}} < 11578. \tag{2.14}$$

Define

$$m(s) := 446 s^{\frac{5}{2}} e^{-\frac{s}{4}}.$$

It is evident that

$$m'(s) = \frac{223}{2} s^{\frac{3}{2}} (10 - s) e^{-\frac{s}{4}}.$$

Since $m'(s) < 0$ for $s > 10$ and $m'(s) > 0$ for $0 < s < 10$, we derive that $m(s)$ attains its maximum value at $s = 10$, so

$$m(\mu(n)) \leq m(10) < 11578,$$

and hence (2.14) holds. We therefore obtain (2.12). This completes the proof. ∎

3 An upper bound and a lower bound for $M_k(n)$

In this section, we aim to establish the following upper bound and the lower bound for $M_0(n)$ and $M_1(n)$ (Theorem 3.1). It turns out that the proof of Theorem 3.1 also requires the following effective bound on $p(n)$ due to Locus Dawsey and Masri [19, Lemma 4.2]. For $n \geq 1$,

$$p(n) = \frac{\pi^2}{6\sqrt{3}\mu(n)^2} \left(1 - \frac{1}{\mu(n)}\right) e^{\mu(n)} + E_p(n), \tag{3.1}$$

where

$$|E_p(n)| \leq 1313 e^{\frac{\mu(n)}{2}}. \tag{3.2}$$

Theorem 3.1. *Let*

$$G(n) := \frac{\pi^2}{12\sqrt{3}\mu(n)^2}\left(1-\frac{1}{\mu(n)}\right)e^{\mu(n)}.$$

Then for $k=0,1$ and $\mu(n)\geq 88$,

$$G(n)\left(1-\frac{1}{\mu(n)^6}\right) \leq M_k(n) \leq G(n)\left(1+\frac{1}{\mu(n)^6}\right). \tag{3.3}$$

Proof. We first show that when $k=0$ or 1 and for $n\geq 3$,

$$M_k(n) = \frac{\pi^2}{12\sqrt{3}\mu(n)^2}\left(1-\frac{1}{\mu(n)}\right)e^{\mu(n)} + R_k^c(n), \tag{3.4}$$

where

$$|R_k^c(n)| \leq 689e^{\frac{\mu(n)}{2}}.$$

Applying (2.13) to (3.1) and using Theorem 2.1, we could derive that for $k=0$ or 1 and $\mu(n)\geq 4$,

$$M_k(n) = \frac{\pi^2}{12\sqrt{3}\mu(n)^2}\left(1-\frac{1}{\mu(n)}\right)e^{\mu(n)} + R_k^c(n),$$

where

$$R_k^c(n) = \frac{(-1)^{n+k}\pi}{2\sqrt{6}\mu(n)}e^{\frac{\mu(n)}{2}} + \frac{1}{2}\left(E_p(n)+(-1)^k E_\beta(n)\right).$$

In light of (2.1) and (3.2), we see that for $k=0$ or 1 and $\mu(n)\geq 4$,

$$\begin{aligned} |R_k^c(n)| &\leq \frac{\pi}{2\sqrt{6}\mu(n)}e^{\frac{\mu(n)}{2}} + \frac{1313}{2}e^{\frac{\mu(n)}{2}} + \frac{63}{2}\mu(n)^{\frac{1}{2}}e^{\frac{\mu(n)}{4}} \\ &\leq \left(\frac{\pi}{8\sqrt{6}} + \frac{1313}{2} + \frac{63}{2}\right)e^{\frac{\mu(n)}{2}} \\ &\leq 689e^{\frac{\mu(n)}{2}}. \end{aligned}$$

This completes the proof of (3.4).

Define

$$T(n) := \frac{689e^{\frac{\mu(n)}{2}}}{\frac{\pi^2}{12\sqrt{3}\mu(n)^2}\left(1-\frac{1}{\mu(n)}\right)e^{\mu(n)}} = \frac{8268\sqrt{3}\mu(n)^2}{\pi^2\left(1-\frac{1}{\mu(n)}\right)}e^{-\frac{\mu(n)}{2}}.$$

Using (3.4), we find that for $n\geq 3$,

$$G(n)\left(1-T(n)\right) \leq M_k(n) \leq G(n)\left(1+T(n)\right).$$

To show (3.3), it is enough to prove that for $\mu(n)\geq 88$,

$$T(n) \leq \frac{1}{\mu(n)^6}. \tag{3.5}$$

Note that for $\mu(n) \geq 2$,

$$\left(1 - \frac{1}{\mu(n)}\right)\left(1 + \frac{2}{\mu(n)}\right) = 1 + \mu(n)^{-2}(\mu(n) - 2) \geq 1.$$

Hence for $n \geq 3$,

$$T(n) \leq \frac{8268\sqrt{3}}{\pi^2}\left(\mu(n)^2 + 2\mu(n)\right) e^{-\frac{\mu(n)}{2}}. \tag{3.6}$$

We claim that for $\mu(n) \geq 88$,

$$\frac{8268\sqrt{3}}{\pi^2} e^{-\frac{\mu(n)}{2}} \leq \frac{1}{2\mu(n)^8}. \tag{3.7}$$

Define

$$L(s) := \frac{16536\sqrt{3}}{\pi^2} s^8 e^{-\frac{s}{2}}.$$

It is evident that

$$L'(s) = \frac{8268\sqrt{3}}{\pi^2} e^{-\frac{s}{2}}(16 - s)s^7.$$

Since $L'(s) \leq 0$ for $s \geq 16$, we deduce that $L(s)$ is decreasing when $s \geq 16$, this implies that

$$L(\mu(n)) = \frac{16536\sqrt{3}}{\pi^2}\mu(n)^8 e^{-\frac{\mu(n)}{2}} \leq L(88) < 1$$

for $\mu(n) \geq 88$, and so the claim is proved. Applying (3.7) to (3.6), we obtain (3.5). This completes the proof. ∎

4 The convexity of $M_0(n)$ and $M_1(n)$

The main objective of this section is to establish the convexity of $M_0(n)$ and $M_1(n)$ (Theorem 1.3). We begin by proving the following two inequalities.

Lemma 4.1. *For $\mu(n) \geq 6$,*

$$\frac{G(n-1)}{G(n)} > \left(1 + \frac{2\pi^2}{3\mu(n)^2}\right)\left(1 - \frac{\pi^2}{\mu(n)^3}\right)\left(1 - \frac{\pi^2}{3\mu(n)} + \frac{\pi^4}{18\mu(n)^2} - \frac{18\pi^4 + \pi^6}{162\mu(n)^3}\right) \tag{4.1}$$

and

$$\frac{G(n+1)}{G(n)} > \left(1 - \frac{2\pi^2}{3\mu(n)^2}\right)\left(1 + \frac{\pi^2}{3\mu(n)^3}\right)\left(1 + \frac{\pi^2}{3\mu(n)} + \frac{\pi^4}{18\mu(n)^2} - \frac{\pi^4}{9\mu(n)^3}\right). \tag{4.2}$$

Proof. Recall that

$$G(n) = \frac{\pi^2}{12\sqrt{3}\mu(n)^2}\left(1 - \frac{1}{\mu(n)}\right)e^{\mu(n)}.$$

We have

$$\frac{G(n-1)}{G(n)} = \frac{\mu(n)^2}{\mu(n-1)^2} \cdot \frac{1 - \frac{1}{\mu(n-1)}}{1 - \frac{1}{\mu(n)}} \cdot \exp\left(\mu(n-1) - \mu(n)\right) \tag{4.3}$$

and

$$\frac{G(n+1)}{G(n)} = \frac{\mu(n)^2}{\mu(n+1)^2} \cdot \frac{1 - \frac{1}{\mu(n+1)}}{1 - \frac{1}{\mu(n)}} \cdot \exp\left(\mu(n+1) - \mu(n)\right). \tag{4.4}$$

From the definition (1.4) of $\mu(n)$, we see that for $\mu(n) \geq 3$,

$$\mu(n-1)^2 = \mu(n)^2 - \frac{2\pi^2}{3} \quad \text{and} \quad \mu(n+1)^2 = \mu(n)^2 + \frac{2\pi^2}{3}. \tag{4.5}$$

It is evident that

$$\mu(n)^2 - \left(\mu(n)^2 - \frac{2\pi^2}{3}\right)\left(1 + \frac{2\pi^2}{3\mu(n)^2}\right) = \frac{4\pi^4}{9\mu(n)^2} > 0$$

and

$$\mu(n)^2 - \left(\mu(n)^2 + \frac{2\pi^2}{3}\right)\left(1 - \frac{2\pi^2}{3\mu(n)^2}\right) = \frac{4\pi^4}{9\mu(n)^2} > 0.$$

Hence, we derive that for $\mu(n) \geq 3$,

$$\frac{\mu(n)^2}{\mu(n-1)^2} > 1 + \frac{2\pi^2}{3\mu(n)^2} \tag{4.6}$$

and

$$\frac{\mu(n)^2}{\mu(n+1)^2} > 1 - \frac{2\pi^2}{3\mu(n)^2}. \tag{4.7}$$

We proceed to show that for $\mu(n) \geq 6$,

$$\frac{1 - \frac{1}{\mu(n-1)}}{1 - \frac{1}{\mu(n)}} > 1 - \frac{\pi^2}{\mu(n)^3} \tag{4.8}$$

and

$$\frac{1 - \frac{1}{\mu(n+1)}}{1 - \frac{1}{\mu(n)}} > 1 + \frac{\pi^2}{3\mu(n)^3}. \tag{4.9}$$

By (4.5), we see that for $\mu(n) \geq 3$,

$$\mu(n-1) = \sqrt{\mu(n)^2 - \frac{2\pi^2}{3}} \quad \text{and} \quad \mu(n+1) = \sqrt{\mu(n)^2 + \frac{2\pi^2}{3}}.$$

Then we have

$$\begin{aligned}\mu(n-1) &= \mu(n) - \frac{\pi^2}{3\mu(n)} - \frac{\pi^4}{18\mu(n)^3} - \frac{\pi^6}{54\mu(n)^5} - \frac{5\pi^8}{648\mu(n)^7} + o\left(\frac{1}{\mu(n)^8}\right),\\ \mu(n+1) &= \mu(n) + \frac{\pi^2}{3\mu(n)} - \frac{\pi^4}{18\mu(n)^3} + \frac{\pi^6}{54\mu(n)^5} - \frac{5\pi^8}{648\mu(n)^7} + o\left(\frac{1}{\mu(n)^8}\right).\end{aligned} \tag{4.10}$$

It can be checked that for $\mu(n) \geq 6$,

$$\mu(n-1) > w(n) \quad \text{and} \quad \mu(n+1) > y(n), \tag{4.11}$$

where

$$w(n) = \mu(n) - \frac{\pi^2}{3\mu(n)} - \frac{\pi^4}{9\mu(n)^3}$$

and

$$y(n) = \mu(n) + \frac{\pi^2}{3\mu(n)} - \frac{\pi^4}{18\mu(n)^3}.$$

Hence, we obtain that for $\mu(n) \geq 6$,

$$\frac{1-\frac{1}{\mu(n-1)}}{1-\frac{1}{\mu(n)}} > \frac{1-\frac{1}{w(n)}}{1-\frac{1}{\mu(n)}}$$

and

$$\frac{1-\frac{1}{\mu(n+1)}}{1-\frac{1}{\mu(n)}} > \frac{1-\frac{1}{y(n)}}{1-\frac{1}{\mu(n)}}.$$

To prove (4.8) and (4.9), it is enough to show that for $\mu(n) \geq 6$,

$$\frac{1-\frac{1}{w(n)}}{1-\frac{1}{\mu(n)}} - \left(1 - \frac{\pi^2}{\mu(n)^3}\right) > 0 \tag{4.12}$$

and

$$\frac{1-\frac{1}{y(n)}}{1-\frac{1}{\mu(n)}} - \left(1 + \frac{\pi^2}{3\mu(n)^3}\right) > 0. \tag{4.13}$$

Observe that

$$\frac{1-\frac{1}{w(n)}}{1-\frac{1}{\mu(n)}} - \left(1 - \frac{\pi^2}{\mu(n)^3}\right) = \frac{\phi_1(\mu(n))}{\mu(n)^6(\mu(n)-1)w(n)},$$

where

$$\phi_1(s) = \frac{2\pi^2}{3}s^5 - \pi^2 s^4 - \frac{4\pi^4}{9}s^3 + \frac{\pi^4}{3}s^2 - \frac{\pi^6}{9}s + \frac{\pi^6}{9}.$$

It can be checked that for $s \geq 4$,

$$\begin{cases} \frac{2\pi^2}{3}s^5 - \pi^2 s^4 - \frac{4\pi^4}{9}s^3 > 0, \\ \frac{\pi^4}{3}s^2 - \frac{\pi^6}{9}s + \frac{\pi^6}{9} > 0, \end{cases}$$

which implies that $\phi_1(s) > 0$ for $s \geq 4$, and so $\phi_1(\mu(n)) > 0$ for $\mu(n) \geq 4$. Hence (4.12) holds.

Similarly, we can write

$$\frac{1 - \frac{1}{y(n)}}{1 - \frac{1}{\mu(n)}} - \left(1 + \frac{\pi^2}{3\mu(n)^3}\right) = \frac{\phi_2(\mu(n))}{\mu(n)^6(\mu(n)-1)y(n)},$$

where

$$\phi_2(s) = \frac{\pi^2}{3}s^4 - \frac{\pi^4}{6}s^3 + \frac{\pi^4}{9}s^2 + \frac{\pi^6}{54}s - \frac{\pi^6}{54}.$$

It can be checked that for $s \geq 5$,

$$\frac{\pi^2}{3}s^4 - \frac{\pi^4}{6}s^3 - \frac{\pi^6}{54} > 0,$$

which implies that $\phi_2(s) > 0$ for $s \geq 5$. It follows that $\phi_2(\mu(n)) > 0$ for $\mu(n) \geq 5$. Hence (4.13) is valid.

We proceed to estimate $\exp(\mu(n-1) - \mu(n))$ and $\exp(\mu(n+1) - \mu(n))$. We claim that for $\mu(n) \geq 6$,

$$\exp(\mu(n-1) - \mu(n)) > 1 - \frac{\pi^2}{3\mu(n)} + \frac{\pi^4}{18\mu(n)^2} - \frac{18\pi^4 + \pi^6}{162\mu(n)^3} \tag{4.14}$$

and

$$\exp(\mu(n+1) - \mu(n)) > 1 + \frac{\pi^2}{3\mu(n)} + \frac{\pi^4}{18\mu(n)^2} - \frac{\pi^4}{9\mu(n)^3}. \tag{4.15}$$

Using (4.11), we derive that for $\mu(n) \geq 6$,

$$\exp(\mu(n-1) - \mu(n)) > \exp\left(-\frac{\pi^2}{3\mu(n)} - \frac{\pi^4}{9\mu(n)^3}\right) \tag{4.16}$$

and

$$\exp(\mu(n+1) - \mu(n)) > \exp\left(\frac{\pi^2}{3\mu(n)} - \frac{\pi^4}{18\mu(n)^3}\right). \tag{4.17}$$

Observe that for $s < 0$,

$$e^s > 1 + s + \frac{s^2}{2} + \frac{s^3}{6} \tag{4.18}$$

and for $s > 0$,

$$e^s > 1 + s + \frac{s^2}{2}. \tag{4.19}$$

Hence, by (4.18), we derive that for $\mu(n) \geq 4$,

$$\begin{aligned}
\exp\left(-\frac{\pi^2}{3\mu(n)} - \frac{\pi^4}{9\mu(n)^3}\right) &> 1 - \frac{\pi^2}{3\mu(n)} + \frac{\pi^4}{18\mu(n)^2} - \frac{18\pi^4 + \pi^6}{162\mu(n)^3} + \frac{\pi^6}{27\mu(n)^4} \\
&\quad - \frac{\pi^8}{162\mu(n)^5} + \frac{\pi^8}{162\mu(n)^6} - \frac{\pi^{10}}{486\mu(n)^7} - \frac{\pi^{12}}{4374\mu(n)^9} \\
&> 1 - \frac{\pi^2}{3\mu(n)} + \frac{\pi^4}{18\mu(n)^2} - \frac{18\pi^4 + \pi^6}{162\mu(n)^3}. \qquad (4.20)
\end{aligned}$$

The second inequality follows from the following observation: For $\mu(n) \geq 4$,

$$\begin{cases}
\dfrac{\pi^6}{27\mu(n)^4} - \dfrac{\pi^8}{162\mu(n)^5} > 0, \\
\dfrac{\pi^8}{162\mu(n)^6} - \dfrac{\pi^{10}}{486\mu(n)^7} - \dfrac{\pi^{12}}{4374\mu(n)^9} > 0.
\end{cases}$$

Applying (4.20) to (4.16), we obtain (4.14).

Using (4.19), we derive that for $\mu(n) \geq 4$,

$$\begin{aligned}
\exp\left(\frac{\pi^2}{3\mu(n)} - \frac{\pi^4}{18\mu(n)^3}\right) &> 1 + \frac{\pi^2}{3\mu(n)} + \frac{\pi^4}{18\mu(n)^2} - \frac{\pi^4}{18\mu(n)^3} - \frac{\pi^6}{54\mu(n)^4} + \frac{\pi^8}{648\mu(n)^6} \\
&> 1 + \frac{\pi^2}{3\mu(n)} + \frac{\pi^4}{18\mu(n)^2} - \frac{\pi^4}{9\mu(n)^3}. \qquad (4.21)
\end{aligned}$$

The second inequality is derived from the following observation, that is, for $\mu(n) \geq 4$,

$$\frac{\pi^4}{18\mu(n)^3} - \frac{\pi^6}{54\mu(n)^4} > 0.$$

Combining (4.17) and (4.21) yields (4.15).

Applying (4.6), (4.8) and (4.14) to (4.3), we obtain (4.1). Substituting (4.7), (4.9) and (4.15) into (4.4), we obtain (4.2). Thus we complete the proof of Lemma 4.1. ∎

We are now in a position to give a proof of Theorem 1.3.

Proof of Theorem 1.3. We aim to prove that

$$\frac{M_k(n-1) + M_k(n+1)}{M_k(n)} > 2. \qquad (4.22)$$

In light of Theorem 3.1, we deduce that for $\mu(n) \geq 88$,

$$\begin{aligned}
\frac{M_k(n-1) + M_k(n+1)}{M_k(n)} &\geq \frac{G(n-1)\left(1 - \frac{1}{\mu(n-1)^4}\right) + G(n+1)\left(1 - \frac{1}{\mu(n+1)^4}\right)}{G(n)\left(1 + \frac{1}{\mu(n)^4}\right)} \\
&= \frac{G(n-1)}{G(n)} \cdot \frac{1 - \frac{1}{\mu(n-1)^4}}{1 + \frac{1}{\mu(n)^4}} + \frac{G(n+1)}{G(n)} \cdot \frac{1 - \frac{1}{\mu(n+1)^4}}{1 + \frac{1}{\mu(n)^4}}. \qquad (4.23)
\end{aligned}$$

We proceed to show that for $\mu(n) \geq 5$,

$$\frac{1 - \frac{1}{\mu(n-1)^4}}{1 + \frac{1}{\mu(n)^4}} > 1 - \frac{6}{\mu(n)^4} \tag{4.24}$$

and

$$\frac{1 - \frac{1}{\mu(n+1)^4}}{1 + \frac{1}{\mu(n)^4}} > 1 - \frac{3}{\mu(n)^4}. \tag{4.25}$$

Using (4.5), we derive that for $\mu(n) \geq 3$,

$$\frac{1 - \frac{1}{\mu(n-1)^4}}{1 + \frac{1}{\mu(n)^4}} = \frac{\mu(n)^4\left(\mu(n-1)^4 - 1\right)}{\left(\mu(n)^4 + 1\right)\mu(n-1)^4} = \frac{\mu(n)^4\left(\left(\mu(n)^2 - \frac{2\pi^2}{3}\right)^2 - 1\right)}{\left(\mu(n)^4 + 1\right)\left(\mu(n)^2 - \frac{2\pi^2}{3}\right)^2}. \tag{4.26}$$

It is easy to show that

$$\begin{aligned}\mu(n)^4\left(\left(\mu(n)^2 - \frac{2\pi^2}{3}\right)^2 - 1\right) &= \mu(n)^8 - \frac{4\pi^2}{3}\mu(n)^6 + \left(\frac{4\pi^4}{9} - 1\right)\mu(n)^4 \\ &> \mu(n)^8 - \frac{4\pi^2}{3}\mu(n)^6 + 42\mu(n)^4 \end{aligned}\tag{4.27}$$

and for $\mu(n) \geq 2$,

$$\begin{aligned}&\left(\mu(n)^4 + 1\right)\left(\mu(n)^2 - \frac{2\pi^2}{3}\right)^2 \\ &= \mu(n)^8 - \frac{4\pi^2}{3}\mu(n)^6 + \left(\frac{4\pi^4}{9} + 1\right)\mu(n)^4 - \frac{4\pi^2}{3}\mu(n)^2 + \frac{4\pi^4}{9} \\ &< \mu(n)^8 - \frac{4\pi^2}{3}\mu(n)^6 + \left(\frac{4\pi^4}{9} + 1\right)\mu(n)^4 \\ &< \mu(n)^8 - \frac{4\pi^2}{3}\mu(n)^6 + 45\mu(n)^4. \end{aligned}\tag{4.28}$$

Applying (4.27) and (4.28) to (4.26), we derive that for $\mu(n) \geq 3$,

$$\begin{aligned}\frac{1 - \frac{1}{\mu(n-1)^4}}{1 + \frac{1}{\mu(n)^4}} &> \frac{\mu(n)^8 - \frac{4\pi^2}{3}\mu(n)^6 + 42\mu(n)^4}{\mu(n)^8 - \frac{4\pi^2}{3}\mu(n)^6 + 45\mu(n)^4} \\ &= 1 - \frac{3}{\mu(n)^4 - \frac{4\pi^2}{3}\mu(n)^2 + 45}. \end{aligned}\tag{4.29}$$

It can be checked that for $\mu(n) \geq 5$,

$$\mu(n)^4 - \frac{4\pi^2}{3}\mu(n)^2 + 45 > \frac{1}{2}\mu(n)^4,$$

so we derive from (4.29) that (4.24) holds for $\mu(n) \geq 5$. Similarly,

$$\frac{1-\frac{1}{\mu(n+1)^4}}{1+\frac{1}{\mu(n)^4}} = \frac{\mu(n)^4\left(\mu(n+1)^4-1\right)}{\left(\mu(n)^4+1\right)\mu(n+1)^4} = \frac{\mu(n)^4\left(\left(\mu(n)^2+\frac{2\pi^2}{3}\right)^2-1\right)}{\left(\mu(n)^4+1\right)\left(\mu(n)^2+\frac{2\pi^2}{3}\right)^2}. \tag{4.30}$$

It is evident that

$$\begin{aligned}\mu(n)^4\left(\left(\mu(n)^2+\frac{2\pi^2}{3}\right)^2-1\right) &= \mu(n)^8+\frac{4\pi^2}{3}\mu(n)^6+\left(\frac{4\pi^4}{9}-1\right)\mu(n)^4\\ &> \mu(n)^8+\frac{4\pi^2}{3}\mu(n)^6+42\mu(n)^4 \end{aligned}\tag{4.31}$$

and for $\mu(n) \geq 5$,

$$\begin{aligned}&\left(\mu(n)^4+1\right)\left(\mu(n)^2+\frac{2\pi^2}{3}\right)^2\\ &= \mu(n)^8+\frac{4\pi^2}{3}\mu(n)^6+\left(\frac{4\pi^4}{9}+1\right)\mu(n)^4+\frac{4\pi^2}{3}\mu(n)^2+\frac{4\pi^4}{9}\\ &< \mu(n)^8+\frac{4\pi^2}{3}\mu(n)^6+45\mu(n)^4. \end{aligned}\tag{4.32}$$

Substituting (4.31) and (4.32) to (4.30), we derive that for $\mu(n) \geq 5$,

$$\begin{aligned}\frac{1-\frac{1}{\mu(n-1)^4}}{1+\frac{1}{\mu(n)^4}} &> \frac{\mu(n)^8+\frac{4\pi^2}{3}\mu(n)^6+42\mu(n)^4}{\mu(n)^8+\frac{4\pi^2}{3}\mu(n)^6+45\mu(n)^4}\\ &= 1-\frac{3}{\mu(n)^4+\frac{4\pi^2}{3}\mu(n)^2+45}\\ &> 1-\frac{3}{\mu(n)^4}.\end{aligned}$$

Hence (4.25) holds for $\mu(n) \geq 5$. Applying (4.1), (4.2), (4.24) and (4.25) to (4.23), we derive that for $\mu(n) \geq 88$,

$$\begin{aligned}&\frac{M_k(n-1)+M_k(n+1)}{M_k(n)}\\ &> \left(1+\frac{2\pi^2}{3\mu(n)^2}\right)\left(1-\frac{\pi^2}{\mu(n)^3}\right)\left(1-\frac{\pi^2}{3\mu(n)}+\frac{\pi^4}{18\mu(n)^2}-\frac{18\pi^4+\pi^6}{162\mu(n)^3}\right)\left(1-\frac{6}{\mu(n)^4}\right)\\ &\quad+\left(1-\frac{2\pi^2}{3\mu(n)^2}\right)\left(1+\frac{\pi^2}{3\mu(n)^3}\right)\left(1+\frac{\pi^2}{3\mu(n)}+\frac{\pi^4}{18\mu(n)^2}-\frac{\pi^4}{9\mu(n)^3}\right)\left(1-\frac{3}{\mu(n)^4}\right)\\ &> 2+\frac{\pi^4}{9\mu(n)^2}-\frac{78}{\mu(n)^3}+\frac{34}{\mu(n)^4}-\frac{152}{\mu(n)^5}+\frac{203}{\mu(n)^6}-\frac{92}{\mu(n)^7}+\frac{988}{\mu(n)^8}+\frac{1169}{\mu(n)^9}\\ &\quad-\frac{1954}{\mu(n)^{10}}+\frac{2459}{\mu(n)^{11}}-\frac{7233}{\mu(n)^{12}}.\end{aligned}\tag{4.33}$$

It can be checked that for $\mu(n) \geq 8$,

$$\begin{cases} \dfrac{\pi^4}{9\mu(n)^2} - \dfrac{78}{\mu(n)^3} > 0, \\ \dfrac{34}{\mu(n)^4} - \dfrac{152}{\mu(n)^5} > 0, \\ \dfrac{203}{\mu(n)^6} - \dfrac{92}{\mu(n)^7} > 0, \\ \dfrac{1169}{\mu(n)^9} - \dfrac{1954}{\mu(n)^{10}} > 0, \\ \dfrac{2459}{\mu(n)^{11}} - \dfrac{7233}{\mu(n)^{12}} > 0. \end{cases}$$

Hence, it follows from (4.33) that (4.22) holds for $\mu(n) \geq 88$ (or equivalently, $n \geq 1180$). It can be checked that (4.22) also holds for $39 \leq n \leq 1180$ if $k = 0$ and for $38 \leq n \leq 1180$ if $k = 1$. This completes the proof of Theorem 1.3. ∎

5 Proof of Theorem 1.5

This section is devoted to establishing the upper bound and the lower bound of $M_k(n-1)M_k(n+1)/M_k(n)^2$.

Proof of Theorem 1.5. Recall that

$$Y_k(n) := \frac{M_k(n-1)M_k(n+1)}{M_k(n)^2}$$

and

$$G(n) = \frac{\pi^2}{12\sqrt{3}\mu(n)^2}\left(1 - \frac{1}{\mu(n)}\right)e^{\mu(n)}.$$

Using Theorem 3.1, we see that for $\mu(n) \geq 88$,

$$X(n)L_Y(n) \leq Y_k(n) \leq X(n)R_Y(n), \tag{5.1}$$

where

$$X(n) = \frac{G(n-1)G(n+1)}{G(n)^2},$$

$$L_Y(n) = \frac{\left(1 - \frac{1}{\mu(n-1)^6}\right)\left(1 - \frac{1}{\mu(n+1)^6}\right)}{\left(1 + \frac{1}{\mu(n)^6}\right)^2}$$

and

$$R_Y(n) = \frac{\left(1 + \frac{1}{\mu(n-1)^6}\right)\left(1 + \frac{1}{\mu(n+1)^6}\right)}{\left(1 - \frac{1}{\mu(n)^6}\right)^2}.$$

To prove Theorem 1.5, we proceed to estimate $X(n)$, $L_Y(n)$ and $R_Y(n)$ in terms of $\mu(n)$. We first consider $X(n)$ given by

$$X(n)=\frac{\mu(n)^4}{\mu(n-1)^2\mu(n+1)^2}\frac{\left(1-\frac{1}{\mu(n-1)}\right)\left(1-\frac{1}{\mu(n+1)}\right)}{\left(1-\frac{1}{\mu(n)}\right)^2}e^{\mu(n-1)+\mu(n+1)-2\mu(n)}. \tag{5.2}$$

Invoking (4.5), we find that

$$\begin{aligned}\frac{\mu(n)^4}{\mu(n-1)^2\mu(n+1)^2}&=\frac{\mu(n)^4}{\left(\mu(n)^2-\frac{2\pi^2}{3}\right)\left(\mu(n)^2+\frac{2\pi^2}{3}\right)}\\&=\frac{\mu(n)^4}{\mu(n)^4-\frac{4\pi^4}{9}}.\end{aligned} \tag{5.3}$$

It can be calculated that

$$\mu(n)^4-\left(\mu(n)^4-\frac{4\pi^4}{9}\right)\left(1+\frac{4\pi^4}{9\mu(n)^4}+\frac{16\pi^8}{81\mu(n)^8}\right)=\frac{64\pi^{12}}{729\mu(n)^8}>0$$

and for $\mu(n)\geq 8$,

$$\mu(n)^4-\left(\mu(n)^4-\frac{4\pi^4}{9}\right)\left(1+\frac{4\pi^4}{9\mu(n)^4}+\frac{\pi^8}{5\mu(n)^8}\right)=-\frac{\pi^8}{405\mu(n)^4}+\frac{4\pi^{12}}{45\mu(n)^8}<0.$$

Hence, by (5.3), we obtain

$$1+\frac{4\pi^4}{9\mu(n)^4}+\frac{16\pi^8}{81\mu(n)^8}\leq\frac{\mu(n)^4}{\mu(n-1)^2\mu(n+1)^2}\leq 1+\frac{4\pi^4}{9\mu(n)^4}+\frac{\pi^8}{5\mu(n)^8}. \tag{5.4}$$

We proceed to estimate the remaining parts on the right-hand side of (5.2). Using (4.10), it is readily checked that for $\mu(n)\geq 6$,

$$\tilde{w}(n)<\mu(n-1)<\hat{w}(n), \tag{5.5}$$

$$\tilde{y}(n)<\mu(n+1)<\hat{y}(n), \tag{5.6}$$

where

$$\begin{aligned}\tilde{w}(n)&=\mu(n)-\frac{\pi^2}{3\mu(n)}-\frac{\pi^4}{18\mu(n)^3}-\frac{\pi^6}{54\mu(n)^5}-\frac{5\pi^8}{324\mu(n)^7},\\\hat{w}(n)&=\mu(n)-\frac{\pi^2}{3\mu(n)}-\frac{\pi^4}{18\mu(n)^3}-\frac{\pi^6}{54\mu(n)^5},\\\tilde{y}(n)&=\mu(n)+\frac{\pi^2}{3\mu(n)}-\frac{\pi^4}{18\mu(n)^3}+\frac{\pi^6}{54\mu(n)^5}-\frac{5\pi^8}{324\mu(n)^7},\\\hat{y}(n)&=\mu(n)+\frac{\pi^2}{3\mu(n)}-\frac{\pi^4}{18\mu(n)^3}+\frac{\pi^6}{54\mu(n)^5}.\end{aligned} \tag{5.7}$$

We next show that for $\mu(n) \geq 44$,

$$1 - \frac{\pi^4}{3\mu(n)^5} - \frac{45}{\mu(n)^6} \leq \frac{\left(1 - \frac{1}{\mu(n-1)}\right)\left(1 - \frac{1}{\mu(n+1)}\right)}{\left(1 - \frac{1}{\mu(n)}\right)^2} \leq 1 - \frac{\pi^4}{3\mu(n)^5}. \tag{5.8}$$

Applying (5.5) and (5.6), we deduce that for $\mu(n) \geq 6$,

$$\frac{\left(1 - \frac{1}{\tilde{w}(n)}\right)\left(1 - \frac{1}{\tilde{y}(n)}\right)}{\left(1 - \frac{1}{\mu(n)}\right)^2} \leq \frac{\left(1 - \frac{1}{\mu(n-1)}\right)\left(1 - \frac{1}{\mu(n+1)}\right)}{\left(1 - \frac{1}{\mu(n)}\right)^2} \leq \frac{\left(1 - \frac{1}{\hat{w}(n)}\right)\left(1 - \frac{1}{\hat{y}(n)}\right)}{\left(1 - \frac{1}{\mu(n)}\right)^2}.$$

Hence to show (5.8), it is enough to show that for $\mu(n) \geq 44$,

$$\frac{\left(1 - \frac{1}{\tilde{w}(n)}\right)\left(1 - \frac{1}{\tilde{y}(n)}\right)}{\left(1 - \frac{1}{\mu(n)}\right)^2} - \left(1 - \frac{\pi^4}{3\mu(n)^5} - \frac{45}{\mu(n)^6}\right) \geq 0 \tag{5.9}$$

and

$$\frac{\left(1 - \frac{1}{\hat{w}(n)}\right)\left(1 - \frac{1}{\hat{y}(n)}\right)}{\left(1 - \frac{1}{\mu(n)}\right)^2} - \left(1 - \frac{\pi^4}{3\mu(n)^5}\right) \leq 0. \tag{5.10}$$

Observe that

$$\frac{\left(1 - \frac{1}{\tilde{w}(n)}\right)\left(1 - \frac{1}{\tilde{y}(n)}\right)}{\left(1 - \frac{1}{\mu(n)}\right)^2} - \left(1 - \frac{\pi^4}{3\mu(n)^5} - \frac{45}{\mu(n)^6}\right) = \frac{\varphi_1(\mu(n))}{\mu(n)^{20}(\mu(n)-1)^2\tilde{w}(n)\tilde{y}(n)},$$

where

$$\begin{aligned}\varphi_1(\mu(n)) &= \mu(n)^{22}(\tilde{w}(n)-1)(\tilde{y}(n)-1)\\ &\qquad - \mu(n)^{14}\left(\mu(n)^6 - \frac{\pi^4}{3}\mu(n) - 45\right)(\mu(n)-1)^2\tilde{w}(n)\tilde{y}(n).\end{aligned}$$

We claim that for $\mu(n) \geq 44$,

$$\varphi_1(\mu(n)) \geq 0. \tag{5.11}$$

It can be checked that $\varphi_1(\mu(n))$ is a polynomial in $\mu(n)$ with degree 18, so we could express

$$\varphi_1(\mu(n)) = \sum_{j=0}^{18} a_j \mu(n)^j.$$

Clearly,

$$\varphi_1(\mu(n)) \geq -\sum_{j=0}^{16} |a_j|\mu(n)^j + a_{17}\mu(n)^{17} + a_{18}\mu(n)^{18}.$$

Moreover, numerical evidence indicates that for $0 \le j \le 15$ and $\mu(n) \ge 27$,

$$-|a_j|\mu(n)^j \ge -|a_{16}|\mu(n)^{16}$$

and

$$a_{16} = 45, \quad a_{17} = -90 + \frac{\pi^4}{3}, \quad a_{18} = 45 - \frac{4\pi^4}{9}.$$

It is readily checked that for $\mu(n) \ge 44$,

$$a_{18}\mu(n)^2 + a_{17}\mu(n) - 17|a_{16}| \ge 0.$$

Assembling all these results above, we conclude that for $\mu(n) \ge 44$,

$$\varphi_1(\mu(n)) \ge \left(a_{18}\mu(n)^2 + a_{17}\mu(n) - 17|a_{16}|\right)\mu(n)^{16} \ge 0.$$

This proves (5.11) and so (5.9) is valid. Similarly, observe that

$$\frac{\left(1-\frac{1}{\hat{w}(n)}\right)\left(1-\frac{1}{\hat{y}(n)}\right)}{\left(1-\frac{1}{\mu(n)}\right)^2} - \left(1 - \frac{\pi^4}{3\mu(n)^5}\right) = \frac{-\varphi_2(\mu(n))}{8748\mu(n)^{15}(\mu(n)-1)^2\hat{w}(n)\hat{y}(n)}, \quad (5.12)$$

where

$$\varphi_2(s) = 3888\pi^4 s^{13} - 2916\pi^4 s^{12} + 810\pi^8 s^{10} - 1377\pi^8 s^9 + 648\pi^8 s^8 \\ + 33\pi^{12}s^6 - 57\pi^{12}s^5 + 27\pi^{12}s^4 + \pi^{16}s^2 - 2\pi^{16}s + \pi^{16}.$$

It can be readily checked that for $s \ge 2$,

$$\begin{cases} 3888\pi^4 s^{13} - 2916\pi^4 s^{12} \ge 0, \\ 810\pi^8 s^{10} - 1377\pi^8 s^9 \ge 0, \\ 33\pi^{12}s^6 - 57\pi^{12}s^5 \ge 0, \\ \pi^{16}s^2 - 2\pi^{16}s \ge 0, \end{cases}$$

which implies that $\varphi_2(s) \ge 0$ for $s \ge 2$, thus we have that for $\mu(n) \ge 2$,

$$\varphi_2(\mu(n)) \ge 0. \quad (5.13)$$

Hence (5.10) is confirmed by applying (5.13) to (5.12). Combining (5.9) and (5.10), we obtain (5.8).

We proceed to estimate $\exp(\mu(n-1) + \mu(n+1) - 2\mu(n))$. Applying (5.5)–(5.7), we find that for $\mu(n) \ge 6$,

$$-\frac{\pi^4}{9\mu(n)^3} - \frac{5\pi^8}{162\mu(n)^7} < \mu(n-1) + \mu(n+1) - 2\mu(n) < -\frac{\pi^4}{9\mu(n)^3}.$$

It follows that

$$\exp\left(-\frac{\pi^4}{9\mu(n)^3}-\frac{5\pi^8}{162\mu(n)^7}\right) < \exp\left(\mu(n-1)+\mu(n+1)-2\mu(n)\right) < \exp\left(-\frac{\pi^4}{9\mu(n)^3}\right). \tag{5.14}$$

Since for $s<0$,

$$1+s<e^s<1+s+s^2,$$

we derive that

$$\exp\left(-\frac{\pi^4}{9\mu(n)^3}\right) < 1-\frac{\pi^4}{9\mu(n)^3}+\frac{\pi^8}{81\mu(n)^6} \tag{5.15}$$

and

$$\exp\left(-\frac{\pi^4}{9\mu(n)^3}-\frac{5\pi^8}{162\mu(n)^7}\right) > 1-\frac{\pi^4}{9\mu(n)^3}-\frac{5\pi^8}{162\mu(n)^7}. \tag{5.16}$$

Applying (5.15) and (5.16) to (5.14), we derive that for $\mu(n)\geq 6$,

$$1-\frac{\pi^4}{9\mu(n)^3}-\frac{5\pi^8}{162\mu(n)^7} < \exp(\mu(n-1)+\mu(n+1)-2\mu(n)) < 1-\frac{\pi^4}{9\mu(n)^3}+\frac{\pi^8}{81\mu(n)^6}. \tag{5.17}$$

Applying (5.4), (5.8) and (5.17) to (5.2), we obtain that for $k=0$ or 1 and $\mu(n)\geq 44$,

$$X(n)\leq\left(1+\frac{4\pi^4}{9\mu(n)^4}+\frac{\pi^8}{5\mu(n)^8}\right)\left(1-\frac{\pi^4}{3\mu(n)^5}\right)\left(1-\frac{\pi^4}{9\mu(n)^3}+\frac{\pi^8}{81\mu(n)^6}\right) \tag{5.18}$$

and

$$X(n)\geq\left(1+\frac{4\pi^4}{9\mu(n)^4}+\frac{16\pi^8}{81\mu(n)^8}\right)\left(1-\frac{\pi^4}{3\mu(n)^5}-\frac{45}{\mu(n)^6}\right)\left(1-\frac{\pi^4}{9\mu(n)^3}-\frac{5\pi^8}{162\mu(n)^7}\right). \tag{5.19}$$

Finally we estimate $L_Y(n)$ and $R_Y(n)$. We claim that for $\mu(n)\geq 16$,

$$L_Y(n)\geq 1-\frac{5}{\mu(n)^6} \quad\text{and}\quad R_Y(n)\leq 1+\frac{5}{\mu(n)^6}. \tag{5.20}$$

Invoking (4.5), we obtain that

$$L_Y(n)=\frac{\mu(n)^{12}\left(\left(\mu(n)^2+\frac{2\pi^2}{3}\right)^3-1\right)\left(\left(\mu(n)^2-\frac{2\pi^2}{3}\right)^3-1\right)}{\left(\mu(n)^6+1\right)^2\left(\mu(n)^4-\frac{4\pi^4}{9}\right)^3}$$

and

$$R_Y(n)=\frac{\mu(n)^{12}\left(\left(\mu(n)^2-\frac{2\pi^2}{3}\right)^3+1\right)\left(\left(\mu(n)^2+\frac{2\pi^2}{3}\right)^3+1\right)}{\left(\mu(n)^6-1\right)^2\left(\mu(n)^4-\frac{4\pi^4}{9}\right)^3}.$$

Hence,

$$L_Y(n) - \left(1 - \frac{5}{\mu(n)^6}\right) = \frac{\psi_1\left(\mu(n)\right)}{\mu(n)^6\left(9\mu(n)^4 - 4\pi^4\right)^3\left(\mu(n)^6+1\right)^2} \tag{5.21}$$

and

$$R_Y(n) - \left(1 + \frac{5}{\mu(n)^6}\right) = \frac{-\psi_2\left(\mu(n)\right)}{\mu(n)^6\left(9\mu(n)^4 - 4\pi^4\right)^3\left(\mu(n)^6-1\right)^2}, \tag{5.22}$$

where

$$\begin{aligned}\psi_1(s) = {} & 729s^{24} - 4860\pi^4 s^{20} + 7290s^{18} + 1296\pi^8 s^{16} - 8748\pi^4 s^{14} + \left(3645 - 192\pi^{12}\right)s^{12}\\ & + 3888\pi^8 s^{10} - 4860\pi^4 s^8 - 576\pi^{12}s^6 + 2160\pi^8 s^4 - 320\pi^{12}\end{aligned}$$

and

$$\begin{aligned}\psi_2(s) = {} & 729s^{24} - 4860\pi^4 s^{20} - 7290s^{18} + 1296\pi^8 s^{16} + 8748\pi^4 s^{14} + \left(3645 - 192\pi^{12}\right)s^{12}\\ & - 3888\pi^8 s^{10} - 4860\pi^4 s^8 + 576\pi^{12}s^6 + 2160\pi^8 s^4 - 320\pi^{12}.\end{aligned}$$

It can be readily checked that for $s \geq 16$,

$$\begin{cases} 729s^{24} - 4860\pi^4 s^{20} - 7290s^{18} \geq 0,\\ 8748\pi^4 s^{14} + \left(3645 - 192\pi^{12} - 3888\pi^8 - 4860\pi^4\right)s^{12} \geq 0,\\ 2160\pi^8 s^4 - 320\pi^{12} \geq 0,\end{cases}$$

which implies that for $s \geq 16$,

$$\psi_2(s) \geq 0.$$

We note that for $s \geq 4$,

$$\psi_1(s) > \psi_2(s).$$

Hence for $\mu(n) \geq 16$,

$$\psi_1\left(\mu(n)\right) \geq \psi_2\left(\mu(n)\right) \geq 0, \tag{5.23}$$

and so (5.20) is verified by applying (5.23) to (5.21) and (5.22) respectively.

Substituting (5.18), (5.19) and (5.20) into (5.1), we derive that for $k = 0,\ 1$ and $\mu(n) \geq 88$,

$$\begin{aligned}Y_k(n) \leq {} & \left(1 + \frac{4\pi^4}{9\mu(n)^4} + \frac{\pi^8}{5\mu(n)^8}\right)\left(1 - \frac{\pi^4}{3\mu(n)^5}\right)\\ & \times\left(1 - \frac{\pi^4}{9\mu(n)^3} + \frac{\pi^8}{81\mu(n)^6}\right)\left(1 + \frac{5}{\mu(n)^6}\right)\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}Y_k(n) \geq {} & \left(1 + \frac{4\pi^4}{9\mu(n)^4} + \frac{16\pi^8}{81\mu(n)^8}\right)\left(1 - \frac{\pi^4}{3\mu(n)^5} - \frac{45}{\mu(n)^6}\right)\\ & \times\left(1 - \frac{\pi^4}{9\mu(n)^3} - \frac{5\pi^8}{162\mu(n)^7}\right)\left(1 - \frac{5}{\mu(n)^6}\right).\end{aligned} \tag{5.25}$$

To prove Theorem 1.5, it is enough to show that for $\mu(n) \geq 115$,

$$\left(1+\frac{4\pi^4}{9\mu(n)^4}+\frac{\pi^8}{5\mu(n)^8}\right)\left(1-\frac{\pi^4}{3\mu(n)^5}\right)$$
$$\times\left(1-\frac{\pi^4}{9\mu(n)^3}+\frac{\pi^8}{81\mu(n)^6}\right)\left(1+\frac{5}{\mu(n)^6}\right)$$
$$<1-\frac{\pi^4}{9\mu(n)^3}+\frac{4\pi^4}{9\mu(n)^4}-\frac{\pi^4}{3\mu(n)^5}+\frac{\frac{\pi^8}{81}+5}{\mu(n)^6} \tag{5.26}$$

and

$$\left(1+\frac{4\pi^4}{9\mu(n)^4}+\frac{16\pi^8}{81\mu(n)^8}\right)\left(1-\frac{\pi^4}{3\mu(n)^5}-\frac{45}{\mu(n)^6}\right)$$
$$\times\left(1-\frac{\pi^4}{9\mu(n)^3}-\frac{5\pi^8}{162\mu(n)^7}\right)\left(1-\frac{5}{\mu(n)^6}\right)$$
$$>1-\frac{\pi^4}{9\mu(n)^3}+\frac{4\pi^4}{9\mu(n)^4}-\frac{\pi^4}{3\mu(n)^5}-\frac{60}{\mu(n)^6}. \tag{5.27}$$

Observe that

$$\left(1+\frac{4\pi^4}{9\mu(n)^4}+\frac{\pi^8}{5\mu(n)^8}\right)\left(1-\frac{\pi^4}{3\mu(n)^5}\right)$$
$$\times\left(1-\frac{\pi^4}{9\mu(n)^3}+\frac{\pi^8}{81\mu(n)^6}\right)\left(1+\frac{5}{\mu(n)^6}\right)$$
$$-\left(1-\frac{\pi^4}{9\mu(n)^3}+\frac{4\pi^4}{9\mu(n)^4}-\frac{\pi^4}{3\mu(n)^5}+\frac{\frac{\pi^8}{81}+5}{\mu(n)^6}\right)$$
$$=-\frac{1}{10935\mu(n)^{25}}\sum_{j=0}^{18} b_j\mu(n)^j,$$

where b_j are real numbers. Here we just list the values of b_{16}, b_{17}, b_{18}:

$$b_{16}=6075\pi^4+1620\pi^8, \quad b_{17}=-2592\pi^8, \quad b_{18}=540\pi^8.$$

Clearly

$$\sum_{j=0}^{18} b_j\mu(n)^j \geq -\sum_{j=0}^{16}|b_j|\mu(n)^j+b_{17}\mu(n)^{17}+b_{18}\mu(n)^{18}.$$

Moreover, it can be checked that for $0 \leq j \leq 15$ and $\mu(n) \geq 5$,

$$-|b_j|\mu(n)^j \geq -|b_{16}|\mu(n)^{16}$$

and for $\mu(n) \geq 11$,

$$b_{18}\mu(n)^2+b_{17}\mu(n)-17|b_{16}|>0.$$

Assembling all these results above, we conclude that for $\mu(n) \geq 115$,

$$\sum_{j=0}^{18} b_j \mu(n)^j \geq \left(b_{18}\mu(n)^2 + b_{17}\mu(n) - 17|b_{16}|\right) \mu(n)^{16} > 0,$$

and so (5.26) is valid. Similarly, to justify (5.27), we note that

$$\begin{aligned}
&\left(1 + \frac{4\pi^4}{9\mu(n)^4} + \frac{16\pi^8}{81\mu(n)^8}\right)\left(1 - \frac{\pi^4}{3\mu(n)^5} - \frac{45}{\mu(n)^6}\right) \\
&\quad \times \left(1 - \frac{\pi^4}{9\mu(n)^3} - \frac{5\pi^8}{162\mu(n)^7}\right)\left(1 - \frac{5}{\mu(n)^6}\right) \\
&\quad - \left(1 - \frac{\pi^4}{9\mu(n)^3} + \frac{4\pi^4}{9\mu(n)^4} - \frac{\pi^4}{3\mu(n)^5} - \frac{60}{\mu(n)^6}\right) \\
&= \frac{1}{39366\mu(n)^{27}} \sum_{j=0}^{21} c_j \mu(n)^j,
\end{aligned}$$

where c_j are real numbers. Here we also list the values of the last three coefficients:

$$c_{19} = 9234\pi^8, \quad c_{20} = -3159\pi^8, \quad c_{21} = 393660.$$

It is transparent that

$$\sum_{j=0}^{21} c_j \mu(n)^r \geq -\sum_{j=0}^{19} |c_j|\mu(n)^j + c_{20}\mu(n)^{20} + c_{21}\mu(n)^{21}.$$

Moreover, it can be checked that for $0 \leq j \leq 18$ and $\mu(n) \geq 3$,

$$-|c_j|\mu(n)^j \geq -|c_{19}|\mu(n)^{19}$$

and for $\mu(n) \geq 115$,

$$c_{21}\mu(n)^2 + c_{20}\mu(n) - 20|c_{19}| > 0.$$

Hence we conclude that for $\mu(n) \geq 115$,

$$\sum_{j=0}^{21} c_j \mu(n)^j \geq \left(c_{21}\mu(n)^2 + c_{20}\mu(n) - 20|c_{19}|\right) \mu(n)^{19} > 0,$$

and so (5.27) is valid.

Substituting (5.26) and (5.27) into (5.24) and (5.25), we arrive at (1.5) and (1.6). This completes the proof of Theorem 1.5. ∎

6 Proofs of Theorem 1.4 and Theorem 1.6

In this section, we aim to prove that $M_0(n)$ (resp. $M_1(n)$) is log-concave for $n \geq 94$ and satisfies the higher order Turán inequalities for $n \geq 207$ with the aid of Theorem 1.5. We first show that $M_0(n)$ (resp. $M_1(n)$) is log-concave for $n \geq 94$.

Proof of Theorem 1.4. Recall that

$$Y_k(n) := \frac{M_k(n-1)M_k(n+1)}{M_k(n)^2}.$$

To prove Theorem 1.4, it is equivalent to prove that $Y_0(n) < 1$ for $n \geq 94$ and $Y_1(n) < 1$ for $n \geq 93$. It is easy to check that for $\mu(n) \geq 4$,

$$-\frac{\pi^4}{9\mu(n)^3} + \frac{4\pi^4}{9\mu(n)^4} \leq 0$$

and

$$-\frac{\pi^4}{3\mu(n)^5} + \frac{\frac{\pi^8}{81}+5}{\mu(n)^6} \leq 0,$$

and by (1.5), we deduce that $Y_k(n) < 1$ for $k = 0,\ 1$ and $n \geq 2011$. It can be checked that $Y_0(n) < 1$ for $94 \leq n \leq 2011$ and $Y_1(n) < 1$ for $93 \leq n \leq 2011$. Hence we conclude that $M_0(n)$ is log-concave for $n \geq 94$ and $M_1(n)$ is log-concave for $n \geq 93$. This completes the proof of Theorem 1.4. ∎

We conclude this paper with the proof of Theorem 1.6 by employing Theorem 1.5. The proof of Theorem 1.6 also requires the following lemma given by Jia [17].

Lemma 6.1 (Jia). *Let u and v be two positive real numbers such that $\frac{\sqrt{5}-1}{2} \leq u < v < 1$. If*

$$u + \sqrt{(1-u)^3} > v,$$

then we have

$$4(1-u)(1-v) - (1-uv)^2 > 0.$$

Proof of Theorem 1.6. To prove $\{M_0(n)\}_{n\geq 207}$ and $\{M_1(n)\}_{n\geq 206}$ satisfy the higher order Turán inequalities, it is equivalent to show that

$$4(1-Y_k(n))(1-Y_k(n+1)) - (1-Y_k(n)Y_k(n+1))^2 > 0 \tag{6.1}$$

for $n \geq 207$ if $k = 0$ and for $n \geq 206$ if $k = 1$. We first show that (6.1) holds for $k = 0$ or 1 and $n \geq 2011$. From Theorem 1.4, we see that $Y_k(n+1) < 1$ for $n \geq 93$. Hence by Lemma 6.1, it's enough to show that for $k = 0,\ 1$ and $n \geq 2011$,

$$\frac{\sqrt{5}-1}{2} \leq Y_k(n) < Y_k(n+1) \tag{6.2}$$

and

$$Y_k(n+1) < Y_k(n) + \sqrt{(1-Y_k(n))^3}. \tag{6.3}$$

Utilizing (1.6) in Theorem 1.5 , we see that for $k = 0,\, 1$ and $\mu(n) \geq 115$,

$$Y_k(n) > 1 - \frac{\pi^4}{9\mu(n)^3} + \frac{4\pi^4}{9\mu(n)^4} - \frac{\pi^4}{3\mu(n)^5} - \frac{60}{\mu(n)^6}.$$

Note that for $\mu(n) \geq 4$,

$$\frac{4\pi^4}{9\mu(n)^4} - \frac{\pi^4}{3\mu(n)^5} - \frac{60}{\mu(n)^6} > 0$$

and

$$1 - \frac{\pi^4}{9\mu(n)^3} > \frac{\sqrt{5}-1}{2}.$$

Hence we derive that for $\mu(n) \geq 115$,

$$Y_k(n) > \frac{\sqrt{5}-1}{2}.$$

Employing Theorem 1.5 again, we find that for $k = 0,\, 1$ and $\mu(n) \geq 115$,

$$\begin{aligned} Y_k(n+1) - Y_k(n) > &\left(1 - \frac{\pi^4}{9\mu(n+1)^3} + \frac{4\pi^4}{9\mu(n+1)^4} - \frac{\pi^4}{3\mu(n+1)^5} - \frac{60}{\mu(n+1)^6}\right) \\ &- \left(1 - \frac{\pi^4}{9\mu(n)^3} + \frac{4\pi^4}{9\mu(n)^4} - \frac{\pi^4}{3\mu(n)^5} + \frac{\frac{\pi^8}{81}+5}{\mu(n)^6}\right). \end{aligned} \tag{6.4}$$

Note that for $\mu(n) \geq 3$,

$$\begin{cases} \dfrac{1}{\mu(n+1)^3} < \dfrac{1}{\mu(n)^3} - \dfrac{\pi^2}{2\mu(n)^5}, \\ \dfrac{1}{\mu(n+1)^4} > \dfrac{1}{\mu(n)^4} - \dfrac{4\pi^2}{3\mu(n)^6}, \\ \dfrac{1}{\mu(n+1)^5} < \dfrac{1}{\mu(n)^5}, \\ \dfrac{1}{\mu(n+1)^6} < \dfrac{1}{\mu(n)^6}. \end{cases} \tag{6.5}$$

Applying (6.5) to (6.4), we attain that for $k = 0,\, 1$ and $\mu(n) \geq 115$,

$$\begin{aligned} Y_k(n+1) - Y_k(n) > &\left(1 - \frac{\pi^4}{9\mu(n)^3} + \frac{4\pi^4}{9\mu(n)^4} + \frac{-\frac{\pi^4}{3}+\frac{\pi^6}{18}}{\mu(n)^5} - \frac{\frac{16\pi^6}{27}+60}{\mu(n)^6}\right) \\ &- \left(1 - \frac{\pi^4}{9\mu(n)^3} + \frac{4\pi^4}{9\mu(n)^4} - \frac{\pi^4}{3\mu(n)^5} + \frac{\frac{\pi^8}{81}+5}{\mu(n)^6}\right) \\ = &\frac{\pi^6}{18\mu(n)^5} - \frac{65+\frac{16\pi^6}{27}+\frac{\pi^8}{81}}{\mu(n)^6}. \end{aligned}$$

It can be checked that for $\mu(n) \geq 15$,

$$\frac{\pi^6}{18\mu(n)^5} - \frac{65 + \frac{16\pi^6}{27} + \frac{\pi^8}{81}}{\mu(n)^6} > 0.$$

Hence we derive that for $k = 0,\,1$ and $\mu(n) \geq 115$,

$$Y_k(n+1) - Y_k(n) > 0,$$

and so (6.2) holds for $k = 0,\,1$ and $\mu(n) \geq 115$.

To prove (6.3), invoking Theorem 1.5 again, we find that for $k = 0,\,1$ and $\mu(n) \geq 115$,

$$\begin{aligned} Y_k(n+1) - Y_k(n) < & \left(1 - \frac{\pi^4}{9\mu(n+1)^3} + \frac{4\pi^4}{9\mu(n+1)^4} - \frac{\pi^4}{3\mu(n+1)^5} + \frac{\frac{\pi^8}{81}+5}{\mu(n+1)^6}\right) \\ & - \left(1 - \frac{\pi^4}{9\mu(n)^3} + \frac{4\pi^4}{9\mu(n)^4} - \frac{\pi^4}{3\mu(n)^5} - \frac{60}{\mu(n)^6}\right) \\ = & \frac{\pi^4}{9}\left(\frac{1}{\mu(n)^3} - \frac{1}{\mu(n+1)^3}\right) + \frac{4\pi^4}{9}\left(\frac{1}{\mu(n+1)^4} - \frac{1}{\mu(n)^4}\right) \\ & + \frac{\pi^4}{3}\left(\frac{1}{\mu(n)^5} - \frac{1}{\mu(n+1)^5}\right) + \frac{\frac{\pi^8}{81}+5}{x(n+1)^6} + \frac{60}{\mu(n)^6}. \end{aligned} \tag{6.6}$$

It is easy to check that for $\mu(n) > 0$,

$$\begin{cases} \dfrac{1}{\mu(n)^3} - \dfrac{1}{\mu(n+1)^3} < \dfrac{\pi^2}{\mu(n)^5}, \\ \dfrac{1}{\mu(n+1)^4} - \dfrac{1}{\mu(n)^4} < 0, \\ \dfrac{1}{\mu(n)^5} - \dfrac{1}{\mu(n+1)^5} < \dfrac{1}{\mu(n)^5} \end{cases} \tag{6.7}$$

and for $\mu(n) \geq 19$,

$$\frac{\frac{\pi^8}{81}+5}{\mu(n+1)^6} + \frac{60}{\mu(n)^6} < \frac{\frac{\pi^8}{81}+5+60}{\mu(n)^6} < \frac{\pi^2}{\mu(n)^5}. \tag{6.8}$$

Applying (6.7) and (6.8) to (6.6), we deduce that for $k = 0,\,1$ and $\mu(n) \geq 115$,

$$\begin{aligned} Y_k(n+1) - Y_k(n) &< \frac{\pi^4}{9} \cdot \frac{\pi^2}{\mu(n)^5} + \frac{\pi^4}{3} \cdot \frac{1}{\mu(n)^5} + \frac{\pi^2}{\mu(n)^5} \\ &= \frac{\frac{\pi^6}{9} + \frac{\pi^4}{3} + \pi^2}{\mu(n)^5}. \end{aligned}$$

We next show that for $k = 0,\ 1$ and $\mu(n) \geq 115$,

$$\sqrt{(1 - Y_k(n))^3} > \frac{\frac{\pi^6}{9} + \frac{\pi^4}{3} + \pi^2}{\mu(n)^5}. \tag{6.9}$$

From (1.5), we see that for $\mu(n) \geq 115$,

$$\begin{aligned}
1 - Y_k(n) &> \frac{\pi^4}{9\mu(n)^3} - \frac{4\pi^4}{9\mu(n)^4} + \frac{\pi^4}{3\mu(n)^5} - \frac{\frac{\pi^8}{81} + 5}{\mu(n)^6} \\
&> \frac{\pi^4}{9\mu(n)^3} - \frac{4\pi^4}{9\mu(n)^4} - \frac{\frac{\pi^8}{81} + 5}{\mu(n)^4}.
\end{aligned}$$

It can be checked that for $\mu(n) \geq 83$,

$$-\frac{4\pi^4}{9\mu(n)^4} - \frac{\frac{\pi^8}{81} + 5}{\mu(n)^4} > -\frac{2}{\mu(n)^3}.$$

It follows that for $\mu(n) \geq 115$,

$$1 - Y_k(n) > \frac{\pi^4}{9\mu(n)^3} - \frac{2}{\mu(n)^3} = \frac{\pi^4 - 18}{9\mu(n)^3} > 0.$$

On the other hand, it is easy to check that for $\mu(n) \geq 33$,

$$\frac{\sqrt{(\pi^4 - 18)^3}}{27\mu(n)^{\frac{9}{2}}} > \frac{\frac{\pi^6}{9} + \frac{\pi^4}{3} + \pi^2}{\mu(n)^5},$$

and so (6.9) holds. In view of Lemma 6.1, we conclude that (6.1) holds for $k = 0$ or 1 and $n \geq 2011$. It can be directly checked that (6.1) is valid for $207 \leq n \leq 2010$ if $k = 0$ and for $206 \leq n \leq 2010$ if $k = 1$. This completes the proof of Theorem 1.6. ∎

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