0-1 Tableaux and the p, q-Legendre-Stirling numbers of the second kind

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Abstract. We define the p, q-Legendre-Stirling numbers $PS_{p,q}(n, k)$ of the second kind, which reduce to the Legendre-Stirling numbers $PS(n, k)$ of the second kind discovered by Everitt, Littlejohn and Wellman when $p = 1$ and $q = 1$ and the q-Legendre-Stirling numbers $PS_q(n, k)$ of the second kind introduced by Mongelli when $p = 1$. By introducing 0-1 tableaux of shape λ with even multiplicities, we give a combinatorial interpretation of the p, q-Legendre-Stirling numbers of the second kind. As an application of such a combinatorial interpretation, we derive three p, q-Legendre-Stirling identities, two of which can be viewed as p, q-analogues of two results due to Everitt-Littlejohn-Wellman and Andrews-Gawronski-Littlejohn, respectively. We also show that q-Legendre-Stirling numbers $PS_q(n, k)$ of the second kind are strongly q-log-concave by constructing a direct injection based on this combinatorial construction.

Keywords: Legendre-Stirling numbers, q-Stirling numbers, 0-1 tableaux, integer partitions, generating function, combinatorial interpretations, q-log-concavity

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1 Introduction

The main objective of this paper is to introduce the p, q-Legendre-Stirling numbers $PS_{p,q}(n, k)$ of the second kind and to give a combinatorial interpretation of $PS_{p,q}(n, k)$ in terms of 0-1 tableaux. The p, q-Legendre-Stirling numbers of the second kind are defined by the following recurrence: For $n \geq k \geq 1$,

$$PS_{p,q}(n, k) = PS_{p,q}(n - 1, k - 1) + [k]_{p,q}[k + 1]_{p,q}PS_{p,q}(n - 1, k) \tag{1.1}$$

with the initial conditions $PS_{p,q}(0, 0) = 1$ and for $n \geq 1$,

$$PS_{p,q}(n, k) = 0 \quad \text{if } k \leq 0 \text{ or } k > n, \tag{1.2}$$

where for $k \geq 1$,

$$[k]_{p,q} = p^{k-1} + p^{k-2}q + \cdots + pq^{k-2} + q^{k-1}. \tag{1.3}$$

The first few polynomials $PS_{p,q}(n,k)$ $(1 \le n \le 3)$ are given as follows:

$$PS_{p,q}(1,1) = 1$$
$$PS_{p,q}(2,1) = p + q$$
$$PS_{p,q}(2,2) = 1$$
$$PS_{p,q}(3,1) = p^2 + 2pq + q^2$$
$$PS_{p,q}(3,2) = p^3 + 2p^2q + 2pq^2 + q^3 + p + q$$
$$PS_{p,q}(3,3) = 1.$$

When $p = 1$ and $q = 1$, $PS_{p,q}(n,k)$ reduce to the Legendre-Stirling numbers $PS(n,k)$ of the second kind, which were defined by Everitt, Littlejohn and Wellman [8] as the coefficients in the integral Lagrangian symmetric powers of the classical Legendre second-order differential expression. Everitt, Littlejohn and Wellman [8], Andrews, Gawronski and Littlejohn [3] and Egge [7] have shown that the Legendre-Stirling numbers of the second kind possess many properties with the classical Stirling numbers of the second kind, such as similar recurrences and generating functions. Throughout the paper, we adopt the notation used by Stanley [20] for the Stirling numbers $S(n,k)$ of the second kind. Since Legendre polynomials are typically written $\{P_n\}$, we adopt the notation $PS(n,k)$ for the Legendre-Stirling numbers of the second kind. When $p = 1$, $PS_{p,q}(n,k)$ become the q-Legendre-Stirling numbers $PS_q(n,k)$ of the second kind introduced by Mongelli [16].

Andrews and Littlejohn [2] and Mongelli [16] provided combinatorial interpretations of the Legendre-Stirling numbers of the second kind and q-Legendre-Stirling numbers of the second kind in terms of set partitions, respectively. In this paper, we give a combinatorial interpretation of the p,q-Legendre-Stirling numbers of the second kind in terms of 0-1 tableaux of shape λ with even parts (or 0-1 even tableaux, for short).

It should be noted that the p,q-Stirling numbers of the second kind was introduced by Wachs and White [21]. For $n \ge k \ge 1$,

$$S_{p,q}(n,k) = p^{k-1}S_{p,q}(n-1,k-1) + [k]_{p,q}S_{p,q}(n-1,k)$$

with the initial conditions $S_{p,q}(0,0) = 1$ and for $n \ge 1$,

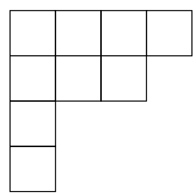
$$S_{p,q}(n,k) = 0 \quad \text{if } k \le 0 \text{ or } k > n.$$

When $p = 1$, $S_{p,q}(n,k)$ reduce to q-Stirling numbers of the second kind introduced by Gould [9]. The combinatorial interpretation of the p,q-Stirling numbers (or the q-Stirling numbers) in terms of 0-1 tableaux was explored by Leroux [12] and Medicis and Leroux [13]. For other combinatorial interpretations of q-Stirling numbers, see, for example, Cai and Readdy [5], Cai, Ehrenborg and Readdy [6], Gould [9], Milne [14, 15] and Wachs and White [21].

We recall some common notation and terminology on partitions as used in [1, Chapter 1]. Recall that a partition λ of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. Then λ_i are called the parts of λ and λ_1 is its largest part. The number of parts of λ is called the length of λ, denoted by $l(\lambda)$. In this paper, we adopt the multiplicity notation of the partition λ. Let λ be the partition with the largest part k. Then we write

$$\lambda = (k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 1^{m_1}),$$

where exactly m_i parts of λ are equal to i. We call the m_i the multiplicity of the part of size i in λ. The Young diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a left-justified array of squares, with λ_i squares in the i-th row. For instance, below is the Young diagram of shape $\lambda = (4, 3, 1, 1)$.



Let $\mathcal{P}_e(k, n-k)$ denote the set of partitions into exactly $2n-2k$ parts with the largest part being less than or equal to k such that each multiplicity m_i is even for $1 \leq i \leq k$.

Definition 1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_{2n-2k}) \in \mathcal{P}_e(k, n-k)$. A 0-1 tableau of shape λ (or 0-1 even tableau, for short) is a tableau obtained by filling 0 or 1 into the Young diagram of λ such that there is exactly one* 1 *in the $(2i-1)$st row and there is at most one* 1 *in the $2i$-th row for $1 \leq i \leq n-k$.*

Fig. 1 is a 0-1 even tableau of shape $\lambda = (4^2, 2^4, 1^2)$.

0	1	0	0
0	0	0	1
0	1		
0	0		
1	0		
0	1		
1			
0			

Fig. 1: A 0-1 even tableau of shape $\lambda = (4^2, 2^4, 1^2)$.

Let $T = (T_1, \ldots, T_{2n-2k})$ be a 0-1 even tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_{2n-2k}) \in \mathcal{P}_e(k, n-k)$, where T_i represents the i-th row of T. We define the shape of T_i as the number of squares in its corresponding row of T, denoted $sh(T_i)$. From the above assumption, we see that $sh(T_i) = \lambda_i$ for $1 \le i \le 2n-2k$. To give a combinatorial interpretation of $PS_{p,q}(n,k)$, we need to define inversions and non-inversions of a 0-1 even tableau.

The inversion number of T, denoted by $\mathrm{inv}(T)$, is defined by

$$\mathrm{inv}(T) = \sum_{i=1}^{2n-2k} \mathrm{inv}_i(T), \tag{1.4}$$

where $\mathrm{inv}_i(T)$ counts the number of 0's to the right side of the 1 occurring in the i-th row of T with the convention that $\mathrm{inv}_i(T) = \lambda_i$ if there does not exist a 1 in the i-th row of T. Here we assume that $\mathrm{inv}(\emptyset) = 0$.

The non-inversion number of T, denoted by $\mathrm{ninv}(T)$, is defined by

$$\mathrm{ninv}(T) = \sum_{i=1}^{2n-2k} \mathrm{ninv}_i(T), \tag{1.5}$$

where $\mathrm{ninv}_{2i-1}(T)$ counts the number of 0's to the left side of the 1 occurring in the $(2i-1)$st row of T and $\mathrm{ninv}_{2i}(T)$ counts the number of 0's and 1 to the left side of the 1 occurring in the $2i$-th row of T. We make the convention that $\mathrm{ninv}_i(T) = 0$ if there does not exist a 1 in the i-th row of T. When $T = \emptyset$, $\mathrm{ninv}(T) = 0$.

For example, for the 0-1 even tableau T given in Fig. 1, we have

$$\mathrm{inv}(T) = 2+0+0+2+1+0+0+1 = 6$$

and

$$\mathrm{ninv}(T) = 1+4+1+0+0+2+0+0 = 8.$$

We have the following combinatorial interpretation of the p,q-Legendre-Stirling numbers of the second kind.

Theorem 1.2. *For $n \ge k \ge 1$, let $\mathcal{TB}(n,k)$ denote the set of 0-1 even tableaux of shape $\lambda \in \mathcal{P}_e(k, n-k)$. Then*

$$PS_{p,q}(n,k) = \sum_{T \in \mathcal{TB}(n,k)} p^{\mathrm{ninv}(T)} q^{\mathrm{inv}(T)}.$$

For example, when $n = 3$ and $k = 2$, there are two partitions in $\mathcal{P}_e(2,1)$, namely (2^2) and (1^2) and there are eight 0-1 tableaux in $\mathcal{TB}(3,2)$, see Fig. 2.

It is evident that

$$\sum_{T \in \mathcal{TB}(3,2)} p^{\mathrm{ninv}(T)} q^{\mathrm{inv}(T)} = p^3 + 2p^2q + 2pq^2 + q^3 + p + q.$$

When $p,q = 1$, we obtain a new combinatorial interpretation of the Legendre-Stirling numbers $PS(n,k)$ of the second kind.

$$
\begin{array}{cccc}
\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \\
\mathrm{inv}(T) = 3 & \mathrm{inv}(T) = 2 & \mathrm{inv}(T) = 2 & \mathrm{inv}(T) = 1 \\
\mathrm{ninv}(T) = 0 & \mathrm{ninv}(T) = 1 & \mathrm{ninv}(T) = 1 & \mathrm{ninv}(T) = 2 \\
\\
\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 1 \\ \hline \end{array} &
\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline \end{array} &
\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array} \\
\mathrm{inv}(T) = 1 & \mathrm{inv}(T) = 0 & \mathrm{inv}(T) = 1 & \mathrm{inv}(T) = 0 \\
\mathrm{ninv}(T) = 2 & \mathrm{ninv}(T) = 3 & \mathrm{ninv}(T) = 0 & \mathrm{ninv}(T) = 1
\end{array}
$$

Fig. 2: 0-1 even tableau $T \in \mathcal{TB}(3,2)$

Corollary 1.3. *For $n \geq k \geq 1$, $PS(n,k)$ counts the number of 0-1 even tableaux of shape $\lambda \in \mathcal{P}_e(k, n-k)$.*

As applications, we derive three p,q-Legendre-Stirling identities based on this combinatorial construction of $PS_{p,q}(n,k)$; see Section 3 for details.

We also establish the strong q-log-concavity of the q-Legendre-Stirling numbers $PS_q(n,k)$ of the second kind by building an injection based on this combinatorial construction. The notion of q-log-concavity was introduced by Stanley [19] and Sagan [17] later introduced the notion of the strong q-log-concavity. A sequence of polynomials $(f_n(q))_{n\geq 0}$ over the field of real numbers is called *q-log-concave* if the difference

$$f_m(q)^2 - f_{m-1}(q)f_{m+1}(q)$$

has nonnegative coefficients as a polynomial in q for all $m \geq 1$. We say that a sequence of polynomials $(f_n(q))_{n\geq 0}$ is *strongly q-log-concave* if

$$f_n(q)f_m(q) - f_{n-1}(q)f_{m+1}(q)$$

has nonnegative coefficients as a polynomial in q for all $m \geq n \geq 1$.

In the literature, q-analogues of many well-known combinatorial numbers have been shown to be strongly q-log-concave, see, for example, q-binomial coefficients [4, 11], q-Stirling numbers of the first kind and the second kind [12, 17] and q-Kaplansky number [10]. In particular, Sagan [18] gave a unified approach to prove the strong q-log-concavities of q-binomial coefficients and q-Stirling numbers of the first kind and the second kind by proving that various sequences of elementary and complete homogeneous symmetric functions are log-concave.

Based on Theorem 1.2, we show the following result by constructing a direct injection.

Theorem 1.4. *For $n \geq 1$ and $1 \leq k \leq l \leq n-1$,*

$$PS_{p,q}(n,k)PS_{p,q}(n,l) - PS_{p,q}(n,k-1)PS_{p,q}(n,l+1) \tag{1.6}$$

has nonnegative coefficients as a polynomial in p, q.

For $p = 1$, we have the following corollary.

Corollary 1.5. *Given a positive integer n, the sequence $\{PS_q(n,k)\}_{1\leq k\leq n}$ is strongly q-log-concave.*

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2. In Section 3, we give combinatorial derivations of three p,q-Legendre-Stirling identities relying on Theorem 1.2, two of which can be viewed as p,q-analogues of two results due to Andrews-Gawronski-Littlejohn [3] and Everitt-Littlejohn-Wellman [8]. In Section 4, we give a proof of Theorem 1.4 by building an injection resorting to Theorem 1.2. We conclude this paper by pointing out a connection between p,q-Legendre-Stirling numbers of the second kind and the complete homogeneous symmetric functions in Section 5.

2 The combinatorial interpretation

The main objective of this section is to give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let

$$TB_{p,q}(n,k) = \sum_{T\in\mathcal{TB}(n,k)} p^{\mathrm{ninv}(T)} q^{\mathrm{inv}(T)}. \tag{2.1}$$

We aim to show that $TB_{p,q}(n,k) = PS_{p,q}(n,k)$ for $n \geq k \geq 1$ by induction on n.

If $n = 1$, then $k = 1$, by definition, we see that $\mathcal{TB}(n,k) = \emptyset$, and so $TB_{p,q}(1,1) = 1$, which equals $PS_{p,q}(1,1) = 1$. Assume that for $m < n$, we have

$$TB_{p,q}(m,k) = PS_{p,q}(m,k) \tag{2.2}$$

for $m \geq k \geq 1$. We proceed to show that $TB_{p,q}(n,k) = PS_{p,q}(n,k)$ for $n \geq k \geq 1$. By the recurrence (1.1) of $PS_{p,q}(n,k)$ and (2.2), it suffices to show that

$$TB_{p,q}(n,k) = TB_{p,q}(n-1,k-1) + [k]_{p,q}[k+1]_{p,q}TB_{p,q}(n-1,k). \tag{2.3}$$

Let $T = (T_1, \ldots, T_{2n-2k}) \in \mathcal{TB}(n,k)$, where $sh(T_i) = \lambda_i$ for $1 \leq i \leq 2n-2k$. By definition, we see that $\lambda_1 \leq k$. Let $\mathcal{TB}(n,=k)$ denote the set of 0-1 even tableaux $T = (T_1, \ldots, T_{2n-2k})$ in $\mathcal{TB}(n,k)$ such that $sh(T_1) = \lambda_1 = k$. Evidently,

$$\mathcal{TB}(n,k) = \mathcal{TB}(n-1,k-1) \cup \mathcal{TB}(n,=k).$$

Using the notation (2.1), we derive that

$$TB_{p,q}(n,k) = TB_{p,q}(n-1,k-1) + \sum_{T\in\mathcal{TB}(n,=k)} p^{\mathrm{ninv}(T)}q^{\mathrm{inv}(T)}. \tag{2.4}$$

Let $T = (T_1,\ldots,T_{2n-2k}) \in \mathcal{TB}(n,=k)$, where $sh(T_1) = \lambda_1 = k$. Assume that $\mathrm{inv}_1(T) = r$ and $\mathrm{inv}_2(T) = s$. Since $\lambda_1 = \lambda_2 = k$, we see that $0 \le r \le k-1$ and $0 \le s \le k$. Moreover, $\mathrm{ninv}_1(T) = k-r-1$ and $\mathrm{ninv}_2(T) = k-s$. Define

$$\varphi(T) = (T_3,\ldots,T_{2n-2k}).$$

It is clear that $\varphi(T) \in \mathcal{TB}(n-1,k)$ such that

$$\mathrm{inv}(T) = \mathrm{inv}(\varphi(T)) + r + s$$

and

$$\mathrm{ninv}(T) = \mathrm{ninv}(\varphi(T)) + k - r - 1 + k - s.$$

Therefore,

$$\begin{aligned}
\sum_{T\in\mathcal{TB}(n,=k)} p^{\mathrm{ninv}(T)}q^{\mathrm{inv}(T)} &= \sum_{r=0}^{k-1}\sum_{s=0}^{k} \sum_{\substack{T\in\mathcal{TB}(n,=k)\\ \mathrm{inv}_1(T)=r,\mathrm{inv}_2(T)=s}} p^{\mathrm{ninv}(T)}q^{\mathrm{inv}(T)} \\
&= \sum_{r=0}^{k-1}\sum_{s=0}^{k} \sum_{\varphi(T)\in\mathcal{TB}(n-1,k)} p^{\mathrm{ninv}(\varphi(T))+k-r-1+k-s}q^{\mathrm{inv}(\varphi(T))+r+s} \\
&= \sum_{\varphi(T)\in\mathcal{TB}(n-1,k)} p^{\mathrm{ninv}(\varphi(T))}q^{\mathrm{inv}(\varphi(T))} \left(\sum_{r=0}^{k-1} p^{k-r-1}q^r\right)\left(\sum_{s=0}^{k} p^{k-s}q^s\right) \\
&= [k]_{p,q}[k+1]_{p,q}TB_{p,q}(n-1,k). \qquad (2.5)
\end{aligned}$$

Substituting (2.5) into (2.4), we obtain (2.3). Using the induction hypothesis (2.2) and the recurrence (1.1) of $PS_{p,q}(n,k)$, we conclude from (2.3) that $TB_{p,q}(n,k) = PS_{p,q}(n,k)$ for $n \ge k \ge 1$, and the theorem is verified. ∎

3 p,q-Legendre-Stirling identities

In this section, we give combinatorial proofs of three p,q-Legendre-Stirling identities using the combinatorial construction stated in Theorem 1.2. Before doing this, we establish the following generating function for 0-1 even tableaux of a regular shape partition.

Lemma 3.1. *Let $\mathcal{S}(i,l)$ denote the set of 0-1 even tableaux of shape (i^{2l}). Then*

$$\sum_{T\in\mathcal{S}(i,l)} p^{\mathrm{ninv}(T)}q^{\mathrm{inv}(T)} = \left([i]_{p,q}[i+1]_{p,q}\right)^l.$$

Proof. Let $T = (T_1, \ldots, T_{2l}) \in \mathcal{S}(i,l)$, where $sh(T_j) = i$ for $1 \leq j \leq 2l$. Assume that $\mathrm{inv}_j(T) = r_j$. Under the assumption that $T \in \mathcal{S}(i,l)$, we derive that $0 \leq r_{2j-1} \leq i-1$ and $0 \leq r_{2j} \leq i$ for $1 \leq j \leq l$. Moreover, $\mathrm{ninv}_{2j-1}(T) = i - 1 - r_{2j-1}$ and $\mathrm{ninv}_{2j}(T) = i - r_{2j}$. Therefore,

$$\begin{aligned}
&\sum_{T \in \mathcal{S}(i,l)} p^{\mathrm{ninv}(T)} q^{\mathrm{inv}(T)} \\
&= \sum_{j=1}^{l} \sum_{r_{2j-1}=0}^{i-1} \sum_{r_{2j}=0}^{i} \prod_{j=1}^{l} \left(p^{i-1-r_{2j-1}+i-r_{2j}} q^{r_{2j-1}+r_{2j}}\right) \\
&= \prod_{j=1}^{l} \left(\sum_{r_{2j-1}=0}^{i-1} p^{i-1-r_{2j-1}} q^{r_{2j-1}} \right) \left(\sum_{r_{2j}=0}^{i} p^{i-r_{2j}} q^{r_{2j}} \right) \\
&= \prod_{j=1}^{l} [i]_{p,q} [i+1]_{p,q} \\
&= ([i]_{p,q}[i+1]_{p,q})^l,
\end{aligned}$$

as desired. This completes the proof. ∎

We next establish the generating function for 0-1 even tableaux of arbitrarily-shaped partitions. When $p = 1$ and $q = 1$, Theorem 3.2 reduces to the rational generating function of $PS(n,k)$ due to Everitt, Littlejohn and Wellman [8].

Theorem 3.2. *For $k \geq 1$,*

$$\sum_{n \geq k} PS_{p,q}(n,k) t^n = \frac{t^k}{\prod_{i=1}^{k} (1 - ([i]_{p,q}[i+1]_{p,q}) \cdot t)}. \tag{3.1}$$

Proof. Let $T = (T_1, \ldots, T_{2n-2k}) \in \mathcal{TB}(n,k)$, where $sh(T_i) = \lambda_i$ for $1 \leq i \leq 2n-2k$. Since $\lambda = (\lambda_1, \ldots, \lambda_{2n-2k}) \in \mathcal{P}_e(k, n-k)$, we may assume that

$$\lambda = (k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 1^{m_1}),$$

where m_i is even for $1 \leq i \leq k$. Let $T^{(1)} = (T_1, \ldots, T_{m_k})$ and for $2 \leq i \leq k$, $T^{(i)} = (T_{m_k + m_{k-1} + \cdots + m_{k-i+2}+1}, \ldots, T_{m_k + m_{k-1} + \cdots + m_{k-i+1}})$. Obviously, $T^{(i)}$ is a 0-1 even tableau of shape $(k-i+1)^{m_{k-i+1}}$. Moreover,

$$T = \bigcup_{i=1}^{k} T^{(i)}, \quad \mathrm{inv}(T) = \sum_{i=1}^{k} \mathrm{inv}(T^{(i)}), \quad \text{and} \quad \mathrm{ninv}(T) = \sum_{i=1}^{k} \mathrm{ninv}(T^{(i)}).$$

The assumption that m_i is even allows us to assume that $m_i = 2\overline{m_i}$ for $1 \leq i \leq k$. Then

$$\overline{m}_1 + \cdots + \overline{m}_k = n - k.$$

Hence, by Theorem 1.2, we derive that

$$\begin{aligned}
& PS_{p,q}(n,k)t^n \\
&= \sum_{T\in\mathcal{TB}(n,k)} p^{\mathrm{ninv}(T)}q^{\mathrm{inv}(T)}t^n \\
&= \sum_{\substack{T^{(1)},\ldots,T^{(k)} \\ \overline{m}_1+\cdots+\overline{m}_k=n-k}} t^{\overline{m}_1+\cdots+\overline{m}_k+k} p^{\sum_{i=1}^k \mathrm{ninv}(T^{(i)})} q^{\sum_{i=1}^k \mathrm{inv}(T^{(i)})} \\
&= \sum_{\substack{T^{(1)},\ldots,T^{(k)} \\ \overline{m}_1+\cdots+\overline{m}_k=n-k}} t^{\overline{m}_1+\cdots+\overline{m}_k+k} \prod_{i=1}^{k} p^{\mathrm{ninv}(T^{(i)})} q^{\mathrm{inv}(T^{(i)})} \\
&= \sum_{\overline{m}_1+\cdots+\overline{m}_k+k=n} t^{\overline{m}_1+\cdots+\overline{m}_k+k} \prod_{i=1}^{k} \left(\sum_{T^{(i)}} p^{\mathrm{ninv}(T^{(i)})} q^{\mathrm{inv}(T^{(i)})} \right). \qquad (3.2)
\end{aligned}$$

By definition, we see that $T^{(i)}$ is a 0-1 even tableau of shape $(k-i+1)^{m_{k-i+1}}$ for $1\leq i\leq k$. It follows from Lemma 3.1 that for $1\leq i\leq k$,

$$\sum_{T^{(i)}} p^{\mathrm{ninv}(T^{(i)})} q^{\mathrm{inv}(T^{(i)})} = ([k-i+1]_{p,q}[k-i+2]_{p,q})^{\overline{m}_{k-i+1}}. \qquad (3.3)$$

Substituting (3.3) into (3.2), we obtain

$$\begin{aligned}
& \sum_{n\geq k} PS_{p,q}(n,k)t^n \\
&= t^k \sum_{\overline{m}_1,\overline{m}_2,\ldots,\overline{m}_k\geq 0} ([1]_{p,q}[2]_{p,q})^{\overline{m}_1}\,([2]_{p,q}[3]_{p,q})^{\overline{m}_2}\cdots([k]_{p,q}[k+1]_{p,q})^{\overline{m}_k}\, t^{\overline{m}_1+\overline{m}_2+\cdots+\overline{m}_k} \\
&= t^k \prod_{i=1}^{k} \sum_{n\geq 0} ([i]_{p,q}[i+1]_{p,q})^n\, t^n \\
&= \frac{t^k}{\prod_{i=1}^{k}(1-([i]_{p,q}[i+1]_{p,q})\cdot t)},
\end{aligned}$$

as desired. This completes the proof. ∎

The following recurrence relation can be viewed as a p,q-analogue of a vertical recurrence relation of $PS(n,k)$ due to Andrews, Gawronski, Littlejohn [3]. In fact, Theorem 3.3 can also be deduced from Theorem 3.2 using generating function approach. Here we aim to give a combinatorial proof of Theorem 3.3 based on Theorem 1.2.

Theorem 3.3. *For $n\geq k\geq 1$,*

$$PS_{p,q}(n+1,k+1) = \sum_{j=k}^{n} ([k+1]_{p,q}[k+2]_{p,q})^{n-j} PS_{p,q}(j,k). \qquad (3.4)$$

Proof. Let $T = (T_1, \ldots, T_{2n-2k}) \in \mathcal{TB}(n+1, k+1)$, where $sh(T_i) = \lambda_i$ for $1 \le i \le 2n-2k$. Under the fact that $\lambda = (\lambda_1, \ldots, \lambda_{2n-2k}) \in \mathcal{P}_e(k+1, n-k)$, we assume that

$$\lambda = ((k+1)^{m_{k+1}}, k^{m_k}, \ldots, 1^{m_1}),$$

where m_i is even for $1 \le i \le k+1$. We divide T into two disjoint 0-1 even tableaux $T^{(1)}$ and $T^{(2)}$, where

$$T^{(1)} = (T_1, \ldots, T_{m_{k+1}})$$

and

$$T^{(2)} = (T_{m_{k+1}+1}, \ldots, T_{2n-2k}).$$

Obviously, $T^{(1)}$ is a 0-1 even tableau of shape $(k+1)^{m_{k+1}}$ and $T^{(2)}$ is a 0-1 even tableau of shape $\hat{\lambda} = (k^{m_k}, (k-1)^{m_{k-1}}, \ldots, 1^{m_1})$ satisfying

$$\mathrm{inv}(T) = \mathrm{inv}(T^{(1)}) + \mathrm{inv}(T^{(2)}) \quad \text{and} \quad \mathrm{ninv}(T) = \mathrm{ninv}(T^{(1)}) + \mathrm{ninv}(T^{(2)}).$$

Since m_{k+1} is even, we may assume that $m_{k+1} = 2j$, where $0 \le j \le n-k$. Then $\hat{\lambda} \in \mathcal{P}_e(k, n-k-j)$. Hence, by Theorem 1.2, we derive that

$$\begin{aligned} &PS_{p,q}(n+1, k+1) \\ &= \sum_{T \in \mathcal{TB}(n+1,k+1)} p^{\mathrm{ninv}(T)} q^{\mathrm{inv}(T)} \\ &= \sum_{j=0}^{n-k} \left(\sum_{T^{(1)} \text{of shape } (k+1)^{2j}} p^{\mathrm{ninv}(T^{(1)})} q^{\mathrm{inv}(T^{(1)})} \right) \left(\sum_{T^{(2)} \text{of shape } \hat{\lambda} \in \mathcal{P}_e(k,n-k-j)} p^{\mathrm{ninv}(T^{(2)})} q^{\mathrm{inv}(T^{(2)})} \right). \end{aligned} \tag{3.5}$$

In view of Theorem 1.2, we have

$$\sum_{T^{(2)} \text{ of shape } \hat{\lambda} \in \mathcal{P}_e(k,n-k-j)} p^{\mathrm{ninv}(T^{(2)})} q^{\mathrm{inv}(T^{(2)})} = PS_{p,q}(n-j, k). \tag{3.6}$$

Using Lemma 3.1, we get

$$\sum_{T^{(1)} \text{ of shape } (k+1)^{2j}} p^{\mathrm{ninv}(T^{(1)})} q^{\mathrm{inv}(T^{(1)})} = ([k+1]_{p,q}[k+2]_{p,q})^j. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we obtain (3.4). This completes the proof. ∎

We finish this section by giving a combinatorial derivation of the following result. In fact, Theorem 3.4 can also be obtained from Theorem 3.2 using generating function techniques.

Theorem 3.4. *For $n \ge k \ge 1$,*

$$(n-k)PS_{p,q}(n,k) = \sum_{j=1}^{n-k} PS_{p,q}(n-j,k) \cdot (([1]_{p,q}[2]_{p,q})^j + \cdots + ([k]_{p,q}[k+1]_{p,q})^j). \tag{3.8}$$

Proof. Let $T = (T_1, \ldots, T_{2n-2k}) \in \mathcal{TB}(n,k)$, where $sh(T_i) = \lambda_i$ for $1 \le i \le 2n-2k$. A 0-1 even tableau T is called a colored 0-1 even tableau if there exists a colored $2i$-th part T_{2i}. Let $\mathcal{TB}^c(n,k)$ denote the set of colored 0-1 even tableaux of shape $\lambda \in \mathcal{P}_e(k, n-k)$. By the definition of colored 0-1 even tableaux, we see that

$$\sum_{T^c \in \mathcal{TB}^c(n,k)} p^{\mathrm{ninv}(T^c)} q^{\mathrm{inv}(T^c)} = (n-k) \sum_{T \in \mathcal{TB}(n,k)} p^{\mathrm{ninv}(T)} q^{\mathrm{inv}(T)}. \tag{3.9}$$

Hence it follows from Theorem 1.2 that

$$(n-k) PS_{p,q}(n,k) = \sum_{T^c \in \mathcal{TB}^c(n,k)} p^{\mathrm{ninv}(T^c)} q^{\mathrm{inv}(T^c)}. \tag{3.10}$$

We proceed to show that

$$\begin{aligned}&\sum_{T^c \in \mathcal{TB}^c(n,k)} p^{\mathrm{ninv}(T^c)} q^{\mathrm{inv}(T^c)} \\ &\quad = \sum_{j=1}^{n-k} PS_{p,q}(n-j,k) \cdot (([1]_{p,q}[2]_{p,q})^j + \cdots + ([k]_{p,q}[k+1]_{p,q})^j).\end{aligned} \tag{3.11}$$

Recall that $\mathcal{S}(i,j)$ denotes the set of 0-1 even tableaux of shape (i^{2j}). To prove (3.11), we will build a bijection ψ between $\mathcal{TB}^c(n,k)$ and $\cup_{j=1}^{n-k} \cup_{i=1}^{k} \mathcal{TB}(n-j,k) \times \mathcal{S}(i,j)$.

Let $T^c = (T_1, \ldots, T_{2n-2k}) \in \mathcal{TB}^c(n,k)$, where T_{2m} is colored $(1 \le m \le n-k)$. Assume that $sh(T_{2m}) = i$ and t is the smallest integer such that $sh(T_{2t+1}) = i$. Define

$$R = (T_1, \ldots, T_{2t}, T_{2m+1}, \ldots, T_{2n-2k}) \text{ and } S = (T_{2t+1}, \ldots, T_{2m}).$$

Set $m = t + j$. It is clear that $S \in \mathcal{S}(i,j)$ and $R \in \mathcal{TB}(n-j,k)$ and this process is reversed. Moreover,

$$\mathrm{inv}(T^c) = \mathrm{inv}(R) + \mathrm{inv}(S) \quad \text{and} \quad \mathrm{ninv}(T^c) = \mathrm{ninv}(R) + \mathrm{ninv}(S).$$

Hence

$$\begin{aligned}&\sum_{T^c \in \mathcal{TB}^c(n,k)} p^{\mathrm{ninv}(T^c)} q^{\mathrm{inv}(T^c)} \\ &= \sum_{j=1}^{n-k} \sum_{i=1}^{k} \sum_{(R,S) \in \mathcal{TB}(n-j,k) \times \mathcal{S}(i,j)} p^{\mathrm{ninv}(R)+\mathrm{ninv}(S)} q^{\mathrm{inv}(R)+\mathrm{inv}(S)} \\ &= \left(\sum_{j=1}^{n-k} \sum_{R \in \mathcal{TB}(n-j,k)} p^{\mathrm{ninv}(R)} q^{\mathrm{inv}(R)} \right) \left(\sum_{i=1}^{k} \sum_{S \in \mathcal{S}(i,j)} p^{\mathrm{ninv}(S)} q^{\mathrm{inv}(S)} \right).\end{aligned} \tag{3.12}$$

Invoking Theorem 1.2, we get

$$\sum_{R\in\mathcal{TB}(n-j,k)} p^{\mathrm{ninv}(R)}q^{\mathrm{inv}(R)} = PS_{p,q}(n-j,k). \tag{3.13}$$

In light of Lemma 3.1, we have

$$\sum_{S\in\mathcal{S}(i,j)} p^{\mathrm{ninv}(S)}q^{\mathrm{inv}(S)} = \left([i]_{p,q}[i+1]_{p,q}\right)^j. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we obtain (3.11). Combining (3.9) and (3.11) yields (3.8). Thus, we complete the proof. ∎

4 q-Log-concavity

The goal of this section is to give a combinatorial proof of Theorem 1.4. In view of Theorem 1.2, it suffices to show the following combinatorial assertion. Butler's bijection [4] for the strong q-log-concavity of q-binomial coefficients and Leroux's bijection [12] for the strong q-log-concavity of q-Stirling numbers of the second kind figure prominently in our proof of the following result.

Theorem 4.1. *For $n\geq 1$ and $1\leq k\leq l\leq n-1$, there is a weight preserving injection Φ from $\mathcal{TB}(n,k-1)\times\mathcal{TB}(n,l+1)$ to $\mathcal{TB}(n,k)\times\mathcal{TB}(n,l)$. More precisely, for $(T,W)\in\mathcal{TB}(n,k-1)\times\mathcal{TB}(n,l+1)$, we have*

$$\Phi(T,W)=(\widehat{T},\widehat{W})\in\mathcal{TB}(n,k)\times\mathcal{TB}(n,l)$$

such that

$$\mathrm{ninv}(\widehat{T})+\mathrm{ninv}(\widehat{W})=\mathrm{ninv}(T)+\mathrm{ninv}(W) \tag{4.1}$$

and

$$\mathrm{inv}(\widehat{T})+\mathrm{inv}(\widehat{W})=\mathrm{inv}(T)+\mathrm{inv}(W). \tag{4.2}$$

Proof. Let $T=(T_1,\ldots,T_{2n-2k+2})\in\mathcal{TB}(n,k-1)$, where $sh(T_i)=\lambda_i$ for $1\leq i\leq 2n-2k+2$ and let $W=(W_1,\ldots,W_{2n-2l-2})\in\mathcal{TB}(n,l+1)$, where $sh(W_i)=\mu_i$ for $1\leq i\leq 2n-2l-2$. We proceed to construct $\Phi(T,W)=(\widehat{T},\widehat{W})\in\mathcal{TB}(n,k)\times\mathcal{TB}(n,l)$ satisfying conditions (4.1) and (4.2) via two steps.

We first construct $(\overline{T},\overline{W})\in\mathcal{TB}(n+1,k)\times\mathcal{TB}(n-1,l)$ based on $(T,W)\in\mathcal{TB}(n,k-1)\times\mathcal{TB}(n,l+1)$ such that

$$\mathrm{ninv}(\overline{T})+\mathrm{ninv}(\overline{W})=\mathrm{ninv}(T)+\mathrm{ninv}(W) \tag{4.3}$$

and

$$\mathrm{inv}(\overline{T})+\mathrm{inv}(\overline{W})=\mathrm{inv}(T)+\mathrm{inv}(W). \tag{4.4}$$

We then construct $(\widehat{T},\widehat{W}) \in \mathcal{TB}(n,k)\times\mathcal{TB}(n,l)$ relying on $(\overline{T},\overline{W}) \in \mathcal{TB}(n+1,k)\times \mathcal{TB}(n-1,l)$ such that

$$\mathrm{ninv}(\widehat{T}) + \mathrm{ninv}(\widehat{W}) = \mathrm{ninv}(\overline{T}) + \mathrm{ninv}(\overline{W}) \tag{4.5}$$

and

$$\mathrm{inv}(\widehat{T}) + \mathrm{inv}(\widehat{W}) = \mathrm{inv}(\overline{T}) + \mathrm{inv}(\overline{W}). \tag{4.6}$$

Combining (4.3), (4.4), (4.5) and (4.6), we could conclude that $(\widehat{T},\widehat{W}) \in \mathcal{TB}(n,k)\times \mathcal{TB}(n,l)$ satisfying conditions (4.1) and (4.2).

Step 1. There are two cases.

Case 1.1. If $sh(W_1) = \mu_1 \le l$, then let $\overline{T} = T$ and $\overline{W} = W$. Clearly, $\overline{T} \in \mathcal{TB}(n+1,k)$ and $\overline{W} \in \mathcal{TB}(n-1,l)$, which satisfy the conditions (4.3) and (4.4).

Case 2.1. If $sh(W_1) = \mu_1 = l+1$, then we assume that

$$W_i = W_{i,1}W_{i,2}\cdots W_{i,\mu_i},$$

where $W_{i,j} = 0$ or 1 for $1 \le j \le \mu_i$. We now divide W_i into two parts $W_i^{(1)}$ and $W_i^{(2)}$, where

$$W_i^{(1)} = \begin{cases} W_{i,1}W_{i,2}\cdots W_{i,l-k+1}, & \text{if } \mu_i > l-k+1, \\ W_{i,1}W_{i,2}\cdots W_{i,\mu_i}, & \text{if } \mu_i \le l-k+1, \end{cases}$$

and

$$W_i^{(2)} = \begin{cases} W_{i,l-k+2}W_{i,l-k+3}\cdots W_{i,\mu_i}, & \text{if } \mu_i > l-k+1, \\ \emptyset, & \text{if } \mu_i \le l-k+1. \end{cases}$$

Let $W^{(2)} = (W_1^{(2)},\dots,W_{2n-2l-2}^{(2)})$, where $sh(W_i^{(2)}) = \mu_i^{(2)}$. Then

$$\mu^{(2)} = (\mu_1^{(2)},\dots,\mu_{2n-2l-2}^{(2)}) \tag{4.7}$$

is a partition with even multiplicities such that $\mu_1^{(2)} = k$ and $l(\mu^{(2)}) \le 2n-2l-2$. Let $T = (T_1,\dots,T_{2n-2k+2}) \in \mathcal{TB}(n,k-1)$, where $sh(T_i) = \lambda_i$ for $1 \le i \le 2n-2k+2$. By definition, we have

$$\lambda = (\lambda_1,\dots,\lambda_{2n-2k+2}) \in \mathcal{P}_e(k-1,n-k+1). \tag{4.8}$$

We further assume that

$$T_i = T_{i,1}T_{i,2}\cdots T_{i,\lambda_i},$$

where $T_{i,j} = 0$ or 1 for $1 \le j \le \lambda_i$. Assume that J is the minimum integer such that $\lambda_{J-1} \ge \mu_J^{(2)}$. We make the convention that $\lambda_0 = 0$. It is easy to see that such J exists since $2n-2k+2 \ge 2n-2l-2$ under the assumption that $k \le l$. Combining the minimum of J and the fact that λ and $\mu^{(2)}$ are partitions with even multiplicities, we derive that J must be odd. In this case, we assume that $J = 2r-1$ for $2 \le r \le n-l$. Moreover, the minimum of J implies that $\lambda_i \le \lambda_{i-1} < \mu_i^{(2)}$ for $1 \le i \le J-1 = 2r-2$.

Define

$$\overline{W}^{(2)} = (\overline{W}^{(2)}_1, \ldots, \overline{W}^{(2)}_{2r-2}, \overline{W}^{(2)}_{2r-1}, \ldots \overline{W}^{(2)}_{2n-2l-2})$$

and

$$\overline{T} = (\overline{T}_1, \ldots, \overline{T}_{2r-2}, \overline{T}_{2r-1}, \ldots, \overline{T}_{2n-2k+2}),$$

where

$$\overline{W}^{(2)}_i = \begin{cases} T_{i,1}\, T_{i,2} \cdots T_{i,\lambda_i}, & \text{if } 1 \le i \le 2r-2 \text{ and } 1 \in W^{(2)}_i \text{ and } 1 \in T_i, \\ W^{(2)}_{i,1}\ W^{(2)}_{i,2} \cdots W^{(2)}_{i,\lambda_i}, & \text{if } 1 \le i \le 2r-2 \text{ and } 1 \in W^{(1)}_i \\ & \ \text{or } 1 \le i \le 2r-2 \text{ and } 1 \notin W_i \\ & \ \text{or } 1 \le i \le 2r-2 \text{ and } 1 \notin T_i \text{ and } 1 \in W^{(2)}_i, \\ W^{(2)}_{i,1}\ W^{(2)}_{i,2} \cdots W^{(2)}_{i,\mu^{(2)}_i}, & \text{if } 2r-1 \le i \le 2n-2l-2, \end{cases} \tag{4.9}$$

and

$$\overline{T}_i = \begin{cases} W^{(2)}_{i,1} \ \cdots\ W^{(2)}_{i,\mu^{(2)}_i}, & \text{if } 1 \le i \le 2r-2 \text{ and } 1 \in W^{(2)}_i \text{ and } 1 \in T_i, \\ T_{i,1}\, T_{i,2} \ \cdots T_{i,\lambda_i}\, W^{(2)}_{i,\lambda_i+1} \cdots\ W^{(2)}_{i,\mu^{(2)}_i}, & \text{if } 1 \le i \le 2r-2 \text{ and } 1 \in W^{(1)}_i \\ & \ \text{or } 1 \le i \le 2r-2 \text{ and } 1 \notin W_i \\ & \ \text{or } 1 \le i \le 2r-2 \text{ and } 1 \notin T_i \text{ and } 1 \in W^{(2)}_i, \\ T_{i,1}\, T_{i,2} \ \cdots\ T_{i,\lambda_i}, & \text{if } 2r-1 \le i \le 2n-2k+2. \end{cases} \tag{4.10}$$

Here $1 \in S_i$ means there exists $1 \le j \le l(S_i)$ such that $S_{i,j} = 1$.

Let $\overline{W} = (\overline{W}_1, \ldots, \overline{W}_{2n-2l-2})$, where

$$\overline{W}_i = \begin{cases} W_{i,1}\ W_{i,2} \cdots\ W_{i,l-k}\ 1 \cup \overline{W}^{(2)}_i, & \text{if } 1 \notin T_i \text{ and } W^{(2)}_{i,j} = 1 \text{ for } \lambda_i + 1 \le j \le \mu_i, \\ W^{(1)}_i \cup \overline{W}^{(2)}_i, & \text{otherwise.} \end{cases} \tag{4.11}$$

We proceed to show that $(\overline{T}, \overline{W}) \in \mathcal{TB}(n+1,k) \times \mathcal{TB}(n-1,l)$ satisfying the conditions (4.3) and (4.4).

First, by definition, we see that the shape of $\overline{T}$ is

$$\overline{\lambda} = (\mu^{(2)}_1, \ldots, \mu^{(2)}_{2r-2}, \lambda_{2r-1}, \lambda_{2r}, \ldots, \lambda_{2n-2k+2})$$

and the shape of $\overline{W}^{(2)}$ is

$$\overline{\mu}^{(2)} = (\lambda_1, \ldots, \lambda_{2r-2}, \mu^{(2)}_{2r-1}, \mu^{(2)}_{2r}, \ldots, \mu^{(2)}_{2n-2l-2}).$$

Since $J = 2r-1$ is the minimum integer such that $\lambda_{J-1} \ge \mu^{(2)}_J$, we deduce that

$$\lambda_{2r-1} \le \lambda_{2r-3} = \lambda_{J-2} < \mu^{(2)}_{J-1} = \mu^{(2)}_{2r-2}.$$

It implies that $\overline{\lambda}$ and $\overline{\mu}$ are partitions. Clearly,

$$\ell(\overline{\lambda}) = 2n - 2k + 2, \quad \overline{\lambda}_1 = \mu_1^{(2)} = k,$$

and so

$$\overline{\lambda} \in \mathcal{P}_e(k, n-k+1).$$

Moreover,

$$\ell(\overline{\mu}^{(2)}) \leq 2n - 2l - 2, \quad \overline{\mu}_1^{(2)} = \lambda_1 \leq k - 1.$$

From the construction (4.11), we see that the shape of $\overline{W}$ is $\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_{2n-2l-2})$, where $\overline{\mu}_1 = \overline{\mu}_1^{(2)} + l - k + 1$. Hence we derive that

$$\overline{\mu} \in \mathcal{P}_e(l, n-l-1).$$

By the constructions (4.9) and (4.10), it is not hard to check that $\overline{T} \in \mathcal{TB}(n+1, k)$ and $\overline{W} \in \mathcal{TB}(n-1, l)$, which satisfy (4.3) and (4.4). Moreover, it can be checked that this process is reversible.

Step 2. We intend to construct $(\widehat{T}, \widehat{W}) \in \mathcal{TB}(n,k) \times \mathcal{TB}(n,l)$ relying on $(\overline{T}, \overline{W})$ obtained in Step 1. Let

$$\overline{T} = (\overline{T}_1, \ldots, \overline{T}_{2n-2k+2}) \in \mathcal{TB}(n+1, k),$$

where $sh(\overline{T}_i) = \overline{\lambda}_i$ for $1 \leq i \leq 2n - 2k + 2$ and let

$$\overline{W} = (\overline{W}_1, \ldots, \overline{W}_{2n-2l-2}) \in \mathcal{TB}(n-1, l),$$

where $sh(\overline{W}_i) = \overline{\mu}_i$ for $1 \leq i \leq 2n - 2l - 2$. By definition, we have

$$\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_{2n-2k+2}) \in \mathcal{P}_e(k, n-k+1)$$

and

$$\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_{2n-2l-2}) \in \mathcal{P}_e(l, n-l-1).$$

Let I be the minimum integer such that $\overline{\lambda}_{2l-2k+2+I-1} \geq \overline{\mu}_I$. It is easy to see that such I exists since $2n - 2k + 2 > 2n - 2l - 2$ under the assumption that $k \leq l$. Moreover, from the fact that $\overline{\lambda}$ and $\overline{\mu}$ are partitions with even multiplicities and the minimum of I, we deduce that I must be odd. In this case, we could assume that $I = 2s - 1$ for $1 \leq s \leq n - l$.

Define

$$\widehat{T} = (\overline{T}_1, \ldots, \overline{T}_{2l-2k+2s}, \overline{W}_{2s-1}, \overline{W}_{2s}, \ldots, \overline{W}_{2n-2l-2}) \tag{4.12}$$

and

$$\widehat{W} = (\overline{W}_1, \ldots, \overline{W}_{2s-2}, \overline{T}_{2l-2k+2+2s-1}, \ldots, \overline{T}_{2n-2k+2}) \tag{4.13}$$

We next show that $\widehat{T} \in \mathcal{TB}(n,k)$ and $\widehat{W} \in \mathcal{TB}(n,l)$ satisfying (4.5) and (4.6).

First, by definition, we see that the shape of $\widehat{T}$ is

$$\widehat{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_{2l-2k+2s}, \overline{\mu}_{2s-1}, \overline{\mu}_{2s}, \ldots, \overline{\mu}_{2n-2l-2})$$

and the shape of $\widehat{W}$ is

$$\widehat{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_{2s-2}, \overline{\lambda}_{2l-2k+2s+1}, \ldots, \overline{\lambda}_{2n-2k+2}).$$

Since $I(= 2s-1)$ is the minimum integer such that $\overline{\lambda}_{2l-2k+2+I-1} \geq \overline{\mu}_I$, we derive that

$$\overline{\lambda}_{2l-2k+2s+1} \leq \overline{\lambda}_{2l-2k+2s-1} = \overline{\lambda}_{2l-2k+2+I-2} < \overline{\mu}_{I-1} = \overline{\mu}_{2s-2}.$$

It implies that $\widehat{\lambda}$ and $\widehat{\mu}$ are partitions. Moreover, it is evident that

$$\ell(\widehat{\lambda}) = 2n-2k, \quad \widehat{\lambda}_1 = \overline{\lambda}_1 \leq k$$

and

$$\ell(\widehat{\mu}) = 2n-2l, \quad \widehat{\mu}_1 \leq \overline{\mu}_1 \leq l.$$

Hence we derive that

$$\widehat{\lambda} \in \mathcal{P}_e(k, n-k) \quad \text{and} \quad \widehat{\mu} \in \mathcal{P}_e(l, n-l).$$

From the constructions (4.12) and (4.13), we conclude that $\widehat{T} \in \mathcal{TB}(n,k)$ and $\widehat{W} \in \mathcal{TB}(n,l)$, which satisfy (4.5) and (4.6). Moreover, this process is reversible. Thus, we complete the proof. ∎

5 Concluding remarks

Recently, Mongelli [16] observed that the Legendre-Stirling numbers of the second kind are specializations of the complete homogeneous symmetric functions. Recall that the k-th complete homogeneous symmetric function $h_k(x_1, x_2, \ldots, x_n)$ is given by

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

for $k \geq 1$ and $h_0(x_1, \ldots, x_n) = 1$ by convention. Mongelli [16] observed that

$$PS(n,k) = h_{n-k}(2, 6, \ldots, k(k+1)),$$

where $PS(n,k)$ is the Legendre-Stirling numbers of the second kind. Mongelli [16] further defined the q-Legendre-Stirling numbers by

$$PS_q(n,k) = H_{j,n}^{x(x+1)}(q) = h_{n-k}([1]_q[2]_q, [2]_q[3]_q, \ldots, [k]_q[k+1]_q),$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$ and gave a combinatorial interpretation of $PS_q(n,k)$ in terms of a specialized set partitions.

We would like to point out that the p,q-Legendre-Stirling numbers of the second kind can also be expressed in terms of the complete homogeneous symmetric functions, namely,

$$PS_{p,q}(n,k) = h_{n-k}([1]_{p,q}[2]_{p,q}, \ldots, [k]_{p,q}[k+1]_{p,q}). \tag{5.1}$$

In this way, Theorem 3.2 is a specialization of the generating function of $h_k(x_1, x_2, \ldots, x_n)$ and Theorem 1.4 is a specialization of a result due to Sagan [18, Theorem 2.6]. As mentioned in the introduction, the main objective of this paper is to give a combinatorial interpretation of $PS_{p,q}(n, k)$ in terms of 0-1 even tableaux. This combinatorial construction enables us to give combinatorial derivations of three p, q-Legendre-Stirling identities and a direct combinatorial proof of q-log-concavity of $PS_q(n, k)$. It would be interesting to see other applications of this combinatorial construction of $PS_{p,q}(n, k)$.

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