Higher order Turán inequalities for the distinct partition function

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Abstract. We prove that the number $q(n)$ of partitions of n with distinct parts is log-concave for $n \geq 33$ and satisfies the third-order Turán inequalities for $n \geq 121$ conjectured by Craig and Pun. In doing so, we establish explicit error terms for $q(n)$ and for $q(n-1)q(n+1)/q(n)^2$ based on Chern's asymptotic formulas for η-quotients.

Keywords: Log-concavity, the third-order Turán inequalities, partitions into distinct parts, the first modified Bessel function of the first kind

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1 Introduction

The objective of this paper is to explore the log-concavity and its generalizations for the distinct partition function. A sequence $\{\alpha_n\}_{n\geq 0}$ of real numbers is said to be log-concave if for $n \geq 1$,

$$\alpha_n^2 \geq \alpha_{n-1}\alpha_{n+1}. \tag{1.1}$$

Note that log-concavity is also known as the Turán inequalities. There are many generalizations of log-concavity. One prominent generalization involves the theory of Jensen polynomials, see, for example, [10–12,29]. The Jensen polynomials $J_\alpha^{d,n}(X)$ of degree d and shift n associated to the sequence $\{\alpha_n\}_{n\geq 0}$ are defined by

$$J_\alpha^{d,n}(X) = \sum_{i=0}^{d} \binom{d}{i} \alpha_{n+i} X^i.$$

When $d = 2$ and shift $n-1$, the Jensen polynomial $J_\alpha^{2,n-1}(X)$ reduces to

$$J_\alpha^{2,n-1}(X) = \alpha_{n-1} + 2\alpha_n X + \alpha_{n+1}X^2.$$

It is clear that $\{\alpha_n\}_{n\geq 0}$ is log-concave at n if and only if $J_\alpha^{2,n-1}(X)$ has only real roots. In general, we say that the sequence $\{\alpha_n\}_{n\geq 0}$ satisfies the order $d \geq 3$ Turán inequality at n if and only if $J_\alpha^{d,n-1}(X)$ has only real roots. In particular, when

$d = 3$, the sequence $\{\alpha_n\}_{n\geq 0}$ is said to satisfy the third-order Turán inequalities if for $n \geq 1$,

$$4(\alpha_n^2 - \alpha_{n-1}\alpha_{n+1})(\alpha_{n+1}^2 - \alpha_n\alpha_{n+2}) \geq (\alpha_n\alpha_{n+1} - \alpha_{n-1}\alpha_{n+2})^2. \tag{1.2}$$

It should be noted that the Turán inequalities and its generalizations arise in the study of the Maclaurin coefficients of real entire functions in Laguerre-Pólya class, see, for example, [14], [28] and [29]. The Turán inequalities and the third-order Turán inequalities for the partition function were initially investigated by Chen [6], DeSalvo and Pak [13] and Nicolas [24]. Recall that a partition of n is a finite list of nondecreasing positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Let $p(n)$ denote the number of partitions of n. DeSalvo and Pak [13] and Nicolas [24] independently proved that the partition function $p(n)$ is log-concave for $n \geq 26$. Chen [6] conjectured that $p(n)$ satisfies the third-order Turán inequalities for $n \geq 95$, which was proved by Chen, Jia, and Wang [7]. Chen, Jia, and Wang [7] further conjectured that for $d \geq 4$, there exists a positive integer $N_p(d)$ such that $p(n)$ satisfies the order d Turán inequalities for $n \geq N_p(d)$, that is, the Jensen polynomial $J_p^{d,n-1}(X)$ associated to $p(n)$ has only real roots for $n \geq N_p(d)$. Griffin, Ono, Rolen, and Zagier [18] showed that for all $d \geq 1$, the Jensen polynomial $J_p^{d,n-1}(X)$ associated to $p(n)$ has only real roots when $n \to \infty$. More recently, Turán inequalities for other partition functions have been extensively investigated, see, for example, Bringmann, Kane, Rolen, and Tripp [5], Dong, Ji, and Jia [15], Engel [16], Liu and Zhang [23], Jia [21] and Ono, Pujahari, and Rolen [25].

The goal of this paper is to investigate the Turán inequalities and the third-order Turán inequalities for the distinct partition function. Let $q(n)$ denote the number of partitions of n with distinct parts. For example, there are eight partitions of 9 with distinct parts:

$$(9),\ (8,1),\ (7,2),\ (6,3),\ (6,2,1),\ (5,4),\ (5,3,1),\ (4,3,2).$$

It is known from Euler's partition theorem that $q(n)$ also counts the number of partitions of n with odd parts, see Andrews [2, Chapter 1] or Euler [17] .

The generating function for $q(n)$ is given by

$$\sum_{n\geq 0} q(n)q^n = \prod_{j=1}^{\infty}(1+q^j) = \prod_{j=0}^{\infty}\frac{1}{1-q^{2j+1}}.$$

Using the circle method, Hagis [19] and Hua [20] established a Rademacher-type formula for $q(n)$ in terms of Kloosterman sums and Bessel functions. Based on this formula, Craig and Pun [9] showed that $q(n)$ satisfies the order d Turán inequalities for sufficiently large n by employing a general result of Griffin, Ono, Rolen, and Zagier [18]. They also made the following conjecture.

Conjecture 1.1 (Craig-Pun). *The function $q(n)$ is log-concave for $n \geq 33$ and satisfies the third-order Turán inequalities for $n \geq 121$.*

The main objective of this paper is to confirm Conjecture 1.1. Instead of using the Rademacher-type formula for $q(n)$ due to Hagis [19] and Hua [20], we explore the utility of Chern's asymptotic formulas for η-quotients [8] in establishing the proof of Conjecture 1.1.

Appealing to Chern's asymptotic formulas, we obtain an asymptotic formula for $q(n)$ with an effective bound on the error term. To state our bound, we adopt the following notation:

$$\nu(n) = \frac{\pi\sqrt{24n+1}}{6\sqrt{2}}. \tag{1.3}$$

We have the following asymptotic formula for $q(n)$.

Theorem 1.2. *For $\nu(n) \geq 21$, or equivalently, $n \geq 135$,*

$$q(n) = \frac{\sqrt{2}\pi^2}{12\nu(n)} I_1(\nu(n)) + R(n), \tag{1.4}$$

where $I_1(s)$ is the first modified Bessel function of the first kind defined as

$$I_1(s) = \frac{s}{\pi}\int_{-1}^{1}(1-t^2)^{\frac{1}{2}}e^{st}\mathrm{d}t, \tag{1.5}$$

and

$$|R(n)| \leq \frac{\sqrt{3}\pi^{\frac{3}{2}}}{6\nu(n)^{\frac{1}{2}}}\exp\left(\frac{\nu(n)}{3}\right). \tag{1.6}$$

Using Theorem 1.2, we establish an upper bound and a lower bound for $q(n)$ which are required in the proof of Conjecture 1.1.

Theorem 1.3. *Let*

$$M(n) := \frac{\sqrt{2}\pi^2}{12\nu(n)} I_1(\nu(n)). \tag{1.7}$$

For $\nu(n) \geq 43$,

$$M(n)\left(1 - \frac{1}{\nu(n)^6}\right) \leq q(n) \leq M(n)\left(1 + \frac{1}{\nu(n)^6}\right). \tag{1.8}$$

Let

$$Q(n) = \frac{q(n-1)q(n+1)}{q(n)^2}. \tag{1.9}$$

It is evident from (1.1) that $q(n)$ is log-concave for $n \geq 33$ is equivalent to $Q(n) \leq 1$ for $n \geq 33$. Using (1.2), one can check that $q(n)$ satisfies the third order Turán inequalities for $n \geq 121$ whenever

$$4(1-Q(n))(1-Q(n+1)) - (1-Q(n)Q(n+1))^2 \geq 0 \tag{1.10}$$

for $n \geq 121$. Hence to prove Conjecture 1.1, it suffices to establish efficient lower and upper bounds for $Q(n)$. By resorting to Theorem 1.3, we obtain

Theorem 1.4. *Let*

$$E_Q(n) := 1 - \frac{\pi^4}{36\nu(n)^3} + \frac{\pi^4}{12\nu(n)^4} - \frac{\pi^4}{32\nu(n)^5}. \tag{1.11}$$

Then for $\nu(n) \geq 67$*,*

$$E_Q(n) - \frac{135}{\nu(n)^6} < Q(n) < E_Q(n) + \frac{126 + \frac{\pi^8}{1296}}{\nu(n)^6}. \tag{1.12}$$

Subsequently, we demonstrate that Conjecture 1.1 can be inferred from Theorem 1.4. As a result, We arrive at the following consequence.

Theorem 1.5. *For* $n \geq 121$*, the cubic polynomial*

$$q(n-1) + 3q(n)x + 3q(n+1)x^2 + q(n+2)x^3$$

has only real roots.

The paper is organized as follows. In Section 2, we derive some inequalities involving the first modified Bessel function of the first kind, which are necessary in the proof of Theorem 1.4. In Section 3, we first prove Theorem 1.2 with the aid of Chern's asymptotic formulas for η-quotients and then use Theorem 1.2 to prove Theorem 1.3. Section 4 is dedicated to deriving Theorem 1.4 through the utilization of Theorem 1.3 and the inequalities on the first modified Bessel function of the first kind established in Section 2. In Section 5, we confirm Conjecture 1.1 with the aid of Theorem 1.4. We conclude in Section 6 with some problems for further investigation.

2 Explicit bounds for $I_1(s)$

To apply the inequality stated in Theorem 1.3 to the proof of Conjecture 1.1, we need to establish specific inequalities related to the first modified Bessel function $I_1(s)$ of the first kind. Before doing this, let us first recall the definitions of the Gamma function $\Gamma(a)$ and the upper incomplete Gamma function $\Gamma(a,s)$, see [1, Chapter 6].

The Gamma function $\Gamma(a)$ is defined by

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\mathrm{d}t.$$

It is known that

$$\Gamma(a+1) = a\Gamma(a) \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

see [27, pp. 32–34].

The upper incomplete Gamma function $\Gamma(a,s)$ is defined by

$$\Gamma(a,s)=\int_s^{\infty} t^{a-1}e^{-t}\mathrm{d}t.$$

The following estimate on $\Gamma(a,s)$ can be derived from the proof of Proposition 2.6 of Pinelis [26] which is required in the proof of Lemma 2.2. For $a\geq 1$ and $s\geq a$,

$$\Gamma(a,s)\leq as^{a-1}e^{-s}. \tag{2.1}$$

The first inequality on $I_1(s)$ for the proof of Conjecture 1.1 is due to Bringmann, Kane, Rolen, and Tripp [5, Lemma 2.2].

Lemma 2.1 (Bringmann-Kane-Rolen-Tripp)**.** *For $s\geq 1$,*

$$I_1(s)\leq \sqrt{\frac{2}{\pi s}}e^{s}. \tag{2.2}$$

We also need the further estimate on $I_1(s)$.

Lemma 2.2. *Let*

$$E_I(s):=1-\frac{3}{8s}-\frac{15}{128s^2}-\frac{105}{1024s^3}-\frac{4725}{32768s^4}-\frac{72765}{262144s^5}. \tag{2.3}$$

Then for $s\geq 26$,

$$\frac{e^s}{\sqrt{2\pi s}}\left(E_I(s)-\frac{31}{s^6}\right)\leq I_1(s)\leq \frac{e^s}{\sqrt{2\pi s}}\left(E_I(s)+\frac{31}{s^6}\right). \tag{2.4}$$

Proof. We start with the integral definition (1.5) of $I_1(s)$,

$$I_1(s)=\frac{s}{\pi}\int_0^1(1-t^2)^{\frac{1}{2}}e^{st}\mathrm{d}t+\frac{s}{\pi}\int_{-1}^0(1-t^2)^{\frac{1}{2}}e^{st}\mathrm{d}t. \tag{2.5}$$

It is clear that

$$\left|\frac{s}{\pi}\int_{-1}^0(1-t^2)^{\frac{1}{2}}e^{st}\mathrm{d}t\right|\leq\frac{s}{\pi}. \tag{2.6}$$

We next estimate the first integral in (2.5). Setting $u=1-t$, we have

$$\frac{s}{\pi}\int_0^1(1-t^2)^{\frac{1}{2}}e^{st}\mathrm{d}t=\frac{se^s}{\pi}\int_0^1(2-u)^{\frac{1}{2}}u^{\frac{1}{2}}e^{-su}\mathrm{d}u. \tag{2.7}$$

Using Taylor's formula, we find that

$$(2-u)^{\frac{1}{2}}=\sqrt{2}-\frac{u}{2\sqrt{2}}-\frac{u^2}{16\sqrt{2}}-\frac{u^3}{64\sqrt{2}}-\frac{5u^4}{1024\sqrt{2}}-\frac{7u^5}{4096\sqrt{2}}+c(\xi)u^6, \tag{2.8}$$

where

$$c(\xi) = \frac{1}{6!}\left(\frac{\mathrm{d}^6}{\mathrm{d}u^6}(2-u)^{\frac{1}{2}}\right)_{u=\xi} = -\frac{21}{1024}(2-\xi)^{-\frac{11}{2}} \quad \text{for some} \quad \xi \in (0,1). \tag{2.9}$$

Substituting (2.8) into (2.7), we obtain

$$\begin{aligned}
&\frac{\sqrt{2}se^s}{\pi}\int_0^1 \left(u^{\frac{1}{2}} - \frac{1}{4}u^{\frac{3}{2}} - \frac{1}{32}u^{\frac{5}{2}} - \frac{1}{128}u^{\frac{7}{2}} - \frac{5}{2048}u^{\frac{9}{2}} - \frac{7}{8192}u^{\frac{11}{2}}\right)e^{-su}\mathrm{d}u \\
&\quad + \frac{se^s}{\pi}\int_0^1 c(\xi)u^{\frac{13}{2}}e^{-su}\mathrm{d}u \\
&= \frac{\sqrt{2}se^s}{\pi}\left(\int_0^\infty - \int_1^\infty\right)\left(u^{\frac{1}{2}} - \frac{1}{4}u^{\frac{3}{2}} - \frac{1}{32}u^{\frac{5}{2}} - \frac{1}{128}u^{\frac{7}{2}} - \frac{5}{2048}u^{\frac{9}{2}} - \frac{7}{8192}u^{\frac{11}{2}}\right)e^{-su}\mathrm{d}u \\
&\quad + \frac{se^s}{\pi}\int_0^1 c(\xi)u^{\frac{13}{2}}e^{-su}\mathrm{d}u := I_1^{(1)}(s) + I_1^{(2)}(s) + I_1^{(3)}(s).
\end{aligned} \tag{2.10}$$

Evaluating the first integral $I_1^{(1)}(s)$ in (2.10) yields the main term:

$$\begin{aligned}
I_1^{(1)}(s) =& \frac{\sqrt{2}se^s}{\pi}\int_0^\infty \left(u^{\frac{1}{2}} - \frac{1}{4}u^{\frac{3}{2}} - \frac{1}{32}u^{\frac{5}{2}} - \frac{1}{128}u^{\frac{7}{2}} - \frac{5}{2048}u^{\frac{9}{2}} - \frac{7}{8192}u^{\frac{11}{2}}\right)e^{-su}\mathrm{d}u \\
=& \frac{\sqrt{2}e^s}{\sqrt{s}\pi}\int_0^\infty \Bigg((su)^{\frac{1}{2}} - \frac{1}{4s}(su)^{\frac{3}{2}} - \frac{1}{32s^2}(su)^{\frac{5}{2}} - \frac{1}{128s^3}(su)^{\frac{7}{2}} \\
&\qquad - \frac{5}{2048s^4}(su)^{\frac{9}{2}} - \frac{7}{8192s^5}(su)^{\frac{11}{2}}\Bigg)e^{-su}\mathrm{d}(su) \\
=& \frac{\sqrt{2}e^s}{\sqrt{s}\pi}\Bigg(\Gamma\left(\frac{3}{2}\right) - \frac{1}{4s}\Gamma\left(\frac{5}{2}\right) - \frac{1}{32s^2}\Gamma\left(\frac{7}{2}\right) - \frac{1}{128s^3}\Gamma\left(\frac{9}{2}\right) \\
&\qquad - \frac{5}{2048s^4}\Gamma\left(\frac{11}{2}\right) - \frac{7}{8192s^5}\Gamma\left(\frac{13}{2}\right)\Bigg) \\
=& \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{3}{8s} - \frac{15}{128s^2} - \frac{105}{1024s^3} - \frac{4725}{32768s^4} - \frac{72765}{262144s^5}\right).
\end{aligned} \tag{2.11}$$

We proceed to evaluate the second integral $I_1^{(2)}(s)$ in (2.10):

$$
\begin{aligned}
I_1^{(2)}(s) =& -\frac{\sqrt{2}se^s}{\pi}\int_1^\infty \left(u^{\frac{1}{2}} - \frac{1}{4}u^{\frac{3}{2}} - \frac{1}{32}u^{\frac{5}{2}} - \frac{1}{128}u^{\frac{7}{2}} - \frac{5}{2048}u^{\frac{9}{2}} - \frac{7}{8192}u^{\frac{11}{2}}\right)e^{-su}\mathrm{d}u \\
&= -\frac{\sqrt{2}e^s}{\sqrt{s}\pi}\int_s^\infty \Bigg((su)^{\frac{1}{2}} - \frac{1}{4s}(su)^{\frac{3}{2}} - \frac{1}{32s^2}(su)^{\frac{5}{2}} - \frac{1}{128s^3}(su)^{\frac{7}{2}} - \\
&\qquad - \frac{5}{2048s^4}(su)^{\frac{9}{2}} - \frac{7}{8192s^5}(su)^{\frac{11}{2}}\Bigg)e^{-su}\mathrm{d}(su) \\
&= \frac{\sqrt{2}e^s}{\sqrt{s}\pi}\Bigg(-\Gamma\left(\frac{3}{2},s\right) + \frac{1}{4s}\Gamma\left(\frac{5}{2},s\right) + \frac{1}{32s^2}\Gamma\left(\frac{7}{2},s\right) + \frac{1}{128s^3}\Gamma\left(\frac{9}{2},s\right) \\
&\qquad + \frac{5}{2048s^4}\Gamma\left(\frac{11}{2},s\right) + \frac{7}{8192s^5}\Gamma\left(\frac{13}{2},s\right)\Bigg), \\
&\overset{(2.1)}{\le} \frac{\sqrt{2}e^s}{\sqrt{s}\pi}\left(\frac{3}{2} + \frac{1}{4}\cdot\frac{5}{2} + \frac{1}{32}\cdot\frac{7}{2} + \frac{1}{128}\cdot\frac{9}{2} + \frac{5}{2048}\cdot\frac{11}{2} + \frac{7}{8192}\cdot\frac{13}{2}\right)\sqrt{s}e^{-s} \\
&= \frac{37495\sqrt{2}}{16384\pi} \qquad \text{for} \quad s \ge \frac{13}{2}.
\end{aligned} \tag{2.12}
$$

It remains to estimate $I_1^{(3)}(s)$ in (2.10). From (2.9), we see that

$$
\begin{aligned}
|I_1^{(3)}(s)| = \left|\frac{se^s}{\pi}\int_0^1 c(\xi)u^{\frac{13}{2}}e^{-su}\mathrm{d}u\right| &\le \frac{21s^{-\frac{13}{2}}e^s}{1024\pi}\int_0^\infty (su)^{\frac{13}{2}}e^{-su}\mathrm{d}(su) \\
&= \frac{21s^{-\frac{13}{2}}e^s}{1024\pi}\Gamma\left(\frac{15}{2}\right) \\
&= \frac{2837835}{131072\sqrt{\pi}}s^{-\frac{13}{2}}e^s.
\end{aligned} \tag{2.13}
$$

Combining (2.6), (2.11), (2.12) and (2.13), we derive that for $s \ge \frac{13}{2}$,

$$
I_1(s) = \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{3}{8s} - \frac{15}{128s^2} - \frac{105}{1024s^3} - \frac{4725}{32768s^4} - \frac{72765}{262144s^5}\right) + r(s), \tag{2.14}
$$

where

$$
\begin{aligned}
|r(s)| &\le \frac{s}{\pi} + \frac{37495\sqrt{2}}{16384\pi} + \frac{2837835}{131072\sqrt{\pi}}s^{-\frac{13}{2}}e^s \\
&= \frac{e^s}{\sqrt{2\pi s}}\cdot\frac{1}{s^6}\left(\left(\frac{\sqrt{2}s}{\sqrt{\pi}} + \frac{37495}{8192\sqrt{\pi}}\right)s^{\frac{13}{2}}e^{-s} + \frac{2837835\sqrt{2}}{131072}\right).
\end{aligned}
$$

To prove (2.4), it suffices to show that for $s \ge 26$,

$$
|r(s)| \le \frac{e^s}{\sqrt{2\pi s}}\cdot\frac{31}{s^6}. \tag{2.15}
$$

Define

$$f(s) := \left(\frac{\sqrt{2}s}{\sqrt{\pi}} + \frac{37495}{8192\sqrt{\pi}}\right) s^{\frac{13}{2}} e^{-s} + \frac{2837835\sqrt{2}}{131072}.$$

Observe that

$$f'(s) = s^{\frac{11}{2}} e^{-s} \left(-\frac{\sqrt{2}}{\sqrt{\pi}}s^2 + \frac{5\left(12288\sqrt{2}-7499\right)}{8192\sqrt{\pi}}s + \frac{487435}{16384\sqrt{\pi}}\right).$$

Since $f'(s) < 0$ for $s \geq 8$, we deduce that $f(s)$ is decreasing when $s \geq 8$. This implies that when $s \geq 26$,

$$f(s) \leq f(26) \approx 30.8068 < 31.$$

Hence the inequality (2.15) is valid. Combining (2.14) and (2.15), we are led to (2.4) in Lemma 2.2. This completes the proof. ∎

By utilizing Lemma 2.2, we derive the following inequalities on $I_1(\nu(n-1))I_1(\nu(n+1))/I_1(\nu(n))^2$, which are essential for establishing Theorem 1.4.

Lemma 2.3. *For $\nu(n) \geq 60$,*

$$\frac{I_1(\nu(n-1))I_1(\nu(n+1))}{I_1(\nu(n))^2} \geq \frac{\nu(n)}{\sqrt{\nu(n-1)\nu(n+1)}}\left(1 - \frac{\pi^4}{36\nu(n)^3} - \frac{5\pi^8}{2592\nu(n)^7}\right) \times \left(1 - \frac{\pi^4}{32\nu(n)^5} - \frac{129}{\nu(n)^6}\right) \tag{2.16}$$

and

$$\frac{I_1(\nu(n-1))I_1(\nu(n+1))}{I_1(\nu(n))^2} \leq \frac{\nu(n)}{\sqrt{\nu(n-1)\nu(n+1)}}\left(1 - \frac{\pi^4}{36\nu(n)^3} + \frac{\pi^8}{1296\nu(n)^6}\right) \times \left(1 - \frac{\pi^4}{32\nu(n)^5} + \frac{121}{\nu(n)^6}\right). \tag{2.17}$$

Proof. Using (2.4), we find that for $\nu(n) \geq 26$,

$$\frac{I_1(\nu(n-1))I_1(\nu(n+1))}{I_1(\nu(n))^2} \geq \frac{\nu(n)}{\sqrt{\nu(n-1)\nu(n+1)}} e^{\nu(n-1)+\nu(n+1)-2\nu(n)} L(n) \tag{2.18}$$

and

$$\frac{I_1(\nu(n-1))I_1(\nu(n+1))}{I_1(\nu(n))^2} \leq \frac{\nu(n)}{\sqrt{\nu(n-1)\nu(n+1)}} e^{\nu(n-1)+\nu(n+1)-2\nu(n)} R(n), \tag{2.19}$$

where $\nu(n)$ and $E_I(s)$ are defined as (1.3) and (2.3) respectively,

$$L(n) = \frac{\left(E_I(\nu(n-1)) - \frac{31}{\nu(n-1)^6}\right)\left(E_I(\nu(n+1)) - \frac{31}{\nu(n+1)^6}\right)}{\left(E_I(\nu(n)) + \frac{31}{\nu(n)^6}\right)^2} \tag{2.20}$$

and

$$R(n)=\frac{\left(E_I(\nu(n-1))+\frac{31}{\nu(n-1)^6}\right)\left(E_I(\nu(n+1))+\frac{31}{\nu(n+1)^6}\right)}{\left(E_I(\nu(n))-\frac{31}{\nu(n)^6}\right)^2}. \tag{2.21}$$

To obtain (2.16) and (2.17), we intend to estimate $\exp(\nu(n-1)+\nu(n+1)-2\nu(n))$, $L(n)$ and $R(n)$ in terms of $\nu(n)$. From the definition (1.3) of $\nu(n)$, we see that for $\nu(n)\geq 2$,

$$\nu(n-1)=\sqrt{\nu(n)^2-\frac{\pi^2}{3}},\quad \nu(n+1)=\sqrt{\nu(n)^2+\frac{\pi^2}{3}}. \tag{2.22}$$

Observe that for $\nu(n)\geq 2$,

$$\nu(n-1)=\nu(n)-\frac{\pi^2}{6\nu(n)}-\frac{\pi^4}{72\nu(n)^3}-\frac{\pi^6}{432\nu(n)^5}-\frac{5\pi^8}{10368\nu(n)^7}+O\left(\frac{1}{\nu(n)^8}\right)$$

and

$$\nu(n+1)=\nu(n)+\frac{\pi^2}{6\nu(n)}-\frac{\pi^4}{72\nu(n)^3}+\frac{\pi^6}{432\nu^5(n)}-\frac{5\pi^8}{10368\nu(n)^7}+O\left(\frac{1}{\nu(n)^8}\right),$$

so it is readily checked that for $\nu(n)\geq 3$,

$$d_v(n)<\nu(n-1)<u_v(n), \tag{2.23}$$

$$\bar{d}_v(n)<\nu(n+1)<\bar{u}_v(n), \tag{2.24}$$

where

$$\begin{aligned} d_v(n)&=\nu(n)-\frac{\pi^2}{6\nu(n)}-\frac{\pi^4}{72\nu(n)^3}-\frac{\pi^6}{432\nu(n)^5}-\frac{5\pi^8}{5184\nu(n)^7},\\ u_v(n)&=\nu(n)-\frac{\pi^2}{6\nu(n)}-\frac{\pi^4}{72\nu(n)^3}-\frac{\pi^6}{432\nu(n)^5},\\ \bar{d}_v(n)&=\nu(n)+\frac{\pi^2}{6\nu(n)}-\frac{\pi^4}{72\nu(n)^3}+\frac{\pi^6}{432\nu(n)^5}-\frac{5\pi^8}{5184\nu(n)^7},\\ \bar{u}_v(n)&=\nu(n)+\frac{\pi^2}{6\nu(n)}-\frac{\pi^4}{72\nu(n)^3}+\frac{\pi^6}{432\nu(n)^5}. \end{aligned} \tag{2.25}$$

With (2.23) and (2.24) in hands, we are now in a position to bound $\exp(\nu(n-1)$ $+\nu(n+1)-2\nu(n))$, $L(n)$ and $R(n)$ in terms of $\nu(n)$.

We first estimate $\exp(\nu(n-1)+\nu(n+1)-2\nu(n))$. Applying (2.23) and (2.24), we find that for $\nu(n)\geq 3$,

$$-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}<\nu(n-1)+\nu(n+1)-2\nu(n)<-\frac{\pi^4}{36\nu(n)^3}.$$

It follows that for $\nu(n) \geq 3$,

$$\exp(\nu(n-1)+\nu(n+1)-2\nu(n)) < \exp\left(-\frac{\pi^4}{36\nu(n)^3}\right) \tag{2.26}$$

and

$$\exp(\nu(n-1)+\nu(n+1)-2\nu(n)) > \exp\left(-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}\right). \tag{2.27}$$

Note that for $s<0$,

$$1+s<e^s<1+s+s^2.$$

Hence we derive that

$$\exp\left(-\frac{\pi^4}{36\nu(n)^3}\right) < 1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^8}{1296\nu(n)^6} \tag{2.28}$$

and

$$\exp\left(-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}\right) > 1-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}. \tag{2.29}$$

Combining (2.26) and (2.28) yields that for $\nu(n) \geq 3$,

$$\exp\left(\nu(n-1)+\nu(n+1)-2\nu(n)\right) < 1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^8}{1296\nu(n)^6}. \tag{2.30}$$

Using (2.27) together with (2.29), we find that for $\nu(n) \geq 3$,

$$\exp\left(\nu(n-1)+\nu(n+1)-2\nu(n)\right) > 1-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}. \tag{2.31}$$

Next, we estimate $L(n)$ and $R(n)$. Let

$$\begin{aligned} P_l(n) = & \frac{1}{\nu(n-1)^6\nu(n+1)^6}\left(\nu(n-1)^6-\frac{3}{8}\nu(n-1)^4u_v(n)-\frac{15}{128}\nu(n-1)^4\right. \\ & \left.-\frac{105}{1024}\nu(n-1)^2u_v(n)-\frac{4725}{32768}\nu(n-1)^2-\frac{72765}{262144}u_v(n)-31\right) \\ & \times\left(\nu(n+1)^6-\frac{3}{8}\nu(n+1)^4\bar{u}_v(n)-\frac{15}{128}\nu(n+1)^4-\frac{105}{1024}\nu(n+1)^2\bar{u}_v(n)\right. \\ & \left.-\frac{4725}{32768}\nu(n+1)^2-\frac{72765}{262144}\bar{u}_v(n)-31\right) \end{aligned} \tag{2.32}$$

and

$$P_r(n) = \frac{1}{\nu(n-1)^6\nu(n+1)^6}\left(\nu(n-1)^6 - \frac{3}{8}\nu(n-1)^4 d_v(n) - \frac{15}{128}\nu(n-1)^4 \right.$$
$$- \frac{105}{1024}\nu(n-1)^2 d_v(n) - \frac{4725}{32768}\nu(n-1)^2 - \left.\frac{72765}{262144}d_v(n) + 31\right)$$
$$\times \left(\nu(n+1)^6 - \frac{3}{8}\nu(n+1)^4 \bar{d}_v(n) - \frac{15}{128}\nu(n+1)^4 - \frac{105}{1024}\nu(n+1)^2 \bar{d}_v(n)\right.$$
$$\left. - \frac{4725}{32768}\nu(n+1)^2 - \frac{72765}{262144}\bar{d}_v(n) + 31\right). \tag{2.33}$$

Applying (2.23) and (2.24) into (2.20) and (2.21), it is not difficult to check that for $\nu(n) \geq 3$,

$$L(n) \geq \frac{P_l(n)}{\left(E_I(\nu(n)) + \frac{31}{\nu(n)^6}\right)^2} \tag{2.34}$$

and

$$R(n) \leq \frac{P_r(n)}{\left(E_I(\nu(n)) - \frac{31}{\nu(n)^6}\right)^2}. \tag{2.35}$$

To bound $L(n)$ and $R(n)$ in terms of $\nu(n)$, we shall show that for $\nu(n) \geq 60$,

$$\frac{P_l(n)}{\left(E_I(\nu(n)) + \frac{31}{\nu(n)^6}\right)^2} \geq 1 - \frac{\pi^4}{32\nu(n)^5} - \frac{129}{\nu(n)^6}$$
$$= \frac{32\nu(n)^6 - \pi^4\nu(n) - 4128}{32\nu(n)^6} \tag{2.36}$$

and

$$\frac{P_r(n)}{\left(E_I(\nu(n)) - \frac{31}{\nu(n)^6}\right)^2} \leq 1 - \frac{\pi^4}{32\nu(n)^5} + \frac{121}{\nu(n)^6}$$
$$= \frac{32\nu(n)^6 - \pi^4\nu(n) + 3872}{32\nu(n)^6}, \tag{2.37}$$

which are equivalent to showing that for $\nu(n) \geq 60$,

$$32\nu(n)^6 P_l(n) - \left(32\nu(n)^6 - \pi^4\nu(n) - 4128\right)\left(E_I(\nu(n)) + \frac{31}{\nu(n)^6}\right)^2 \geq 0 \tag{2.38}$$

and

$$\left(32\nu(n)^6 - \pi^4\nu(n) + 3872\right)\left(E_I(\nu(n)) - \frac{31}{\nu(n)^6}\right)^2 - 32\nu(n)^6 P_r(n) \geq 0. \tag{2.39}$$

Substituting (2.22) and (2.25) into (2.32) and (2.33), we find that

$$32\nu(n)^6P_l(n) - \left(32\nu(n)^6 - \pi^4\nu(n) - 4128\right)\left(E_I(\nu(n)) + \frac{31}{\nu(n)^6}\right)^2$$
$$= \frac{\sum_{j=0}^{26} a_j\nu(n)^j}{\nu(n)^{14}\nu(n-1)^6\nu(n+1)^6} \tag{2.40}$$

and

$$\left(32\nu(n)^6 - \pi^4\nu(n) + 3872\right)\left(E_I(\nu(n)) - \frac{31}{\nu(n)^6}\right)^2 - 32\nu(n)^6P_r(n)$$
$$= \frac{\sum_{j=0}^{26} b_j\nu(n)^j}{\nu(n)^{14}\nu(n-1)^6\nu(n+1)^6}, \tag{2.41}$$

where a_j and b_j are real numbers. Here we just list the values of a_{24}, a_{25}, a_{26}, b_{24}, b_{25} and b_{26}:

$$a_{24} = 78 - \frac{175\pi^4}{64}, \quad a_{25} = -1608 - \frac{19\pi^4}{16}, \quad a_{26} = 160 - \frac{4\pi^4}{3},$$

$$b_{24} = 102 + \frac{175\pi^4}{64}, \quad b_{25} = \frac{19\pi^4}{16} - 1416, \quad b_{26} = \frac{4\pi^4}{3} - 96.$$

It can be readily checked that for any $0 \le j \le 23$ and $\nu(n) \ge 27$,

$$-|a_j|\nu(n)^j \ge -|a_{24}|\nu(n)^{24}$$

and

$$-|b_j|\nu(n)^j \ge -|b_{24}|\nu(n)^{24}.$$

It follows that for $\nu(n) \ge 27$,

$$\sum_{j=0}^{26} a_j\nu(n)^j \ge -\sum_{j=0}^{24}|a_j|\,\nu(n)^j + a_{25}\nu(n)^{25} + a_{26}\nu(n)^{26}$$
$$\ge -25\,|a_{24}|\,\nu(n)^{24} + a_{25}\nu(n)^{25} + a_{26}\nu(n)^{26}$$

and

$$\sum_{j=0}^{26} b_j\nu(n)^j \ge -\sum_{j=0}^{24}|b_j|\,\nu(n)^j + b_{25}\nu(n)^{25} + b_{26}\nu(n)^{26}$$
$$\ge -25\,|b_{24}|\,\nu(n)^{24} + b_{25}\nu(n)^{25} + b_{26}\nu(n)^{26}.$$

Moreover, one can easily check that for $\nu(n) \ge 60$,

$$-25\,|a_{24}|\,\nu(n)^{24} + a_{25}\nu(n)^{25} + a_{26}\nu(n)^{26} \ge 0$$

and

$$-25\,|b_{24}|\,\nu(n)^{24}+b_{25}\nu(n)^{25}+b_{26}\nu(n)^{26}\geq 0.$$

Hence (2.38) and (2.39) hold for $\nu(n)\geq 60$, and so (2.36) and (2.37) hold for $\nu(n)\geq 60$. Substituting (2.36) into (2.34), we derive that for $\nu(n)\geq 60$,

$$L(n)\geq 1-\frac{\pi^4}{32\nu(n)^5}-\frac{129}{\nu(n)^6}. \tag{2.42}$$

Plugging (2.37) into (2.35), we deduce that for $\nu(n)\geq 60$,

$$R(n)\leq 1-\frac{\pi^4}{32\nu(n)^5}+\frac{121}{\nu(n)^6}. \tag{2.43}$$

Applying (2.30), (2.31), (2.42) and (2.43) into (2.18) and (2.19), we are lead to (2.16) and (2.17). This completes the proof. ∎

3 Proofs of Theorem 1.2 and Theorem 1.3

To prove Theorem 1.2, we first derive an asymptotic formula for $q(n)$ with an explicit bound by specializing an asymptotic formula for η-quotients $G(q)$ due to Chern [8]. Define

$$G(q)=G(e^{2\pi i\tau}):=\prod_{r=1}^{R}(q^{m_r};q^{m_r})_\infty^{\delta_r}, \tag{3.1}$$

where $\mathbf{m}=(m_1,\ldots,m_R)$ is a sequence of R distinct positive integers and $\delta=(\delta_1,\ldots,\delta_R)$ is a sequence of R non-zero integers. Here and throughout this paper, we have adopted the standard notation on q-series [2].

$$(a;q)_n=\prod_{j=0}^{n-1}(1-aq^j)\quad\text{and}\quad(a;q)_\infty=\prod_{j=0}^{\infty}(1-aq^j).$$

In order to state Chern's result, we need a few preliminary definitions. Assume that h and j are positive integers with $\gcd(h,j)=1$, set

$$\Delta_1=-\frac{1}{2}\sum_{r=1}^{R}\delta_r,\qquad \Delta_2=\sum_{r=1}^{R}m_r\delta_r,$$

$$\Delta_3(k)=-\sum_{r=1}^{R}\frac{\delta_r\gcd^2(m_r,k)}{m_r},\qquad \Delta_4(k)=\prod_{r=1}^{R}\left(\frac{m_r}{\gcd(m_r,k)}\right)^{-\frac{\delta_r}{2}},$$

$$\widehat{A}_k(n) = \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} \exp\left(-\frac{2\pi nhi}{k} - \pi i \sum_{r=1}^{R} \delta_r s\left(\frac{m_r h}{\gcd(m_r,k)}, \frac{k}{\gcd(m_r,k)}\right)\right), \quad (3.2)$$

where $s(h,j)$ is the Dedekind sum defined by

$$s(h,j) = \sum_{r=1}^{j-1} \left(\frac{r}{j} - \left[\frac{r}{j}\right] - \frac{1}{2}\right)\left(\frac{hr}{j} - \left[\frac{hr}{j}\right] - \frac{1}{2}\right).$$

Let $L = \mathrm{lcm}(m_1,\dots,m_R)$. We divide the set $\{1, 2\cdots, L\}$ into two disjoint subsets:

$$\mathcal{L}_{>0} := \{1 \le l \le L : \Delta_3(l) > 0\},$$
$$\mathcal{L}_{\le 0} := \{1 \le l \le L : \Delta_3(l) \le 0\}.$$

We write

$$G(q) = \sum_{n\ge 0} g(n) q^n.$$

Chern [8] obtained an asymptotic formula for $g(n)$ with $\Delta_1 \le 0$.

Define

$$\mathbb{E}_{\Delta_1}(s) := \begin{cases} 1, & \Delta_1 = 0, \\ 2\sqrt{s}, & \Delta_1 = -\frac{1}{2}, \\ s\log(s+1), & \Delta_1 = -1, \\ s^{-2\Delta_1-1}\zeta(-\Delta_1), & \text{otherwise}, \end{cases} \quad (3.3)$$

where $\zeta(\cdot)$ is Riemann zeta-function.

Theorem 3.1 (Chern)**.** *If $\Delta_1 \le 0$ and the inequality*

$$\min_{1\le r\le R}\left(\frac{\gcd^2(m_r,l)}{m_r}\right) \ge \frac{\Delta_3(l)}{24} \quad (3.4)$$

holds for all $1 \le l \le L$, then for positive integers N and $n > -\frac{\Delta_2}{24}$, we have

$$g(n) = E(n) + \sum_{l\in\mathcal{L}_{>0}} 2\pi\Delta_4(l)\left(\frac{24n+\Delta_2}{\Delta_3(l)}\right)^{-\frac{\Delta_1+1}{2}}$$
$$\times \sum_{\substack{1\le k\le N \\ k\equiv_L l}} \frac{I_{-\Delta_1-1}\left(\frac{\pi}{6k}\sqrt{\Delta_3(l)(24n+\Delta_2)}\right)}{k}\widehat{A}_k(n), \quad (3.5)$$

where

$$|E(n)| \leq \frac{2^{-\Delta_1}\pi^{-1}N^{-\Delta_1+2}}{n+\frac{\Delta_2}{24}} \exp\left(2\pi\left(n+\frac{\Delta_2}{24}\right)N^{-2}\right) \sum_{l\in\mathcal{L}_{>0}} \Delta_4(l)\exp\left(\frac{\Delta_3(l)\pi}{3}\right)$$

$$+ 2\exp\left(2\pi\left(n+\frac{\Delta_2}{24}\right)N^{-2}\right)\mathbb{E}_{\Delta_1}(N)$$

$$\times\left(\sum_{1\leq l\leq L}\Delta_4(l)\exp\left(\frac{\pi\Delta_3(l)}{24}+\sum_{r=1}^{R}\frac{|\Delta_r|\exp\left(-\pi\gcd^2(m_r,l)/m_r\right)}{\left(1-\exp\left(-\pi\gcd^2(m_r,l)/m_r\right)\right)^2}\right)\right.$$

$$\left.-\sum_{l\in\mathcal{L}_{>0}}\Delta_4(l)\exp\left(\frac{\pi\Delta_3(l)}{24}\right)\right),$$

and $I_\nu(s)$ is the ν-th modified Bessel function of the first kind.

We are now in a position to prove Theorem 1.2 by means of Theorem 3.1.

Proof of Theorem 1.2. Recall that

$$\sum_{n=0}^{\infty} q(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty},$$

so we have $\mathbf{m} = (1,2)$ and $\delta = (-1,1)$. It is straightforward to compute that $\Delta_1 = 0$, and $\Delta_2 = 1$. We also have $L = 2$. The values of $\Delta_3(l)$ and $\Delta_4(l)$ for $1 \leq l \leq L$ are listed in Table 1. Hence $\mathcal{L}_{>0} = \{1\}$. It can be readily checked that (3.4) is always true for $1 \leq l \leq 2$.

Table 1: The values of $\Delta_3(l)$ and $\Delta_4(l)$ for $1 \leq l \leq 2$.

l	1	2
$\Delta_3(l)$	$\frac{1}{2}$	-1
$\Delta_4(l)$	$\frac{\sqrt{2}}{2}$	1

Hence, by Theorem 3.1, we have

$$q(n) = E(n) + \frac{\sqrt{2}\pi^2}{12\nu(n)} \sum_{\substack{1\leq k\leq N\\ 2\nmid k}} I_1\left(\frac{\nu(n)}{k}\right)\frac{\widehat{A}_k(n)}{k},$$

where $\nu(n)$ is defined as in (1.3), and

$$|E(n)| \le \frac{\pi^{-1}N^2}{n+\frac{1}{24}} \exp\left(2\pi\left(n+\frac{1}{24}\right)N^{-2}\right)\cdot\frac{\sqrt{2}}{2}\exp\left(\frac{\pi}{6}\right) + 2\exp\left(2\pi\left(n+\frac{1}{24}\right)N^{-2}\right)$$
$$\times\left\{\frac{\sqrt{2}}{2}\exp\left(\frac{\pi}{48}+\frac{\exp(-\pi)}{(1-\exp(-\pi))^2}+\frac{\exp(-\pi/2)}{(1-\exp(-\pi/2))^2}\right)\right.$$
$$\left.+\exp\left(-\frac{\pi}{24}+\frac{\exp(-\pi)}{(1-\exp(-\pi))^2}+\frac{\exp(-2\pi)}{(1-\exp(-2\pi))^2}\right)-\frac{\sqrt{2}}{2}\exp\left(\frac{\pi}{48}\right)\right\}.$$

Assume that $N=\lfloor\nu(n)\rfloor$. Observe that

$$\frac{\lfloor\nu(n)\rfloor^2}{\nu(n)^2}\le 1 \quad\text{and}\quad \frac{\nu(n)^2}{\lfloor\nu(n)\rfloor^2}<\frac{\nu(n)^2}{(\nu(n)-1)^2}<2 \quad \text{for } \nu(n)\ge 4,$$

we deduce that

$$|E(n)| \le \frac{\sqrt{2}\pi}{6}\exp\left(\frac{12}{\pi}+\frac{\pi}{6}\right)$$
$$+2\exp\left(\frac{12}{\pi}\right)\left\{\frac{\sqrt{2}}{2}\exp\left(\frac{\pi}{48}+\frac{\exp(-\pi)}{(1-\exp(-\pi))^2}+\frac{\exp(-\pi/2)}{(1-\exp(-\pi/2))^2}\right)\right.$$
$$\left.+\exp\left(-\frac{\pi}{24}+\frac{\exp(-\pi)}{(1-\exp(-\pi))^2}+\frac{\exp(-2\pi)}{(1-\exp(-2\pi))^2}\right)-\frac{\sqrt{2}}{2}\exp\left(\frac{\pi}{48}\right)\right\}\le 173.$$

Hence, we conclude that for $\nu(n)\ge 4$,

$$q(n)=E(n)+\frac{\sqrt{2}\pi^2}{12\nu(n)}\sum_{\substack{1\le k\le\lfloor\nu(n)\rfloor\\ 2\nmid k}} I_1\left(\frac{\nu(n)}{k}\right)\frac{\widehat{A}_k(n)}{k}, \tag{3.6}$$

where $|E(n)|\le 173$.

Observing that $\widehat{A}_1(n)=1$, so by (3.6), we have

$$q(n)=\frac{\sqrt{2}\pi^2}{12\nu(n)}I_1(\nu(n))+R(n),$$

where

$$R(n)=E(n)+\frac{\sqrt{2}\pi^2}{12\nu(n)}\sum_{\substack{3\le k\le\lfloor\nu(n)\rfloor\\ 2\nmid k}} I_1\left(\frac{\nu(n)}{k}\right)\frac{\widehat{A}_k(n)}{k}. \tag{3.7}$$

Hence, in order to show Theorem 1.2, it suffices to show that for $\nu(n)\ge 21$,

$$|R(n)|\le\frac{\sqrt{3}\pi^{\frac{3}{2}}}{6\nu(n)^{\frac{1}{2}}}\exp\left(\frac{\nu(n)}{3}\right). \tag{3.8}$$

By the definition of $\widehat{A}_k(n)$, we derive that for $n \geq 0$ and $k \geq 1$,

$$|\widehat{A}_k(n)| \leq k,$$

since $|e^{2\pi s i}| = 1$ for any $s \in \mathbb{R}$. It yields

$$\begin{aligned}\left| \frac{\sqrt{2}\pi^2}{12\nu(n)} \sum_{\substack{3\leq k\leq \lfloor \nu(n) \rfloor \\ 2\nmid k}} I_1\left(\frac{\nu(n)}{k}\right) \frac{\widehat{A}_k(n)}{k} \right| &\leq \frac{\sqrt{2}\pi^2}{12\nu(n)} \sum_{\substack{3\leq k\leq \lfloor \nu(n) \rfloor \\ 2\nmid k}} I_1\left(\frac{\nu(n)}{k}\right) \\ &\leq \frac{\sqrt{2}\pi^2}{12\nu(n)} \frac{\lfloor \nu(n) \rfloor}{2} I_1\left(\frac{\nu(n)}{3}\right) \\ &\leq \frac{\sqrt{2}\pi^2}{24} I_1\left(\frac{\nu(n)}{3}\right).\end{aligned}$$

Invoking (2.2), we derive that

$$\left| \frac{\sqrt{2}\pi^2}{12\nu(n)} \sum_{\substack{3\leq k\leq N \\ 2\nmid k}} I_1\left(\frac{\nu(n)}{k}\right) \frac{\widehat{A}_k(n)}{k} \right| \leq \frac{\sqrt{3}\pi^{\frac{3}{2}}}{12\nu(n)^{\frac{1}{2}}} \exp\left(\frac{\nu(n)}{3}\right). \tag{3.9}$$

Substituting (3.9) into (3.7), we deduce that for $\nu(n) \geq 4$,

$$|R(n)| \leq 173 + \frac{\sqrt{3}\pi^{\frac{3}{2}}}{12\nu(n)^{\frac{1}{2}}} \exp\left(\frac{\nu(n)}{3}\right). \tag{3.10}$$

We proceed to show that for $\nu(n) \geq 21$,

$$\frac{\sqrt{3}\pi^{\frac{3}{2}}}{12\nu(n)^{\frac{1}{2}}} \exp\left(\frac{\nu(n)}{3}\right) > 173. \tag{3.11}$$

Define

$$r(s) := \frac{692\sqrt{3}}{\pi^{\frac{3}{2}}} s^{\frac{1}{2}} \exp\left(-\frac{s}{3}\right).$$

It is evident that

$$r'(s) = \frac{346}{\sqrt{3}\pi^{\frac{3}{2}} s^{\frac{1}{2}}} (-2s+3) \exp\left(-\frac{s}{3}\right).$$

Since $r'(s) \leq 0$ when $s \geq \frac{3}{2}$, we deduce that $r(s)$ is decreasing when $s \geq \frac{3}{2}$. This implies that

$$r(\nu(n)) \leq r(21) < 1$$

for $\nu(n) \geq 21$. So the inequality (3.11) is valid. Applying (3.11) to (3.10), we are led to (3.8). This completes the proof. ∎

We conclude this section with the proof of Theorem 1.3 with the aid of Theorem 1.2.

Proof of Theorem 1.3. Define

$$G(n) := \frac{\frac{\sqrt{3}\pi^{\frac{3}{2}}}{6\nu(n)^{\frac{1}{2}}}\exp\left(\frac{\nu(n)}{3}\right)}{\frac{\sqrt{2}\pi^2}{12\nu(n)}I_1(\nu(n))} = \sqrt{\frac{6\nu(n)}{\pi}} \cdot \frac{\exp\left(\frac{\nu(n)}{3}\right)}{I_1(\nu(n))}. \tag{3.12}$$

Thanks to Theorem 1.2, we have

$$M(n)(1 - G(n)) \leq q(n) \leq M(n)(1 + G(n)).$$

To show (1.8), it is enough to prove that for $\nu(n) \geq 38$,

$$G(n) \leq \frac{1}{\nu(n)^6}. \tag{3.13}$$

Using Lemma 2.2, we find that for $s \geq 26$,

$$I_1(s) \geq \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{3}{8s} - \frac{15}{128s^2} - \frac{105}{1024s^3} - \frac{4725}{32768s^4} - \frac{72765}{262144s^5} - \frac{31}{s^6}\right).$$

Note that for $s \geq 4$

$$\frac{1}{8s} - \frac{15}{128s^2} - \frac{105}{1024s^3} - \frac{4725}{32768s^4} - \frac{72765}{262144s^5} - \frac{31}{s^6} \geq 0,$$

so for $s \geq 26$,

$$I_1(s) \geq \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{1}{2s}\right). \tag{3.14}$$

Substituting (3.14) into (3.12), we derive that for $\nu(n) \geq 26$,

$$G(n) \leq \frac{2\sqrt{3}\nu(n)}{1 - \frac{1}{2\nu(n)}}\exp\left(-\frac{2\nu(n)}{3}\right). \tag{3.15}$$

Based on the following observation:

$$\left(1 - \frac{1}{2\nu(n)}\right)\left(1 + \frac{1}{\nu(n)}\right) = 1 + \frac{1}{2\nu(n)^2}(\nu(n) - 1) \geq 1 \quad \text{for} \quad \nu(n) \geq 1,$$

we find that (3.15) can be further bounded by

$$G(n) \leq 2\sqrt{3}\nu(n)\left(1 + \frac{1}{\nu(n)}\right)\exp\left(-\frac{2\nu(n)}{3}\right). \tag{3.16}$$

We claim that for $\nu(n) \geq 43$,

$$2\sqrt{3}\exp\left(-\frac{2\nu(n)}{3}\right) \leq \frac{1}{2\nu(n)^7}, \tag{3.17}$$

which can be recast as

$$4\sqrt{3}\nu(n)^7 \exp\left(-\frac{2\nu(n)}{3}\right) \leq 1.$$

Define

$$L(s) := 4\sqrt{3}s^7 \exp\left(-\frac{2s}{3}\right).$$

Since for $s \geq \frac{21}{2}$,

$$L'(s) = 4\sqrt{3}\exp\left(-\frac{2s}{3}\right) s^6 \left(-\frac{2}{3}s + 7\right) \leq 0,$$

we deduce that $L(s)$ is decreasing when $s \geq \frac{21}{2}$. It follows that for $\nu(n) \geq 43$,

$$L(\nu(n)) = 4\sqrt{3}\nu(n)^7 \exp\left(-\frac{2\nu(n)}{3}\right) \leq L(43) < 1,$$

and so (3.17) holds when $\nu(n) \geq 43$. Hence the claim is verified.

Applying (3.17) to (3.16), we derive that for $\nu(n) \geq 43$,

$$G(n) \leq \nu(n)\left(1 + \frac{1}{\nu(n)}\right) \cdot \frac{1}{2\nu(n)^7} < \frac{1}{\nu(n)^6}.$$

This completes the proof. ∎

4 Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4 with the aid of Theorem 1.3 and Lemma 2.3.

Proof of Theorem 1.4. Recall that

$$Q(n) = \frac{q(n-1)q(n+1)}{q(n)^2}.$$

Define

$$A(n) = \frac{M(n-1)M(n+1)}{M(n)^2}, \tag{4.1}$$

where $M(n)$ is defined as in (1.7). From Theorem 1.3, we see that for $\nu(n) \geq 43$,

$$A(n)L_Q(n) \leq Q(n) \leq A(n)R_Q(n), \tag{4.2}$$

where

$$L_Q(n) = \frac{\left(1 - \frac{1}{\nu(n-1)^6}\right)\left(1 - \frac{1}{\nu(n+1)^6}\right)}{\left(1 + \frac{1}{\nu(n)^6}\right)^2} \tag{4.3}$$

and

$$R_Q(n) = \frac{\left(1+\frac{1}{\nu(n-1)^6}\right)\left(1+\frac{1}{\nu(n+1)^6}\right)}{\left(1-\frac{1}{\nu(n)^6}\right)^2}. \tag{4.4}$$

To obtain (1.12), we proceed to estimate $A(n)$, $L_Q(n)$ and $R_Q(n)$ in terms of $\nu(n)$. We first consider $A(n)$. Substituting (1.7) into (4.1), we find that

$$A(n) = \frac{\nu(n)^2 I_1(\nu(n-1)) I_1(\nu(n+1))}{\nu(n-1)\nu(n+1) I_1(\nu(n))^2}. \tag{4.5}$$

Applying Lemma 2.3 to (4.5), we deduce that for $\nu(n) \geq 60$,

$$\begin{aligned} A(n) \geq & \frac{\nu(n)^3}{\sqrt{\nu(n-1)^3\nu(n+1)^3}}\left(1-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}\right) \\ & \times\left(1-\frac{\pi^4}{32\nu(n)^5}-\frac{129}{\nu(n)^6}\right) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} A(n) \leq & \frac{\nu(n)^3}{\sqrt{\nu(n-1)^3\nu(n+1)^3}}\left(1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^8}{1296\nu(n)^6}\right) \\ & \times\left(1-\frac{\pi^4}{32\nu(n)^5}+\frac{121}{\nu(n)^6}\right). \end{aligned} \tag{4.7}$$

We claim that for $\nu(n) \geq 8$,

$$1+\frac{\pi^4}{12\nu(n)^4}+\frac{7\pi^8}{864\nu(n)^8} \leq \frac{\nu(n)^3}{\sqrt{\nu(n-1)^3\nu(n+1)^3}} \leq 1+\frac{\pi^4}{12\nu(n)^4}+\frac{\pi^8}{123\nu(n)^8}, \tag{4.8}$$

which is equivalent to

$$\begin{cases} \nu(n)^{12}-\nu(n-1)^6\nu(n+1)^6\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{7\pi^8}{864\nu(n)^8}\right)^4 \geq 0, \\ \nu(n)^{12}-\nu(n-1)^6\nu(n+1)^6\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{\pi^8}{123\nu(n)^8}\right)^4 \leq 0. \end{cases} \tag{4.9}$$

Recall that

$$\nu(n-1) = \sqrt{\nu(n)^2-\frac{\pi^2}{3}} \quad \text{and} \quad \nu(n+1) = \sqrt{\nu(n)^2+\frac{\pi^2}{3}}.$$

It can be calculated that

$$\begin{aligned} & \nu(n)^{12}-\nu(n-1)^6\nu(n+1)^6\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{7\pi^8}{864\nu(n)^8}\right)^4 \\ = & \frac{\pi^{12}}{406239826673664\nu(n)^{32}}\big(1340897918976\nu(n)^{32}+27935373312\pi^4\nu(n)^{28} \\ & +1551965184\pi^8\nu(n)^{24}-1551965184\pi^{12}\nu(n)^{20}-60816096\pi^{16}\nu(n)^{16} \\ & -3873177\pi^{20}\nu(n)^{12}+625779\pi^{24}\nu(n)^8+33957\pi^{28}\nu(n)^4+2401\pi^{32}\big) \end{aligned} \tag{4.10}$$

and

$$\nu(n)^{12}-\nu(n-1)^6\nu(n+1)^6\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{\pi^8}{123\nu(n)^8}\right)^4$$

$$=-\frac{\pi^8}{42715740489984\nu(n)^{32}}\left(4823367264\nu(n)^{36}-141396118128\pi^4\nu(n)^{32}\right.$$
$$-2942756919\pi^8\nu(n)^{28}-175420755\pi^{12}\nu(n)^{24}+163918779\pi^{16}\nu(n)^{20}$$
$$+6413999\pi^{20}\nu(n)^{16}+418192\pi^{24}\nu(n)^{12}-66144\pi^{28}\nu(n)^8$$
$$\left.-3584\pi^{32}\nu(n)^4-256\pi^{36}\right). \tag{4.11}$$

Note that for $\nu(n)\geq 4$,

$$1551965184\pi^8\nu(n)^{24}-1551965184\pi^{12}\nu(n)^{20}$$
$$-60816096\pi^{16}\nu(n)^{16}-3873177\pi^{20}\nu(n)^{12}\geq 0, \tag{4.12}$$

and for $\nu(n)\geq 8$,

$$\begin{cases} 4823367264\nu(n)^{36}-141396118128\pi^4\nu(n)^{32} \\ \quad -2942756919\pi^8\nu(n)^{28}-175420755\pi^{12}\nu(n)^{24}\geq 0, \\ 418192\pi^{24}\nu(n)^{12}-66144\pi^{28}\nu(n)^8-3584\pi^{32}\nu(n)^4-256\pi^{36}\geq 0. \end{cases} \tag{4.13}$$

Applying (4.12) to (4.10) and applying (4.13) to (4.11), we conclude that (4.9) holds for $\nu(n)\geq 8$, which implies (4.8) holds for $\nu(n)\geq 8$, and so the claim is verified. Substituting (4.8) into (4.6) and (4.7), we get that for $\nu(n)\geq 60$,

$$A(n)\geq\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{7\pi^8}{864\nu(n)^8}\right)\left(1-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}\right)$$
$$\times\left(1-\frac{\pi^4}{32\nu(n)^5}-\frac{129}{\nu(n)^6}\right) \tag{4.14}$$

and

$$A(n)\leq\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{\pi^8}{123\nu(n)^8}\right)\left(1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^8}{1296\nu(n)^6}\right)$$
$$\times\left(1-\frac{\pi^4}{32\nu(n)^5}+\frac{121}{\nu(n)^6}\right). \tag{4.15}$$

We proceed to show that for $\nu(n)\geq 4$,

$$L_Q(n)\geq 1-\frac{5}{\nu(n)^6}\quad\text{and}\quad R_Q(n)\leq 1+\frac{5}{\nu(n)^6}. \tag{4.16}$$

Applying (2.22) into (4.3) and (4.4), we find that

$$L_Q(n)=\frac{\nu(n)^{12}\left(\left(\nu(n)^2+\frac{\pi^2}{3}\right)^3-1\right)\left(\left(\nu(n)^2-\frac{\pi^2}{3}\right)^3-1\right)}{\left(\nu(n)^6+1\right)^2\left(\nu(n)^4-\frac{\pi^4}{9}\right)^3}$$

and

$$R_Q(n)=\frac{\nu(n)^{12}\left(\left(\nu(n)^2+\frac{\pi^2}{3}\right)^3+1\right)\left(\left(\nu(n)^2-\frac{\pi^2}{3}\right)^3+1\right)}{\left(\nu(n)^6-1\right)^2\left(\nu(n)^4-\frac{\pi^4}{9}\right)^3}.$$

Assume that

$$\begin{aligned}\phi(s)=&729s^{24}-1215\pi^4s^{20}+7290s^{18}+81\pi^8s^{16}-2187\pi^4s^{14}+\left(3645-3\pi^{12}\right)s^{12}\\&+243\pi^8s^{10}-1215\pi^4s^8-9\pi^{12}s^6+135\pi^8s^4-5\pi^{12}\end{aligned}$$

and

$$\begin{aligned}\psi(s)=&729s^{24}-1215\pi^4s^{20}-7290s^{18}+81\pi^8s^{16}+2187\pi^4s^{14}+\left(3645-3\pi^{12}\right)s^{12}\\&-243\pi^8s^{10}-1215\pi^4s^8+9\pi^{12}s^6+135\pi^8s^4-5\pi^{12}.\end{aligned}$$

It can be checked that

$$L_Q(n)-\left(1-\frac{5}{\nu(n)^6}\right)=\frac{\phi\left(\nu(n)\right)}{\nu(n)^6\left(9\nu(n)^4-\pi^4\right)^3\left(\nu(n)^6+1\right)^2} \tag{4.17}$$

and

$$R_Q(n)-\left(1+\frac{5}{\nu(n)^6}\right)=\frac{-\psi\left(\nu(n)\right)}{\nu(n)^6\left(9\nu(n)^4-\pi^4\right)^3\left(\nu(n)^6-1\right)^2}. \tag{4.18}$$

Moreover, it is not difficult to show that $\psi(s)\geq 0$ for $s\geq 4$ and

$$\phi(s)-\psi(s)=14580s^{18}-4374\pi^4s^{14}+486\pi^8s^{10}-18\pi^{12}s^6>0$$

for $s\geq 2$. Hence we derive that for $\nu(n)\geq 4$,

$$\phi\left(\nu(n)\right)>\psi\left(\nu(n)\right)\geq 0. \tag{4.19}$$

It follows that (4.16) is valid.

Substituting (4.14), (4.15) and (4.16) into (4.2), we derive that for $\nu(n)\geq 60$,

$$\begin{aligned}Q(n)\geq&\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{7\pi^8}{864\nu(n)^8}\right)\left(1-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}\right)\\&\times\left(1-\frac{\pi^4}{32\nu(n)^5}-\frac{129}{\nu(n)^6}\right)\left(1-\frac{5}{\nu(n)^6}\right)\end{aligned} \tag{4.20}$$

and

$$Q(n) \leq \left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{\pi^8}{123\nu(n)^8}\right)\left(1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^8}{1296\nu(n)^6}\right)$$
$$\times\left(1-\frac{\pi^4}{32\nu(n)^5}+\frac{121}{\nu(n)^6}\right)\left(1+\frac{5}{\nu(n)^6}\right). \tag{4.21}$$

To prove Theorem 1.4, it is enough to show that for $\nu(n) \geq 67$,

$$\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{7\pi^8}{864\nu(n)^8}\right)\left(1-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}\right)$$
$$\times\left(1-\frac{\pi^4}{32\nu(n)^5}-\frac{129}{\nu(n)^6}\right)\left(1-\frac{5}{\nu(n)^6}\right)$$
$$> 1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^4}{12\nu(n)^4}-\frac{\pi^4}{32\nu(n)^5}-\frac{135}{\nu(n)^6} \tag{4.22}$$

and

$$\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{\pi^8}{123\nu(n)^8}\right)\left(1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^8}{1296\nu(n)^6}\right)$$
$$\times\left(1-\frac{\pi^4}{32\nu(n)^5}+\frac{121}{\nu(n)^6}\right)\left(1+\frac{5}{\nu(n)^6}\right)$$
$$< 1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^4}{12\nu(n)^4}-\frac{\pi^4}{32\nu(n)^5}+\frac{126+\frac{\pi^8}{1296}}{\nu(n)^6}. \tag{4.23}$$

We first show (4.22). Observe that

$$\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{7\pi^8}{864\nu(n)^8}\right)\left(1-\frac{\pi^4}{36\nu(n)^3}-\frac{5\pi^8}{2592\nu(n)^7}\right)$$
$$\times\left(1-\frac{\pi^4}{32\nu(n)^5}-\frac{129}{\nu(n)^6}\right)\left(1-\frac{5}{\nu(n)^6}\right)$$
$$-\left(1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^4}{12\nu(n)^4}-\frac{\pi^4}{32\nu(n)^5}-\frac{135}{\nu(n)^6}\right)$$
$$=\frac{1}{71663616\nu(n)^{27}}\sum_{j=0}^{21}c_j\nu(n)^j, \tag{4.24}$$

where c_j are real numbers. Here we just list the values of c_{19}, c_{20}, c_{21}:

$$c_{19} = 642816\pi^8, \quad c_{20} = -304128\pi^8, \quad c_{21} = 71663616.$$

Clearly,

$$\sum_{j=0}^{21}c_j\nu(n)^j \geq -\sum_{j=0}^{19}|c_j|\nu(n)^j + c_{20}\nu(n)^{20} + c_{21}\nu(n)^{21}.$$

Moreover, it can be checked that for $0 \le j \le 18$ and $\nu(n) \ge 4$,

$$-|c_j|\nu(n)^j \ge -|c_{19}|\nu(n)^{19}.$$

On the other hand, It is not difficult to check that for $\nu(n) \ge 67$,

$$c_{21}\nu(n)^2 + c_{20}\nu(n) - 20|c_{19}| > 0.$$

Assembling all these results above, we conclude that for $\nu(n) \ge 67$,

$$\sum_{j=0}^{21} c_j\nu(n)^j \ge \left(c_{21}\nu(n)^2 + c_{20}\nu(n) - 20|c_{19}|\right)\nu(n)^{19} > 0.$$

This proves (4.22).

Similarly, to justify (4.23), we first note that

$$\begin{aligned}
&\left(1+\frac{\pi^4}{12\nu(n)^4}+\frac{\pi^8}{123\nu(n)^8}\right)\left(1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^8}{1296\nu(n)^6}\right)\\
&\quad\times\left(1-\frac{\pi^4}{32\nu(n)^5}+\frac{121}{\nu(n)^6}\right)\left(1+\frac{5}{\nu(n)^6}\right)\\
&\quad-\left(1-\frac{\pi^4}{36\nu(n)^3}+\frac{\pi^4}{12\nu(n)^4}-\frac{\pi^4}{32\nu(n)^5}+\frac{126+\frac{\pi^8}{1296}}{\nu(n)^6}\right)\\
&=-\frac{1}{20404224\nu(n)^{26}}\sum_{j=0}^{19} d_j\nu(n)^j, \qquad (4.25)
\end{aligned}$$

where d_j are real numbers. Here we also list the values of the last three coefficients:

$$d_{17} = 53136\pi^8, \quad d_{18} = -183600\pi^8, \quad d_{19} = 47232\pi^8. \qquad (4.26)$$

It is transparent that

$$\sum_{j=0}^{19} d_j\nu(n)^j \ge -\sum_{j=0}^{17}|d_j|\nu(n)^j + d_{18}\nu(n)^{18} + d_{19}\nu(n)^{19}. \qquad (4.27)$$

Moreover, it can be checked that for $0 \le j \le 16$ and $\nu(n) \ge 2$,

$$-|d_j|\nu(n)^j \ge -|d_{17}|\nu(n)^{17}$$

and for $\nu(n) \ge 7$,

$$d_{19}\nu(n)^2 + d_{18}\nu(n) - 18|d_{17}| > 0.$$

Hence we conclude that for $\nu(n) \ge 67$,

$$\sum_{j=0}^{19} d_j\nu(n)^j \ge \left(d_{19}\nu(n)^2 + d_{18}\nu(n) - 18|d_{17}|\right)\nu(n)^{17} > 0,$$

and so (4.23) is valid.

Substituting (4.22) and (4.23) into (4.20) and (4.21), we arrive at (1.12). This completes the proof. ∎

5 Proof of Conjecture 1.1

In this section, we confirm Conjecture 1.1 by resorting to Theorem 1.4. Before doing this, we need to recall the following lemma given by Jia [21].

Lemma 5.1 (Jia). *Let u and v be two positive real numbers such that $\frac{\sqrt{5}-1}{2} \le u < v < 1$. If*

$$u + \sqrt{(1-u)^3} > v,$$

then we have

$$4(1-u)(1-v) - (1-uv)^2 > 0.$$

Proof of Conjecture 1.1. We first prove that $q(n)$ is log-concave for $n \ge 33$. It is equivalent to proving that for $n \ge 33$,

$$Q(n) \le 1.$$

By Theorem 1.4, we see that for $\nu(n) \ge 67$,

$$Q(n) < 1 - \frac{\pi^4}{36\nu(n)^3} + \frac{\pi^4}{12\nu(n)^4} - \frac{\pi^4}{32\nu(n)^5} + \frac{126 + \frac{\pi^8}{1296}}{\nu(n)^6}. \tag{5.1}$$

It is easy to check that for $\nu(n) \ge 44$,

$$-\frac{\pi^4}{36\nu(n)^3} + \frac{\pi^4}{12\nu(n)^4} \le 0$$

and

$$-\frac{\pi^4}{32\nu(n)^5} + \frac{126 + \frac{\pi^8}{1296}}{\nu(n)^6} \le 0.$$

Hence, we conclude that $Q(n) < 1$ for $n \ge 1365$. It can be checked that $Q(n) < 1$ for $33 \le n \le 1365$. Therefore, we derive that $Q(n) < 1$ for $n \ge 33$, and so $q(n)$ is log-concave for $n \ge 33$.

To prove that $q(n)$ satisfies the third order Turán inequalities for $n \ge 121$, it is equivalent to demonstrating that for $n \ge 121$,

$$4(1 - Q(n))(1 - Q(n+1)) - (1 - Q(n)Q(n+1))^2 > 0. \tag{5.2}$$

It can be directly checked that (5.2) is true when $121 \le n \le 1365$, so it is enough to prove that (5.2) holds for $n \ge 1365$. Since $Q(n+1) < 1$ for $n \ge 32$, and by Lemma 5.1, we see that it suffices to prove that for $n \ge 1365$,

$$\frac{\sqrt{5}-1}{2} \le Q(n) < Q(n+1) \tag{5.3}$$

and

$$Q(n+1) < Q(n) + \sqrt{(1 - Q(n))^3}. \tag{5.4}$$

Using Theorem 1.4, we see that for $\nu(n) \geq 67$,

$$Q(n) > 1 - \frac{\pi^4}{36\nu(n)^3} + \frac{\pi^4}{12\nu(n)^4} - \frac{\pi^4}{32\nu(n)^5} - \frac{135}{\nu(n)^6}.$$

It is easy to check that for $\nu(n) \geq 5$,

$$\frac{\pi^4}{12\nu(n)^4} - \frac{\pi^4}{32\nu(n)^5} - \frac{135}{\nu(n)^6} > 0$$

and

$$1 - \frac{\pi^4}{36\nu(n)^3} \geq 1 - \frac{\pi^4}{36 \cdot 5^3} > \frac{\sqrt{5}-1}{2}.$$

Hence we deduce that for $\nu(n) \geq 67$,

$$Q(n) > \frac{\sqrt{5}-1}{2}.$$

Using Theorem 1.4 again, we find that for $\nu(n) \geq 67$,

$$\begin{aligned} Q(n+1) - Q(n) > &\left(1 - \frac{\pi^4}{36\nu(n+1)^3} + \frac{\pi^4}{12\nu(n+1)^4} - \frac{\pi^4}{32\nu(n+1)^5} - \frac{135}{\nu(n+1)^6}\right) \\ &- \left(1 - \frac{\pi^4}{36\nu(n)^3} + \frac{\pi^4}{12\nu(n)^4} - \frac{\pi^4}{32\nu(n)^5} + \frac{126 + \frac{\pi^8}{1296}}{\nu(n)^6}\right). \end{aligned} \tag{5.5}$$

Note that for $\nu(n) \geq 3$,

$$\begin{cases} \dfrac{1}{\nu(n+1)^3} < \dfrac{1}{\nu(n)^3} - \dfrac{\pi^2}{4\nu(n)^5}, \\ \dfrac{1}{\nu(n+1)^4} > \dfrac{1}{\nu(n)^4} - \dfrac{2\pi^2}{3\nu(n)^6}, \\ \dfrac{1}{\nu(n+1)^5} < \dfrac{1}{\nu(n)^5}, \\ \dfrac{1}{\nu(n+1)^6} < \dfrac{1}{\nu(n)^6}. \end{cases} \tag{5.6}$$

Applying (5.6) to (5.5), we derive that for $\nu(n) \geq 67$,

$$\begin{aligned} Q(n+1) - Q(n) > &\left(1 - \frac{\pi^4}{36\nu(n)^3} + \frac{\pi^4}{12\nu(n)^4} + \frac{-\frac{\pi^4}{32} + \frac{\pi^6}{144}}{\nu(n)^5} - \frac{\frac{\pi^6}{18} + 135}{\nu(n)^6}\right) \\ &- \left(1 - \frac{\pi^4}{36\nu(n)^3} + \frac{\pi^4}{12\nu(n)^4} - \frac{\pi^4}{32\nu(n)^5} + \frac{126 + \frac{\pi^8}{1296}}{\nu(n)^6}\right) \\ = &\frac{\pi^6}{144\nu(n)^5} - \frac{261 + \frac{\pi^6}{18} + \frac{\pi^8}{1296}}{\nu(n)^6}. \end{aligned}$$

It can be checked that for $\nu(n) \geq 49$,

$$\frac{\pi^6}{144\nu(n)^5} - \frac{261 + \frac{\pi^6}{18} + \frac{\pi^8}{1296}}{\nu(n)^6} > 0,$$

so we get that for $\nu(n) \geq 67$,

$$Q(n+1) - Q(n) > 0,$$

and (5.3) is verified since $\nu(n) = 67$ whenever $n = 1365$.

To prove (5.4), using Theorem 1.4 again, we find that for $\nu(n) \geq 67$,

$$\begin{aligned}
Q(n+1) - Q(n) &< \left(1 - \frac{\pi^4}{36\nu(n+1)^3} + \frac{\pi^4}{12\nu(n+1)^4} - \frac{\pi^4}{32\nu(n+1)^5} + \frac{126 + \frac{\pi^8}{1296}}{\nu(n+1)^6}\right) \\
&\quad - \left(1 - \frac{\pi^4}{36\nu(n)^3} + \frac{\pi^4}{12\nu(n)^4} - \frac{\pi^4}{32\nu(n)^5} - \frac{135}{\nu(n)^6}\right) \\
&= \frac{\pi^4}{36}\left(\frac{1}{\nu(n)^3} - \frac{1}{\nu(n+1)^3}\right) - \frac{\pi^4}{12}\left(\frac{1}{\nu(n)^4} - \frac{1}{\nu(n+1)^4}\right) \\
&\quad + \frac{\pi^4}{32}\left(\frac{1}{\nu(n)^5} - \frac{1}{\nu(n+1)^5}\right) + \frac{126 + \frac{\pi^8}{1296}}{\nu(n+1)^6} + \frac{135}{\nu(n)^6}.
\end{aligned} \tag{5.7}$$

It can be checked that for $\nu(n) > 0$,

$$\begin{cases}
\dfrac{1}{\nu(n)^3} - \dfrac{1}{\nu(n+1)^3} < \dfrac{\pi^2}{2\nu(n)^5}, \\
\dfrac{1}{\nu(n+1)^4} - \dfrac{1}{\nu(n)^4} < 0, \\
\dfrac{1}{\nu(n)^5} - \dfrac{1}{\nu(n+1)^5} < \dfrac{1}{\nu(n)^5},
\end{cases} \tag{5.8}$$

and for $\nu(n) \geq 55$,

$$\frac{126 + \frac{\pi^8}{1296}}{\nu(n+1)^6} + \frac{135}{\nu(n)^6} < \frac{126 + \frac{\pi^8}{1296} + 135}{\nu(n)^6} < \frac{\pi^2}{2\nu(n)^5}. \tag{5.9}$$

Applying (5.8) and (5.9) to (5.7), we derive that for $\nu(n) \geq 67$,

$$\begin{aligned}
Q(n+1) - Q(n) &< \frac{\pi^4}{36} \cdot \frac{\pi^2}{2\nu(n)^5} + \frac{\pi^4}{32} \cdot \frac{1}{\nu(n)^5} + \frac{\pi^2}{2\nu(n)^5} \\
&= \frac{\frac{\pi^6}{72} + \frac{\pi^4}{32} + \frac{\pi^2}{2}}{\nu(n)^5}.
\end{aligned} \tag{5.10}$$

It remains to show that for $\nu(n) \geq 67$,

$$\sqrt{(1-Q(n))^3} > \frac{\frac{\pi^6}{72}+\frac{\pi^4}{32}+\frac{\pi^2}{2}}{\nu(n)^5}. \tag{5.11}$$

Since for $\nu(n) \geq 67$,

$$1-Q(n) > \frac{\pi^4}{36\nu(n)^3} - \frac{\pi^4}{12\nu(n)^4} + \frac{\pi^4}{32\nu(n)^5} - \frac{126+\frac{\pi^8}{1296}}{\nu(n)^6},$$

and for $\nu(n) \geq 44$, it can be checked that

$$\frac{\pi^4}{32\nu(n)^5} - \frac{126+\frac{\pi^8}{1296}}{\nu(n)^6} > 0$$

and

$$-\frac{\pi^4}{12\nu(n)^4} > -\frac{1}{4\nu(n)^3}.$$

Hence we deduce that for $\nu(n) \geq 67$,

$$1-Q(n) > \frac{\pi^4}{36\nu(n)^3} - \frac{1}{4\nu(n)^3} = \frac{\pi^4-9}{36\nu(n)^3} > 0. \tag{5.12}$$

It can be easily checked that for $\nu(n) \geq 31$,

$$\frac{\sqrt{(\pi^4-9)^3}}{216\nu(n)^{\frac{9}{2}}} > \frac{\frac{\pi^6}{72}+\frac{\pi^4}{32}+\frac{\pi^2}{2}}{\nu(n)^5}. \tag{5.13}$$

Combining (5.12) and (5.13), we obtain (5.11), and so (5.4) is valid when $\nu(n) \geq 67$. This completes the proof. ∎

6 Concluding Remarks

Let $p_k(n)$ denote the number of partitions of n in which none of the parts are multiples of k. By definition, we see that

$$\sum_{n\geq 0} p_k(n) q^n = \prod_{n=1}^{\infty} \frac{1-q^{kn}}{1-q^n}.$$

When $k=2$, this partition function $p_k(n)$ reduces to $q(n)$. In [9], Craig and Pun also conjectured that the minimal number N_k and M_k such that $p_k(n)$ is log-concave for $n \geq N_k$ and satisfies the third-order Turán inequalities for $n \geq M_k$, where $k=3,4,5$. More precisely, Craig and Pun [9] conjectured that

$$N_3 = 58, \quad N_4 = 17, \quad N_5 = 42$$

and

$$M_3 = 185, \quad M_4 = 64, \quad M_5 = 137.$$

Based on Chern's asymptotic formulas for η-quotients, and using the similar argument in this paper, we could show that their conjectured values are true. Here we omit the details. However, our method can not be applied to find the values N_k and M_k for any fixed k such that $p_k(n)$ is log-concave for $n \geq N_k$ and satisfies the third-order Turán inequalities for $n \geq M_k$. It would be interesting to find a unified way to determine such N_k and M_k in terms of k.

Last but not least, we would like to mention that while studying the third-order Turán inequalities for $p(n)$, Chen [6] undertook a comprehensive study on inequalities pertaining to invariants of a binary form. In particular, he considered the following three invariants of the quartic binary form

$$\begin{aligned}
A\left(a_0, a_1, a_2, a_3, a_4\right) &= a_0 a_4 - 4a_1 a_3 + 3a_2^2, \\
B\left(a_0, a_1, a_2, a_3, a_4\right) &= -a_0 a_2 a_4 + a_2^3 + a_0 a_3^2 + a_1^2 a_4 - 2a_1 a_2 a_3, \\
I\left(a_0, a_1, a_2, a_3, a_4\right) &= A\left(a_0, a_1, a_2, a_3, a_4\right)^3 - 27B\left(a_0, a_1, a_2, a_3, a_4\right)^2.
\end{aligned}$$

Chen [6] conjectured both the partition function $p(n)$ and the spt-function $\mathrm{spt}(n)$ satisfy the inequalities derived from the invariants of the quartic binary form for large n. For the definition of the spt-function, please see Andrews [3] or Chen [6]. Chen's conjectured inequalities on the partition function $p(n)$ have recently been proved by Banerjee [4], Jia and Wang [22] and Wang and Yang [30].

In the same vein, we will present corresponding conjectures on $q(n)$.

Conjecture 6.1. *Let $a_n = q(n)$, then*

$$\begin{aligned}
A\left(a_{n-1}, a_n, a_{n+1}, a_{n+2}, a_{n+3}\right) &> 0, && \text{for } n \geq 230, \\
B\left(a_{n-1}, a_n, a_{n+1}, a_{n+2}, a_{n+3}\right) &> 0, && \text{for } n \geq 272, \\
I\left(a_{n-1}, a_n, a_{n+1}, a_{n+2}, a_{n+3}\right) &> 0, && \text{for } n \geq 267.
\end{aligned}$$

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