A Combinatorial Proof of a Schmidt Type Theorem of Andrews and Paule

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Abstract. This note is devoted to a combinatorial proof of a Schmidt type theorem due to Andrews and Paule. A four-variable refinement of Andrews and Paule's theorem is also obtained based on this combinatorial construction.

The main objective of this note is to give a combinatorial proof of the following partition theorem due to Andrews and Paule [3]. Sylvester's bijection [4–6] for Euler's partition theorem and Wright's bijection [2, 7, 8] for the Jacobi's triple product identity plays an important role in the combinatorial construction.

Theorem 1 (Andrews-Paule)**.** *Assume that* $n \geq 1$*. Let* $s(n)$ *denote the number of partitions* $a_1 + a_2 + a_3 + \cdots$ *satisfying* $a_1 \geq a_2 \geq a_3 \geq \cdots$ *and* $n = a_1 + a_3 + a_5 + \cdots$*. Let* $t(n)$ *denote the number of two-color partitions of* n*. Then*

$$s(n) = t(n)$$

For example, let $n = 3$. There are ten partitions counted by $s(3)$, which are

$$3, 3+3, 3+2, 3+1, 2+2+1, 2+2+1+1, 2+1+1, 2+1+1+1,$$
$$1+1+1+1+1, 1+1+1+1+1+1.$$

and there are also ten red and green partitions counted by $t(3)$, which are

$$3_r, 3_g, 2_r+1_r, 2_g+1_r, 2_r+1_g, 2_g+1_g, 1_r+1_r+1_r, 1_r+1_r+1_g,$$
$$1_r+1_g+1_g, 1_g+1_g+1_g.$$

Proof. Let $\mathcal{T}(n)$ denote the set of two-color partitions counted by $t(n)$ and let $\mathcal{S}(n)$ denote the set of partitions counted by $s(n)$. We aim to construct a bijection ϕ between $\mathcal{T}(n)$ and $\mathcal{S}(n)$.

Let λ be a two-color partition in $\mathcal{T}(n)$ with r red parts and l green parts. Assume that $m = \max\{r, l\}$. We aim to define $\phi(\lambda) = \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{2m-1}, \gamma_{2m})$ such that $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{2m-1} \geq \gamma_{2m} \geq 0$ and $\gamma_1 + \gamma_3 + \cdots + \gamma_{2m-1} = n$.

Let α be a partition consisting of all red parts in λ and β be a partition consisting of all green parts in λ. First, add 0 at the end of α or β so that they are of the same length depending on which is of smaller length. Assume that $r \le l$, so $m = l$. Then $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r, \underbrace{0, \ldots, 0}_{l-r})$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$.

We next define a pair $(\overline{\alpha}, \overline{\beta})$ of partitions with distinct parts corresponding to (α, β), where $\overline{\alpha} = (\alpha_1 + l - 1, \alpha_2 + l - 2, \ldots, \alpha_r + l - r, l - r - 1, \ldots, 1, 0)$, and $\overline{\beta} = (\beta_1 + l - 1, \beta_2 + l - 2, \ldots, \beta_l)$. Obviously, $|\overline{\alpha}| + |\overline{\beta}| = |\alpha| + |\beta| + l(l-1)$.

We now apply Wright's bijection to represent $(\overline{\alpha}, \overline{\beta})$ as a Young diagram of an ordinary partition $Y(\overline{\alpha}, \overline{\beta})$: put l squares on the diagonal, and then for $j = 1, 2, \ldots, l$, put $\bar{\alpha}_j$ squares in row j to the right of the diagonal and $\bar{\beta}_j$ squares in column j below the diagonal. For example, Figure 1. gives the Young diagram of $(\overline{\alpha}, \overline{\beta})$, where $\overline{\alpha} = (3, 2, 0)$ and $\overline{\beta} = (5, 3, 1)$.

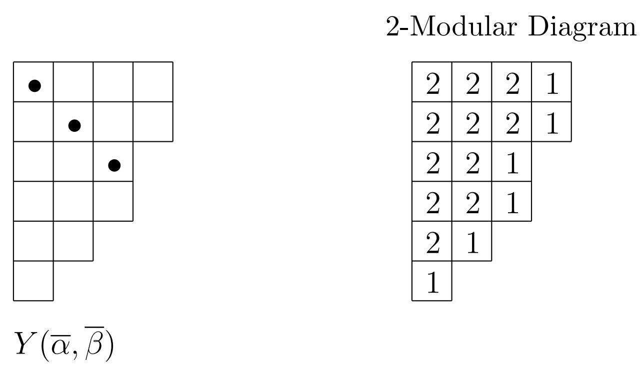

Figure 1: The Young diagram of $(\bar{\alpha}, \bar{\beta})$ and 2-Modular diagram.

For each row in the Young diagram of $(\overline{\alpha}, \overline{\beta})$, write 2 in each box and a 1 at the end of the row to obtain the 2-modular diagram. Decompose the 2-modular diagram into hooks H_1, $H_2, \ldots$ with the diagonal boxes as corners. Let μ_1 be the number of squares in H_1, let μ_2 be the number of 2's in H_1, let μ_3 be the number of squares in H_2, let μ_4 be the number of 2's in H_2, and so on. Set $\mu = (\mu_1, \mu_2, \ldots, \mu_{2l-1}, \mu_{2l})$, see Figure 2. Then μ is clearly a partition with distinct parts. Furthermore, $\mu_1 + \mu_3 + \cdots + \mu_{2l-1} = |\alpha| + |\beta| + l^2$. Hence we may define $\gamma = (\mu_1 - (2l-1), \mu_2 - 2l - 2, \ldots, \mu_{2l-1} - 1, \mu_{2l})$. Clearly, $\gamma_1 + \gamma_3 + \cdots + \gamma_{2l-1} = |\alpha| + |\beta|$, and so $\gamma \in \mathcal{S}(n)$. Furthermore, this process is reversible since Sylvester's bijection and Wright's bijection are reversible. Thus, we complete the proof of Theorem 1. ∎

Applying the above bijection, we get the following correspondence between the set $\mathcal{T}(3)$ and the set $\mathcal{S}(3)$.

$$3_r \leftrightarrows 3+3 \qquad 3_g \leftrightarrows 3 \qquad 2_r + 1_r \leftrightarrows 2+2+1+1$$

$$2_g + 1_r \leftrightarrows 3+1 \qquad 2_r + 1_g \leftrightarrows 3+2 \qquad 2_g + 1_g \leftrightarrows 2+1+1$$

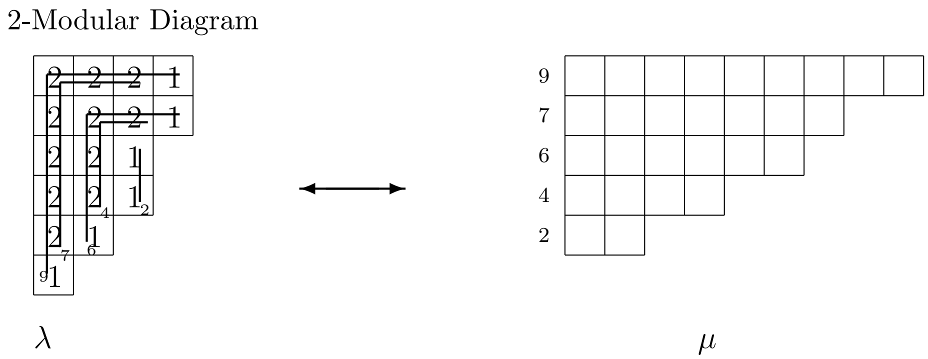

Figure 2: Sylvester's bijection.

$$1_r + 1_r + 1_r \leftrightarrows 1+1+1+1+1+1 \qquad 1_r + 1_r + 1_g \leftrightarrows 2+1+1+1$$

$$1_r + 1_g + 1_g \leftrightarrows 2+2+1 \qquad 1_g + 1_g + 1_g \leftrightarrows 1+1+1+1+1$$

The following result immediately follows from the combinatorial construction of Theorem 1.

Theorem 2. *Assume that $n \geq 1$, $r, l, p, q \geq 1$. Let $s_{r,l,p,q}(n)$ denote the number of partitions $a_1 + a_2 + a_3 + \cdots + a_{2\max\{r,l\}}$ satisfying $p + q \geq a_1 \geq a_2 \geq \cdots \geq a_{2\max\{r,l\}} \geq 0$ and $n = a_1 + a_3 + a_5 + \cdots + a_{2\max\{r,l\}-1}$. Let $t_{r,l,p,q}(n)$ denote the number of two-color partitions of n such that there are r red parts and l blue parts with the largest red part being not bigger than p and the largest blue part being not bigger than q. Then*

$$s_{r,l,p,q}(n) = t_{r,l,p,q}(n)$$

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