The q-Log-Concavity and Unimodality of q-Kaplansky Numbers

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Abstract. q-Kaplansky numbers were considered by Chen and Rota. We find that q-Kaplansky numbers are connected to the symmetric differences of Gaussian polynomials introduced by Reiner and Stanton. Based on the work of Reiner and Stanton, we establish the unimodality of q-Kaplansky numbers. We also show that q-Kaplansky numbers are the generating functions for the inversion number and the major index of two special kinds of $(0,1)$-sequences. Furthermore, we show that q-Kaplansky numbers are strongly q-log-concave.

Keywords: Inversion number, major index, q-log-concavity, unimodality, q-Catalan numbers, Foata's fundamental bijection, integer partitions

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1 Introduction

The main objective of this paper is to give two combinatorial interpretations of q-Kaplansky numbers introduced by Chen and Rota [4] and to establish some properties of q-Kaplansky numbers. Recall that the Kaplansky number $K(n,m)$ is defined by

$$K(n,m) = \frac{n}{n-m}\binom{n-m}{m},$$

for $n \geq 2m \geq 0$. The combinatorial interpretation of $K(n,m)$ was first given by Kaplansky [14], so we call $K(n,m)$ the Kaplansky number. Kaplansky found that $K(n,m)$ counts the number of ways of choosing m nonadjacent elements arranged on a cycle, which can also be interpreted as the number of dissections of type $1^{n-2k}2^k$ of an n-cycle given by Chen, Lih and Yeh [5]. Kaplansky numbers appear in many classical polynomials, such as Chebyshev polynomials of the first kind [17, 18] and Lucas polynomials [15].

q-Kaplansky numbers were introduced by Chen and Rota [4]. For convenience, we adopt the following definition: For $n \geq 1$ and $0 \leq m \leq n$,

$$K_q(n,m) = \frac{1-q^{n+m}}{1-q^n}\begin{bmatrix} n \\ m \end{bmatrix}, \tag{1.1}$$

where $\begin{bmatrix} n \\ m \end{bmatrix}$ is the Gaussian polynomial, also called the q-binomial coefficient, as given by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q)}.$$

By the symmetric property of the Gaussian polynomial, it is not hard to show that $K_q(n,m)$ is a symmetric polynomial of degree $m(n-m)+m$ with nonnegative coefficients.

The first result of this paper is to give two combinatorial interpretations of q-Kaplansky numbers. Let $w = w_1w_2\cdots w_n$ be a $(0,1)$-sequence of length n, the number of inversions of w, denoted $\mathrm{inv}(w)$, is the number of pairs (w_i, w_j) such that $i < j$ and $w_i > w_j$, and the major index of w, denoted $\mathrm{maj}(w)$, is the sum of indices $i < n$ such that $w_i > w_{i+1}$. For example, for $w = 10010110$, we have $\mathrm{inv}(w) = 8$ and $\mathrm{maj}(w) = 1 + 4 + 7 = 12$.

It can be shown that q-Kaplansky numbers are related to two sets $\mathcal{K}(m, n-m+1)$ and $\overline{\mathcal{K}}(m, n-m+1)$ of $(0,1)$-sequences. More precisely, for $n \geq m \geq 0$, let $\mathcal{K}(m, n-m+1)$ denote the set of $(0,1)$-sequences $w = w_1w_2\cdots w_{n+1}$ of length $n+1$ consisting of m copies of 1's and $n-m+1$ copies of 0's such that if $w_{n+1} = 1$, then $w_1 = 0$. For $n \geq m \geq 0$, let $\overline{\mathcal{K}}(m, n-m+1)$ denote the set of $(0,1)$-sequences $w = w_1w_2\cdots w_{n+1}$ of length $n+1$ consisting of m copies of 1's and $n-m+1$ copies of 0's such that if $w_{n+1} = 1$ and $t := \max\{i : w_i = 0\}$, then $t = 1$ or $w_{t-1} = 0$ when $t \geq 2$. We have the following combinatorial interpretations.

Theorem 1.1. *For $n \geq m \geq 0$,*

$$K_q(n,m) = \sum_{w\in\mathcal{K}(m,n-m+1)} q^{\mathrm{inv}(w)} \tag{1.2}$$

$$= \sum_{w\in\overline{\mathcal{K}}(m,n-m+1)} q^{\mathrm{maj}(w)}. \tag{1.3}$$

The second result of this paper is to establish the strong q-log-concavity of $K_q(n,m)$. Recall that a sequence of polynomials $(f_n(q))_{n\geq 0}$ over the field of real numbers is called q-log-concave if the difference

$$f_m(q)^2 - f_{m+1}(q)f_{m-1}(q)$$

has nonnegative coefficients as a polynomial in q for all $m \geq 1$. Sagan [20] also introduced the notion of the strong q-log-concavity. We say that a sequence of polynomials $(f_n(q))_{n\geq 0}$ is strongly q-log-concave if

$$f_n(q)f_m(q) - f_{n-1}(q)f_{m+1}(q)$$

has nonnegative coefficients as a polynomial in q for any $m \geq n \geq 1$.

It is known that q-analogues of many well-known combinatorial numbers are strongly q-log-concave. Butler [2] and Krattenthaler [16] proved the strong q-log-concavity of q-binomial coefficients, respectively. Leroux [12] and Sagan [20] studied the strong q-log-concavity of q-Stirling numbers of the first kind and the second kind. Chen, Wang and Yang [8] have shown that q-Narayana numbers are strongly q-log-concave.

We obtain the following result which implies that q-Kaplansky numbers are strongly q-log-concave.

Theorem 1.2. *For $1 \leq m \leq l < n$ and $0 \leq r \leq 2l - 2m + 2$,*

$$K_q(n,m)K_q(n,l) - q^r K_q(n,m-1)K_q(n,l+1) \tag{1.4}$$

has nonnegative coefficients as a polynomial in q.

Corollary 1.3. *Given a positive integer n, the sequence $(K_q(n,m))_{0 \leq m \leq n}$ is strongly q-log-concave.*

It is easy to check that the degree of $K_q(n,m)K_q(n,l)$ exceeds the degree of $K_q(n,m-1)K_q(n,l+1)$ by $2l - 2m + 2$, so if the difference (1.4) of these two polynomials has nonnegative coefficients, then $r \leq 2l - 2m + 2$.

To conclude the introduction, let us say a few words about the unimodiality of q-Kaplansky numbers. We find that q-Kaplansky numbers are connected to the following symmetric differences of Gaussian polynomials introduced by Reiner and Stanton [19].

$$F_{n,m}(q) = \begin{bmatrix} n+m \\ m \end{bmatrix} - q^n \begin{bmatrix} n+m-2 \\ m-2 \end{bmatrix}. \tag{1.5}$$

The following theorem is due to Reiner and Stanton [19].

Theorem 1.4 (Reiner-Stanton)**.** *When $m \geq 2$ and n is even, the polynomial $F_{n,m}(q)$ is symmetric and unimodal.*

Recently, Chen and Jia [6] provided a simple proof of the unimodality of $F_{n,m}(q)$ by using semi-invariants. According to the following recursions of Gaussian polynomials [1, p.35,Theorem 3.2 (3.3)],

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}, \tag{1.6}$$

$$\begin{bmatrix} n-1 \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} - q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \tag{1.7}$$

we find that

$$\begin{aligned} F_{n,m}(q) &= \begin{bmatrix} n+m \\ m \end{bmatrix} - q^n \begin{bmatrix} n+m-2 \\ m-2 \end{bmatrix} \\ &\overset{(1.6)}{=} \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix} - q^n \begin{bmatrix} n+m-2 \\ m-2 \end{bmatrix} + q^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix} \\ &\overset{(1.7)}{=} \begin{bmatrix} n+m-2 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix} \\ &= \frac{1-q^{n+2m-1}}{1-q^{n+m-1}} \begin{bmatrix} n+m-1 \\ m \end{bmatrix} \end{aligned}$$

$$= \quad K_q(n+m-1, m). \tag{1.8}$$

Combining Theorem 1.4 and (1.8), we have the following result.

Theorem 1.5. *When $n \geq m \geq 2$ and $n-m$ is odd, the q-Kaplansky number $K_q(n,m)$ is symmetric and unimodal.*

It should be noted that $K_q(n,m)$ is not always unimodal for any $n \geq m \geq 2$. For example,

$$K_q(6,2) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10}$$

is not unimodal.

q-Kaplansky numbers are also related to q-Catalan polynomials $C_n(q)$, defined by

$$C_n(q) = \frac{1-q}{1-q^{n+1}} \begin{bmatrix} 2n \\ n \end{bmatrix} = \frac{1-q}{1-q^{2n+1}} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}. \tag{1.9}$$

It is well-known that $C_n(q)$ is a polynomial in q with non-negative coefficients [10]. Combining (1.1) and (1.9), it is readily seen that

$$(1-q)K_q(2n+1, n) = (1-q^{3n+1})C_n(q).$$

Hence, by Theorem 1.5, we obtain the following result.

Theorem 1.6. *When n is even, the polynomial $\frac{1-q^{3n+1}}{1-q}C_n(q)$ is symmetric and unimodal.*

Finally, we would like to state a result of Stanley [22, p.523] about the unimodality of the q-Catalan polynomials and two conjectures on the unimodality of the q-Catalan polynomials due to Chen, Wang and Wang [7] and Xin and Zhong [24, Conjecture 1.2], respectively. Apparently, Conjecture 1.8 implies Conjecture 1.9 when $n \geq 16$.

Theorem 1.7 (Stanley)**.** *For $n \geq 1$, the polynomial $\frac{1+q}{1+q^n}C_n(q)$ is symmetric and unimodal.*

Conjecture 1.8 (Chen, Wang and Wang)**.** *For $n \geq 16$, the q-Catalan polynomial $C_n(q)$ is unimodal.*

Conjecture 1.9 (Xin and Zhong)**.** *For $n \geq 1$, the polynomial $(1+q)C_n(q)$ is unimodal.*

2 Proof of Theorem 1.1

To prove Theorem 1.1, we first recall a result due to MacMahon [13]. For $n \geq m \geq 0$, let $\mathcal{M}(m, n-m)$ be the set of $(0,1)$-sequences of length n consisting of m copies of 1's and $n-m$ copies of 0's. The following well-known result is due to MacMahon (see [1, Chapter 3.4]).

Theorem 2.1 (MacMahon)**.** *For $n \geq m \geq 0$,*

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{w \in \mathcal{M}(m,n-m)} q^{\mathrm{inv}(w)} \tag{2.1}$$

$$= \sum_{w \in \mathcal{M}(m,n-m)} q^{\mathrm{maj}(w)}. \tag{2.2}$$

Foata's fundamental bijection [9] can be used to establish the equivalence of (2.1) and (2.2). There are several ways to describe Foata's fundamental bijection, see, for example, Foata [9], Haglund [11, p.2] and Sagan and Savage [21]. Here we give a description due to Sagan and Savage [21].

Proof of the equivalence between (2.1) *and* (2.2): Let $w = w_1 w_2 \cdots w_n \in \mathcal{M}(m, n-m)$. We aim to construct a $(0,1)$-sequence $\widetilde{w} = \phi(w) = \widetilde{w}_1 \widetilde{w}_2 \cdots \widetilde{w}_n$ in $\mathcal{M}(m, n-m)$ such that $\mathrm{inv}(\widetilde{w}) = \mathrm{maj}(w)$.

Let w be a $(0,1)$-sequence with d descents, so that we can write

$$w = 0^{m_0} 1^{n_0} 0^{m_1} 1^{n_1} 0^{m_2} \cdots 1^{n_{d-1}} 0^{m_d} 1^{n_d}, \tag{2.3}$$

where $m_0 \geq 0$ and $m_i \geq 1$ for $1 \leq i \leq d$, $n_i \geq 1$ for $0 \leq i \leq d-1$ and $n_d \geq 0$.

Define

$$\widetilde{w} = \phi(w) = 0^{m_d - 1} 1 0^{m_{d-1} - 1} 1 \cdots 0^{m_1 - 1} 1 0^{m_0} 1^{n_0 - 1} 0 1^{n_1 - 1} \cdots 0 1^{n_{d-1} - 1} 0 1^{n_d}. \tag{2.4}$$

It has been shown in [21] that $\mathrm{inv}(\widetilde{w}) = \mathrm{maj}(w)$.

The inverse map ϕ^{-1} of ϕ can be described recursively. Let $\widetilde{w} \in \mathcal{M}(m, n-m)$, we may write $\widetilde{w} = 0^a 1 u 0 1^b$ for $a, b \geq 0$, define

$$w = \phi^{-1}(\widetilde{w}) = \phi^{-1}(u) 1 0^{a+1} 1^b. \tag{2.5}$$

It has been proved in [21] that $\phi^{-1}(\phi(w)) = w$ and $\phi(\phi^{-1}(\widetilde{w})) = \widetilde{w}$. Furthermore, $\mathrm{inv}(\widetilde{w}) = \mathrm{maj}(w)$. Hence the map ϕ is a bijection. This completes the proof of the equivalence of (2.1) and (2.2). ∎

For $n \geq m \geq 0$, let $\mathcal{M}_0(m, n-m+1)$ be the set of $(0,1)$-sequences $w = w_1 w_2 \cdots w_{n+1}$ of length $n+1$ consisting of m copies of 1's and $n-m+1$ copies of 0's such that $w_{n+1} = 0$. We have the following result.

Lemma 2.2. *For $n \geq m \geq 0$,*

$$q^m \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{w \in \mathcal{M}_0(m,n-m+1)} q^{\mathrm{inv}(w)} \tag{2.6}$$

$$= \sum_{w \in \mathcal{M}_0(m,n-m+1)} q^{\mathrm{maj}(w)}. \tag{2.7}$$

Proof. By Theorem 2.1, we see that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{w\in\mathcal{M}(m,n-m)} q^{\mathrm{inv}(w)}.$$

To prove (2.6), it suffices to show that

$$\sum_{w\in\mathcal{M}(m,n-m)} q^{\mathrm{inv}(w)+m} = \sum_{w\in\mathcal{M}_0(m,n-m+1)} q^{\mathrm{inv}(w)}. \tag{2.8}$$

We construct a bijection ψ between the set $\mathcal{M}(m,n-m)$ and the set $\mathcal{M}_0(m,n-m+1)$ such that for $w\in\mathcal{M}(m,n-m)$ and $\psi(w)\in\mathcal{M}_0(m,n-m+1)$, we have

$$\mathrm{inv}(w)+m=\mathrm{inv}(\psi(w)). \tag{2.9}$$

Let $w=w_1w_2\cdots w_n$. Define

$$\psi(w)=w_1w_2\cdots w_n0.$$

It is clear that $\psi(w)\in\mathcal{M}_0(m,n-m+1)$ and (2.9) holds. Furthermore, it is easy to see that ψ is reversible. Hence we have(2.8).

We proceed to show that (2.6) and (2.7) are equivalent by using Foata's fundamental bijection ϕ. Let $w=w_1w_2\cdots w_{n+1}$ be in $\mathcal{M}_0(m,n-m+1)$, by definition, we see that $w_{n+1}=0$. Define

$$\widetilde{w}=\phi^{-1}(w)=\widetilde{w}_1\widetilde{w}_2\cdots\widetilde{w}_{n+1},$$

where ϕ^{-1} is defined in (2.5). By (2.5), we see that $\widetilde{w}_{n+1}=0$ since $w_{n+1}=0$. Hence $\widetilde{w}\in\mathcal{M}_0(m,n-m+1)$. Furthermore ϕ^{-1} is reversible and $\mathrm{inv}(w)=\mathrm{maj}(\widetilde{w})$. It follows (2.6) and (2.7) are equivalent, and so (2.7) is valid. ∎

For $n\geq m\geq 1$, let $\mathcal{M}_1(m,n-m+1)$ be the set of $(0,1)$-sequences $w=w_1w_2\cdots w_{n+1}$ of length $n+1$ consisting of m copies of 1's and $n-m+1$ copies of 0's such that $w_1=0$ and $w_{n+1}=1$. For $n\geq m\geq 1$, let $\overline{\mathcal{M}}_1(m,n-m+1)$ be the set of $(0,1)$-sequences $w=w_1w_2\cdots w_{n+1}$ of length $n+1$ consisting of m copies of 1's and $n-m+1$ copies of 0's such that $w_{n+1}=1$, and if $t:=\max\{i:w_i=0\}$, then $t=1$ or $w_{t-1}=0$ when $t\geq 2$. To wit, for $w\in\overline{\mathcal{M}}_1(m,n-m+1)$, if $m\geq 1$ and $n>m$, then w can be written as $u001^{n+1-t}$, where $2\leq t\leq n$ and $u\in\mathcal{M}(m+t-n-1,n-m-1)$, and if $m\geq 1$ and $n=m$, then w can be written as 01^m.

Lemma 2.3. *For $n\geq m\geq 1$,*

$$\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = \sum_{w\in\mathcal{M}_1(m,n-m+1)} q^{\mathrm{inv}(w)} \tag{2.10}$$

$$= \sum_{w\in\overline{\mathcal{M}}_1(m,n-m+1)} q^{\mathrm{maj}(w)}. \tag{2.11}$$

Proof. By Theorem 2.1, we see that

$$\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = \sum_{w\in\mathcal{M}(m-1,n-m)} q^{\mathrm{inv}(w)}.$$

To prove (2.10), it suffices to show that

$$\sum_{w\in\mathcal{M}(m-1,n-m)} q^{\mathrm{inv}(w)} = \sum_{w\in\mathcal{M}_1(m,n-m+1)} q^{\mathrm{inv}(w)}. \tag{2.12}$$

We now construct a bijection φ between the set $\mathcal{M}(m-1,n-m)$ and the set $\mathcal{M}_1(m,n-m+1)$ such that for $w\in\mathcal{M}(m-1,n-m)$ and $\varphi(w)\in\mathcal{M}_1(m,n-m+1)$, we have

$$\mathrm{inv}(w) = \mathrm{inv}(\varphi(w)). \tag{2.13}$$

Let $w = w_1w_2\cdots w_{n-1}$. Define

$$\varphi(w) = 0w_1w_2\cdots w_{n-1}1.$$

It is clear that $\varphi(w)\in\mathcal{M}_1(m,n-m+1)$ and (2.13) holds. Furthermore, ψ is reversible. Hence we have (2.12).

We proceed to show that (2.11) holds. By (2.2), it suffices to show that

$$\sum_{w\in\mathcal{M}(m-1,n-m)} q^{\mathrm{maj}(w)} = \sum_{w\in\overline{\mathcal{M}}_1(m,n-m+1)} q^{\mathrm{maj}(w)}. \tag{2.14}$$

We now construct a bijection τ between the set $\mathcal{M}(m-1,n-m)$ and the set $\overline{\mathcal{M}}_1(m, n-m+1)$ such that for $w\in\mathcal{M}(m-1,n-m)$ and $\tau(w)\in\overline{\mathcal{M}}_1(m,n-m+1)$, we have

$$\mathrm{maj}(w) = \mathrm{maj}(\tau(w)). \tag{2.15}$$

Let $w = w_1w_2\cdots w_{n-1}\in\mathcal{M}(m-1,n-m)$. If $n=m$, then $w=1^{m-1}$, and so define $\tau(w)=01^m$. If $n>m$, then let $t=\max\{i : w_i = 0\}$, obviously, $t\geq 1$. In this case, we may write $w = w_1w_2\cdots w_{t-1}01^{n-t-1}$. Define

$$\widetilde{w} = \tau(w) = \widetilde{w}_1\widetilde{w}_2\cdots\widetilde{w}_{n+1}$$

as follows: set $\widetilde{w}_{n+1}=1$, and set $\widetilde{w}_j = w_j$ for $1\leq j\leq t$, $\widetilde{w}_{t+1}=0$, and set $\widetilde{w}_{j+1}=w_j=1$ for $t+1\leq j\leq n-1$.

From the above construction, it is easy to see that $\widetilde{w}\in\overline{\mathcal{M}}_1(m,n-m+1)$ and (2.15) holds. Furthermore, it can be checked that this construction is reversible, so (2.14) is valid. ∎

We are now in a position to give a proof of Theorem 1.1 based on Lemma 2.2 and Lemma 2.3.

Proof of Theorem 1.1: By the definition of $\mathcal{K}(m,n-m+1)$, we see that

$$\mathcal{K}(m,n-m+1) = \mathcal{M}_0(m,n-m+1)\cup\mathcal{M}_1(m,n-m+1).$$

Combining (2.6) and (2.10), we derive that for $n \geq m \geq 1$,

$$\begin{aligned}\sum_{w \in \mathcal{K}(m,n-m+1)} q^{\mathrm{inv}(w)} &= \sum_{w \in \mathcal{M}_0(m,n-m+1)} q^{\mathrm{inv}(w)} + \sum_{w \in \mathcal{M}_1(m,n-m+1)} q^{\mathrm{inv}(w)} \\ &= q^m \begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \\ &= \frac{1-q^{n+m}}{1-q^n} \begin{bmatrix} n \\ m \end{bmatrix} \\ &= K_q(n,m).\end{aligned}$$

Similarly, by definition, we see that

$$\overline{\mathcal{K}}(m, n-m+1) = \mathcal{M}_0(m, n-m+1) \cup \overline{\mathcal{M}}_1(m, n-m+1).$$

By (2.7) and (2.11), we find that $n \geq m \geq 1$,

$$\begin{aligned}\sum_{w \in \overline{\mathcal{K}}(m,n-m+1)} q^{\mathrm{maj}(w)} &= \sum_{w \in \mathcal{M}_0(m,n-m+1)} q^{\mathrm{maj}(w)} + \sum_{w \in \overline{\mathcal{M}}_1(m,n-m+1)} q^{\mathrm{maj}(w)} \\ &= q^m \begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \\ &= \frac{1-q^{n+m}}{1-q^n} \begin{bmatrix} n \\ m \end{bmatrix} \\ &= K_q(n,m).\end{aligned}$$

Furthermore, it is easy to check that (1.2) and (1.3) are valid when $m = 0$. This completes the proof of Theorem 1.1. ∎

3 Proof of Theorem 1.2

Before we prove Theorem 1.2, it is useful to preset the following result.

Lemma 3.1. *For* $1 \leq m \leq l < N$ *and* $M - m \geq N - l \geq 1$,

$$D_q(M, N, m, l) = \begin{bmatrix} M \\ m \end{bmatrix} \begin{bmatrix} N \\ l \end{bmatrix} - \begin{bmatrix} M \\ m-1 \end{bmatrix} \begin{bmatrix} N \\ l+1 \end{bmatrix}$$

has nonnegative coefficients as a polynomial in q.

Lemma 3.1 reduces to the strong q-log-concavity of Gaussian polynomials when $M = N$. We prove Lemma 3.1 by generalizing Butler's bijection [2]. To describe the proof, we need to recall some notation and terminology on partitions as in [1, Chapter 1]. A partition λ of a positive integer n is a finite nonincreasing sequence of positive integers

$(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. Then λ_i are called the parts of λ and λ_1 is its largest part. The number of parts of λ is called the length of λ, denoted by $l(\lambda)$. The weight of λ is the sum of parts of λ, denoted $|\lambda|$. The conjugate $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$ of a partition λ is defined by setting λ'_i to be the number of parts of λ that are greater than or equal to i. Clearly, $l(\lambda) = \lambda'_1$ and $\lambda_1 = l(\lambda')$.

Let $\mathcal{P}(m, n-m)$ denote the set of partitions λ such that $\ell(\lambda) \leq m$ and $\lambda_1 \leq n-m$. It is well-known that the Gaussian polynomial has the following partition interpretation [1, Theorem 3.1]:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{\lambda \in \mathcal{P}(m,n-m)} q^{|\lambda|}. \tag{3.1}$$

We are now prepared for the proof of Lemma 3.1 based on (3.1).

Proof of Lemma 3.1: For $1 \leq m \leq l < N$ and $M - m \geq N - l \geq 1$, by (3.1), it suffices to construct an injection Φ from $\mathcal{P}(m-1, M-m+1) \times \mathcal{P}(l+1, N-l-1)$ to $\mathcal{P}(m, M-m) \times \mathcal{P}(l, N-l)$ such that if $\Phi(\lambda, \mu) = (\eta, \rho)$, then $|\lambda| + |\mu| = |\eta| + |\rho|$.

Let

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1}) \in \mathcal{P}(m-1, M-m+1)$$

and

$$\mu = (\mu_1, \mu_2, \ldots, \mu_{l+1}) \in \mathcal{P}(l+1, N-l-1),$$

where $\lambda_1 \leq M - m + 1$ and $\mu_1 \leq N - l - 1$.

We aim to construct a pair of partitions

$$(\eta, \rho) \in \mathcal{P}(m, M-m) \times \mathcal{P}(l, N-l).$$

Let I be the largest integer such that $\lambda_I \geq \mu_{I+1} + l - m + M - N + 1$. If no such I exists, then let $I = 0$. In this case, we see that $\lambda_1 < M - m$ and set $\gamma = \lambda$ and $\tau = \mu$. Obviously, $\gamma_1 < M - m$ and $\tau_1 < N - l$. We now assume that $1 \leq I \leq m-1$ and define

$$\gamma = (\mu_1 + (l-m+M-N+1), \ldots, \mu_I + (l-m+M-N+1), \lambda_{I+1}, \ldots, \lambda_{m-1}) \tag{3.2}$$

and

$$\tau = (\lambda_1 - (l-m+M-N+1), \ldots, \lambda_I - (l-m+M-N+1), \mu_{I+1}, \ldots, \mu_{l+1}). \tag{3.3}$$

Since I is the largest integer such that $\lambda_I \geq \mu_{I+1} + (l - m + M - N + 1)$, we get

$$\lambda_{I+1} < \mu_{I+2} + (l - m + M - N + 1) \leq \mu_I + (l - m + M - N + 1).$$

It follows that γ defined in (3.2) and τ defined in (3.3) are partitions. Furthermore,

$$\gamma_1 = \mu_1 + (l - m + M - N + 1) \leq M - m$$

and

$$\tau_1 = \lambda_1 - (l - m + M - N + 1) \leq N - l.$$

Let γ' and τ' be the conjugates of γ and τ, respectively. We see that

$$\ell(\gamma') = \gamma_1 \le M - m \quad \text{and} \quad \ell(\tau') = \tau_1 \le N - l,$$

so we can assume that

$$\gamma' = (\gamma'_1, \gamma'_2, \ldots, \gamma'_{M-m})$$

and

$$\tau' = (\tau'_1, \tau'_2, \ldots, \tau'_{N-l}).$$

Then

$$\gamma'_1 \le m - 1 \quad \text{and} \quad \tau'_1 \le l + 1.$$

Let J be the largest integer such that $\tau'_J \ge \gamma'_{J+1} + l - m + 1$. If no such J exists, let $J = 0$, then $\tau'_1 < l$ and set $\widetilde{\gamma} = \gamma'$, and $\widetilde{\tau} = \tau'$. Obviously, $\widetilde{\gamma}_1 < m$ and $\widetilde{\tau}_1 < l$. We now assume that $1 \le J \le N - l$ and define

$$\widetilde{\gamma} = (\tau'_1 - (l - m + 1), \tau'_2 - (l - m + 1), \ldots, \tau'_J - (l - m + 1), \gamma'_{J+1}, \ldots, \gamma'_{M-m}) \quad (3.4)$$

and

$$\widetilde{\tau} = (\gamma'_1 + (l - m + 1), \gamma'_2 + (l - m + 1), \ldots, \gamma'_J + (l - m + 1), \tau'_{J+1}, \ldots, \tau'_{N-l}). \quad (3.5)$$

Similarly, since J is the largest integer such that $\tau'_J \ge \gamma'_{J+1} + l - m + 1$, we find that

$$\tau'_{J+1} < \gamma'_{J+2} + l - m + 1 \le \gamma'_J + l - m + 1,$$

so $\widetilde{\gamma}$ defined in (3.4) and $\widetilde{\tau}$ defined in (3.5) are partitions. By the constructions of $\widetilde{\gamma}$ and $\widetilde{\tau}$, we see that

$$\widetilde{\gamma}_1 = \tau'_1 - (l - m + 1) \le m$$

and

$$\widetilde{\tau}_1 = \gamma'_1 + (l - m + 1) \le l.$$

Let η and ρ be the conjugates of $\widetilde{\gamma}$ and $\widetilde{\tau}$, respectively. It is easy to check that $\eta \in \mathcal{P}(m, M - m)$ and $\rho \in \mathcal{P}(l, N - l)$. Furthermore, this process is reversible. Thus, we complete the proof of Lemma 3.1. ∎

Combining Lemma 3.1 and the unimodality of Gaussian polynomials, we obtain the following result.

Lemma 3.2. *For $1 \le m \le l < N$, $M - m \ge N - l \ge 1$ and $0 \le r \le M - N + 2l - 2m + 2$,*

$$D_q^r(M, N, m, l) = \begin{bmatrix} M \\ m \end{bmatrix} \begin{bmatrix} N \\ l \end{bmatrix} - q^r \begin{bmatrix} M \\ m-1 \end{bmatrix} \begin{bmatrix} N \\ l+1 \end{bmatrix} \quad (3.6)$$

has nonnegative coefficients as a polynomial in q.

Proof. Let A denote the degree of the polynomial $\begin{bmatrix} M \\ m \end{bmatrix} \begin{bmatrix} N \\ l \end{bmatrix}$ and let B denote the degree of the polynomial $\begin{bmatrix} M \\ m-1 \end{bmatrix} \begin{bmatrix} N \\ l+1 \end{bmatrix}$. We have

$$A = m(M - m) + l(N - l),$$

$$B = (m-1)(M-m+1) + (l+1)(N-l-1).$$

Furthermore,

$$A - B = M - N + 2l - 2m + 2.$$

Let

$$\begin{bmatrix} M \\ m \end{bmatrix}\begin{bmatrix} N \\ l \end{bmatrix} = \sum_{i=0}^{A} a_i q^i, \quad \begin{bmatrix} M \\ m-1 \end{bmatrix}\begin{bmatrix} N \\ l+1 \end{bmatrix} = \sum_{i=0}^{B} b_i q^i$$

and let

$$D_q^r(M,N,m,l) = \begin{bmatrix} M \\ m \end{bmatrix}\begin{bmatrix} N \\ l \end{bmatrix} - q^r \begin{bmatrix} M \\ m-1 \end{bmatrix}\begin{bmatrix} N \\ l+1 \end{bmatrix} = \sum_{i=0}^{A} c_i q^i,$$

where $c_i = a_i$ for $0 \le i < r$, $c_i = a_i - b_{i-r}$ for $r \le i \le B + r$ and $c_i = a_i$ for $B + r + 1 \le i \le A$. It is easy to see that $c_i \ge 0$ for $0 \le i < r$ and $B + r + 1 \le i \le A$. It remains to show that $c_i \ge 0$ for $r \le i \le B + r$.

It is known that the Gaussian polynomial $\begin{bmatrix} M \\ m \end{bmatrix}$ is symmetric and unimodal, see, for example, [1, Theorem 3.10] and [23, Exercise 7.75], so

$$a_i = a_{A-i} \text{ for } 0 \le i \le A, \quad \text{and} \quad b_i = b_{B-i} \text{ for } 0 \le i \le B, \tag{3.7}$$

$$a_0 \le a_1 \le \cdots \le a_{\lfloor A/2 \rfloor} = a_{\lceil A/2 \rceil} \ge \cdots \ge a_{A-1} \ge a_A, \tag{3.8}$$

and

$$b_0 \le b_1 \le \cdots \le b_{\lfloor A/2 \rfloor} = b_{\lceil A/2 \rceil} \ge \cdots \ge b_{B-1} \ge b_B. \tag{3.9}$$

By Lemma 3.1, we see that for $0 \le i \le A$,

$$a_i - b_i \ge 0. \tag{3.10}$$

We consider the following two cases:

Case 1. If $r \le i \le A/2$, then

$$c_i = a_i - b_{i-r} = a_i - a_{i-r} + a_{i-r} - b_{i-r},$$

which is nonnegative by (3.8) and (3.10).

Case 2. If $A/2 \le i \le B + r$, then

$$c_i = a_i - b_{i-r} \overset{(3.7)}{=} a_{A-i} - b_{B-i+r} = a_{A-i} - a_{B-i+r} + a_{B-i+r} - b_{B-i+r},$$

which is nonnegative by (3.8) and (3.10). Thus, we complete the proof of Lemma 3.2. ∎

We conclude this paper with a proof of Theorem 1.2 by using Lemma 3.2.

Proof of Theorem 1.2: Recall that

$$K_q(n,m) = \frac{1-q^{n+m}}{1-q^n}\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} + q^n \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$

Hence

$$
\begin{aligned}
&K_q(n,m)K_q(n,l) - q^r K_q(n,m-1)K_q(n,l+1)\\
&= \left(\begin{bmatrix} n \\ m \end{bmatrix} + q^n \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \right) \left(\begin{bmatrix} n \\ l \end{bmatrix} + q^n \begin{bmatrix} n-1 \\ l-1 \end{bmatrix} \right)\\
&\quad - q^r \left(\begin{bmatrix} n \\ m-1 \end{bmatrix} + q^n \begin{bmatrix} n-1 \\ m-2 \end{bmatrix} \right) \left(\begin{bmatrix} n \\ l+1 \end{bmatrix} + q^n \begin{bmatrix} n-1 \\ l \end{bmatrix} \right)\\
&= \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix} - q^r \begin{bmatrix} n \\ m-1 \end{bmatrix} \begin{bmatrix} n \\ l+1 \end{bmatrix}\\
&\quad + q^n \left(\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix} - q^r \begin{bmatrix} n-1 \\ m-2 \end{bmatrix} \begin{bmatrix} n \\ l+1 \end{bmatrix} \right)\\
&\quad + q^n \left(\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix} - q^r \begin{bmatrix} n \\ m-1 \end{bmatrix} \begin{bmatrix} n-1 \\ l \end{bmatrix} \right)\\
&\quad + q^{2n} \left(\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix} - q^r \begin{bmatrix} n-1 \\ m-2 \end{bmatrix} \begin{bmatrix} n-1 \\ l \end{bmatrix} \right).
\end{aligned}
$$

Using the notation in Lemma 3.2, we see that

$$
\begin{aligned}
&K_q(n,m)K_q(n,l) - q^r K_q(n,m-1)K_q(n,l+1)\\
&= D_q^r(n,n,m,l) + q^n D_q^r(n-1,n,m-1,l) + q^n D_q^r(n,n-1,m,l-1)\\
&\quad + q^{2n} D_q^r(n-1,n-1,m-1,l-1).
\end{aligned}
$$

Applying Lemma 3.2, we find that for $1 \le m \le l < n$ and $0 \le r \le 2l-2m+2$,

$$D_q^r(n,n,m,l),\ D_q^r(n-1,n,m-1,l),\ \text{and}\ D_q^r(n-1,n-1,m-1,l-1)$$

have nonnegative coefficients as polynomials in q, respectively, and for $1 \le m \le l < n$ and $0 \le r \le 2l-2m+1$,

$$D_q^r(n,n-1,m,l-1)$$

has nonnegative coefficients as a polynomial in q. It follows that for $1 \le m \le l < n$ and $0 \le r \le 2l-2m+1$,

$$K_q(n,m)K_q(n,l) - q^r K_q(n,m-1)K_q(n,l+1) \tag{3.11}$$

has nonnegative coefficients as a polynomial in q. Hence it remains to show that the difference (3.11) has nonnegative coefficients as a polynomial in q when $r = 2l-2m+2$. It suffices to show that

$$q^n D_q^{2l-2m+2}(n-1,n-1,m-1,l-1) + D_q^{2l-2m+2}(n,n-1,m,l-1) \tag{3.12}$$

has nonnegative coefficients as a polynomial in q. First, it is easy to check that

$$q^n D_q^{2l-2m+2}(n-1,n-1,m-1,l-1) + D_q^{2l-2m+2}(n,n-1,m,l-1)$$

$$= K_q(n,m)\begin{bmatrix} n-1 \\ l-1 \end{bmatrix} - q^{2l-2m+2}K_q(n,m-1)\begin{bmatrix} n-1 \\ l \end{bmatrix}.$$

Using the following relation:

$$K_q(n,m) = \frac{1-q^{n+m}}{1-q^n}\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m\begin{bmatrix} n \\ m \end{bmatrix},$$

we find that

$$\begin{aligned}
&q^n D_q^{2l-2m+2}(n-1,n-1,m-1,l-1) + D_q^{2l-2m+2}(n,n-1,m,l-1)\\
&\quad= \left(\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m\begin{bmatrix} n \\ m \end{bmatrix}\right)\begin{bmatrix} n-1 \\ l-1 \end{bmatrix} - q^{2l-2m+2}\left(\begin{bmatrix} n-1 \\ m-2 \end{bmatrix} + q^{m-1}\begin{bmatrix} n \\ m-1 \end{bmatrix}\right)\begin{bmatrix} n-1 \\ l \end{bmatrix}\\
&\quad= \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l-1 \end{bmatrix} - q^{2l-2m+2}\begin{bmatrix} n-1 \\ m-2 \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix}\\
&\qquad +q^m\left(\begin{bmatrix} n \\ m \end{bmatrix}\begin{bmatrix} n-1 \\ l-1 \end{bmatrix} - q^{2l-2m+1}\begin{bmatrix} n \\ m-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix}\right)\\
&\quad= D_q^{2l-2m+2}(n-1,n-1,m-1,l-1) + q^m D_q^{2l-2m+1}(n,n-1,m,l-1).
\end{aligned}$$

From Lemma 3.2, we see that

$$D_q^{2l-2m+2}(n-1,n-1,m-1,l-1), \quad \text{and} \quad D_q^{2l-2m+1}(n,n-1,m,l-1)$$

have nonnegative coefficients as a polynomial in q, respectively, and so (3.12) has nonnegative coefficients as a polynomial in q. Thus, we complete the proof of Theorem 1.2. ∎

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