Turán inequalities for the broken k-diamond partition functions

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Abstract. We obtain an asymptotic formula for Andrews and Paule's broken k-diamond partition function $\Delta_k(n)$ where $k = 1$ or 2. Based on this asymptotic formula, we derive that $\Delta_k(n)$ satisfies the order d Turán inequalities for $d \geq 1$ and for sufficiently large n when $k = 1$ or 2 by using a general result of Griffin, Ono, Rolen and Zagier. We also show that Andrews and Paule's broken k-diamond partition function $\Delta_k(n)$ is log-concave for $n \geq 1$ when $k = 1$ or 2, which implies that $\Delta_k(a)\Delta_k(b) \geq \Delta_k(a+b)$ for $a, b \geq 1$ when $k = 1$ or 2.

Keywords: broken k-diamond partition functions, log-concavity, higher order Turán inequalities, Jensen polynomials

AMS Classification: 11P82, 05A19, 30A10

1 Introduction

The main objective of this paper is to establish Turán inequalities for the number of broken k-diamond partitions. The notion of broken k-diamond partitions was introduced by Andrews and Paule [2]. A broken k-diamond partition $\pi = (b_2, \ldots, b_{2k+2}, \ldots, b_{(2k+1)l+1}, a_1, \ldots, a_{2k+2}, \ldots, a_{(2k+1)l+1})$ is a plane partition satisfying the relations illustrated in Figure 1, where a_i, b_i are nonnegative integers and $a_i \rightarrow a_j$ is interpreted as $a_i \geq a_j$. More precisely, each building block in Figure 1 has the same order structure as shown in Figure 2. We call such block a k-elongated partition diamond of length 1. It should be noted that the broken block $(b_2, b_3, \ldots, b_{2k+2})$ is also a k-elongated partition diamond of length 1 from which a source b_1 is deleted.

Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. Andrews and Paule [2] established the following generating function for $\Delta_k(n)$:

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \prod_{n=1}^{\infty} \frac{\left(1-q^{2n}\right)\left(1-q^{(2k+1)n}\right)}{\left(1-q^n\right)^3\left(1-q^{(4k+2)n}\right)}.$$

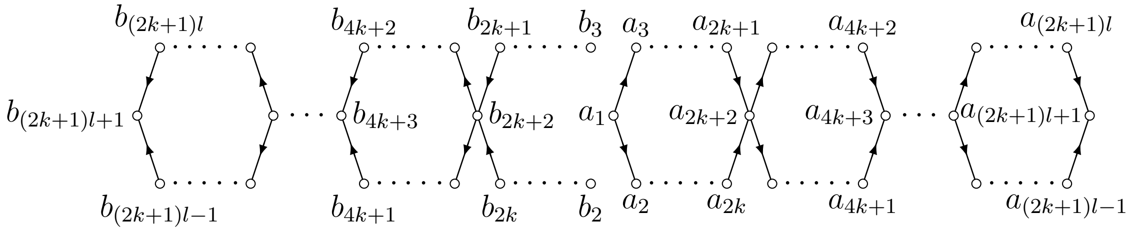

Figure 1: A broken k-diamond of length $2l$.

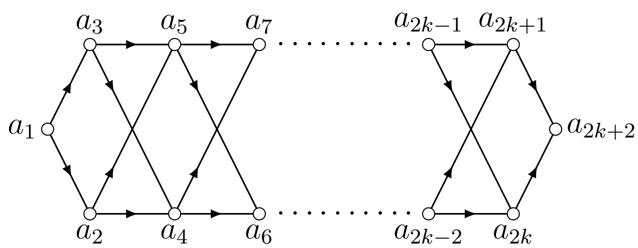

Figure 2: A k-elongated partition diamond of length 1.

It's known that $\Delta_k(n)$ possesses many beautiful arithmetic properties. Many Ramanujan-like congruences satisfied by $\Delta_1(n)$ and $\Delta_2(n)$ were proved by Andrews and Paule [2] and other authors, see, for example, Chan [8], Chen, Fan and Yu [10], Hirschhorn [18], Paule and Radu [22] and so on. It should be noted that $\Delta_k(n)$ are the coefficients of a modular function over $\Gamma_0(4k+2)$.

The Turán inequalities arise in the study of real entire functions in Laguerre-Pólya class, which are closely related to the study of the Riemann hypothesis [15, 24]. A sequence $\{\alpha_n\}_{n\geq 0}$ of real numbers is log-concave if it satisfies the (second order) Turán inequalities $\alpha_n^2 \geq \alpha_{n-1}\alpha_{n+1}$ for $n \geq 1$. We call the sequence $\{\alpha_n\}_{n\geq 0}$ satisfies the third order Turán inequalities if for $n \geq 1$,

$$4(\alpha_n^2 - \alpha_{n-1}\alpha_{n+1})(\alpha_{n+1}^2 - \alpha_n\alpha_{n+2}) \geq (\alpha_n\alpha_{n+1} - \alpha_{n-1}\alpha_{n+2})^2.$$

As stated by Chen, Jia and Wang [12] and Griffin, Ono, Rolen and Zagier [17], the higher order Turán inequalities are conveniently formulated in terms of the Jensen polynomials. The Jensen polynomials $J_\alpha^{d,n}(X)$ of degree d and shift n associated to the sequence $\{\alpha_n\}_{n\geq 0}$ are defined by

$$J_\alpha^{d,n}(X) = \sum_{i=0}^{d} \binom{d}{i} \alpha_{n+i} X^i. \tag{1.1}$$

When $d = 2$ and shift $n-1$, the Jensen polynomial $J_\alpha^{2,n-1}(X)$ reduces to

$$J_\alpha^{2,n-1}(X) = \alpha_{n-1} + 2\alpha_n X + \alpha_{n+1} X^2.$$

It is clear that $\{\alpha_n\}_{n\geq 0}$ is log-concave at n if and only if $J_\alpha^{2,n-1}(X)$ has only real roots. In general, we say that the sequence $\{\alpha_n\}_{n\geq 0}$ satisfies the order d Turán inequality at n if and only if $J_\alpha^{d,n-1}(X)$ is hyperbolic. Recall that a polynomial is hyperbolic if all of its roots are real.

There are several recent works on the Turán inequalities for the partition functions. Nicolas [20] and DeSalvo and Pak [14] proved that the partition function $p(n)$ is log-concave for $n \geq 26$, where $p(n)$ is the number of partitions of n. Chen [11] conjectured that $p(n)$ satisfies the third order Turán inequalities for $n \geq 95$, which was proved by Chen, Jia and Wang [12]. Chen, Jia and Wang [12] further conjectured that for $d \geq 4$, there exists a positive integer $N_p(d)$ such that $p(n)$ satisfies the order d Turán inequalities for $n \geq N_p(d)$, that is, the Jensen polynomial $J_p^{d,n-1}(X)$ associated to $p(n)$ is hyperbolic for $n \geq N_p(d)$. Griffin, Ono, Rolen and Zagier [17] showed that Chen, Jia and Wang's conjecture is true for sufficiently large n. In fact, they obtained the following general result:

Theorem 1.1 (Proof of Theorem 7 of [17])**.** *Let $\{a_f(n)\}_{n\geq 0}$ be a sequence of positive real numbers. If*

$$a_f(n) = A_f n^{\frac{k-1}{2}} I_{k-1}(4\pi\sqrt{mn}) + O(n^C e^{2\pi\sqrt{mn}})$$

as $n \to \infty$ for some non-zero constants A_f, m and C, where $I_\nu(s)$ is the ν-th modified Bessel function of the first kind. Then for $d \geq 1$, the Jensen polynomial $J_{a_f}^{d,n}(X)$ associated to $a_f(n)$ is hyperbolic for sufficiently large n.

Since then, Turán inequalities for other partition functions have been extensively investigated. For example, Engel [16] showed that the overpartition function $\overline{p}(n)$ is log-concave for $n \geq 2$ and Liu and Zhang [19] showed that the overpartition function $\overline{p}(n)$ satisfies the third order Turán inequalities for $n \geq 16$. Recently, Bringmann, Kane, Rolen and Tripp [6] showed that k-colored partition function $p_k(n)$ is log-concave for $n \geq 6$. Craig and Pun [13] showed that the number of the k-regular partitions of n satisfies the order $d \geq 1$ Turán inequalities for sufficiently large n. Ono, Pujahari and Rolen [21] showed that the number of MacMahon's plane partitions of n satisfies the order $d \geq 1$ Turán inequalities for sufficiently large n. It should be noted that Craig and Pun's result can be viewed as a direct consequence of Theorem 1.1.

In this paper, we intend to explore Turán inequalities for broken k-diamond partitions. Appealing to Sussman's Rademacher-type formula for η-quotients [25] , we obtain the following asymptotic formula for $\Delta_k(n)$, where $k = 1$ or 2.

Theorem 1.2. *For $k = 1$ or 2, as $n \to \infty$,*

$$\Delta_k(n) = \frac{(5k+2)\pi^3}{18(2k+1)x_k^2(n)} I_2\left(\sqrt{\frac{5k+2}{2k+1}}x_k(n)\right) + O\left(x_k^{-5/2}(n)\exp\left(\frac{\sqrt{5k+2}}{2\sqrt{2k+1}}x_k(n)\right)\right), \tag{1.2}$$

where $I_2(s)$ is the second modified Bessel function of the first kind, and

$$x_k(n) = \frac{\pi\sqrt{24n-(2k+2)}}{6}. \tag{1.3}$$

Combining Theorem 1.1 and Theorem 1.2, we derive that for $k = 1$ or 2 and $d \geq 1$, $\Delta_k(n)$ satisfies the order d Turán inequalities for sufficiently large n. To wit,

Corollary 1.3. *Let $\Delta_k = \{\Delta_k(n)\}_{n\geq 0}$. For $k = 1$ or 2 and $d \geq 1$, the Jensen polynomial $J^{d,n}_{\Delta_k}(X)$ associated to Δ_k is hyperbolic for sufficiently large n.*

It is worth mentioning that there exists a minimal natural number $N_{\Delta_k}(d)$ such that $J^{d,n}_{\Delta_k}(X)$ is hyperbolic for $n \geq N_{\Delta_k}(d)$. Table 1 provides conjectural values for $N_{\Delta_k}(d)$ for $1 \leq k \leq 2$ and $2 \leq d \leq 13$.

d	2	3	4	5	6	7	8	9	10	11	12	13
$N_{\Delta_1}(d)$	0	4	17	41	72	116	171	238	320	415	525	650
$N_{\Delta_2}(d)$	0	4	17	34	62	99	147	200	272	355	445	552

Table 1: The conjectural values of $N_{\Delta_k}(d)$ for $1 \leq k \leq 2$ and $2 \leq d \leq 13$.

We further prove that $N_{\Delta_k}(2) = 0$ where $k = 1$ or 2. More precisely, we show that

Theorem 1.4. *For $k = 1$ or 2, $\Delta_k(n)$ is log-concave for $n \geq 1$, that is, for $n \geq 1$,*

$$\Delta_k^2(n) \geq \Delta_k(n-1)\Delta_k(n+1). \tag{1.4}$$

As noted by Asai, Kubo and Kuo [3] and Sagan [23], we see that Theorem 1.4 implies the following multiplicative properties of $\Delta_k(n)$.

Corollary 1.5. *For $k = 1$ or 2 and a, $b \geq 1$,*

$$\Delta_k(a)\Delta_k(b) \geq \Delta_k(a+b).$$

It should be noted that multiplicative properties of the partition function $p(n)$ were initially obtained by Bessenrodt and Ono [5]. Subsequently, multiplicative properties of other partition functions have been established, for example, Beckwith and Bessenrodt [4] established multiplicative properties of the k-regular partition functions and Bringmann, Kane, Rolen and Tripp [6] acquired multiplicative properties of the k-colored partition functions, which resolved a conjecture of Chern, Fu and Tang [9].

This article is organized as follows. In Section 2, we prove Theorem 1.2 with the aid of Sussman's Rademacher-type formula for η-quotients. Section 3 is devoted to the proof of Theorem 1.4. To this end, we derive an upper bound and a lower bound for $\Delta_k(n)$ with the aid of Theorem 1.2 and establish an inequality on the second modified Bessel function of the first kind. In Section 4, we pose some questions and remarks for future work.

2 Proof of Theorem 1.2

To prove Theorem 1.2, we first derive Rademacher-type formulas for $\Delta_k(n)$ ($k = 1$ or 2) with the aid of Sussman's Rademacher-type formula for η-quotients [25]. Define

$$G(q) := \prod_{r=1}^{R} (q^{m_r}; q^{m_r})_\infty^{\delta_r}, \tag{2.1}$$

where $\mathbf{m} = (m_1, \ldots, m_R)$ is a sequence of R distinct positive integers and $\delta = (\delta_1, \ldots, \delta_R)$ is a sequence of R non-zero integers. Here and throughout this paper, we have adopted the standard notation on q-series [1]:

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

In order to present Sussman's result, we need a few preliminary definitions. Let

$$n_0 = -\sum_{r=1}^{R} m_r \delta_r.$$

The function $q^{\frac{n_0}{24}} G(q)$ is holomorphic in the open unit disk D, so we may write

$$G(q) = q^{-\frac{n_0}{24}} \sum_{n \geq 0} g(n) q^n$$

for some coefficients $g(n)$. Sussman [25] obtained a Rademacher-type formula for $g(n)$ with $\frac{1}{2}\sum_{r=1}^{R} \delta_r < 0$, which is a special case of Bringmann and Ono [7]. Let

$$c_1 = -\frac{1}{2}\sum_{r=1}^{R} \delta_r, \quad c_2(j) = \prod_{r=1}^{R} \left(\frac{\gcd(m_r, j)}{m_j}\right)^{\frac{\delta_r}{2}}, \quad c_3(j) = -\sum_{r=1}^{R} \frac{\delta_r \gcd^2(m_r, j)}{m_r},$$

$$\hat{A}_j(n) = \sum_{\substack{0 \leq h < j \\ \gcd(h,j)=1}} \exp\left(-\frac{2\pi h i}{j} - \pi i \sum_{r=1}^{R} \delta_r s\left(\frac{m_r h}{\gcd(m_r, j)}, \frac{j}{\gcd(m_r, j)}\right)\right), \tag{2.2}$$

where $s(h, j)$ is the Dedekind sum:

$$s(h,j) = \sum_{r=1}^{j-1} \left(\frac{r}{j} - \left[\frac{r}{j}\right] - \frac{1}{2}\right)\left(\frac{hr}{j} - \left[\frac{hr}{j}\right] - \frac{1}{2}\right).$$

Theorem 2.1 (Sussman). *If $c_1 > 0$ and the periodic function $\beta(j)\colon \mathbb{N} \to \mathbb{R}$ given by*

$$\beta(j) = \min_{1 \leq r \leq R} \frac{\gcd^2(m_r, j)}{m_r} - \frac{c_3(j)}{24} \tag{2.3}$$

is non-negative, then for $n > \frac{n_0}{24}$,

$$g(n) = 2\pi \left(\frac{1}{24n - n_0}\right)^{\frac{c_1+1}{2}} \sum_{\substack{j\geq 1 \\ c_3(j)\geq 0}} c_2(j)c_3(j)^{\frac{c_1+1}{2}} j^{-1}\hat{A}_j(n) I_{c_1+1}\left(\frac{\pi\sqrt{c_3(j)(24n-n_0)}}{6j}\right), \tag{2.4}$$

where $I_\nu(s)$ is the ν-th modified Bessel function of the first kind.

Here and throughout this paper, we adopt the following infinite series definition of the modified Bessel function of the first kind,

$$I_\nu(s) := \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r+\nu+1)} \left(\frac{s}{2}\right)^{2r+\nu}.$$

By invocation of Theorem 2.1, we attain the following Rademacher-type formula for $\Delta_k(n)$ ($k = 1$ or 2).

Theorem 2.2. *For $k = 1$ or 2 and $n \geq 1$,*

$$\Delta_k(n) = \frac{\pi^3}{18x_k^2(n)} \sum_{j\geq 1} \alpha_k(j) j^{-1} \hat{A}_j(n) I_2\left(\frac{\sqrt{\alpha_k(j)}x_k(n)}{j}\right), \tag{2.5}$$

where $x_k(n)$ is defined in (1.3), $\hat{A}_j(n)$ is defined in (2.2), $I_2(s)$ is the second modified Bessel function of the first kind, and

$$\alpha_k(j) := \begin{cases} 1 + \dfrac{\gcd^2(2k+1,j)}{2k+1}, & j \text{ is even}, \\[2ex] \dfrac{5}{2} - \dfrac{\gcd^2(2k+1,j)}{4k+2}, & j \text{ is odd}. \end{cases} \tag{2.6}$$

Proof. Recall that

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2;q^2)_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty^3 (q^{4k+2};q^{4k+2})_\infty}.$$

We have

$$\mathbf{m} = (1, 2, 2k+1, 4k+2) \quad \text{and} \quad \delta = (-3, 1, 1, -1),$$

so that $n_0 = 2k+2$, $c_1 = 1$, and for $j \geq 1$,

$$\begin{aligned} c_2(j) &= \left(\frac{\gcd(2,j)}{2}\right)^{\frac{1}{2}} \left(\frac{\gcd(2k+1,j)}{2k+1}\right)^{\frac{1}{2}} \left(\frac{\gcd(4k+2,j)}{4k+2}\right)^{-\frac{1}{2}} \\ &= \left(\frac{\gcd(2,j)\gcd(2k+1,j)}{\gcd(4k+2,j)}\right)^{\frac{1}{2}} = 1, \end{aligned}$$

$$c_3(j) = 3 - \frac{\gcd^2(2,j)}{2} - \frac{\gcd^2(2k+1,j)}{2k+1} + \frac{\gcd^2(4k+2,j)}{4k+2}$$
$$= \begin{cases} 1 + \dfrac{\gcd^2(2k+1,j)}{2k+1}, & j \text{ is even}, \\ \dfrac{5}{2} - \dfrac{\gcd^2(2k+1,j)}{4k+2}, & j \text{ is odd}, \end{cases} \tag{2.7}$$

and

$$\min_{1\le r\le 4}\left(\frac{\gcd^2(m_r,j)}{m_r}\right) = \min\left\{1, \frac{\gcd^2(2,j)}{2}, \frac{\gcd^2(2k+1,j)}{2k+1}, \frac{\gcd^2(2k+1,j)\gcd^2(2,j)}{4k+2}\right\}$$
$$= \begin{cases} \min\left\{1, \dfrac{\gcd^2(2k+1,j)}{2k+1}\right\}, & j \text{ is even}, \\ \min\left\{\dfrac{1}{2}, \dfrac{\gcd^2(2k+1,j)}{4k+2}\right\}, & j \text{ is odd}. \end{cases} \tag{2.8}$$

Combining (2.7) and (2.8), we get

$$\beta_k(j) = \begin{cases} \min\left\{1, \dfrac{\gcd^2(2k+1,j)}{2k+1}\right\} - \dfrac{1}{24}\left(1 + \dfrac{\gcd^2(2k+1,j)}{2k+1}\right), & j \text{ is even}, \\ \min\left\{\dfrac{1}{2}, \dfrac{\gcd^2(2k+1,j)}{4k+2}\right\} - \dfrac{1}{24}\left(\dfrac{5}{2} - \dfrac{\gcd^2(2k+1,j)}{4k+2}\right), & j \text{ is odd}. \end{cases}$$

Set $\alpha_k(j) = c_3(j)$. From the definitions of $\alpha_k(j)$ and $\beta_k(j)$, we see that $\alpha_k(j)$ and $\beta_k(j)$ are periodic functions with period $4k+2$. The following table gives the values of $\alpha_k(j)$ and $\beta_k(j)$ for $k = 1$ or 2 and $1 \le j \le 4k+2$.

j	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\alpha_1(j)$	$\frac{7}{3}$	$\frac{4}{3}$	1	$\frac{4}{3}$	$\frac{7}{3}$	4	$\frac{7}{3}$	$\frac{4}{3}$	1	$\frac{4}{3}$	$\cdots$
$\alpha_2(j)$	$\frac{12}{5}$	$\frac{6}{5}$	$\frac{12}{5}$	$\frac{6}{5}$	0	$\frac{6}{5}$	$\frac{12}{5}$	$\frac{6}{5}$	$\frac{12}{5}$	6	$\cdots$
$\beta_1(j)$	$\frac{5}{72}$	$\frac{5}{18}$	$\frac{11}{24}$	$\frac{5}{18}$	$\frac{5}{72}$	$\frac{5}{6}$	$\frac{5}{72}$	$\frac{5}{18}$	$\frac{11}{24}$	$\frac{5}{18}$	$\cdots$
$\beta_2(j)$	0	$\frac{3}{20}$	0	$\frac{3}{20}$	$\frac{1}{2}$	$\frac{3}{20}$	0	$\frac{3}{20}$	0	$\frac{3}{4}$	$\cdots$

Table 2: The values of $\alpha_k(j)$ and $\beta_k(j)$ for $k = 1$ or $k = 2$.

From Table 2, we find that $\alpha_k(j) \ge 0$ and $\Delta_k(n)$ satisfies the condition in Theorem 2.1 when $k = 1$ or 2. Hence we derive (2.5) by substituting relevant values into Theorem 2.1. This completes the proof. ∎

It should be noted that Theorem 2.1 could not be applied to derive the explicit formula for $\Delta_k(n)$ when $k \geq 3$. Setting $j=1$, we find that

$$\min_{1\leq r\leq 4}\left(\frac{\gcd^2(m_r,j)}{m_r}\right)=\min\left\{\frac{1}{2},\frac{1}{4k+2}\right\}=\frac{1}{4k+2},$$

and

$$c_3(1)=\frac{5}{2}-\frac{1}{4k+2}.$$

But when $k\geq 3$,

$$\frac{1}{4k+2}<\frac{1}{24}\left(\frac{5}{2}-\frac{1}{4k+2}\right),$$

which implies that $\Delta_k(n)$ does not satisfy the condition in Theorem 2.1 when $k\geq 3$. Here and in the sequel, we assume that $k=1$ or 2.

We are now in a position to prove Theorem 1.2 by means of Theorem 2.2.

Proof of Theorem 1.2. Define

$$M_k(n):=\frac{\alpha_k(1)\pi^3}{18x_k^2(n)}I_2\left(\sqrt{\alpha_k(1)}x_k(n)\right). \tag{2.9}$$

Observing that $\hat{A}_1(n)=1$ and $\alpha_k(1)=\frac{5k+2}{2k+1}$, we obtain from Theorem 2.2 that

$$\Delta_k(n)=M_k(n)+R_k(n), \tag{2.10}$$

where

$$R_k(n)=\frac{\pi^3}{18x_k^2(n)}\sum_{j\geq 2}\alpha_k(j)j^{-1}\hat{A}_j(n)I_2\left(\frac{\sqrt{\alpha_k(j)}x_k(n)}{j}\right).$$

We next establish an upper bound for $|R_k(n)|$.

By the definition of $\hat{A}_j(n)$, we see that for $n\geq 0$ and $j\geq 1$,

$$|\hat{A}_j(n)|\leq j, \tag{2.11}$$

since $|e^{2\pi ri}|=1$ for any $r\in\mathbb{R}$.

In light of the fact that $\alpha_k(j)$ is a periodic function with period $4k+2$, we see from Table 2 that

$$\max_{\substack{j\geq 2\\ k=1,2}}\alpha_k(j)=\max_{\substack{2\leq j\leq 4k+2\\ k=1,2}}\alpha_k(j)=6. \tag{2.12}$$

Combining (2.11) and (2.12), we are led to

$$
\begin{aligned}
|R_k(n)| &\leq \frac{\pi^3}{18x_k^2(n)} \sum_{j\geq 2} |\alpha_k(j)| j^{-1} |\hat{A}_j(n)| I_2\left(\frac{\sqrt{\alpha_k(j)}x_k(n)}{j}\right) \\
&\leq \frac{\pi^3}{3x_k^2(n)} \sum_{j\geq 2} I_2\left(\frac{\sqrt{\alpha_k(j)}x_k(n)}{j}\right) \\
&= \frac{\pi^3}{3x_k^2(n)} \sum_{\substack{j\geq 2\\ 4k+2\nmid j}} I_2\left(\frac{\sqrt{\alpha_k(j)}x_k(n)}{j}\right) + \frac{\pi^3}{3x_k^2(n)} \sum_{\substack{j\geq 2\\ 4k+2\mid j}} I_2\left(\frac{\sqrt{\alpha_k(j)}x_k(n)}{j}\right) \\
&= \frac{\pi^3}{3x_k^2(n)} \sum_{\substack{j\geq 2\\ 4k+2\nmid j}} I_2\left(\frac{\sqrt{\alpha_k(j)}x_k(n)}{j}\right) + \frac{\pi^3}{3x_k^2(n)} \sum_{j\geq 1} I_2\left(\frac{\sqrt{\alpha_k(4k+2)}x_k(n)}{j(4k+2)}\right).
\end{aligned}
\tag{2.13}
$$

It is evident that

$$\max_{\substack{j\geq 2\\ 4k+2\nmid j}} \alpha_k(j) = \max_{2\leq j<4k+2} \alpha_k(j) \leq \alpha_k(1)$$

and

$$\frac{\sqrt{\alpha_k(4k+2)}}{4k+2} \leq \frac{\sqrt{\alpha_k(1)}}{2}.$$

Hence (2.13) can be further bounded by

$$|R_k(n)| \leq \frac{\pi^3}{3x_k^2(n)} \sum_{j\geq 2} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{j}\right) + \frac{\pi^3}{3x_k^2(n)} \sum_{j\geq 1} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2j}\right), \tag{2.14}$$

since $I_2(s)$ is increasing on $(0,\infty)$.

Note that for $N \geq 0$,

$$
\begin{aligned}
\sum_{j\geq N+1} I_2\left(\frac{s}{j}\right) &= \sum_{j\geq N+1} \sum_{m\geq 0} \frac{1}{m!(m+2)!}\left(\frac{s}{2j}\right)^{2m+2} \\
&\leq \int_N^\infty \sum_{m\geq 0} \frac{1}{m!(m+2)!}\left(\frac{s}{2t}\right)^{2m+2} \mathrm{d}t \\
&= \sum_{m\geq 0} \frac{1}{m!(m+2)!} \int_N^\infty \left(\frac{s}{2t}\right)^{2m+2} \mathrm{d}t \\
&= N \sum_{m\geq 0} \frac{1}{(2m+1)m!(m+2)!}\left(\frac{s}{2N}\right)^{2m+2} \\
&\leq N \sum_{m\geq 0} \frac{1}{(m+1)!(m+2)!}\left(\frac{s}{2N}\right)^{2m+2} \\
&= N \sum_{m\geq 1} \frac{1}{m!(m+1)!}\left(\frac{s}{2N}\right)^{2m}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{2N^2}{s}\sum_{m\ge 0}\frac{1}{m!(m+1)!}\left(\frac{s}{2N}\right)^{2m+1}\\
&= \frac{2N^2}{s} I_1\left(\frac{s}{N}\right).
\end{aligned}
$$

Thus we obtain from (2.14) that

$$
\begin{aligned}
|R_k(n)| &\le \frac{\pi^3}{3x_k^2(n)}\sum_{j\ge 2} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{j}\right) + \frac{\pi^3}{3x_k^2(n)}\sum_{j\ge 1} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2j}\right)\\
&= \frac{\pi^3}{3x_k^2(n)}\sum_{j\ge 3} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{j}\right) + \frac{\pi^3}{3x_k^2(n)}\sum_{j\ge 2} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2j}\right)\\
&\quad + \frac{2\pi^3}{3x_k^2(n)} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right)\\
&\le \frac{8\pi^3}{3\sqrt{\alpha_k(1)}x_k^3(n)} I_1\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right) + \frac{4\pi^3}{3\sqrt{\alpha_k(1)}x_k^3(n)} I_1\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right)\\
&\quad + \frac{2\pi^3}{3x_k^2(n)} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right)\\
&\le \frac{4\pi^3}{\sqrt{\alpha_k(1)}x_k^3(n)} I_1\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right) + \frac{2\pi^3}{3x_k^2(n)} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right).
\end{aligned}
$$

Using Lemma 2.2 (1) of Bringmann, Kane, Rolen and Tripp [6], we find that for $s \ge 1$,

$$
I_1(s) \le \sqrt{\frac{2}{\pi s}}e^s \quad \text{and} \quad I_2(s) \le \sqrt{\frac{2}{\pi s}}e^s,
$$

and so

$$
\begin{aligned}
|R_k(n)| &\le \frac{4\pi^3}{\sqrt{\alpha_k(1)}x_k^3(n)} I_1\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right) + \frac{2\pi^3}{3x_k^2(n)} I_2\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right)\\
&\le \left(\frac{4\pi^3}{\sqrt{\alpha_k(1)}x_k^3(n)} + \frac{2\pi^3}{3x_k^2(n)}\right)\cdot \frac{2}{\sqrt{\pi x_k(n)}\alpha_k^{\frac14}(1)} \exp\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right)\\
&= \left(\frac{2}{\sqrt{\alpha_k(1)}x_k(n)} + \frac{1}{3}\right)\cdot \frac{4\pi^{\frac52}}{\alpha_k^{\frac14}(1)x_k^{\frac52}(n)} \exp\left(\frac{\sqrt{\alpha_k(1)}x_k(n)}{2}\right).
\end{aligned}
$$

Since

$$
\frac{2}{\sqrt{\alpha_k(1)}x_k(n)} < \frac{2}{3}
$$

for $n \geq 1$, we find that for $n \geq 1$,

$$|R_k(n)| \leq \frac{4\pi^{\frac{5}{2}}}{\alpha_k^{\frac{1}{4}}(1) x_k^{\frac{5}{2}}(n)} \exp\left(\frac{\sqrt{\alpha_k(1)} x_k(n)}{2}\right). \tag{2.15}$$

Theorem 1.2 immediately follows from (2.10) and (2.15) upon noting that

$$x_k(n) = \frac{\pi\sqrt{24n-(2k+2)}}{6} \to \infty \quad \text{as} \quad n \to \infty,$$

and the proof is complete. ∎

3 Proof of Theorem 1.4

To prove Theorem 1.4, we establish an upper bound and a lower bound for $\Delta_k(n)$ in light of Theorem 2.2 and an inequality on $I_2(s)$.

Theorem 3.1. *Let $x_k(n)$ be defined as in (1.3), let $\alpha_k(n)$ be defined as in (2.6) and let $M_k(n)$ be defined as in (2.9). For $k = 1$ or 2 and $x_k(n) \geq 152$, we have*

$$M_k(n)\left(1 - \frac{1}{x_k^6(n)}\right) \leq \Delta_k(n) \leq M_k(n)\left(1 + \frac{1}{x_k^6(n)}\right). \tag{3.1}$$

Proof. Define

$$G_k(n) := \frac{\frac{4\pi^{\frac{5}{2}}}{\alpha_k^{\frac{1}{4}}(1)x_k^{\frac{5}{2}}(n)} \exp\left(\frac{1}{2}\sqrt{\alpha_k(1)}x_k(n)\right)}{\frac{\alpha_k(1)\pi^3}{18x_k^2(n)} I_2\left(\sqrt{\alpha_k(1)}x_k(n)\right)} = \frac{72}{\alpha_k^{\frac{5}{4}}(1)\sqrt{\pi}x_k^{\frac{1}{2}}(n)} \cdot \frac{\exp\left(\frac{1}{2}\sqrt{\alpha_k(1)}x_k(n)\right)}{I_2\left(\sqrt{\alpha_k(1)}x_k(n)\right)}. \tag{3.2}$$

Using (2.15), we see that

$$M_k(n)(1 - G_k(n)) \leq \Delta_k(n) \leq M_k(n)(1 + G_k(n)).$$

To show (3.1), it is enough to prove that for $x_k(n) \geq 152$,

$$G_k(n) \leq \frac{1}{x_k^6(n)}. \tag{3.3}$$

Invoking Lemma 2.2 (4) of Bringmann, Kane, Rolen and Tripp [6], we know that for $s \geq 231$,

$$\left| I_2(s)e^{-s}\sqrt{2\pi s} - 1 + \frac{15}{8s} - \frac{105}{128s^2} - \frac{315}{1024s^3} \right| \leq \frac{3968}{3s^4}. \tag{3.4}$$

Hence for $s \geq 231$,

$$I_2(s) \geq \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{15}{8s} + \frac{105}{128s^2} + \frac{315}{1024s^3} - \frac{3968}{3s^4}\right).$$

Note that

$$\frac{105}{128s^2} + \frac{315}{1024s^3} - \frac{3968}{3s^4} \geq 0$$

for $s \geq 40$, and so

$$I_2(s) \geq \frac{e^s}{\sqrt{2\pi s}} \left(1 - \frac{15}{8s}\right) \tag{3.5}$$

for $s \geq 231$.

Substituting (3.5) into (3.2), and using the following two observations:

$$\max\left\{\frac{\sqrt{2}}{\alpha_1(1)}, \frac{\sqrt{2}}{\alpha_2(1)}\right\} = \frac{3\sqrt{2}}{7} \approx 0.606 < 1$$

and

$$\max\left\{\frac{15}{8\sqrt{\alpha_1(1)}}, \frac{15}{8\sqrt{\alpha_2(1)}}\right\} = \frac{15\sqrt{21}}{56},$$

we derive that for $x_k(n) \geq 152$,

$$\begin{aligned} G_k(n) &\leq 72 \cdot \frac{\sqrt{2}}{\alpha_k(1)} \cdot \frac{\exp\left(-\frac{1}{2}\sqrt{\alpha_k(1)} x_k(n)\right)}{1 - \frac{15}{8\sqrt{\alpha_k(1)} x_k(n)}} \\ &\leq 72 \cdot \frac{\exp\left(-\frac{\sqrt{21}}{6} x_k(n)\right)}{1 - \frac{15\sqrt{21}}{56 x_k(n)}}. \end{aligned} \tag{3.6}$$

Under the following observation that

$$\left(1 - \frac{15\sqrt{21}}{56x_k(n)}\right)\left(1 + \frac{3}{x_k(n)}\right) = 1 + \frac{3\left(56 - 5\sqrt{21}\right)}{56x_k^2(n)}\left(x_k(n) - \frac{15\sqrt{21}}{56 - 5\sqrt{21}}\right) \geq 1$$

for $x_k(n) \geq 152$, we find that (3.6) can be further bounded by:

$$G_k(n) \leq 72\left(1 + \frac{3}{x_k(n)}\right) \exp\left(-\frac{\sqrt{21}}{6} x_k(n)\right). \tag{3.7}$$

We claim that for $x_k(n) \geq 152$,

$$72 \exp\left(-\frac{\sqrt{21}}{6} x_k(n)\right) \leq \frac{1}{2x_k^6(n)}. \tag{3.8}$$

Define

$$L(s) := 144 s^6 \exp\left(-\frac{\sqrt{21}}{6} s\right).$$

It is evident that

$$L'(s) = 24 \exp\left(-\frac{\sqrt{21}}{6}s\right) s^5 \left(-\sqrt{21}s + 36\right).$$

Since $L'(s) \leq 0$ for $s \geq \frac{36}{\sqrt{21}}$, it follows that $L(s)$ is decreasing when $s \geq \frac{36}{\sqrt{21}}$. This implies that

$$L(x_k(n)) = 144 x_k^6(n) \exp\left(-\frac{\sqrt{21}}{6}x_k(n)\right) \leq L(152) < 1$$

for $x_k(n) \geq 152$. So the claim is proved.

Applying (3.8) to (3.7), we are led to

$$G_k(n) \leq \left(1 + \frac{3}{x_k(n)}\right) \cdot \frac{1}{2x_k^6(n)} < \frac{1}{x_k^6(n)}$$

for $x_k(n) \geq 152$. This completes the proof. ∎

The following inequality on $I_2(s)$ is also required in the proof of Theorem 1.4.

Theorem 3.2. *For $k = 1$ or 2 and $x_k(n) \geq 152$,*

$$\frac{I_2^2\left(\sqrt{\alpha_k(1)}x_k(n)\right)}{I_2\left(\sqrt{\alpha_k(1)}x_k(n-1)\right) I_2\left(\sqrt{\alpha_k(1)}x_k(n+1)\right)} > 1 + \frac{\pi^4\sqrt{\alpha_k(1)}}{9x_k^3(n)} - \frac{1100}{x_k^4(n)}. \tag{3.9}$$

Proof. From (3.4), we see that for $s \geq 231$,

$$I_2(s) \geq \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{15}{8s} + \frac{105}{128s^2} + \frac{315}{1024s^3} - \frac{3968}{3s^4}\right) \tag{3.10}$$

and

$$I_2(s) \leq \frac{e^s}{\sqrt{2\pi s}}\left(1 - \frac{15}{8s} + \frac{105}{128s^2} + \frac{315}{1024s^3} + \frac{3968}{3s^4}\right). \tag{3.11}$$

For convenience, let

$$\gamma_1(k) = \frac{15}{8\sqrt{\alpha_k(1)}}, \quad \gamma_2(k) = \frac{105}{128\alpha_k(1)},$$
$$\gamma_3(k) = \frac{315}{1024\alpha_k^{3/2}(1)}, \quad \gamma_4(k) = \frac{3968}{3\alpha_k^2(1)},$$

and

$$h_k(n) = \frac{\left(1 - \frac{\gamma_1(k)}{x_k(n)} + \frac{\gamma_2(k)}{x_k^2(n)} + \frac{\gamma_3(k)}{x_k^3(n)} - \frac{\gamma_4(k)}{x_k^4(n)}\right)^2}{\left(1 - \frac{\gamma_1(k)}{x_k(n-1)} + \frac{\gamma_2(k)}{x_k^2(n-1)} + \frac{\gamma_3(k)}{x_k^3(n-1)} + \frac{\gamma_4(k)}{x_k^4(n-1)}\right)}$$
$$\times \frac{1}{\left(1 - \frac{\gamma_1(k)}{x_k(n+1)} + \frac{\gamma_2(k)}{x_k^2(n+1)} + \frac{\gamma_3(k)}{x_k^3(n+1)} + \frac{\gamma_4(k)}{x_k^4(n+1)}\right)}. \tag{3.12}$$

Combining (3.10) and (3.11), we derive that for $x_k(n) \geq 152$,

$$\frac{I_2^2\left(\sqrt{\alpha_k(1)}x_k(n)\right)}{I_2\left(\sqrt{\alpha_k(1)}x_k(n-1)\right)I_2\left(\sqrt{\alpha_k(1)}x_k(n+1)\right)} \geq \frac{\sqrt{x_k(n-1)x_k(n+1)}}{x_k(n)}\exp\left(\sqrt{\alpha_k(1)}(2x_k(n)-x_k(n-1)-x_k(n+1))\right)h_k(n). \tag{3.13}$$

From the definition (1.3) of $x_k(n)$, we see that for $n \geq 2$,

$$x_k(n-1) = \sqrt{x_k^2(n) - \frac{2\pi^2}{3}}, \quad x_k(n+1) = \sqrt{x_k^2(n) + \frac{2\pi^2}{3}}. \tag{3.14}$$

This implies that

$$0 < \frac{\sqrt{x_k(n-1)x_k(n+1)}}{x_k(n)} < 1,$$

so

$$\frac{\sqrt{x_k(n-1)x_k(n+1)}}{x_k(n)} > \frac{x_k^2(n-1)x_k^2(n+1)}{x_k^4(n)} = 1 - \frac{4\pi^4}{9x_k^4(n)}. \tag{3.15}$$

To estimate the remaining parts on the right-hand side of (3.13), we plan to establish an upper bound and a lower bound for $x_k(n-1)$ and $x_k(n+1)$ in terms of $x_k(n)$. Observe that for $n \geq 2$,

$$x_k(n-1) = x_k(n) - \frac{\pi^2}{3x_k(n)} - \frac{\pi^4}{18x_k^3(n)} - \frac{\pi^6}{54x_k^5(n)} - \frac{5\pi^8}{648x_k^7(n)} + o\left(\frac{1}{x_k^7(n)}\right)$$

and

$$x_k(n+1) = x_k(n) + \frac{\pi^2}{3x_k(n)} - \frac{\pi^4}{18x_k^3(n)} + \frac{\pi^6}{54x_k^5(n)} - \frac{5\pi^8}{648x_k^7(n)} + o\left(\frac{1}{x_k^7(n)}\right),$$

so it is readily checked that for $x_k(n) \geq 23$,

$$\tilde{w}_k(n) < x_k(n-1) < \hat{w}_k(n) \tag{3.16}$$

and

$$\tilde{y}_k(n) < x_k(n+1) < \hat{y}_k(n), \tag{3.17}$$

where

$$\begin{cases} \tilde{w}_k(n) = x_k(n) - \dfrac{\pi^2}{3x_k(n)} - \dfrac{\pi^4}{18x_k^3(n)} - \dfrac{\pi^6}{x_k^5(n)}, \\ \hat{w}_k(n) = x_k(n) - \dfrac{\pi^2}{3x_k(n)} - \dfrac{\pi^4}{18x_k^3(n)} - \dfrac{\pi^6}{54x_k^5(n)}, \\ \tilde{y}_k(n) = x_k(n) + \dfrac{\pi^2}{3x_k(n)} - \dfrac{\pi^4}{18x_k^3(n)}, \\ \hat{y}_k(n) = x_k(n) + \dfrac{\pi^2}{3x_k(n)} - \dfrac{\pi^4}{18x_k^3(n)} + \dfrac{\pi^6}{54x_k^5(n)}. \end{cases} \tag{3.18}$$

Combining (3.16) and (3.17), we deduce that for $x_k(n) \geq 23$,

$$2x_k(n) - x_k(n-1) - x_k(n+1) > 2x_k(n) - \hat{w}_k(n) - \hat{y}_k(n) = \frac{\pi^4}{9x_k^3(n)} > 0.$$

Hence

$$\begin{aligned}
&\exp\left(\sqrt{\alpha_k(1)}\big(2x_k(n) - x_k(n-1) - x_k(n+1)\big)\right) \\
&\quad > 1 + \sqrt{\alpha_k(1)}(2x_k(n) - x_k(n-1) - x_k(n+1)) \\
&\quad > 1 + \frac{\pi^4\sqrt{\alpha_k(1)}}{9x_k^3(n)}.
\end{aligned} \tag{3.19}$$

We proceed to show that for $x_k(n) \geq 62$,

$$h_k(n) \geq 1 - \frac{1000}{x_k^4(n)}. \tag{3.20}$$

Define

$$P_k(n) := x_k^2(n)x_k^4(n-1)x_k^4(n+1)\left(x_k^4(n) - \gamma_1(k)x_k^3(n) + \gamma_2(k)x_k^2(n) + \gamma_3(k)x_k(n) - \gamma_4(k)\right)^2,$$

$$\begin{aligned}
\tilde{Q}_k(n) :=& x_k^{10}(n)\left(x_k^4(n-1) - \gamma_1(k)x_k^3(n-1) + \gamma_2(k)x_k^2(n-1) + \gamma_3(k)x_k(n-1) + \gamma_4(k)\right) \\
&\times \left(x_k^4(n+1) - \gamma_1(k)x_k^3(n+1) + \gamma_2(k)x_k^2(n+1) + \gamma_3(k)x_k(n+1) + \gamma_4(k)\right)
\end{aligned}$$

and

$$\begin{aligned}
Q_k(n) :=& x_k^{10}(n)\left(x_k^4(n-1) - \gamma_1(k)x_k^2(n-1)\tilde{w}_k(n) + \gamma_2(k)x_k^2(n-1) + \gamma_3(k)\hat{w}_k(n) + \gamma_4(k)\right) \\
&\times \left(x_k^4(n+1) - \gamma_1(k)x_k^2(n+1)\tilde{y}_k(n) + \gamma_2(k)x_k^2(n+1) + \gamma_3(k)\hat{y}_k(n) + \gamma_4(k)\right).
\end{aligned}$$

Form (3.12) we have

$$h_k(n) = \frac{P_k(n)}{\tilde{Q}_k(n)}.$$

Applying (3.16) and (3.17), we find that for $x_k(n) \geq 23$,

$$\tilde{Q}_k(n) \leq Q_k(n).$$

Moreover, it can be checked that for $x_k(n) \geq 3$,

$$\tilde{Q}_k(n) > 0.$$

Hence for $x_k(n) \geq 23$,

$$h_k(n) = \frac{P_k(n)}{\tilde{Q}_k(n)} \geq \frac{P_k(n)}{Q_k(n)}.$$

To prove (3.20), it suffices to show that for $x_k(n) \geq 62$,

$$\frac{P_k(n)}{Q_k(n)} \geq 1 - \frac{1000}{x_k^4(n)}, \tag{3.21}$$

which is equivalent to

$$x_k^4(n)(P_k(n) - Q_k(n)) + 1000Q_k(n) \geq 0 \tag{3.22}$$

for $x_k(n) \geq 62$.

From the definitions of $P_k(n)$ and $Q_k(n)$, together with (3.14) and (3.18), we infer that the left-hand side of (3.22) is a polynomial in $x_k(n)$ with degree 18. So we could write

$$x_k^4(n)(P_k(n) - Q_k(n)) + 1000Q_k(n) = \sum_{j=0}^{18} f_j(k) x_k^j(n).$$

Clearly,

$$x_k^4(n)(P_k(n)-Q_k(n))+1000Q_k(n) \geq -\sum_{j=0}^{16} |f_j(k)| x_k^j(n) + f_{17}(k) x_k^{17}(n) + f_{18}(k) x_k^{18}(n).$$

Moreover, numerical evidence indicates that for $0 \leq j \leq 15$ and $x_k(n) \geq 20$,

$$-|f_j(k)| x_k^j(n) \geq -|f_{16}(k)| x_k^{16}(n)$$

and

$$\begin{aligned}
f_{16}(k) &= -\frac{4340}{\alpha_k^3(1)} + \frac{20625}{4\alpha_k(1)} - \frac{145\pi^4}{96\alpha_k(1)}, \\
f_{17}(k) &= \frac{9920}{\sqrt{\alpha_k^5(1)}} - \frac{3750}{\sqrt{\alpha_k(1)}} + \frac{5\pi^4}{8\sqrt{\alpha_k(1)}}, \\
f_{18}(k) &= 1000 - \frac{15872}{3\alpha_k^2(1)}.
\end{aligned}$$

It is readily checked that for $x_k(n) \geq 62$,

$$f_{18}(k) x_k^2(n) + f_{17}(k) x_k(n) - 17|f_{16}(k)| \geq 0.$$

Assembling all these results above, we conclude that for $x_k(n) \geq 62$,

$$\begin{aligned}
&x_k^4(n)(P_k(n) - Q_k(n)) + 1000Q_k(n) \\
&\qquad \geq (f_{18}(k) x_k^2(n) + f_{17}(k) x_k(n) - 17|f_{16}(k)|) x_k^{16}(n) \geq 0,
\end{aligned}$$

and so (3.20) is valid.

Applying (3.15), (3.19) and (3.20) to (3.13), we obtain that for $x_k(n) \geq 152$,

$$\begin{aligned}
&\frac{I_2^2\left(\sqrt{\alpha_k(1)}x_k(n)\right)}{I_2\left(\sqrt{\alpha_k(1)}x_k(n-1)\right)I_2\left(\sqrt{\alpha_k(1)}x_k(n+1)\right)}\\
&\geq \left(1-\frac{4\pi^4}{9x_k^4(n)}\right)\left(1+\frac{\pi^4\sqrt{\alpha_k(1)}}{9x_k^3(n)}\right)\left(1-\frac{1000}{x_k^4(n)}\right)\\
&= 1+\frac{\pi^4\sqrt{\alpha_k(1)}}{9x_k^3(n)}-\frac{1000+\frac{4\pi^4}{9}}{x_k^4(n)}-\frac{\frac{4}{81}\pi^8\sqrt{\alpha_k(1)}+\frac{1000}{9}\pi^4\sqrt{\alpha_k(1)}}{x_k^7(n)}\\
&\quad +\frac{4000\pi^4}{9x_k^8(n)}+\frac{4000\pi^8\sqrt{\alpha_k(1)}}{81x_k^{11}(n)}.
\end{aligned}$$

It's easy to check that for $x_k(n) \geq 7$,

$$\frac{100-\frac{4\pi^4}{9}}{x_k^4(n)}-\frac{\frac{4}{81}\pi^8\sqrt{\alpha_k(1)}+\frac{1000}{9}\pi^4\sqrt{\alpha_k(1)}}{x_k^7(n)} \geq 0.$$

Therefore, we arrive at

$$\frac{I_2^2\left(\sqrt{\alpha_k(1)}x_k\right)}{I_2\left(\sqrt{\alpha_k(1)}x_k(n-1)\right)I_2\left(\sqrt{\alpha_k(1)}x_k(n+1)\right)} \geq 1+\frac{\pi^4\sqrt{\alpha_k(1)}}{9x_k^3(n)}-\frac{1100}{x_k^4(n)} \tag{3.23}$$

for $x_k(n) \geq 152$. This completes the proof. ∎

With Theorem 3.1 and Theorem 3.2 in hand, we are now in a position to give a proof of Theorem 1.4.

Proof of Theorem 1.4. To prove (1.4), it is enough to show that

$$\frac{\Delta_k^2(n)}{\Delta_k(n-1)\Delta_k(n+1)} \geq 1. \tag{3.24}$$

Utilizing Theorem 3.1, we find that for $x_k(n) \geq 152$,

$$\begin{aligned}
&\frac{\Delta_k^2(n)}{\Delta_k(n-1)\Delta_k(n+1)}\\
&\geq \frac{M_k^2(n)\left(1-\frac{1}{x_k^6(n)}\right)^2}{M_k(n-1)M_k(n+1)\left(1+\frac{1}{x_k^6(n-1)}\right)\left(1+\frac{1}{x_k^6(n+1)}\right)}\\
&\geq \frac{x_k^2(n-1)x_k^2(n+1)}{x_k^4(n)}\cdot\frac{I_2^2\left(\sqrt{\alpha_k(1)}x_k(n)\right)}{I_2\left(\sqrt{\alpha_k(1)}x_k(n-1)\right)I_2\left(\sqrt{\alpha_k(1)}x_k(n+1)\right)}\cdot g_k(n),
\end{aligned} \tag{3.25}$$

where

$$g_k(n) := \frac{\left(1 - \frac{1}{x_k^6(n)}\right)^2}{\left(1 + \frac{1}{x_k^6(n-1)}\right)\left(1 + \frac{1}{x_k^6(n+1)}\right)}.$$

We claim that for $x_k(n) \geq 75$,

$$g_k(n) \geq 1 - \frac{10}{x_k^6(n)}. \tag{3.26}$$

From (3.14), we see that

$$\frac{x_k^2(n-1)x_k^2(n+1)}{x_k^4(n)} = \frac{(x_k^2(n) - \frac{2\pi^2}{3})(x_k^2(n) + \frac{2\pi^2}{3})}{x_k^4(n)} = 1 - \frac{4\pi^4}{9x_k^4(n)}. \tag{3.27}$$

Hence $g_k(n)$ can be simplified as:

$$g_k(n) = \frac{\left(x_k^6(n) - 1\right)^2 \left(x_k^4(n) - \frac{4\pi^4}{9}\right)^3}{x_k^{12}(n)\left(\left(x_k^2(n) - \frac{2\pi^2}{3}\right)^3 + 1\right)\left(\left(x_k^2(n) + \frac{2\pi^2}{3}\right)^3 + 1\right)}. \tag{3.28}$$

It is easy to show that for $x_k(n) \geq 75$,

$$\begin{aligned}
&\left(x_k^6(n) - 1\right)^2 \left(x_k^4(n) - \frac{4\pi^4}{9}\right)^3 \\
&\geq \left(x_k^{12}(n) - 2x_k^6(n)\right)\left(x_k^8(n) - \frac{8\pi^4}{9}x_k^4(n)\right)\left(x_k^4(n) - \frac{4\pi^4}{9}\right) \\
&= x_k^{24}(n) - \frac{4\pi^4}{3}x_k^{20}(n) - 2x_k^{18}(n) + \frac{32\pi^8}{81}x_k^{16}(n) + \frac{8\pi^4}{3}x_k^{14}(n) - \frac{64\pi^8}{81}x_k^{10}(n) \\
&\geq x_k^{24}(n) - \frac{4\pi^4}{3}x_k^{20}(n) - 2x_k^{18}(n)
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
&x_k^{12}(n)\left(\left(x_k^2(n) - \frac{2\pi^2}{3}\right)^3 + 1\right)\left(\left(x_k^2(n) + \frac{2\pi^2}{3}\right)^3 + 1\right) \\
&= x_k^{24}(n) - \frac{4\pi^4}{3}x_k^{20}(n) + 2x_k^{18}(n) + \frac{16\pi^8}{27}x_k^{16}(n) + \frac{8\pi^4}{3}x_k^{14}(n) - \left(\frac{64\pi^{12}}{729} - 1\right)x_k^{12}(n) \\
&\leq x_k^{24}(n) - \frac{4\pi^4}{3}x_k^{20}(n) + 3x_k^{18}(n).
\end{aligned} \tag{3.30}$$

Applying (3.29) and (3.30) to (3.28), we see that for $x_k(n) \geq 75$,

$$
\begin{aligned}
g_k(n) &\geq \frac{x_k^{24}(n) - \frac{4\pi^4}{3}x_k^{20}(n) - 2x_k^{18}(n)}{x_k^{24}(n) - \frac{4\pi^4}{3}x_k^{20}(n) + 3x_k^{18}(n)} \\
&= 1 - \frac{5}{x_k^6(n) - \frac{4\pi^4}{3}x_k^2(n) + 3} \\
&\geq 1 - \frac{10}{x_k^6(n)},
\end{aligned}
$$

so the claim is proved.

Substituting (3.9), (3.26) and (3.27) to (3.25), we see that for $x_k(n) \geq 152$,

$$
\begin{aligned}
\frac{\Delta_k(n)^2}{\Delta_k(n-1)\Delta_k(n+1)} &\geq \left(1 - \frac{4\pi^4}{9x_k^4(n)}\right)\left(1 + \frac{\pi^4\sqrt{\alpha_k(1)}}{9x_k^3(n)} - \frac{1100}{x_k^4(n)}\right)\left(1 - \frac{10}{x_k^6(n)}\right) \\
&= 1 + \frac{\pi^4\sqrt{\alpha_k(1)}}{9x_k^3(n)} - \frac{1100 + \frac{4\pi^4}{9}}{x_k^4(n)} - \frac{10}{x_k^6(n)} - \frac{4\pi^8\sqrt{\alpha_k(1)}}{81x_k^7(n)} \\
&\quad + \frac{4400\pi^4}{9x_k^8(n)} - \frac{10\pi^4\sqrt{\alpha_k(1)}}{9x_k^9(n)} + \frac{11000 + \frac{40\pi^4}{9}}{x_k^{10}(n)} \\
&\quad + \frac{40\pi^8\sqrt{\alpha_k(1)}}{81x_k^{13}(n)} - \frac{44000\pi^4}{9x_k^{14}(n)}.
\end{aligned}
$$

It is readily checked that for $x_k(n) \geq 73$,

$$
\begin{aligned}
&\frac{\pi^4\sqrt{\alpha_k(1)}}{9x_k^3(n)} - \frac{1200}{x_k^4(n)} \geq 0, \\
&\frac{100 - \frac{4\pi^4}{9}}{x_k^4(n)} - \frac{10}{x_k^6(n)} - \frac{4\pi^8\sqrt{\alpha_k(1)}}{81x_k^7(n)} \geq 0, \\
&\frac{4400\pi^4}{9x_k^8(n)} - \frac{10\pi^4\sqrt{\alpha_k(1)}}{9x_k^9(n)} \geq 0
\end{aligned}
$$

and

$$
\frac{40\pi^8\sqrt{\alpha_k(1)}}{81x_k^{13}(n)} - \frac{44000\pi^4}{9x_k^{14}(n)} \geq 0.
$$

Assembling all these results, we conclude that for $x_k(n) \geq 152$ (that is, $n \geq 3512$),

$$
\frac{\Delta_k^2(n)}{\Delta_k(n-1)\Delta_k(n+1)} \geq 1. \tag{3.31}
$$

It is routine to check that (3.31) is true for $1 \leq n \leq 3512$, and hence the proof is complete. ∎

4 Concluding remarks

To conclude, we mention some questions and remarks for further investigation. The main objection of this paper is to dig into the Turán inequalities for the broken k-diamond partition function where $k = 1$ or 2. But the numerical evidence suggests that the main results in this paper are also valid for all $k \geq 1$. To wit,

Conjecture 4.1. *For $k \geq 3$, $\Delta_k(n)$ is log-concave for $n \geq 1$, that is,*

$$\Delta_k^2(n) \geq \Delta_k(n-1)\Delta_k(n+1). \tag{4.1}$$

More generally, we conjectured that for $k \geq 3$ and $d \geq 1$, the Jensen polynomial $J_{\Delta_k}^{d,n}(X)$ associated to $\Delta_k(n)$ is hyperbolic for sufficiently large n.

As alluded to after the proof of Theorem 2.2 in Section 2, Sussman's formula could not be applied to derive the explicit formula for $\Delta_k(n)$ when $k \geq 3$. Therefore, the crucial point to solve these two conjectures is to establish explicit formulas of $\Delta_k(n)$ for $k \geq 3$.

Acknowledgment. This work was supported by the National Science Foundation of China.

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