A Cyclic Analogue of Stanley's Shuffling Theorem

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Abstract. We introduce the cyclic major index of a cyclic permutation and give a bivariate analogue of the enumerative formula for the cyclic shuffles with a given cyclic descent number due to Adin, Gessel, Reiner and Roichman, which can be viewed as a cyclic analogue of Stanley's shuffling theorem. This gives an answer to a question of Adin, Gessel, Reiner and Roichman, which has been posed by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema again.

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1 Introduction

The main theme of this note is to establish a cyclic analogue of Stanley's shuffling theorem. Recall that Stanley's shuffling theorem establishes an explicit expression for the generating function of the number of shuffles of two disjoint permutations σ and π with a given cyclic descent number and a given major index. Here we adopt some common notation and terminology on permutations as used in [13, Chapter 1]. We say that $\pi = \pi_1\pi_2\cdots\pi_n$ is a permutation of length n if it is a sequence of n distinct letters (not necessarily from 1 to n). For example, $\pi = 9\,2\,8\,10\,12\,3\,7$ is a permutation of length 7. Let $\mathfrak{S}_n$ denote the set of all permutations of length n.

Let $\pi \in \mathfrak{S}_n$. We say that $1 \le i \le n-1$ is a descent of π if $\pi_i > \pi_{i+1}$. The set of descents of π is called the descent set of π, denoted $\mathrm{Des}(\pi)$, viz.,

$$\mathrm{Des}(\pi) := \{1 \le i \le n-1 : \pi_i > \pi_{i-1}\}.$$

The number of its descents is called the descent number, denoted $\mathrm{des}(\pi)$, namely,

$$\mathrm{des}(\pi) := \#\mathrm{Des}(\pi),$$

where the hash symbol $\#\mathcal{T}$ stands for the cardinality of a set $\mathcal{T}$. The major index of π, denoted $\mathrm{maj}(\pi)$, is defined to be the sum of its descents. To wit,

$$\mathrm{maj}(\pi) := \sum_{k \in \mathrm{Des}(\pi)} k.$$

Let $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$ be disjoint permutations, that is, permutations with no letters in common. We say that $\alpha \in \mathfrak{S}_{n+m}$ is a shuffle of σ and π if both σ and π are subsequences of α. The set of shuffles of σ

and π is denoted $\mathcal{S}(\sigma,\pi)$. For example,

$$\mathcal{S}(6\,3,1\,4)=\{6\,3\,1\,4,6\,1\,3\,4,6\,1\,4\,3,1\,4\,6\,3,1\,6\,3\,4,1\,6\,4\,3\}.$$

Clearly, the number of permutations in $\mathcal{S}(\sigma,\pi)$ is $\binom{m+n}{n}$ for two disjoint permutations $\sigma\in\mathfrak{S}_n$ and $\pi\in\mathfrak{S}_m$.

Stanley's shuffling theorem states that

Theorem 1.1. *Let $\sigma\in\mathfrak{S}_m$ and $\pi\in\mathfrak{S}_n$ be disjoint permutations, where $\mathrm{des}(\sigma)=r$ and $\mathrm{des}(\pi)=s$. Then*

$$\sum_{\substack{\alpha\in\mathcal{S}(\sigma,\pi)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)}=\begin{bmatrix}m-r+s\\ k-r\end{bmatrix}\begin{bmatrix}n-s+r\\ k-s\end{bmatrix}q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+(k-s)(k-r)}. \tag{1.1}$$

Here

$$\begin{bmatrix}n\\ m\end{bmatrix}=\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q)}$$

is the Gaussian polynomial (also called the q-binomial coefficient), see Andrews [2, Chapter 1].

Stanley [12] obtained the above expression in light of the q-Pfaff-Saalschütz identity in his setting of P-partitions. Bijective proofs of Stanley's shuffling theorem have been given by Goulden [6], Stadler [11], Ji and Zhang [10].

Recently, Adin, Gessel, Reiner and Roichman [1] introduced a cyclic version of quasisymmetric functions with a corresponding cyclic shuffle operation. A cyclic permutation $[\pi]$ of length n is the set of all rotations of a permutation $\pi=\pi_1\pi_2\cdots\pi_n$, i.e,

$$[\pi]=\{\pi_1\pi_2\cdots\pi_n,\pi_2\pi_3\cdots\pi_n\pi_1,\ldots,\pi_n\pi_1\cdots\pi_{n-1}\}.$$

For example,

$$[4\,2\,3\,1]=\{4\,2\,3\,1,2\,3\,1\,4,3\,1\,4\,2,1\,4\,2\,3\} \tag{1.2}$$

is a cyclic permutation of length 4, where

$$[4\,2\,3\,1]=[2\,3\,1\,4]=[3\,1\,4\,2]=[1\,4\,2\,3].$$

Let π_l be the largest element in $[\pi]$. The linear permutation $\hat{\pi}=\pi_l\pi_{l+1}\cdots\pi_n\pi_1\cdots\pi_{l-1}$ corresponding to the cyclic permutation $[\pi]$ is called the representative of the cyclic permutation $[\pi]$. For the example above, $4\,2\,3\,1$ is the representative of the cyclic permutation $[4\,2\,3\,1]$. Here and in the sequel, we use the representative to represent each cyclic permutation $[\pi]$. For example, we use $[4\,2\,3\,1]$ to represent the cyclic permutation in (1.2). In this way, all cyclic permutations of $\{1,2,3,4\}$ are listed as follows:

$$[4\,1\,2\,3],[4\,3\,1\,2],[4\,1\,3\,2],[4\,2\,1\,3],[4\,2\,3\,1],[4\,3\,2\,1].$$

Let $\mathfrak{S}_n^c$ denote the set of all cyclic permutations of length n and let $[\sigma]\in\mathfrak{S}_n^c$ and $[\pi]\in\mathfrak{S}_m^c$ be disjoint cyclic permutations, that is, cyclic permutations with no letters in common. We say that $[\alpha]\in\mathfrak{S}_{n+m}^c$ is a cyclic shuffle of two cyclic permutations $[\sigma]$ and $[\pi]$ if both $[\sigma]$ and $[\pi]$ are circular subsequences of $[\alpha]$. Recall that a cyclic permutation $[\pi]$ is called a circular subsequence of $[\alpha]$ if there exists a rotation of $[\alpha]$, which contains π linearly. The set of cyclic shuffles of $[\sigma]$ and $[\pi]$ is denoted $\mathcal{S}^c([\sigma],[\pi])$. For example,

$$\mathcal{S}^c([6\,3],[4\,1])=\{[6\,3\,\mathbf{1}\,\mathbf{4}],[6\,3\,\mathbf{4}\,\mathbf{1}],[6\,\mathbf{1}\,\mathbf{4}\,3],[6\,\mathbf{4}\,\mathbf{1}\,3],[6\,\mathbf{1}\,3\,\mathbf{4}],[6\,\mathbf{4}\,3\,\mathbf{1}]\}. \tag{1.3}$$

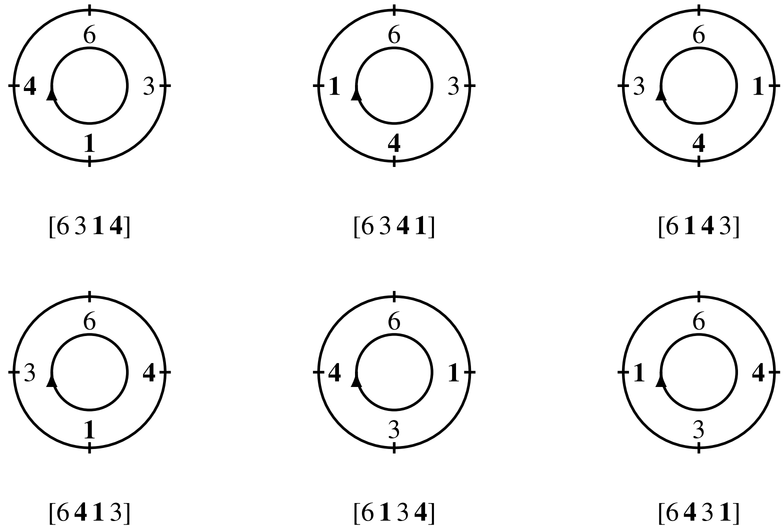

Figure 1: The circular representations of cyclic shuffles of $[6\,3]$ and $[4\,1]$.

The elements of $[\pi]$ in $[\alpha]$ are in boldface to distinguish them from the elements of $[\sigma]$. Figure 1 lays out the circular representations of cyclic shuffles of $[6\,3]$ and $[4\,1]$.

Evidently,

$$\#\mathcal{S}^c([\sigma],[\pi]) = (m+n-1)\binom{m+n-2}{m-1}, \tag{1.4}$$

for two disjoint cyclic permutations $[\sigma] \in \mathfrak{S}_n^c$ and $[\pi] \in \mathfrak{S}_m^c$, see [5, Eq. (7)].

In order to study Solomon's descent algebra, Cellini [3, 4] introduced the cyclic descent set. Let $\pi = \pi_1\pi_2\ldots\pi_n$ be a linear permutation. The cyclic descent set of π is defined to be

$$\mathrm{cDes}(\pi) = \{1 \le i \le n \colon \pi_i > \pi_{i+1}\}$$

with the convention $\pi_{n+1} = \pi_1$. The number of its cyclic descents is called the cyclic descent number, denoted $\mathrm{cdes}(\pi)$, viz.,

$$\mathrm{cdes}(\pi) := \#\mathrm{cDes}(\pi).$$

Let $[\pi]$ be a cyclic permutation of length n. Note that all linear permutations corresponding to $[\pi]$ have the same number of cyclic descents, so we may define the cyclic descent number of $[\pi]$ as

$$\mathrm{cdes}\left([\pi]\right) = \mathrm{cdes}\left(\pi\right), \tag{1.5}$$

where π is any linear permutation corresponding to $[\pi]$.

Based on their setting of cyclic quasi-symmetric functions, Adin, Gessel, Reiner and Roichman [1] established the following enumerative formula for the cyclic shuffles with a given cyclic descent number.

Theorem 1.2 (Adin-Gessel-Reiner-Roichman)**.** *Let $[\sigma] \in \mathfrak{S}_m^c$ and $[\pi] \in \mathfrak{S}_n^c$ be disjoint cyclic permutations, where $\mathrm{cdes}([\sigma]) = r$ and $\mathrm{cdes}([\pi]) = s$. Let $\mathcal{S}^c([\sigma],[\pi],k)$ denote the set of cyclic shuffles of $[\sigma]$ and $[\pi]$ with cyclic descent number k. Then*

$$\#\mathcal{S}^c([\sigma],[\pi],k) = \frac{k(m-r)(n-s)+(m+n-k)rs}{(m-r+s)(n-s+r)}\binom{m-r+s}{k-r}\binom{n-s+r}{k-s}. \tag{1.6}$$

Summing (1.6) over all k gives (1.4) upon using the Chu-Vandermonde identity [13, p. 135, Ex. 100]. At the end of their paper, Adin, Gessel, Reiner and Roichman [1] asked a question about looking for a notion of cyclic major index, which provides a bivariate analogue of Theorem 1.2. This question has been posed by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema in [5, Question 4.1] again.

In this paper, we introduce the cyclic major index of a cyclic permutation $[\pi]$. Let $[\pi]$ be a cyclic permutation of length n. Suppose that the representative of $[\pi]$ is $\hat{\pi} = \hat{\pi}_1\hat{\pi}_2\cdots\hat{\pi}_n$, where $\hat{\pi}_1$ is the largest element in $[\pi]$. The cyclic major index of the cyclic permutation $[\pi]$ is defined to be

$$\mathrm{maj}([\pi]) = \mathrm{maj}(\hat{\pi}). \tag{1.7}$$

For example, the representative of the cyclic permutation $[4\,1\,3\,2]$ is $\hat{\pi} = 4\,1\,3\,2$, and so its cyclic major index is defined to be the major index of $\hat{\pi} = 4\,1\,3\,2$. It gives that $\mathrm{maj}([4\,1\,3\,2]) = 1 + 3 = 4$.

In order to state the cyclic analogue of Stanley's shuffling theorem, we will need to introduce the cyclic descent-bottom set of a cyclic permutation and recall the splitting map S_i defined by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema in [5], which maps a cyclic permutation to a linear permutation. Let $[\pi]$ be a cyclic permutation of length n. The cyclic descent-bottom set of $[\pi]$ is defined as:

$$\mathrm{cB_d}([\pi]) = \{\pi_{i+1} \colon \pi_i > \pi_{i+1}, \text{ for } 1 \le i \le n\} \tag{1.8}$$

with the convention $\pi_{n+1} = \pi_1$. It should be mentioned that the descent-bottom set of a linear permutation has been studied by Haglund and Visontai [7] and Hall and Remmel [8, 9].

It is manifest from (1.5) and (1.8) that

$$\#\mathrm{cB_d}([\pi]) = \mathrm{cdes}([\pi]).$$

For example,

$$\mathrm{cB_d}([6\,4\,1\,3]) = \{1, 4\}.$$

Let $[\pi]$ be a cyclic permutation of length n. For $i \in [\pi]$, Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema [5] defined the map $S_i([\pi])$ to be the unique permutation corresponding to $[\pi]$ which starts with i. For example,

$$S_5([5\,1\,3\,4]) = 5\,1\,3\,4,\ S_1([5\,1\,3\,4]) = 1\,3\,4\,5,\ S_3([5\,1\,3\,4]) = 3\,4\,5\,1,$$

and

$$S_4([5\,1\,3\,4]) = 4\,5\,1\,3.$$

We obtain the following generating function of the number of cyclic shuffles of two disjoint cyclic permutations with a given cyclic descent number and a given cyclic major index.

Theorem 1.3 (Cyclic Stanley's shuffling theorem)**.** *Let $[\sigma] \in \mathfrak{S}_m^c$ and let $[\pi] \in \mathfrak{S}_n^c$ be disjoint cyclic permutations, where $\mathrm{cdes}([\sigma]) = r$ and $\mathrm{cdes}([\pi]) = s$. Suppose that the largest element of $[\sigma]$ and $[\pi]$ is in $[\sigma]$. Then*

$$\sum_{\substack{[\alpha]\in\mathcal{S}^c([\sigma],[\pi])\\ \mathrm{cdes}([\alpha])=k}} q^{\mathrm{maj}([\alpha])}$$

$$= \begin{bmatrix} m-r+s \\ k-r \end{bmatrix} \begin{bmatrix} n-s+r-1 \\ k-s-1 \end{bmatrix} q^{\mathrm{maj}([\sigma])+(k-s)(k-r)} \sum_{i \notin \mathrm{cB_d}([\pi])} q^{\mathrm{maj}(S_i([\pi]))}$$

$$+ \begin{bmatrix} m-r+s-1 \\ k-r \end{bmatrix} \begin{bmatrix} n-s+r \\ k-s \end{bmatrix} q^{\mathrm{maj}([\sigma])+(k-s+1)(k-r)} \sum_{i \in \mathrm{cB_d}([\pi])} q^{\mathrm{maj}(S_i([\pi]))}. \tag{1.9}$$

Setting $q \to 1$ in Theorem 1.3, we obtain (1.6), that is,

$$\begin{aligned} &\#\mathcal{S}^c([\sigma],[\pi],k) \\ &= \sum_{i \notin \mathrm{cB_d}[\pi]} \binom{m-r+s}{k-r}\binom{n-s+r-1}{n-k+r} + \sum_{i \in \mathrm{cB_d}[\pi]} \binom{m-r+s-1}{k-r}\binom{n-s+r}{n-k+r} \\ &= (n-s)\binom{m-r+s}{k-r}\binom{n-s+r-1}{n-k+r} + s\binom{m-r+s-1}{k-r}\binom{n-s+r}{n-k+r} \\ &= \frac{k(m-r)(n-s)+(m+n-k)rs}{(m-r+s)(n-s+r)}\binom{m-r+s}{k-r}\binom{n-s+r}{k-s}. \end{aligned}$$

2 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3 with the aid of Stanley's shuffling theorem.

Proof of Theorem 1.3. Let $[\sigma] \in \mathfrak{S}_m^c$ and let $[\pi] \in \mathfrak{S}_n^c$ be two disjoint cyclic permutations, where $\mathrm{cdes}([\sigma]) = r$ and $\mathrm{cdes}([\pi]) = s$. Suppose that the largest element of $[\sigma]$ and $[\pi]$ is in $[\sigma]$. Let $\hat{\sigma} = \hat{\sigma}_1\hat{\sigma}_2\cdots\hat{\sigma}_m$ be the representative of the cyclic permutation $[\sigma]$, that is, $\hat{\sigma}_1$ is the largest element of $[\sigma]$. Under the hypothesis of this theorem, we see that $\hat{\sigma}_1$ is greater than all elements in $[\pi]$. Define

$$\hat{\sigma}' = \hat{\sigma}_2\cdots\hat{\sigma}_m. \tag{2.1}$$

Obviously,

$$\mathrm{cdes}([\sigma]) = \mathrm{des}(\hat{\sigma}') + 1 \tag{2.2}$$

and

$$\mathrm{maj}([\sigma]) = \mathrm{maj}(\hat{\sigma}') + \mathrm{des}(\hat{\sigma}') + 1. \tag{2.3}$$

Let $\mathcal{S}^c([\sigma],[\pi])$ denote the set of cyclic shuffles of $[\sigma]$ and $[\pi]$, and let $\mathcal{S}(\hat{\sigma}', S_i([\pi]))$ denote the set of linear shuffles of $\hat{\sigma}'$ and $S_i([\pi])$, where σ' is defined in (2.1) and $S_i([\pi])$ is the unique permutation corresponding to $[\pi]$ which starts with $i \in [\pi]$. We claim that there is a bijection ψ between the set $\mathcal{S}^c([\sigma],[\pi])$ and the set $\bigcup_{i\in[\pi]} \mathcal{S}(\hat{\sigma}', S_i([\pi]))$. Moreover, for $[\alpha] \in \mathcal{S}^c([\sigma],[\pi])$, we have $\psi(\alpha) = \hat{\alpha}'$ such that

$$\mathrm{cdes}([\alpha]) = \mathrm{des}(\hat{\alpha}') + 1 \tag{2.4}$$

and

$$\mathrm{maj}([\alpha]) = \mathrm{maj}(\hat{\alpha}') + \mathrm{des}(\hat{\alpha}') + 1. \tag{2.5}$$

Let $[\alpha] \in \mathcal{S}^c([\sigma],[\pi])$ and let $\hat{\alpha} = \hat{\alpha}_1\hat{\alpha}_2\cdots\hat{\alpha}_{n+m}$ be the representative of $[\alpha]$, which is a linear permutation corresponding to $[\alpha]$ such that $\hat{\alpha}_1$ is the largest element in $[\alpha]$. Since $\hat{\sigma}_1$ is the largest element in $[\sigma]$ and $[\pi]$, we deduce that $\hat{\alpha}_1 = \hat{\sigma}_1$ and $\mathrm{cdes}([\alpha]) = \mathrm{des}(\hat{\alpha})$. Define

$$\hat{\alpha}' = \hat{\alpha}_2\hat{\alpha}_3\cdots\hat{\alpha}_{n+m}.$$

From the construction of $\hat{\alpha}'$, it is evident that $\hat{\alpha}' \in \bigcup_{i\in[\pi]} \mathcal{S}(\hat{\sigma}', S_i([\pi]))$ and $[\alpha]$ and $\hat{\alpha}'$ satisfy (2.4) and (2.5). Moreover, this process is clearly reversible. This proved the claim. We therefore obtain

$$\begin{aligned}
&\sum_{\substack{[\alpha]\in\mathcal{S}^c([\sigma],[\pi])\\ \mathrm{cdes}([\alpha])=k}} q^{\mathrm{maj}([\alpha])}\\
&= \sum_{i\in[\pi]} \sum_{\substack{\hat{\alpha}'\in\mathcal{S}(\hat{\sigma}',S_i([\pi]))\\ \mathrm{des}(\hat{\alpha}')=k-1}} q^{\mathrm{maj}(\hat{\alpha}')+k}\\
&= \sum_{i\notin \mathrm{cB_d}([\pi])} \sum_{\substack{\hat{\alpha}'\in\mathcal{S}(\hat{\sigma}',S_i([\pi]))\\ \mathrm{des}(\hat{\alpha}')=k-1}} q^{\mathrm{maj}(\hat{\alpha}')+k} + \sum_{i\in \mathrm{cB_d}([\pi])} \sum_{\substack{\hat{\alpha}'\in\mathcal{S}(\hat{\sigma}',S_i([\pi]))\\ \mathrm{des}(\hat{\alpha}')=k-1}} q^{\mathrm{maj}(\hat{\alpha}')+k}.
\end{aligned} \tag{2.6}$$

By (2.2) and (2.3), we see that

$$\mathrm{des}(\hat{\sigma}') = \mathrm{cdes}([\sigma]) - 1 = r-1 \quad \textit{and} \quad \mathrm{maj}(\hat{\sigma}') = \mathrm{maj}([\sigma]) - r. \tag{2.7}$$

Observe that $\mathrm{des}(S_i([\pi])) = \mathrm{cdes}([\pi]) = s$ if $i \notin \mathrm{cB_d}([\pi])$. Hence, by Theorem 1.1, we obtain

$$\begin{aligned}
&\sum_{i\notin \mathrm{cB_d}([\pi])} \sum_{\substack{\hat{\alpha}'\in\mathcal{S}(\hat{\sigma}',S_i([\pi]))\\ \mathrm{des}(\hat{\alpha}')=k-1}} q^{\mathrm{maj}(\hat{\alpha}')+k}\\
&= \sum_{i\notin \mathrm{cB_d}([\pi])} \begin{bmatrix} m-r+s\\ k-r\end{bmatrix}\begin{bmatrix} n-s+r-1\\ k-s-1\end{bmatrix} q^{\mathrm{maj}(\hat{\sigma}')+\mathrm{maj}(S_i([\pi]))+(k-s-1)(k-r)+k}\\
&\overset{(2.7)}{=} \begin{bmatrix} m-r+s\\ k-r\end{bmatrix}\begin{bmatrix} n-s+r-1\\ k-s-1\end{bmatrix} q^{(k-s)(k-r)+\mathrm{maj}([\sigma])} \sum_{i\notin \mathrm{cB_d}([\pi])} q^{\mathrm{maj}(S_i([\pi]))}.
\end{aligned} \tag{2.8}$$

Since $\mathrm{des}(S_i([\pi])) = \mathrm{cdes}([\pi]) - 1 = s-1$ when $i \in \mathrm{cB_d}([\pi])$, it follows from Theorem 1.1 that

$$\begin{aligned}
&\sum_{i\in \mathrm{cB_d}([\pi])} \sum_{\substack{\hat{\alpha}'\in\mathcal{S}(\hat{\sigma}',S_i([\pi]))\\ \mathrm{des}(\hat{\alpha}')=k-1}} q^{\mathrm{maj}(\hat{\alpha}')+k}\\
&= \sum_{i\in \mathrm{cB_d}([\pi])} \begin{bmatrix} m-r+s-1\\ k-r\end{bmatrix}\begin{bmatrix} n-s+r\\ k-s\end{bmatrix} q^{\mathrm{maj}(\hat{\sigma}')+\mathrm{maj}(S_i([\pi]))+(k-s)(k-r)+k}\\
&\overset{(2.7)}{=} \begin{bmatrix} m-r+s-1\\ k-r\end{bmatrix}\begin{bmatrix} n-s+r\\ k-s\end{bmatrix} q^{(k-s+1)(k-r)+\mathrm{maj}([\sigma])} \sum_{i\in \mathrm{cB_d}([\pi])} q^{\mathrm{maj}(S_i([\pi]))}.
\end{aligned} \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.6), we obtain (1.9). This completes the proof. ∎

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