Some Refinements of Stanley's Shuffle Theorem

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Abstract. We first give a combinatorial proof of Stanley's shuffle theorem by using the insertion lemma of Haglund, Loehr and Remmel. Based on this combinatorial construction, we establish several refinements of Stanley's shuffle theorem.

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1 Introduction

Stanley's shuffle theorem gives an explicit expression for the generating function of the number of shufflings of two disjoint permutation σ and π with k descents and the major index being t. Let us recall some common notation and terminology on permutations as used in [17, Chapter 1]. We say that $\pi = \pi_1 \cdots \pi_n$ is a permutation of length n if it is a sequence of n distinct lettersnot necessarily from 1 to n. Let $\ell(\pi)$ denote the length of the permutation π. For example, $\pi = 9\,3\,8\,10\,12\,3\,7$ is a permutation of length 7, and so $\ell(\pi) = 7$. Let $\mathfrak{S}_n$ denote the set of all permutations of length n. We say that $1 \leq i \leq n-1$ is a descent of $\pi \in \mathfrak{S}_n$ if $\pi_i > \pi_{i+1}$ and $1 \leq i \leq n-1$ is an ascent of $\pi \in \mathfrak{S}_n$ if $\pi_i < \pi_{i+1}$. The set of descents of π is called the descent set of π, denoted $\mathrm{Des}(\pi)$ and the number of its descents is called the descent number, denoted $\mathrm{des}(\pi)$. The major index of π, denoted $\mathrm{maj}(\pi)$, is defined to be the sum of its descents. To wit,

$$\mathrm{maj}(\pi) := \sum_{k \in \mathrm{Des}(\pi)} k.$$

Let $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$ be two disjoint permutations, that is, permutations with no letters in common. We say that $\alpha \in \mathfrak{S}_{n+m}$ is a shuffle of σ and π if both σ and π are subsequences of α. The set of shuffles of σ and π is denoted $\mathfrak{S}(\sigma, \pi)$. For example,

$$\begin{aligned}\mathfrak{S}(2\,6\,3, 1\,4) = \{&2\,6\,3\,1\,4, 2\,6\,1\,3\,4, 2\,6\,1\,4\,3, 2\,1\,4\,6\,3, 2\,1\,6\,3\,4,\ 2\,1\,6\,4\,3, 1\,2\,4\,6\,3,\\ &1\,4\,2\,6\,3,\ 1\,2\,6\,3\,4, 1\,2\,6\,4\,3\}.\end{aligned}$$

It is easy to see that the number of permutations in $\mathfrak{S}(\sigma, \pi)$ is $\binom{m+n}{n}$.

Define

$$S_{k,q}(\sigma,\pi) = \sum_{\substack{\alpha\in\mathfrak{S}(\sigma,\pi)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)}.$$

In light of the q-Pfaff-Saalschütz identity in his setting of P-partitions, Stanley [16] obtained a compact expression for $S_{k,q}(\sigma,\pi)$ in terms of the Gaussian polynomial (also called the q-binomial coefficients), as given by

$$\begin{bmatrix} n+m \\ n \end{bmatrix} = \begin{cases} \dfrac{(1-q^{n+m})(1-q^{n+m-1})\cdots(1-q^{m+1})}{(1-q^n)(1-q^{n-1})\cdots(1-q)}, & \text{for } n,m\geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for n,m are non-negative integers, see Andrews [2, Chapter 1]. More precisely,

Theorem 1.1 (Stanley's shuffle theorem)**.** *Let σ and π be two disjoint permutations. Then*

$$\sum_{\substack{\alpha\in\mathfrak{S}(\sigma,\pi)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} \ell(\sigma)-\mathrm{des}(\sigma)+\mathrm{des}(\pi) \\ k-\mathrm{des}(\sigma) \end{bmatrix} \begin{bmatrix} \ell(\pi)-\mathrm{des}(\pi)+\mathrm{des}(\sigma) \\ k-\mathrm{des}(\pi) \end{bmatrix} \times q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+(k-\mathrm{des}(\pi))(k-\mathrm{des}(\sigma))}. \tag{1.1}$$

Stanley asked for a proof of Theorem 1.1 which avoids the use of the q-Pfaff-Saalschütz identity (see [2, Eq.3.3.11]). Bijective proofs of Stanley's shuffle theorem have been given by Goulden [9] and Stadler [15]. Goulden's proof is obtained by finding bijections for lattice path representations of shuffles which reduce σ and π to canonical permutations, for which the generating function is easily given. Stadler's bijection is more elementary, but the inverse of Stadler's map is not very explicit. In this paper, we first give an explicit bijective proof of Theorem 1.1 by using the insertion lemma of Haglund, Loehr and Remmel [10]. It turns out that the insertion lemma of Haglund, Loehr and Remmel is equivalent to Stanley's shuffle theorem in the case $\ell(\pi)=1$. It should be mentioned that Novick [13] used the insertion lemma of Haglund, Loehr and Remmel to give a bijective proof of the following theorem due to Garsia and Gessel [6].

Theorem 1.2 (Garsia and Gessel)**.** *Let σ and π be two disjoint permutations. Then*

$$\sum_{\alpha\in\mathfrak{S}(\sigma,\pi)} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} \ell(\sigma)+\ell(\pi) \\ \ell(\pi) \end{bmatrix} q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)}. \tag{1.2}$$

In fact, Theorem 1.2 can be derived from Theorem 1.1 by employing q-analogue of the Chu-Vandermonde summation (see [2, Eq.3.3.10]),

$$\sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} m+n \\ h \end{bmatrix}.$$

Based on this combinatorial construction, we obtain four refinements of Stanley's shuffle theorem, see Theorem 4.1, Theorem 4.4, Theorem 4.8 and Theorem 4.11 in Section 4. As immediate consequences of these four refinements, we obtain two more general refinements. More precisely, let $\mathfrak{S}^{la,\,sa,\,sb}(\sigma,\pi)$ denote the set of shuffles $\alpha = \alpha_1 \cdots \alpha_{\ell(\alpha)}$ of two disjoint permutations $\sigma = \sigma_1 \cdots \sigma_{\ell(\sigma)}$ and $\pi = \pi_1 \cdots \pi_{\ell(\pi)}$ such that $\alpha_{\ell(\alpha)} = \min\{\sigma_{\ell(\sigma)}, \pi_{\ell(\pi)}\}$, satisfying the conditions in Definition 4.3 and Definition 4.7. For example,

$$\mathfrak{S}^{sb,\,la,\,sa}(2\,6\,3, 1\,4) = \{2\,1\,6\,4\,3\}.$$

Combining Theorem 4.1, Theorem 4.4 and Theorem 4.11, we obtain the following refinement of Stanley's shuffle theorem.

Theorem 1.3. *Let σ and π be two disjoint permutations. Then*

$$\sum_{\substack{\alpha\in\mathfrak{S}^{la,\,sa,\,sb}(\sigma,\pi)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix}\ell(\sigma)-\mathrm{des}(\sigma)+\mathrm{des}(\pi)-2\\ k-\mathrm{des}(\sigma)-2\end{bmatrix}\begin{bmatrix}\ell(\pi)-\mathrm{des}(\pi)+\mathrm{des}(\sigma)-1\\ k-\mathrm{des}(\pi)-1\end{bmatrix}$$
$$\times\, q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+\ell(\sigma)+\ell(\pi)-1+(k-\mathrm{des}(\pi)-1)(k-\mathrm{des}(\sigma)-1)}.$$

Let $\mathfrak{S}^{la,\,lb,\,sb}(\sigma,\pi)$ denote the set of shuffles $\alpha = \alpha_1 \cdots \alpha_{\ell(\alpha)}$ of two disjoint permutations $\sigma = \sigma_1 \cdots \sigma_{\ell(\sigma)}$ and $\pi = \pi_1 \cdots \pi_{\ell(\pi)}$ such that $\alpha_{\ell(\alpha)} = \min\{\sigma_{\ell(\sigma)}, \pi_{\ell(\pi)}\}$, satisfying the conditions in Definition 4.3 and Definition 4.10. For example,

$$\mathfrak{S}^{la,\,lb,\,sb}(2\,6\,3, 1\,4) = \{1\,2\,6\,4\,3\}.$$

Using Theorem 4.1 and Theorem 4.4, as well as Theorem 4.8, we obtain the following refinement of Stanley's shuffle theorem.

Theorem 1.4. *Let σ and π be two disjoint permutations. Then*

$$\sum_{\substack{\alpha\in\mathfrak{S}^{la,\,lb,\,sb}(\sigma,\pi)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix}\ell(\sigma)-\mathrm{des}(\sigma)+\mathrm{des}(\pi)-1\\ k-\mathrm{des}(\sigma)-1\end{bmatrix}\begin{bmatrix}\ell(\pi)-\mathrm{des}(\pi)+\mathrm{des}(\sigma)-2\\ k-\mathrm{des}(\pi)-1\end{bmatrix}$$
$$\times\, q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+\ell(\sigma)+\ell(\pi)-1+(k-\mathrm{des}(\pi)-1)(k-\mathrm{des}(\sigma))}.$$

To conclude the introduction, let us say a few words on the recent work that has built upon Stanley's shuffle theorem. Inspired by Stanley's shuffle theorem, Gessel and Zhuang [7] introduced the concept of shuffle compatible and stated that Stanley's shuffle theorem imply that maj and des are shuffle compatible. Gessel and Zhuang [7] further investigated the shuffle compatibility property of other permutation statistics and posed several conjectures involving the shuffle compatibility. Some of these conjectures were confirmed by Baker-Jarvis and Sagan [3], Grinberg [8] and Yang and Yan [18], respectively. In particular, Baker-Jarvis and Sagan [3] provided unified bijective techniques to give demonstration of shuffle compatibility. Cyclic shuffle and cyclic shuffle compatibility were investigated by Adin, Gessel, Reiner and Roichman [1] and Domagalski, Liang, Minnich and Sagan [4], respectively. In [11], we established a cyclic analogue of Theorem 1.1.

Last but not least, we would like to mention one interesting consequence of Stanley's shuffle theorem, which was explicitly stated by Sagan and Savage [14, Corollary 2.4] and proved by using Foata's fundamental bijection [5].

Theorem 1.5 (Sagan and Savage). *For $m, n \geq 1$,*

$$\sum_{\substack{\alpha \in \mathfrak{S}(1^m,\, 2^n) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2}. \tag{1.3}$$

It should be noted that a specialization of our combinatorial construction provides an alternative proof of Theorem 1.5.

The paper is organized as follows. In Section 2, we recall the insertion lemma of Haglund, Loehr and Remmel and demonstrate that the insertion lemma of Haglund, Loehr and Remmel is equivalent to Stanley's shuffle theorem in the case $\ell(\pi) = 1$. Section 3 is devoted to the bijective proof of Theorem 1.1 based on the insertion lemma of Haglund, Loehr and Remmel. In Section 4, we establish four refinements of Stanley's shuffle theorem relying on this combinatorial construction.

2 The insertion lemma of Haglund, Loehr and Remmel

This section is devoted to illustrating the insertion lemma of Haglund, Loehr and Remmel [10]. We follow the terminology, notation and the example in Section 4 of their paper.

Assume that $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ and $r \notin \sigma$ (that is, there does not exist $1 \leq j \leq n$ such that $\sigma_j = r$). For $0 \leq i \leq n$, let $\sigma^{(i)}(r)$ denote the permutation of length $n+1$ obtained by inserting r into σ before σ_{i+1}. Here we assume that $\sigma^{(n)}(r)$ denotes the permutation of length $n+1$ obtained by inserting r into σ after σ_n. The insertion lemma of Haglund, Loehr and Remmel [10] showed that no matter what the relative value of r is with respect to the elements in σ,

$$\sum_{i=0}^{n} q^{\mathrm{maj}(\sigma^{(i)}(r))} = (1 + q + \cdots + q^n) q^{\mathrm{maj}(\sigma)}. \tag{2.1}$$

This relation can be used to establish the following celebrated formula due to MacMahon [12].

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!, \tag{2.2}$$

where $[n]_q! = [1]_q [2]_q \cdots [n]_q$ with $[k]_q = 1 + q + \cdots + q^{k-1}$.

Haglund, Loehr and Remmel [10] also classified the possible spaces where they can insert r into σ into two sets called the right-to-left spaces which they denoted as RL-spaces and the left-to-right spaces which they denoted as LR-spaces. That is, a space i is called a RL-space of σ relative to r if

1. $i = n$ and $\sigma_n < r$,
2. $i = 0$ and $r < \sigma_1$,
3. $0 < i < n$ and $\sigma_i > \sigma_{i+1} > r$,
4. $0 < i < n$ and $r > \sigma_i > \sigma_{i+1}$, or
5. $0 < i < n$ and $\sigma_i < r < \sigma_{i+1}$.

Then a space i is a LR-space of σ relative to r if it is not a RL-space of σ relative to r. Assume that there are l RL-spaces for σ relative to r. Then label the RL-spaces from right to left with $0, \ldots, l-1$ and label the LR-spaces from left to right with $l, \ldots, n$ and call this labeling the canonical labeling for σ relative to r. For example suppose that $r = 5$ and $\sigma = 10\ 1\ 9\ 8\ 2\ 7\ 4\ 3\ 6$ is a permutation in $\mathfrak{S}_9$. By definition, we see the RL-spaces of σ relative to 5 are $0, 2, 3, 5, 7$ and 8 and the LR-spaces of σ relative to 5 are $1, 4, 6$ and 9. The canonical labeling of σ relative to r is

$$_{\mathbf{5}}10\,_{6}1\,_{\mathbf{4}}9\,_{\mathbf{3}}8\,_{7}2\,_{\mathbf{2}}7\,_{8}4\,_{\mathbf{1}}3\,_{\mathbf{0}}6\,_{9},$$

where the bold number in the subscript represents the labeling of the RL-spaces of σ relative to 5.

Haglund, Loehr and Remmel [10] established the following insertion lemma.

Lemma 2.1 (The insertion lemma)**.** *Assume that $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ and $r \notin \sigma$, and let $\sigma^{(i)}(r)$ denote the permutation obtained by inserting r into σ before σ_{i+1}. If the label at the i-th space in the cannonical labeling of σ relative to r is equal to a, then*

$$\mathrm{maj}(\sigma^{(i)}(r)) = a + \mathrm{maj}(\sigma).$$

With a careful examination of the definitions of the RL-spaces and the LR-spaces, we obtain the following lemma, which is useful in the proof of Stanley's shuffle theorem. We denote by $d_k(\sigma)$ the number of descents in σ greater than or equal to k. Obviously, $d_1(\sigma) = \mathrm{des}(\sigma)$ and $\mathrm{maj}(\sigma) = \sum_{k=1}^{n} d_k(n)$.

Lemma 2.2. *Let σ, r and $\sigma^{(i)}(r)$ be given in Lemma 2.1. If i is a RL-space of σ relative to r, then $\mathrm{des}(\sigma^{(i)}(r)) = \mathrm{des}(\sigma)$. If i is a LR-space of σ relative to r, then $\mathrm{des}(\sigma^{(i)}(r)) = \mathrm{des}(\sigma) + 1$. Moreover, the number of RL-spaces of σ relative to r is one more than the number of* des*cents in σ.*

Proof. Assume that there are k descents in σ. By the definitions of RL-spaces and LR-spaces, we find that if i is a RL-space of σ relative to r, then $\mathrm{des}(\sigma^{(i)}(r)) =$

$\mathrm{des}(\sigma) = k$. Moreover, the major increment $\mathrm{maj}(\sigma^{(i)}(r)) - \mathrm{maj}(\sigma) = d_{i+1}(\sigma^{(i)}(r))$, and hence $\mathrm{maj}(\sigma^{(i)}(r)) - \mathrm{maj}(\sigma) \leq k$. If i is a LR-space of σ relative to r, then $\mathrm{des}(\sigma^{(i)}(r)) = \mathrm{des}(\sigma) + 1$. Moreover, the major increment $\mathrm{maj}(\sigma^{(i)}(r)) - \mathrm{maj}(\sigma) \geq i + d_{i+1}(\sigma) \geq k$. Assume that the number of RL-spaces of σ relative to r is n_r and the number of LR-spaces of σ relative to r is n_l. Clearly, $n_r + n_l = n + 1$. By Lemma 2.1, we see that the major increment at each space of σ is different, so we conclude that $n_r \leq k + 1$ and $n_l \leq n - k$. Since $n_r + n_l = n + 1$, we derive that $n_r = k + 1$ and $n_l = n - k$. This completes the proof. ∎

We conclude this section with the proof of Stanley's shuffle theorem in the case $\ell(\pi) = 1$ in view of Lemma 2.1 and Lemma 2.2.

If $\ell(\pi) = 1$ in Theorem 1.1 , we have $\mathrm{des}(\pi) = 0$ and $\mathrm{maj}(\pi) = 0$, and thus Theorem 1.1 reads as follows:

$$\sum_{\substack{\alpha \in \mathfrak{S}(\sigma,\pi) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} \ell(\sigma) - \mathrm{des}(\sigma) \\ k - \mathrm{des}(\sigma) \end{bmatrix} \begin{bmatrix} \mathrm{des}(\sigma) + 1 \\ k \end{bmatrix} q^{\mathrm{maj}(\sigma) + k(k - \mathrm{des}(\sigma))}. \tag{2.3}$$

Assume that $\ell(\sigma) = m$ and $\mathrm{des}(\sigma) = r$. In this case, we see that the right-hand side of (2.3) is non-zero if and only if $k = r$ or $k = r + 1$. Hence (2.3) can be written as

$$\sum_{\substack{\alpha \in \mathfrak{S}(\sigma,\pi) \\ \mathrm{des}(\alpha)=r}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} r + 1 \\ r \end{bmatrix} q^{\mathrm{maj}(\pi)} = (1 + q + \cdots + q^r)\, q^{\mathrm{maj}(\sigma)}. \tag{2.4}$$

and

$$\sum_{\substack{\alpha \in \mathfrak{S}(\sigma,\pi) \\ \mathrm{des}(\alpha)=r+1}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} m - r \\ 1 \end{bmatrix} q^{\mathrm{maj}(\pi)+r+1} = q^{r+1}(1 + q + \cdots + q^{m-r-1})\, q^{\mathrm{maj}(\sigma)}. \tag{2.5}$$

Obviously, the identity (2.4) and the identity (2.5) are immediate consequences of Lemma 2.1 and Lemma 2.2.

3 The bijection

In this section, we will give a proof of Theorem 1.1 in the general case by means of Lemma 2.1 and Lemma 2.2. To state the proof, we need to recall some notation and terminology on partitions as in [2, Chapter 1]. A partition λ of a positive integer n is a finite nonincreasing sequence of nonnegative integers $(\lambda_1, \ldots, \lambda_s)$ such that $\sum_{i=1}^{s} \lambda_i = n$. Then λ_i are called the parts of λ, where λ_1 is its largest part and λ_s is its smallest part. The number of parts of λ is called the length of λ, denoted by $\ell(\lambda)$. The weight of λ is the sum of parts of λ, denoted $|\lambda|$. Let $\mathcal{P}_{\leq n}(m)$ denote the set of partitions λ such that $\ell(\lambda) \leq n$ and $\lambda_1 \leq m$. It is well-known that the Gaussian polynomial has the following partition interpretation [2, Theorem 3.1]:

$$\begin{bmatrix} n + m \\ n \end{bmatrix} = \sum_{\lambda \in \mathcal{P}_{\leq n}(m)} q^{|\lambda|}. \tag{3.1}$$

In general, let $\mathcal{P}_n(t,m)$ denote the set of partitions λ such that $\ell(\lambda)=n$, $\lambda_n \geq t$ and $\lambda_1 \leq m$, we have

$$q^{nt}\begin{bmatrix} n+m-t \\ n \end{bmatrix} = \sum_{\lambda \in \mathcal{P}_n(t,m)} q^{|\lambda|}. \tag{3.2}$$

When $t=0$, we see that (3.2) coincides with (3.1).

Using (3.2), we see that Theorem 1.1 is equivalent to the following combinatorial statement.

Theorem 3.1. *Assume that $\sigma \in \mathfrak{S}_m$ and $\pi \in \mathfrak{S}_n$ are two disjoint permutations, where $\mathrm{des}(\sigma)=r$ and $\mathrm{des}(\pi)=s$. Let $\mathfrak{S}(\sigma,\pi|k)$ denote the set of all shuffles of σ and π with k descents. Then there is a bijection Φ between $\mathfrak{S}(\sigma,\pi|k)$ and $\mathcal{P}_{k-r}(k-s,m)\times\mathcal{P}_{n-k+r}(0,\,k-s)$, namely, for $\alpha \in \mathfrak{S}(\sigma,\pi|k)$, we have $(\lambda,\mu)=\Phi(\alpha)\in\mathcal{P}_{k-r}(k-s,m)\times\mathcal{P}_{n-k+r}(0,\,k-s)$ such that*

$$\mathrm{maj}(\alpha)=|\lambda|+|\mu|+\mathrm{maj}(\sigma)+\mathrm{maj}(\pi). \tag{3.3}$$

To prove Theorem 3.1, we first give a description of the map Φ and then we show that the map Φ is a bijection as desired in Theorem 3.1.

Definition 3.2 (The map Φ)**.** *Let $\sigma=\sigma_1\cdots\sigma_m$ be a permutation with r descents and let $\pi=\pi_1\cdots\pi_n$ be a permutation with s descents. Assume that $\alpha=\alpha_1\cdots\alpha_{n+m}$ is the shuffle of σ and π with k descents. The pair of partitions $(\lambda,\mu)=\Phi(\alpha)$ can be constructed as follows: Let $\alpha^{(i)}$ denote the permutation obtained by removing $\pi_1,\pi_2,\ldots,\pi_i$ from α. Obviously, $\alpha^{(n)}=\sigma$. Here we assume that $\alpha^{(0)}=\alpha$. For $1\leq i\leq n$, define*

$$t(i)=\mathrm{maj}(\alpha^{(i-1)})-\mathrm{maj}(\alpha^{(i)})-d_i(\pi). \tag{3.4}$$

Since there are k descents in α and there are r descents in σ, it follows that there exists $k-r$ permutations in $\alpha^{(1)},\ldots,\alpha^{(n)}$, denoted by $\alpha^{(i_1)},\ldots,\alpha^{(i_{k-r})}$ where $1\leq i_1<i_2<\cdots<i_{k-r}\leq n$, such that $\mathrm{des}(\alpha^{(i_l-1)})=\mathrm{des}(\alpha^{(i_l)})+1$ for $1\leq l\leq k-r$. Let $\{j_1,\ldots,j_{n-k+r}\}\in\{1,\ldots,n\}\backslash\{i_1,i_2,\ldots,i_{k-r}\}$, where $1\leq j_1<j_2<\cdots<j_{n-k+r}\leq n$. Then $\mathrm{des}(\alpha^{(j_l-1)})=\mathrm{des}(\alpha^{(j_l)})$ for $1\leq l\leq n-k+r$. The pair of partitions $(\lambda,\mu)=\Phi(\alpha)$ is defined by

$$\lambda=(t(i_{k-r}),t(i_{k-r-1}),\ldots,t(i_1)), \tag{3.5}$$

and

$$\mu=(t(j_1),t(j_2),\ldots,t(j_{n-k+r})). \tag{3.6}$$

For example, let

$$\sigma = 9\,3\,8\,10\,12\,4\,7, \quad \pi = \mathbf{1\,2\,6\,5\,13\,11}, \quad \text{and} \quad \alpha = \mathbf{1}\,9\,\mathbf{2\,6}\,3\,\mathbf{5\,13}\,8\,10\,12\,\mathbf{11}\,4\,7,$$

where $m=7$, $r=2$, $n=6$, $s=2$ and $k=5$. The elements of π in $\alpha^{(i)}$ are in boldface to distinguish them from the elements of σ. The pairs of partitions $(\lambda,\mu)=\Phi(\alpha)$ can be constructed as follows:

i	$\alpha^{(i)}$	$d_i(\pi)$	$\mathrm{maj}(\alpha^{(i-1)}) - \mathrm{maj}(\alpha^{(i)})$	$t(i)$	$\mathrm{des}(\alpha^{(i-1)}) - \mathrm{des}(\alpha^{(i)})$
6	9 3 8 10 12 4 7	0	6	6	1
5	9 3 8 10 12 **11** 4 7	1	5	4	1
4	9 3 **13** 8 10 12 **11** 4 7	1	3	2	0
3	9 3 **5** **13** 8 10 12 **11** 4 7	2	5	3	1
2	9 **6** 3 **5** **13** 8 10 12 **11** 4 7	2	4	2	0
1	9 **2** **6** 3 **5** **13** 8 10 12 **11** 4 7	2	5	3	0
0	**1** 9 **2** **6** 3 **5** **13** 8 10 12 **11** 4 7				

Hence $\lambda = (6, 4, 3)$ and $\mu = (3, 2, 2)$.

In order to prove that the map Φ is a bijection, we shall reformulate the insertion lemma of Haglund, Loehr and Remmel. To this end, we first recall the notation of the major increment sequence introduced by Novick [13]. Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ and $r \notin \sigma$. Recall that $\sigma^{(i)}(r)$ denotes the permutation obtained by inserting r before σ_{i+1} (or after σ_i if $i = n$). For $0 \le i \le n$, define the major increment

$$\mathrm{im}(\sigma, i, r) = \mathrm{maj}(\sigma^{(i)}(r)) - \mathrm{maj}(\sigma)$$

and the major increment sequence

$$MIS(\sigma, r) = (\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, n, r)).$$

Combining Lemma 2.1 and Lemma 2.2, we have the following corollary.

Corollary 3.3. *Let $\sigma \in \mathfrak{S}_n$ with k descents and $r \notin \sigma$. Then $MIS(\sigma, r)$ is a shuffling of $k+1, k+2, \ldots, n$ and $k, \ldots, 1, 0$. In particular, $\mathrm{im}(\sigma, i, r)$ is either $\min\{\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, i-1, r)\} - 1$ or $\max\{\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, i-1, r)\} + 1$. More precisely, if $\mathrm{des}(\sigma^{(i)}(r)) = \mathrm{des}(\sigma) + 1$, then*

$$\mathrm{im}(\sigma, i, r) = \max\{\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, i-1, r)\} + 1,$$

otherwise,

$$\mathrm{im}(\sigma, i, r) = \min\{\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, i-1, r)\} - 1.$$

For example, let $\sigma = 5\,1\,6\,2\,4 \in \mathfrak{S}_5$ and $r = 3$. Then $\mathrm{des}(\sigma) = 2$, $\mathrm{maj}(\sigma) = 1+3 = 4$ and

i	$\sigma^{(i)}(3)$	$\mathrm{maj}(\sigma^{(i)}(3))$	$\mathrm{im}(\sigma, i, 3)$	$\mathrm{des}(\sigma^{(i)}(3)) - \mathrm{des}(\sigma)$
0	**3** 5 1 6 2 4	6	2	0
1	5 **3** 1 6 2 4	7	3	1
2	5 1 **3** 6 2 4	5	1	0
3	5 1 6 **3** 2 4	8	4	1
4	5 1 6 2 **3** 4	4	0	0
5	5 1 6 2 4 **3**	9	5	1

so $MIS(\sigma, r) = (2, 3, 1, 4, 0, 5)$ which is a shuffle of $3, 4, 5$ and $2, 1, 0$.

Let $MIS_i(\sigma, r) = (\mathrm{im}(\sigma, 0, r), \ldots, \mathrm{im}(\sigma, i-1, r))$ be the first i elements of $MIS(\sigma, r)$. Employing the insertion lemma of Haglund, Loehr and Remmel, Novick [13] found the following interesting proposition about $MIS_i(\sigma, r)$. It turns out that this proposition plays an important role in the proof that the map Φ is a bijection. For completeness, we provide a simple proof of this proposition with the aid of Corollary 3.3. Here we use the common notation $\chi(T) = 1$ if the statement T is true and $\chi(T) = 0$ otherwise.

Proposition 3.4 (Novick)**.** *Assume that σ is a permutation of length m with r descents. Let $p, q \notin \sigma$ and let $\sigma^{(i-1)}(p)$ be the permutation by inserting p before σ_i. Then $MIS_i(\sigma^{(i-1)}(p), q)$ is a permutation of the set $\{\mathrm{im}(\sigma, j, p) + \chi(q > p) | 0 \leq j < i\}$.*

Proof. Let $\sigma[i] = \sigma_1\sigma_2 \cdots \sigma_i$ be the permutation of first i elements of σ. By Corollary 3.3, we find that $MIS(\sigma[i], p)$ is a permutation of the set $\{0, 1, \ldots, i\}$. Note that $\mathrm{im}(\sigma[i], i, p) = i\chi(\sigma_i > p)$, so $MIS_i(\sigma[i], p)$ is a permutation of the set $\{j-1+\chi(p > \sigma_i) | 0 < j \leq i\}$. By the definition of descents, we find that

$$\mathrm{im}(\sigma[i], j, p) + d_i(\sigma) = \mathrm{im}(\sigma, j, p) \quad \text{for} \quad 0 \leq j < i.$$

Hence $MIS_i(\sigma, p)$ is a permutation of the set $\{j - 1 + \chi(p > \sigma_i) + d_i(\sigma) | 0 < j \leq i\}$. Using the same argument, we derive that $MIS_i(\sigma^{(i-1)}(p), q)$ is a permutation of the set $\{j - 1 + \chi(q > p) + d_i(\sigma^{(i-1)}(p)) | 0 < j \leq i\}$, where $\sigma_i^{(i-1)}(p) = p$. Note that $d_i(\sigma^{(i-1)}(p)) = \chi(p > \sigma_i) + d_i(\sigma)$, so we conclude that $MIS_i(\sigma^{(i-1)}(p), q)$ is a permutation of the set $\{j - 1 + \chi(q > p) + \chi(p > \sigma_i) + d_i(\sigma) | 0 < j \leq i\}$. Then the proposition follows immediately by comparing $MIS_i(\sigma^{(i-1)}(p), q)$ with $MIS_i(\sigma, p)$. This completes the proof. ∎

For example, let $\sigma = 5\,8\,1\,4\,6\,2 \in \mathfrak{S}_6$, $p = 7$, $q = 9$ and $i = 5$. Note that $\sigma^{(4)}(7) = 5\,8\,1\,4\,7\,6\,2 \in \mathfrak{S}_7$, and it can be evaluated that

$$MIS(\sigma, 7) = (3,\ 2,\ 4,\ 5,\ 6,\ 1,\ 0)$$

and

$$MIS(\sigma^{(4)}(7), 9) = (4,\ 5,\ 3,\ 6,\ 7,\ 2,\ 1,\ 0).$$

We find that $MIS_5(\sigma, 7)$ is a permutation of $\{2,\ 3,\ 4,\ 5,\ 6\}$ while $MIS_5(\sigma^{(4)}(7), 9)$ is a permutation of $\{3,\ 4,\ 5,\ 6,\ 7\}$. On the other hand,

$$MIS(\sigma, 9) = (3,\ 4,\ 2,\ 5,\ 6,\ 1,\ 0),$$

and $\sigma^{(4)}(9) = 5\,8\,1\,4\,9\,6\,2 \in \mathfrak{S}_7$

$$MIS(\sigma^{(4)}(9), 7) = (4,\ 3,\ 5,\ 6,\ 2,\ 7,\ 1,\ 0).$$

It is clear that both $MIS_5(\sigma, 9)$ and $MIS_5(\sigma^{(4)}(9), 7)$ are permutations of $\{2,\ 3,\ 4,\ 5,\ 6\}$.

With Corollary 3.3 and Proposition 3.4 in hand, we can now show that the map Φ in Definition 3.2 is a map from $\mathfrak{S}(\sigma, \pi|k)$ to $\mathcal{P}_{k-r}(k - s, m) \times \mathcal{P}_{n-k+r}(0,\ k - s)$.

Lemma 3.5. *Assume that $\sigma \in \mathfrak{S}_m$ and $\pi \in \mathfrak{S}_n$ are two disjoint permutations, where $\operatorname{des}(\sigma) = r$ and $\operatorname{des}(\pi) = s$. Let $\alpha \in \mathfrak{S}(\sigma, \pi|k)$ and $(\lambda, \mu) = \Phi(\alpha)$. Then $\lambda \in \mathcal{P}_{k-r}(k-s, m)$ and $\mu \in \mathcal{P}_{n-k+r}(0,\, k-s)$. Furthermore, $\operatorname{maj}(\alpha) = |\lambda| + |\mu| + \operatorname{maj}(\sigma) + \operatorname{maj}(\pi)$.*

Proof. Recall that $\sigma = \sigma_1 \cdots \sigma_m$ is a permutation with r descents and $\pi = \pi_1 \cdots \pi_n$ be a permutation with s descents. Assume that $\alpha = \alpha_1 \cdots \alpha_{n+m}$ is the shuffle of σ and π with k descents. For $1 \le i \le n$, let $\alpha^{(i)}$ denote the permutation obtained by removing $\pi_1, \pi_2, \ldots, \pi_i$ from α and let k_i be the position at which π_i is inserted into $\alpha^{(i)}$ to yield $\alpha^{(i-1)}$. More precisely, $\alpha^{(i-1)}$ is obtained by inserting π_i into $\alpha^{(i)}$ before $\alpha^{(i)}_{k_i}$. Since α is a shuffle of σ and π, we deduce that $k_1 \le k_2 \le \cdots \le k_n$. Define

$$T^{(i)} = (\operatorname{im}(\alpha^{(i)}, 0, \pi_i) - d_i(\pi), \ldots, \operatorname{im}(\alpha^{(i)}, k_i - 1, \pi_i) - d_i(\pi)) \tag{3.7}$$

for $1 \le i \le n$. From the definition (3.4) of $t(i)$, it's clear that $t(i)$ is the final element of $T^{(i)}$. By Corollary 3.3, we see that the elements of $T^{(i)}$ are distinct. So we may assume that $T^{(i)}$ is a permutation of a set $ST^{(i)}$. In terms of Proposition 3.4, Novick [13] showed that

$$ST^{(1)} \subseteq ST^{(2)} \subseteq \cdots \subseteq ST^{(n)} \subseteq \{0, 1, \ldots, m\}. \tag{3.8}$$

It implies that $0 \le t(l) \le m$ for any $1 \le l \le n$.

Recall that $\alpha^{(i_1)}, \ldots, \alpha^{(i_{k-r})}$ are $k-r$ permutations such that

$$\operatorname{des}(\alpha^{(i_l)}) = \operatorname{des}(\alpha^{(i_l+1)}) + 1 \quad \text{for} \quad 1 \le l \le k-r,$$

where $1 \le i_1 < i_2 < \cdots < i_{k-r} \le n$ and $\alpha^{(j_1)}, \ldots, \alpha^{(j_{n-k+r})}$ are permutations such that

$$\operatorname{des}(\alpha^{(j_l-1)}) = \operatorname{des}(\alpha^{(j_l)}) \quad \text{for} \quad 1 \le l \le n-k+r,$$

where $1 \le j_1 < j_2 < \cdots < j_{n-k+r} \le n$. From Corollary 3.3, we see that if $\operatorname{des}(\sigma^{(i)}(r)) = \operatorname{des}(\sigma) + 1$, then $\operatorname{im}(\sigma, i, r) = \max\{\operatorname{im}(\sigma, 0, r), \ldots, \operatorname{im}(\sigma, i-1, r)\} + 1$, otherwise, $\operatorname{im}(\sigma, i, r) = \min\{\operatorname{im}(\sigma, 0, r), \ldots, \operatorname{im}(\sigma, i-1, r)\} - 1$. Since $t(i)$ is the final element of T_i, it follows that $t(i)$ is either the largest element of T_i or the smallest element of T_i. Hence, by (3.8), we derive that

$$m \ge t(i_{k-r}) \ge \cdots \ge t(i_1) \ge t(j_1) \ge \cdots \ge t(j_{n-k+r}) \ge 0. \tag{3.9}$$

To prove that $\lambda \in \mathcal{P}_{k-r}(k-s, m)$ and $\mu \in \mathcal{P}_{n-k+r}(0,\, k-s)$, by (3.5) and (3.6), it suffices to show that $t(j_1) \le k-s \le t(i_1)$. By the definition of i_1, we see that $\operatorname{des}(\alpha^{(i_1-1)}) = \operatorname{des}(\alpha) = k$ and $\operatorname{des}(\alpha^{(i_1)}) = k-1$, and so $\operatorname{maj}(\alpha^{(i_1-1)}) - \operatorname{maj}(\alpha^{(i_1)}) \ge k$. But $d_{i_1}(\pi) \le \operatorname{des}(\pi) = s$, it follows that

$$t(i_1) = \operatorname{maj}(\alpha^{(i_1-1)}) - \operatorname{maj}(\alpha^{(i_1)}) - d_{i_1}(\pi) \ge k-s. \tag{3.10}$$

Combining (3.9) and (3.10), we conclude that $\lambda = (t(i_{k-r}), t(i_{k-r-1}), \ldots, t(i_1))$ is a partition in $\mathcal{P}_{k-r}(k-s, m)$. From the definition of j_1, we have $\operatorname{des}(\alpha^{(j_1-1)}) =$

$\text{des}(\alpha^{(j_1)}) = k - j_1 + 1$, and so $\text{maj}(\alpha^{(j_1-1)}) - \text{maj}(\alpha^{(j_1)}) \le k - j_1 + 1$. Since $d_{j_1}(\pi) \ge \text{des}(\pi) - j_1 + 1 = s - j_1 + 1$, we have

$$t(j_1) = \text{maj}(\alpha^{(j_1-1)}) - \text{maj}(\alpha^{(j_1)}) - d_{j_1}(\pi) \le k - s. \tag{3.11}$$

Combining (3.9) and (3.11), we arrive at $\mu = (t(j_1), t(j_2), \ldots, t(j_{n-k+r}))$ is a partition in $\mathcal{P}_{n-k+r}(0,\, k-s)$. Moreover, it is evident from (3.4), (3.5) and (3.6) that $\text{maj}(\alpha) - \text{maj}(\sigma) = \text{maj}(\alpha^{(0)}) - \text{maj}(\sigma^{(n)}) = \sum_{i=1}^{n} t(i) + \text{maj}(\pi) = |\lambda| + |\mu| + \text{maj}(\pi)$. This completes the proof. ∎

To prove Theorem 3.1, we also need to define the inverse map of Φ.

Definition 3.6 (The map Ψ)**.** *Assume that σ and π are given in Theorem 3.1. Let $\lambda = (\lambda_1, \ldots, \lambda_{k-r}) \in \mathcal{P}_{k-r}(k-s, m)$ and $\mu = (\mu_1, \ldots, \mu_{n-k+r}) \in \mathcal{P}_{n-k+r}(0,\, k-s)$. Let*

$$M^{(n)} = \{\lambda_1, \ldots, \lambda_{k-r}, \mu_1, \ldots, \mu_{n-k+r}\} \tag{3.12}$$

be a multi-set consisting of all parts of λ and μ. All of the elements in $M^{(n)}$ are listed in non-increasing order. The map $\Psi\colon (\lambda, \mu) \to \alpha$ is defined as follows: Assume that $\alpha^{(n)} = \sigma$ and $k_{n+1} = m + 1$. Set $b = 0$ and carry out the following procedure.

(A) *Define*

$$T^{(n-b)} = (\text{im}(\alpha^{(n-b)}, 0, \pi_{n-b}) - d_{n-b}(\pi), \ldots, \text{im}(\alpha^{(n-b)}, k_{n-b+1} - 1, \pi_{n-b}) - d_{n-b}(\pi)). \tag{3.13}$$

We use $T_i^{(n-b)}$ to denote the i-th element of $T^{(n-b)}$. Let k_{n-b} be the largest positive integer such that $T_{k_{n-b}}^{(n-b)} \in M^{(n-b)}$ (the existence of k_{n-b} will be proved in Lemma 3.7). Let $\alpha^{(n-b-1)}$ be the permutation obtained by inserting π_{n-b} into $\alpha^{(n-b)}$ before $\alpha_{k_{n-b}}^{(n-b)}$. Define

$$M^{(n-b-1)} = M^{(n-b)} \setminus \{T_{k_{n-b}}^{(n-b)}\}, \tag{3.14}$$

which is a multiset of length $n - b - 1$.

(B) *Replace b by $b + 1$. If $b = n$, then we are done. Otherwise, go back to* (A).

Then $\Psi(\lambda, \mu) = \alpha^{(0)}$.

For example, let $\sigma = 9\,3\,8\,10\,12\,4\,7 \in \mathfrak{S}_7$, $\pi = 1\,2\,6\,5\,13\,11 \in \mathfrak{S}_6$, where $\text{des}(\sigma) = 2$ and $\text{des}(\pi) = 2$. Given $k = 5$, $\lambda = (6, 4, 3)$ and $\mu = (3, 2, 2)$, we will recover the shuffle α of σ and π as follows. The elements of π in $\alpha^{(i)}$ are in boldface to distinguish them from the elements of σ.

i	π_i	$T^{(i)}$	$M^{(i)}$	k_i	$\alpha^{(i)}$
6	11	$(3, 2, 4, 5, 1, \mathbf{6}, 7, 0)$	$\{6, 4, 3, 3, 2, 2\}$	6	$9\,3\,8\,10\,12\,4\,7$
5	13	$(3, 2, \mathbf{4}, 5, 6, 1, \ldots)$	$\{4, 3, 3, 2, 2\}$	3	$9\,3\,8\,10\,12\,\mathbf{11}\,4\,7$
4	5	$(3, 4, \mathbf{2}, \ldots)$	$\{3, 3, 2, 2\}$	3	$9\,3\,\mathbf{13}\,8\,10\,12\,\mathbf{11}\,4\,7$
3	6	$(2, \mathbf{3}, 4, \ldots)$	$\{3, 3, 2\}$	2	$9\,3\,\mathbf{5}\,\mathbf{13}\,8\,10\,12\,\mathbf{11}\,4\,7$
2	2	$(3, \mathbf{2}, \ldots)$	$\{3, 2\}$	2	$9\,\mathbf{6}\,3\,\mathbf{5}\,\mathbf{13}\,8\,10\,12\,\mathbf{11}\,4\,7$
1	1	$(\mathbf{3}, 2, \ldots)$	$\{3\}$	1	$9\,\mathbf{2}\,\mathbf{6}\,3\,\mathbf{5}\,\mathbf{13}\,8\,10\,12\,\mathbf{11}\,4\,7$
0			$\emptyset$		$\mathbf{1}\,9\,\mathbf{2}\,\mathbf{6}\,3\,\mathbf{5}\,\mathbf{13}\,8\,10\,12\,\mathbf{11}\,4\,7$

Hence $\alpha = \alpha^{(0)} = 1\,9\,2\,6\,3\,5\,13\,8\,10\,12\,11\,4\,7$.

We proceed to prove that the map Ψ defined in Definition 3.6 is a map from $\mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$ to $\mathfrak{S}(\sigma, \pi|k)$.

Lemma 3.7. *Assume that σ and π are given in Theorem 3.1. Let $(\lambda, \mu) \in \mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$ and let $\alpha = \Psi(\lambda, \mu)$. Then $\alpha \in \mathfrak{S}(\sigma, \pi|k)$. Moreover,* $\mathrm{maj}(\alpha) = |\lambda| + |\mu| + \mathrm{maj}(\sigma) + \mathrm{maj}(\pi)$.

Proof. We first show that α is a shuffle of σ and π. To this end, we need to show that k_{n-b} in Definition 3.6 exists and $k_{n-b} \leq k_{n-b+1}$ for $0 \leq b \leq n-1$.

Using Corollary 3.3, we find that when $b = 0$, $T^{(n)}$ is a permutation of the set $\{0, 1, \ldots, m\}$ since $d_n(\pi) = 0$. From (3.12), we see that all elements in $M^{(n)}$ are in $T^{(n)}$, and so k_n exists. For $1 \leq b \leq n-1$, Assume that k_{n-b+1} exists, we proceed to show that k_{n-b} exists. In light of Proposition 3.4, we derive that the elements in $T^{(n-b)}$ are the same as the first k_{n-b+1} elements in $T^{(n-b+1)}$. Since k_{n-b+1} is the largest integer such that $T^{(n-b+1)}_{k_{n-b+1}} \in M^{(n-b+1)}$, we deduce that all of the elements in $M^{(n-b)}$ are located to the right of $T^{(n-b+1)}_{k_{n-b+1}}$ in $T^{(n-b+1)}$, and so all of the elements in $M^{(n-b)}$ are also in $T^{(n-b)}$. It follows that k_{n-b} exists. Moreover, by definition, it is easy to see that $k_{n-b} \leq k_{n-b+1}$. Hence we have proven that α is a shuffle of σ and π.

We next show that there are k descents in α. Suppose to the contrary that $\mathrm{des}(\alpha) \neq k$. Assume that $\mathrm{des}(\alpha) = l < k$. Let $(\overline{\lambda}, \overline{\mu}) = \Phi(\alpha^{(0)})$, by Lemma 3.5, we derive that $\overline{\lambda} \in \mathcal{P}_{l-r}(l-s, m)$ and $\overline{\mu} \in \mathcal{P}_{\leq n-l+r}(l-s)$, that is,

$$m \geq \overline{\lambda}_1 \geq \cdots \geq \overline{\lambda}_{l-r} \geq l-s \geq \overline{\mu}_1 \geq \cdots \geq \overline{\mu}_{n-l+r} \geq 0.$$

Let

$$\overline{M}^{(n)} = \{\overline{\lambda}_1, \ldots, \overline{\lambda}_{l-r}, \overline{\mu}_1, \ldots \overline{\mu}_{n-l+r}\}$$

be a multiset consisting of all parts of $\overline{\lambda}$ and $\overline{\mu}$. All of the elements in $\overline{M}^{(n)}$ are listed in non-increasing order. From the definitions of Φ and Ψ, it is easy to see that $\overline{M}^{(n)}$ equals $M^{(n)}$ defined in (3.12). Since $k > l$, we derive that $\overline{\mu}_{k-l} = \lambda_{k-r} \geq k-s$ which contradicts to the fact that $\overline{\mu}_{k-l} \leq l-s < k-s$. Applying the same argument yields that $\mathrm{des}(\alpha^{(0)}) = l > k$ is also impossible. Hence we arrive at the conclusion that $\mathrm{des}(\alpha^{(0)}) = k$. Therefore, Ψ is a map from $\mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$ to $\mathfrak{S}(\sigma, \pi|k)$.

It remains to show that $\mathrm{maj}(\alpha) = |\lambda| + |\mu| + \mathrm{maj}(\sigma) + \mathrm{maj}(\pi)$. Let $\alpha^{(n-b-1)}$ be the permutation obtained by inserting π_{n-b} into $\alpha^{(n-b)}$ before $\alpha^{(n-b)}_{k_{n-b}}$. From (3.13), we find that

$$\mathrm{maj}(\alpha^{(n-b-1)}) - \mathrm{maj}(\alpha^{(n-b)}) = T^{(n-b)}_{k_{n-b}} + d_{n-b}(\pi). \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\mathrm{maj}(\alpha) - \mathrm{maj}(\sigma) = \mathrm{maj}(\alpha^{(0)}) - \mathrm{maj}(\alpha^{(n)}) = |\lambda| + |\mu| - \sum_{b=1}^{n} d_b(\pi) = |\lambda| + |\mu| + \mathrm{maj}(\pi),$$

as desired. This completes the proof. ∎

We are finally in a position to give a proof of Theorem 3.1 based on Lemma 3.5 and Lemma 3.7.

Proof of Theorem 3.1: Let $\alpha \in \mathfrak{S}(\sigma, \pi|k)$. Utilizing Lemma 3.5, we find that $\Phi(\alpha)$ belongs to $\mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$. Combining the definition of Φ and the definition of Ψ, we deduce that $\Psi(\Phi(\alpha)) = \alpha$.

Conversely, let $(\lambda, \mu) \in \mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$. Invoking Lemma 3.7, we know that $\Psi(\lambda, \mu) \in \mathfrak{S}(\sigma, \pi|k)$. By virtue of Definition 3.2 and Definition 3.6, we obtain that $\Phi(\Psi(\lambda, \mu)) = (\lambda, \mu)$.

Therefore, the map Φ is a bijection between $\mathfrak{S}(\sigma, \pi|k)$ and $\mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$. This completes the proof. ∎

4 Refinements

In this section, we first state four refinements of Stanley's shuffle theorem and then give proofs of these refinements by using Lemma 3.5 and Lemma 3.7.

Assume that $\sigma = \sigma_1 \cdots \sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ are two disjoint permutations. From Theorem 3.1, we see that Stanley's shuffle theorem is equivalent to the statement that there is a bijection Φ such that for $\alpha \in \mathfrak{S}(\sigma, \pi|k)$, we have $(\lambda, \mu) = \Phi(\alpha) \in \mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$ satisfying the relation (3.3).

In our first refinement, we restrict our attention to the subset of $\mathfrak{S}(\sigma, \pi|k)$, denoted by $\mathfrak{S}^{sb}(\sigma, \pi|k)$ such that for $\alpha \in \mathfrak{S}^{sb}(\sigma, \pi|k)$, we have $(\lambda, \mu) = \Phi(\alpha)$ where $\mu_{n-k+r} \geq 1$. This is also the reason that we denote this subset by $\mathfrak{S}^{sb}(\sigma, \pi|k)$. The images of $\alpha \in \mathfrak{S}^{sb}(\sigma, \pi|k)$ under the bijection Φ give more restrictions on the smallest part of the partition μ.

In the same vein, our second refinement is defined on the subset $\mathfrak{S}^{la}(\sigma, \pi|k)$ of $\mathfrak{S}(\sigma, \pi|k)$ and the third refinement is defined the subset $\mathfrak{S}^{sa}(\sigma, \pi|k)$. For $\alpha \in \mathfrak{S}^{la}(\sigma, \pi|k)$, we have $(\lambda, \mu) = \Phi(\alpha)$ satisfying $\lambda_1 = m$, whereas, for $\alpha \in \mathfrak{S}^{sa}(\sigma, \pi|k)$, we have $(\lambda, \mu) = \Phi(\alpha)$ such that $\lambda_{k-r} = k-s$. The fourth refinement involves the subset $\mathfrak{S}^{lb}(\sigma, \pi|k)$ such that for $\alpha \in \mathfrak{S}^{sa}(\sigma, \pi|k)$, we have $(\lambda, \mu) = \Phi(\alpha)$ satisfying $\mu_1 = k-s$.

To express these four refinements in terms of generating function, we need to consider the generating functions of the following two special sets of partitions. Let $\mathcal{P}_n(t, = m)$ denote the set of partitions λ such that $\ell(\lambda) = n$, $\lambda_n \geq t$ and $\lambda_1 = m$. Using (3.1) as a starting point, it's not difficult to derive that

$$q^{(n-1)t+m} \begin{bmatrix} n+m-t-1 \\ n-1 \end{bmatrix} = \sum_{\lambda \in \mathcal{P}_n(t,=m)} q^{|\lambda|}. \tag{4.1}$$

Let $\mathcal{P}_n(=t, m)$ denote the set of partitions λ such that $\ell(\lambda) = n$, $\lambda_n = t$ and $\lambda_1 \leq m$. The following generating function is required in the proof of the third refinement of Stanley's shuffle theorem.

$$q^{nt}\begin{bmatrix} n+m-t-1 \\ n-1 \end{bmatrix} = \sum_{\lambda\in\mathcal{P}_n(=t,m)} q^{|\lambda|}. \tag{4.2}$$

4.1 Statements of refinements

Assume that $\sigma=\sigma_1\cdots\sigma_m\in\mathfrak{S}_m$ and $\pi=\pi_1\cdots\pi_n\in\mathfrak{S}_n$ are two disjoint permutations. Let $\mathfrak{S}^{sb}(\sigma,\pi)$ denote the set of shuffles $\alpha=\alpha_1\cdots\alpha_{n+m}$ of σ and π such that $\alpha_{n+m}=\min\{\pi_n,\sigma_m\}$. For example,

$$\mathfrak{S}^{sb}(2\,6\,3,\mathbf{1\,4})=\{2\,6\,\mathbf{1\,4}\,3,2\,\mathbf{1}\,6\,\mathbf{4}\,3,\mathbf{1}\,2\,6\,\mathbf{4}\,3,2\,\mathbf{1\,4}\,6\,3,\mathbf{1}\,2\,\mathbf{4}\,6\,3,\mathbf{1\,4}\,2\,6\,3\}$$

and

$$\mathfrak{S}^{sb}(2\,8\,5,\mathbf{1\,4})=\{2\,8\,5\,\mathbf{1\,4},2\,8\,\mathbf{1}\,5\,\mathbf{4},2\,\mathbf{1}\,8\,5\,\mathbf{4},\mathbf{1}\,2\,8\,5\,\mathbf{4}\}.$$

Our first refinement of Stanley's shuffle theorem is the following.

Theorem 4.1 (The first refinement)**.** *Let $\sigma=\sigma_1\cdots\sigma_m\in\mathfrak{S}_m$ and $\pi=\pi_1\cdots\pi_n\in\mathfrak{S}_n$ be two disjoint permutations.*

- *If $\pi_n>\sigma_m$, then*

$$\sum_{\substack{\alpha\in\mathfrak{S}^{sb}(\sigma,\pi)\\ \operatorname{des}(\alpha)=k}} q^{\operatorname{maj}(\alpha)} = \begin{bmatrix} \ell(\sigma)-\operatorname{des}(\sigma)+\operatorname{des}(\pi) \\ k-\operatorname{des}(\sigma) \end{bmatrix}\begin{bmatrix} \ell(\pi)-\operatorname{des}(\pi)+\operatorname{des}(\sigma)-1 \\ k-\operatorname{des}(\pi)-1 \end{bmatrix}$$
$$\times\, q^{\operatorname{maj}(\sigma)+\operatorname{maj}(\pi)+\ell(\pi)+(k-\operatorname{des}(\pi)-1)(k-\operatorname{des}(\sigma))}. \tag{4.3}$$

- *If $\pi_n<\sigma_m$, then*

$$\sum_{\substack{\alpha\in\mathfrak{S}^{sb}(\pi,\sigma)\\ \operatorname{des}(\alpha)=k}} q^{\operatorname{maj}(\alpha)} = \begin{bmatrix} \ell(\sigma)-\operatorname{des}(\sigma)+\operatorname{des}(\pi)-1 \\ k-\operatorname{des}(\sigma)-1 \end{bmatrix}\begin{bmatrix} \ell(\pi)-\operatorname{des}(\pi)+\operatorname{des}(\sigma) \\ k-\operatorname{des}(\pi) \end{bmatrix}$$
$$\times\, q^{\operatorname{maj}(\sigma)+\operatorname{maj}(\pi)+\ell(\sigma)+(k-\operatorname{des}(\pi))(k-\operatorname{des}(\sigma)-1)}. \tag{4.4}$$

As an immediate corollary of Theorem 4.1, we obtain the following result.

Corollary 4.2. *Assume that $\sigma=\sigma_1\cdots\sigma_m\in\mathfrak{S}_m$ and $\pi=\pi_1\cdots\pi_n\in\mathfrak{S}_n$ are two disjoint permutations. Let $\mathfrak{S}^{sb}(\sigma,\pi|k)$ denote the set of shuffles $\alpha=\alpha_1\cdots\alpha_{n+m}$ of σ and π such that $\alpha_{n+m}=\min\{\sigma_m,\pi_n\}$ with k descents.*

- *If $\pi_n>\sigma_m$, then*

$$\#\mathfrak{S}^{sb}(\sigma,\pi|k)=\binom{\ell(\sigma)-\operatorname{des}(\sigma)+\operatorname{des}(\pi)}{k-\operatorname{des}(\sigma)}\binom{\ell(\pi)-\operatorname{des}(\pi)+\operatorname{des}(\sigma)-1}{k-\operatorname{des}(\pi)-1}.$$

- *If $\pi_n < \sigma_m$, then*

$$\#\mathfrak{S}^{sb}(\sigma,\pi|k) = \binom{\ell(\sigma)-\mathrm{des}(\sigma)+\mathrm{des}(\pi)-1}{k-\mathrm{des}(\sigma)-1}\binom{\ell(\pi)-\mathrm{des}(\pi)+\mathrm{des}(\sigma)}{k-\mathrm{des}(\pi)}.$$

We now turn to state the second refinement of Stanley's shuffle theorem. In doing so, we need to introduce a new shuffle subset.

Definition 4.3. *Let $\sigma = \sigma_1\cdots\sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1\cdots\pi_n \in \mathfrak{S}_n$ be two disjoint permutations and let α be a shuffle of σ and π. If $\pi_n > \sigma_m$, then set $\gamma = \pi$ and $\delta = \sigma$. If $\pi_n < \sigma_m$, then set $\gamma = \sigma$ and $\delta = \pi$. Let a_i be the largest number such that δ_{a_i} is before γ_i in α. Then α can be represented in the following form:*

$$\alpha = \delta_1\cdots\delta_{a_1}\gamma_1\delta_{a_1+1}\cdots\delta_{a_2}\gamma_2\delta_{a_2+1}\cdots\delta_{a_{\ell(\gamma)}}\gamma_{\ell(\gamma)}\delta_{a_{\ell(\gamma)}+1}\cdots\delta_{\ell(\delta)}. \tag{4.5}$$

Here we assume that $\delta_0 = 0$, $\delta_{a_{\ell(\gamma)+1}} = 0$ and $\gamma_{\ell(\gamma)+1} = +\infty$. A shuffle α of σ and π is defined to be in the set $\mathfrak{S}^{la}(\sigma,\pi)$ if there exists $\mathrm{ld}(\gamma) \le j \le \ell(\gamma)$ (where $\mathrm{ld}(\gamma)$ is the largest descent of γ if it exists, and $\ell(\gamma)$ otherwise) in (4.5) satisfying the following four conditions:

1. $\gamma_{j+1} > \delta_{a_{j+1}} > \delta_{a_{j+1}+1} > \cdots > \delta_{\ell(\delta)}$.
2. *If $\gamma_j < \delta_{a_{j+1}}$, then $\delta_{a_j} = \delta_{a_{j+1}}$.*
3. *If $j = i$, then $\delta_{a_j} = \delta_{a_{j+1}}$.*
4. *If $\gamma_j > \delta_{a_{j+1}}$ and $j > i$, then $\delta_{a_j} \neq \delta_{a_{j+1}}$ and $\gamma_j > \delta_{a_j+1} > \delta_{a_j+2} > \cdots > \delta_{a_{j+1}}$. Moreover* (a) *$\delta_{a_j} > \gamma_j$; or* (b) *$\delta_{a_j} < \delta_{a_j+1}$.*

For example, let $\sigma = 2\,1\,3\,8\,5\,3\,1 \in \mathfrak{S}_6$ and $\pi = 6\,9\,7\,11\,12 \in \mathfrak{S}_5$. Since $\sigma_6 = 1 < \pi_5 = 12$, set $\gamma = \pi = \mathbf{6\,9\,7\,11\,12}$ and $\delta = \sigma = 2\,1\,3\,8\,5\,3\,1$. It can be checked that

$$\alpha = 2\,1\,3\,\mathbf{6}\,\mathbf{9}\,8\,\underline{\mathbf{7}}\,\mathbf{11}\,5\,3\,\mathbf{12}\,1 \in \mathfrak{S}^{la}(\sigma,\pi)$$

and

$$\overline{\alpha} = 2\,1\,3\,\mathbf{6}\,\mathbf{9}\,\mathbf{7}\,8\,\mathbf{11}\,5\,3\,\mathbf{12}\,1 \notin \mathfrak{S}^{la}(\sigma,\pi).$$

As other examples, we find that

$$\mathfrak{S}^{la}(2\,6\,3,\mathbf{1\,4}) = \{2\,6\,3\,\underline{\mathbf{1}}\,\mathbf{4}, 2\,6\,\mathbf{1}\,\underline{\mathbf{4}}\,3, 2\,\mathbf{1}\,6\,\underline{\mathbf{4}}\,3, \mathbf{1}\,2\,6\,\underline{\mathbf{4}}\,3\}$$

and

$$\mathfrak{S}^{la}(\mathbf{2\,8\,5},1\;\;4) = \{\mathbf{2}\,\mathbf{8}\,1\,\underline{\mathbf{5}}\,4, \mathbf{2}\,1\,\mathbf{8}\,\underline{\mathbf{5}}\,4, 1\,\mathbf{2}\,\mathbf{8}\,\underline{\mathbf{5}}\,4, \mathbf{2}\,1\,4\,\underline{\mathbf{8}}\,\mathbf{5}, 1\,\mathbf{2}\,4\,\underline{\mathbf{8}}\,\mathbf{5}, 1\,4\,\mathbf{2}\,\underline{\mathbf{8}}\,\mathbf{5}\}.$$

Here the elements of γ in the examples above are in boldface. The element γ_j in these examples satisfying the conditions in Definition 4.3 is underlined.

The second refinement of Stanley's shuffle theorem can be stated as follows.

Theorem 4.4 (The second refinement)**.** *Let $\sigma = \sigma_1 \cdots \sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ be two disjoint permutations.*

- *If $\pi_n > \sigma_m$, then*

$$\sum_{\substack{\alpha \in \mathfrak{S}^{la}(\sigma,\pi) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} \ell(\sigma) - \mathrm{des}(\sigma) + \mathrm{des}(\pi) - 1 \\ k - \mathrm{des}(\sigma) - 1 \end{bmatrix} \begin{bmatrix} \ell(\pi) - \mathrm{des}(\pi) + \mathrm{des}(\sigma) \\ k - \mathrm{des}(\pi) \end{bmatrix}$$
$$\times\, q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+\ell(\sigma)+(k-\mathrm{des}(\pi))(k-\mathrm{des}(\sigma)-1)}. \tag{4.6}$$

- *If $\pi_n < \sigma_m$, then*

$$\sum_{\substack{\alpha \in \mathfrak{S}^{la}(\sigma,\pi) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} \ell(\sigma) - \mathrm{des}(\sigma) + \mathrm{des}(\pi) \\ k - \mathrm{des}(\sigma) \end{bmatrix} \begin{bmatrix} \ell(\pi) - \mathrm{des}(\pi) + \mathrm{des}(\sigma) - 1 \\ k - \mathrm{des}(\pi) - 1 \end{bmatrix}$$
$$\times\, q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+\ell(\pi)+(k-\mathrm{des}(\pi)-1)(k-\mathrm{des}(\sigma))}. \tag{4.7}$$

The following result is immediate from Theorem 4.4.

Corollary 4.5. *Assume that $\sigma = \sigma_1 \cdots \sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ are two disjoint permutations. Let $\mathfrak{S}^{la}(\sigma,\pi|k)$ denote the set of shuffles of σ and π with k descents, satisfying the conditions in Definition 4.3.*

- *If $\pi_n > \sigma_m$, then*

$$\#\mathfrak{S}^{la}(\sigma,\pi|k) = \binom{\ell(\sigma) - \mathrm{des}(\sigma) + \mathrm{des}(\pi) - 1}{k - \mathrm{des}(\sigma) - 1} \binom{\ell(\pi) - \mathrm{des}(\pi) + \mathrm{des}(\sigma)}{k - \mathrm{des}(\pi)}.$$

- *If $\pi_n < \sigma_m$, then*

$$\#\mathfrak{S}^{la}(\sigma,\pi|k) = \binom{\ell(\sigma) - \mathrm{des}(\sigma) + \mathrm{des}(\pi)}{k - \mathrm{des}(\sigma)} \binom{\ell(\pi) - \mathrm{des}(\pi) + \mathrm{des}(\sigma) - 1}{k - \mathrm{des}(\pi) - 1}.$$

The combination of Theorem 4.1 and Theorem 4.4 leads to the following result, which seems to be of interest.

Theorem 4.6. *Assume that $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$. Let $\sigma = \sigma_1 \cdots \sigma_m$ and $\sigma' = \sigma'_1 \cdots \sigma'_m$ are two permutations in $\mathfrak{S}_m$ such that $\mathrm{des}(\sigma) = \mathrm{des}(\sigma')$ and $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma')$. If $\sigma_m < \pi_n < \sigma'_m$, then*

$$\sum_{\substack{\alpha \in \mathfrak{S}^{sb}(\sigma',\pi) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \sum_{\substack{\alpha \in \mathfrak{S}^{la}(\sigma,\pi) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} \tag{4.8}$$

and

$$\sum_{\substack{\alpha \in \mathfrak{S}^{sb}(\sigma,\pi) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \sum_{\substack{\alpha \in \mathfrak{S}^{la}(\sigma',\pi) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)}. \tag{4.9}$$

It would be interesting to give a direct combinatorial proof of Theorem 4.6.

In order to present the third refinement of Stanley's shuffle theorem, we need to define the following shuffle subset.

Definition 4.7. *Let $\sigma = \sigma_1 \cdots \sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ be two disjoint permutations and let $\alpha = \alpha_1 \cdots \alpha_{n+m}$ be a shuffle of σ and π. For $1 \le j \le \mathrm{sd}(\pi)$ (where $\mathrm{sd}(\pi)$ is the smallest descent of π if it exists, and n otherwise), assume that π_j is the p-th element of α, that is, $\alpha_p = \pi_j$, and there are q elements from σ before α_p in α, say $\sigma_1, \sigma_2, \ldots, \sigma_q$ (q could be zero). If $\sigma_1 > \cdots > \sigma_q > \alpha_p$ and at least one of the following conditions holds: (1) when $p < n$, $\sigma_q < \alpha_{p+1}$ (2) when $p < n$, $\alpha_p > \alpha_{p+1}$, (3) $\alpha_{n+m} = \pi_j$, then we say the shuffle α is in the set $\mathfrak{S}^{sa}(\sigma, \pi)$.*

For example,

$$\mathfrak{S}^{sa}(2\,6\,3, \mathbf{1}\,\mathbf{4}) = \{2\,\underline{\mathbf{1}}\,\mathbf{4}\,6\,3, 2\,\underline{\mathbf{1}}\,6\,\mathbf{4}\,3, 2\,\underline{\mathbf{1}}\,6\,3\,\mathbf{4}, \mathbf{1}\,\underline{\mathbf{4}}\,2\,6\,3\},$$

where the element satisfying the conditions in Definition 4.7 is underlined.

We obtain the following refinement of Stanley's shuffle theorem involving the set $\mathfrak{S}^{sa}(\sigma, \pi)$.

Theorem 4.8 (The third refinement)**.** *Let σ and π be two disjoint permutations. Then*

$$\sum_{\substack{\alpha \in \mathfrak{S}^{sa}(\sigma,\pi) \\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix} \ell(\sigma) - \mathrm{des}(\sigma) + \mathrm{des}(\pi) - 1 \\ k - \mathrm{des}(\sigma) - 1 \end{bmatrix} \begin{bmatrix} \ell(\pi) - \mathrm{des}(\pi) + \mathrm{des}(\sigma) \\ k - \mathrm{des}(\pi) \end{bmatrix}$$
$$\times\, q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+(k-\mathrm{des}(\pi))(k-\mathrm{des}(\sigma))}. \tag{4.10}$$

Let $q \to 1$ in Theorem 4.8, we obtain the following consequence.

Corollary 4.9. *Assume that σ and π are two disjoint permutations. Let $\mathfrak{S}^{sa}(\sigma, \pi|k)$ denote the set of shuffles of σ and π with k descents, satisfying the conditions in Definition 4.7. Then*

$$\#\mathfrak{S}^{sa}(\sigma, \pi|k) = \binom{\ell(\sigma) - \mathrm{des}(\sigma) + \mathrm{des}(\pi) - 1}{k - \mathrm{des}(\sigma) - 1} \binom{\ell(\pi) - \mathrm{des}(\pi) + \mathrm{des}(\sigma)}{k - \mathrm{des}(\pi)}.$$

Our last refinement of Stanley's shuffle theorem is defined on the following shuffle subset.

Definition 4.10. *Let $\sigma = \sigma_1 \cdots \sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ be two disjoint permutations and let $\alpha = \alpha_1 \cdots \alpha_{n+m}$ be a shuffle of σ and π. For $1 \le j \le \mathrm{sa}(\pi)$ (where $\mathrm{sa}(\pi)$ is the smallest ascent of π), assume that π_j is the p-th element of α (that is, $\alpha_p = \pi_j$) and there are q elements from σ before α_p in α (q could be zero), say $\sigma_1, \sigma_2, \ldots, \sigma_q$. If $\sigma_1 < \cdots < \sigma_q < \alpha_p$ and at least one of the following conditions holds: (1) when $p < n$, $\sigma_q > \alpha_{p+1}$, (2) when $p < n$, $\alpha_p < \alpha_{p+1}$, (3) $\alpha_{n+m} = \pi_j$, then we say the shuffle α is in the set $\mathfrak{S}^{lb}(\sigma, \pi)$.*

For example,

$$\mathfrak{S}^{lb}(2\,6\,3, \mathbf{1}\,\mathbf{4}) = \{\underline{\mathbf{1}}\,\mathbf{4}\,2\,6\,3, \underline{\mathbf{1}}\,2\,\mathbf{4}\,6\,3, \underline{\mathbf{1}}\,2\,6\,\mathbf{4}\,3, \underline{\mathbf{1}}\,2\,6\,3\,\mathbf{4}\},$$

Here we underline the element that meets the conditions in Definition 4.10.

By considering the set $\mathfrak{S}^{lb}(\sigma,\pi)$, we can refine Stanley's shuffle theorem as follows.

Theorem 4.11 (The fourth refinement)**.** *Let σ and π be two disjoint permutations. Then*

$$\sum_{\substack{\alpha\in\mathfrak{S}^{lb}(\sigma,\pi)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \begin{bmatrix}\ell(\sigma)-\mathrm{des}(\sigma)+\mathrm{des}(\pi)\\ k-\mathrm{des}(\sigma)\end{bmatrix}\begin{bmatrix}\ell(\pi)-\mathrm{des}(\pi)+\mathrm{des}(\sigma)-1\\ k-\mathrm{des}(\pi)\end{bmatrix} \times q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\pi)+(k-\mathrm{des}(\pi))(k-\mathrm{des}(\sigma)+1)}. \tag{4.11}$$

Using Theorem 4.8, we obtain the following corollary.

Corollary 4.12. *Assume that σ and π are two disjoint permutations. Let $\mathfrak{S}^{lb}(\sigma,\pi|k)$ denote the set of shuffles of σ and π with k descents, satisfying the conditions in Definition 4.10. Then*

$$\#\mathfrak{S}^{lb}(\sigma,\pi|k) = \binom{\ell(\sigma)-\mathrm{des}(\sigma)+\mathrm{des}(\pi)}{k-\mathrm{des}(\sigma)}\binom{\ell(\pi)-\mathrm{des}(\pi)+\mathrm{des}(\sigma)-1}{k-\mathrm{des}(\pi)}.$$

Combining Theorem 4.8 and Theorem 4.11, together with Theorem 3.1, we arrive at the following consequence. It would be interesting to give a direct combinatorial proof of Corollary 4.13.

Corollary 4.13. *Let σ and π be two disjoint permutations. Then*

$$\sum_{\substack{\alpha\in\mathfrak{S}^{lb}(\sigma,\pi)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} + \sum_{\substack{\alpha\in\mathfrak{S}^{sa}(\pi,\sigma)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)} = \sum_{\substack{\alpha\in\mathfrak{S}(\sigma,\pi)\\ \mathrm{des}(\alpha)=k}} q^{\mathrm{maj}(\alpha)}.$$

4.2 Proofs

In this subsection, we will prove Theorem 4.1, Theorem 4.4 , Theorem 4.8 and Theorem 4.11 by refining the map Φ given in Theorem 3.1 on their corresponding shuffle subsets.

Based on (3.2) and using Theorem 3.1, we see that the proof of Theorem 4.1 is equivalent to the proof of the following combinatorial statement.

Theorem 4.14. *Assume that $\sigma=\sigma_1\cdots\sigma_m\in\mathfrak{S}_m$ and $\pi=\pi_1\cdots\pi_n\in\mathfrak{S}_n$ are two disjoint permutations, where $\mathrm{des}(\sigma)=r$ and $\mathrm{des}(\pi)=s$. If $\pi_n>\sigma_m$, then the map Φ given in Definition 3.2 is a bijection between $\mathfrak{S}^{sb}(\sigma,\pi|k)$ and $\mathcal{P}_{k-r}(k-s,m)\times\mathcal{P}_{n-k+r}(1,k-s)$. If $\pi_n<\sigma_m$, then the map Φ given in Definition 3.2 is a bijection between $\mathfrak{S}^{sb}(\sigma,\pi|k)$ and $\mathcal{P}_{k-s}(k-r,n)\times\mathcal{P}_{m-k+s}(1,k-r)$.*

Proof. Without loss of generality, we may prove this theorem in the case $\pi_n > \sigma_m$. The case $\pi_n < \sigma_m$ can be justified with the same argument by exchanging σ and π.

Let $\alpha \in \mathfrak{S}^{sb}(\sigma,\pi|k)$. Assume that $\Phi(\alpha) = (\lambda,\mu)$. From Lemma 3.5, we see that $\lambda \in \mathcal{P}_{k-r}(k-s,m)$ and $\mu \in \mathcal{P}_{n-k+r}(0,\, k-s)$, namely,

$$m \geq \lambda_1 \geq \cdots \geq \lambda_{k-r} \geq k-s \geq \mu_1 \geq \cdots \geq \mu_{n-k+r} \geq 0.$$

We proceed to show that $\mu_{n-k+r} \geq 1$ if $\mu \neq \emptyset$. Recall that $T^{(i)}$ is defined as (3.7) for $1 \leq i \leq n$. Since $\alpha \in \mathfrak{S}^{sb}(\sigma,\pi|k)$, it follows from Corollary 3.3 that $0 \notin T^{(n)}$. Using the relation (3.8), we find that $0 \notin T^{(i)}$ for $1 \leq i \leq n$. By the definition of Φ, we see that there exists i such that μ_{n-k+r} is the final element of $T^{(i)}$, it implies that $\mu_{n-k+r} \geq 1$, and so $\Phi(\alpha) = (\lambda,\mu) \in \mathcal{P}_{k-r}(k-s,m) \times \mathcal{P}_{n-k+r}(1,k-s)$.

Conversely, let $\lambda \in \mathcal{P}_{k-r}(k-s,m)$ and $\mu \in \mathcal{P}_{n-k+r}(1,k-s)$. Assume that $\Psi(\lambda,\mu) = \overline{\alpha}$, where the map Ψ is given in Definition 3.6. In light of Lemma 3.7, we derive that $\overline{\alpha} = \overline{\alpha}_1 \cdots \overline{\alpha}_{n+m}$ is a shuffle of σ and π with k descents. To prove that $\overline{\alpha} \in \mathfrak{S}^{sb}(\sigma,\pi)$, it suffices to show that $\overline{\alpha}_{n+m} = \sigma_m$. Suppose to the contrary that $\overline{\alpha}_{n+m} \neq \sigma_m$, that is, $\overline{\alpha}_{n+m} = \pi_n$. Since Ψ and Φ are inverses of each other, we have $\Phi(\overline{\alpha}) = \Phi(\Psi(\lambda,\mu)) = (\lambda,\mu)$. Let $\overline{\alpha}^{(i)}$ denote the permutation obtained by removing $\pi_1,\ldots,\pi_i$ from $\overline{\alpha}$. Clearly, $\overline{\alpha}^{(n)} = \sigma$. Let k_i be the position at which π_i is inserted into $\overline{\alpha}^{(i)}$ to yield $\overline{\alpha}^{(i-1)}$. Since $\overline{\alpha}_{n+m} = \pi_n$, so $k_n = m+1$. Under the condition that $\sigma_m < \pi_n$, we find that $\mathrm{des}(\overline{\alpha}^{(n-1)}) = \mathrm{des}(\overline{\alpha}^{(n)})$, and by the definition of Φ, we deduce that $\mu_{n-k+r} = \mathrm{maj}(\overline{\alpha}^{(n-1)}) - \mathrm{maj}(\overline{\alpha}^{(n)}) - d_n(\pi) = 0$, which contradicts the condition that $\mu \in \mathcal{P}_{n-k+r}(1,k-s)$, that is, $\mu_{n-k+r} \geq 1$. Hence $\overline{\alpha}_{n+m} = \sigma_m$, and so $\overline{\alpha} \in \mathfrak{S}^{sb}(\sigma,\pi)$. This completes the proof. ∎

We next give a proof of Theorem 4.4. According to (3.2) and (4.1), and by Theorem 3.1, it suffices to prove the following combinatorial statement.

Theorem 4.15. *Assume that $\sigma = \sigma_1 \cdots \sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ are two disjoint permutations, where $\mathrm{des}(\sigma) = r$ and $\mathrm{des}(\pi) = s$. If $\pi_n > \sigma_m$, then the map Φ given in Definition 3.2 is a bijection between $\mathfrak{S}^{la}(\sigma,\pi|k)$ and $\mathcal{P}_{k-r}(k-s,=m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$. If $\pi_n < \sigma_m$, then the map Φ given in Definition 3.2 is a bijection between $\mathfrak{S}^{la}(\sigma,\pi|k)$ and $\mathcal{P}_{k-s}(k-r,=n) \times \mathcal{P}_{n-k+s}(0,k-r)$.*

Proof. Similar to Theorem 4.14, it suffices to prove this theorem in the case $\pi_n > \sigma_m$. Let $\alpha \in \mathfrak{S}^{la}(\sigma,\pi|k)$. Assume that $\Phi(\alpha) = (\lambda,\mu)$. Using Lemma 3.5, we find that $\lambda \in \mathcal{P}_{k-r}(k-s,m)$ and $\mu \in \mathcal{P}_{n-k+r}(0,\, k-s)$. We next show that $\lambda_1 = m$. Recall that $\alpha^{(i)}$ denotes the permutation obtained by removing $\pi_1,\ldots,\pi_i$ from α. From the definition of Φ (that is, Definition 3.2), we see that

$$\lambda_1 = t(i_{k-r}) = \mathrm{maj}(\alpha^{(i_{k-r}-1)}) - \mathrm{maj}(\alpha^{(i_{k-r})}) - d_{i_{k-r}}(\pi),$$

where i_{k-r} is the largest number such that $\mathrm{des}(\alpha^{(i_{k-r}-1)}) = \mathrm{des}(\alpha^{(i_{k-r})})+1$. Under the precondition that $\alpha \in \mathfrak{S}^{la}(\sigma,\pi|k)$, we know that there exists $\mathrm{ld}(\pi) \leq j \leq n$ (where $\mathrm{ld}(\pi)$ is the largest descent of π if it exists, and n otherwise) satisfying the conditions

(1)–(4) in Definition 4.3. It is easy to check that $\mathrm{des}(\alpha^{(j-1)}) = \mathrm{des}(\alpha^{(j)})+1$. Moreover, for $p > j$, we have $\mathrm{des}(\alpha^{(p-1)}) = \mathrm{des}(\alpha^{(p)})$. It implies that $i_{k-r} = j$. Moreover,

$$\mathrm{maj}(\alpha^{(i_{k-r}-1)}) - \mathrm{maj}(\alpha^{(i_{k-r})}) - d_{i_{k-r}}(\pi) = m,$$

and so $\lambda_1 = t(i_{k-r}) = m$. Hence we derive that $\Phi(\alpha) = (\lambda, \mu) \in \mathcal{P}_{k-r}(k-s, = m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$.

To prove that the map Φ is the desired bijection, we proceed to show that the inverse map Ψ given in Definition 3.6 is also a map from $\mathcal{P}_{k-r}(k-s, = m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$ to $\mathfrak{S}^{la}(\sigma, \pi|k)$. Let $\lambda \in \mathcal{P}_{k-r}(k-s, = m)$ and $\mu \in \mathcal{P}_{n-k+r}(0,\, k-s)$. Assume that $\Psi(\lambda, \mu) = \overline{\alpha}$. We aim to show that $\overline{\alpha} \in \mathfrak{S}^{la}(\sigma, \pi)$.

With the help of Lemma 3.7, we know that $\overline{\alpha} \in \mathfrak{S}(\sigma, \pi|k)$. It remains to show that there exists j satisfies the conditions in Definition 4.3. Recall that $M^{(n)}$ is given by (3.12) and $T^{(n-b)}$ is given by (3.13) for $0 \le b \le n$. Let k_{n-b} be the largest positive integer such that $T^{(n-b)}_{k_{n-b}} \in M^{(n-b)}$. Then $M^{(n-b-1)}$ is obtained from $M^{(n-b)}$ by removing one $T^{(n-b)}_{k_{n-b}}$ and the permutation $\alpha^{(n-b-1)}$ is obtained by inserting π_{n-b} into $\alpha^{(n-b)}$ before $\alpha^{(n-b)}_{k_{n-b}}$.

Since $\lambda_1 = m$, we find that $m \in M^{(n)}$. From Corollary 3.3, we deduce that $m \in T^{(n)}$. Assume that j is the largest number such that $T^{(j)}_{k_j} = m$. We proceed to prove that j satisfies the conditions in the definition of $\mathfrak{S}^{la}(\sigma, \pi|k)$.

Since j the largest number such that $T^{(j)}_{k_j} = m$, that means that for $n \ge p > j$, $T^{(p)}_{k_p} \le k-s$, it implies that $\mathrm{des}(\overline{\alpha}^{(p-1)}) = \mathrm{des}(\overline{\alpha}^{(p)})$, and so $\mathrm{des}(\overline{\alpha}^{(j)}) = \mathrm{des}(\overline{\alpha}^{(n)}) = \mathrm{des}(\sigma)$. Observe that $\overline{\alpha}^{(j-1)}$ is obtained by inserting π_j into $\overline{\alpha}^{(j)}$ before $\overline{\alpha}^{(j)}_{k_j+1}$, so all the elements in $\overline{\alpha}^{(j-1)}$ before π_i are all from σ, that is, $\sigma_1, \ldots, \sigma_{k_j-1}$. Hence it follows that

$$d_{k_j}(\overline{\alpha}^{(j)}) = d_{k_j}(\sigma) \tag{4.12}$$

since $\mathrm{des}(\overline{\alpha}^{(j)}) = \mathrm{des}(\sigma)$. Recall that $T^{(j)}_{k_j} = m$, by definition, we have

$$\mathrm{maj}(\overline{\alpha}^{(j-1)}) - \mathrm{maj}(\overline{\alpha}^{(j)}) - d_j(\pi) = m. \tag{4.13}$$

It means that $\mathrm{des}(\overline{\alpha}^{(j-1)}) = \mathrm{des}(\overline{\alpha}^{(j)}) + 1$, and the new generated descent of $\overline{\alpha}^{(j-1)}$ would be k_j or $k_j + 1$. Hence, from (4.12), we deduce that there are only three possible cases where (4.13) holds.

Case 1. k_j is the new generated descent of $\overline{\alpha}^{(j-1)}$, $d_{k_j}(\overline{\alpha}^{(j)}) = m - k_j$ and $d_j(\pi) = 0$; or

Case 2. $k_j + 1$ is the new generated descent of $\overline{\alpha}^{(j-1)}$, $d_{k_j}(\overline{\alpha}^{(j)}) = m - k_j - 1$ and $d_j(\pi) = 0$; or

Case 3. $k_j + 1$ is the new generated descent of $\overline{\alpha}^{(j-1)}$, $d_{k_j}(\overline{\alpha}^{(j)}) = m - k_j$ and $d_j(\pi) = 1$.

It can be check that j satisfies the conditions in the definition of $\mathfrak{S}^{la}(\sigma, \pi)$ under the condition that $\mathrm{des}(\overline{\alpha}^{(j)}) = \mathrm{des}(\overline{\alpha}^{(j+1)}) = \cdots = \mathrm{des}(\overline{\alpha}^{(n)}) = \mathrm{des}(\overline{\alpha})$. More precisely, if

$\overline{\alpha}^{(j-1)}$ satisfies the conditions stated in Case 1, it is easy to check that j satisfies the conditions (1) and (2) in Definition 4.3. When $\overline{\alpha}^{(j-1)}$ belongs to Case 2, we find that j satisfies the conditions (1) and (4) in Definition 4.3. If $\overline{\alpha}^{(j-1)}$ meets the conditions in Case 3, we find that j satisfies the conditions (1) and (3).

Since $\overline{\alpha}$ is obtained from $\overline{\alpha}^{(j-1)}$ by inserting $\pi_1, \ldots, \pi_{j-1}$ before π_j, it does not affect the aforementioned conclusion that j satisfies the conditions in the definition of $\mathfrak{S}^{la}(\sigma, \pi)$. It follows that $\Psi(\lambda, \mu) = \overline{\alpha} \in \mathfrak{S}^{la}(\sigma, \pi)$. Hence we conclude that the map Ψ is a map from $\mathcal{P}_{k-r}(k-s, = m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$ to $\mathfrak{S}^{la}(\sigma, \pi|k)$. Since the map Φ and the map Ψ are inverse to each other, we can deduce that the map Φ is the bijection as desired. This completes the proof. ∎

Applying (3.2) and (4.2), as well as Theorem 3.1, we find that the proof of Theorem 4.8 is equivalent to the proof of the following combinatorial assertion.

Theorem 4.16. *Assume that $\sigma \in \mathfrak{S}_m$ and $\pi \in \mathfrak{S}_n$ are two disjoint permutations, where $\mathrm{des}(\sigma) = r$ and $\mathrm{des}(\pi) = s$. Then the map Φ given in Definition 3.2 is a bijection between $\mathfrak{S}^{sa}(\sigma, \pi|k)$ and $\mathcal{P}_{k-r}(= k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$.*

Proof. Let $\alpha \in \mathfrak{S}^{sa}(\sigma, \pi|k)$. Assume that $\Phi(\alpha) = (\lambda, \mu)$. According to Lemma 3.5, we see that $\lambda \in \mathcal{P}_{k-r}(k-s, m)$ and $\mu \in \mathcal{P}_{n-k+r}(0,\, k-s)$, namely,

$$m \geq \lambda_1 \geq \cdots \geq \lambda_{k-r} \geq k-s \geq \mu_1 \geq \cdots \geq \mu_{n-k+r} \geq 0.$$

We proceed to show that $\lambda_{k-r} = k-s$. Since $\alpha \in \mathfrak{S}^{sa}(\sigma, \pi|k)$, there exists $1 \leq j \leq \mathrm{sd}(\pi)$ (where $\mathrm{sd}(\pi)$ is the smallest descent of π if it exists, and n otherwise) satisfying the conditions in Definition 4.7. Recall that $\alpha^{(i)}$ is the permutation obtained by removing $\pi_1, \ldots, \pi_i$ from α. From Definition 4.7, we see that $\mathrm{des}(\alpha^{(j-1)}) = \mathrm{des}(\alpha^{(j)}) + 1$, $d_j(\pi) = \mathrm{des}(\pi) = s$. Assume that i_1 is the integer j satisfying the conditions in Definition 4.7. It means that for $p < i_1$, $\mathrm{des}(\alpha^{(p-1)}) = \mathrm{des}(\alpha^{(p)})$. Hence we conclude that $\mathrm{des}(\alpha^{(i_1-1)}) = \mathrm{des}(\alpha^{(0)}) = \mathrm{des}(\alpha) = k$, and by $\mathrm{des}(\alpha^{(i_1-1)}) = \mathrm{des}(\alpha^{(i_1)}) + 1$, we deduce that $\mathrm{des}(\alpha^{(i_1)}) = k-1$. Since i_1 meets the conditions in Definition 4.7, we derive that elements before π_{i_1} in $\alpha^{(i_1-1)}$ are from σ, we may assume that $\sigma_1, \ldots, \sigma_q$. Then $\sigma_1 > \cdots > \sigma_q > \alpha_p$, so we arrive at $\mathrm{maj}(\alpha^{(i_1-1)}) - \mathrm{maj}(\alpha^{(i_1)}) = k$. Moreover, by the definition of Φ, we see that

$$\lambda_{k-r} = t(i_1) = \mathrm{maj}(\alpha^{(i_1-1)}) - \mathrm{maj}(\alpha^{(i_1)}) - d_{i_1}(\pi) = k-s.$$

It yields that $\lambda \in \mathcal{P}_{k-r}(= k-s, m)$.

We proceed to show that the inverse map Ψ given in Definition 3.6 is also a map from $\mathcal{P}_{k-r}(= k-s, m) \times \mathcal{P}_{n-k+r}(0,\, k-s)$ to $\mathfrak{S}^{sa}(\sigma, \pi|k)$. Let $\lambda \in \mathcal{P}_{k-r}(= k-s, m)$ and $\mu \in \mathcal{P}_{n-k+r}(0,\, k-s)$. Assume that $\Psi(\lambda, \mu) = \overline{\alpha}$. From Lemma 3.7, we derive that $\overline{\alpha} = \overline{\alpha}_1 \cdots \overline{\alpha}_{n+m}$ is a shuffle of σ and π with k descents. Assume that $\overline{\alpha}^{(i-1)}$ is the permutation obtained from $\overline{\alpha}^{(i)}$ by inserting π_i before $\overline{\alpha}^{(i)}_{k_i+1}$ and i_1 is the smallest integer such that $\mathrm{des}(\overline{\alpha}^{(i_1-1)}) = \mathrm{des}(\overline{\alpha}^{(i_1)}) + 1$. We proceed to show that i_1 satisfies the conditions in Definition 4.7. Set $k_{i_1} = q$. Then $\sigma_1, \ldots, \sigma_q$ must appear before π_{i_1} in $\overline{\alpha}^{(i_1-1)}$. Under the assumption that i_1 is the smallest integer such that

$\text{des}(\overline{\alpha}^{(i_1-1)}) = \text{des}(\overline{\alpha}^{(i_1)}) + 1$, we derive that $\text{des}(\overline{\alpha}^{(i_1-1)}) = \text{des}(\overline{\alpha}) = k$, and so $\text{des}(\overline{\alpha}^{(i_1)}) = k-1$. Consequently, $\text{maj}(\overline{\alpha}^{(i_1-1)}) - \text{maj}(\overline{\alpha}^{(i_1)}) \geq k$. In particular, we deduce that if $\text{maj}(\overline{\alpha}^{(i_1-1)}) - \text{maj}(\overline{\alpha}^{(i_1)}) = k$, then $\sigma_1 > \cdots > \sigma_q > \pi_{i_1}$ and either $\sigma_q < \overline{\alpha}_{q+1}^{(i_1-1)}$ or $\pi_{i_1} > \overline{\alpha}_{q+1}^{(i_1)}$. From the proofs of Lemma 3.5 and Lemma 3.7, we find that $\lambda_{k-r} = t(i_1) = \text{maj}(\overline{\alpha}^{(i_1-1)}) - \text{maj}(\overline{\alpha}^{(i_1)}) - d_{i_1}(\pi)$. Since $\lambda_{k-r} = k-s$, we see that

$$\text{maj}(\overline{\alpha}^{(i_1-1)}) - \text{maj}(\overline{\alpha}^{(i_1)}) - d_{i_1}(\pi) = k-s. \tag{4.14}$$

Observe that $\text{maj}(\overline{\alpha}^{(i_1-1)}) - \text{maj}(\overline{\alpha}^{(i_1)}) \geq k$ and $d_{i_1}(\pi) \leq s$, in order for (4.14) to be valid, it's necessary that

$$\text{maj}(\overline{\alpha}^{(i_1-1)}) - \text{maj}(\overline{\alpha}^{(i_1)}) = k \quad \text{and} \quad d_{i_1}(\pi) = s.$$

Hence we conclude that i_1 satisfies the condition in Definition 4.7, and so $\overline{\alpha} \in \mathfrak{S}^{sa}(\sigma, \pi)$. This completes the proof. ∎

We conclude this paper with the proof of Theorem 4.11. In light of (3.2) and (4.1), together with Theorem 3.1, it is necessary to prove the following combinatorial statement.

Theorem 4.17. *Assume that $\sigma \in \mathfrak{S}_m$ and $\pi \in \mathfrak{S}_n$ are two disjoint permutations, where $\text{des}(\sigma) = r$ and $\text{des}(\pi) = s$. Then the map Φ given in Definition 3.2 is a bijection between $\mathfrak{S}^{lb}(\sigma, \pi|k)$ and $\mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0, = k-s)$.*

Proof. Let $\alpha \in \mathfrak{S}^{lb}(\sigma, \pi|k)$. Assume that $\Phi(\alpha) = (\lambda, \mu)$. From Lemma 3.5, we see that $\lambda \in \mathcal{P}_{k-r}(k-s, m)$ and $\mu \in \mathcal{P}_{n-k+r}(0,\ k-s)$, namely,

$$m \geq \lambda_1 \geq \cdots \geq \lambda_{k-r} \geq k-s \geq \mu_1 \geq \cdots \geq \mu_{n-k+r} \geq 0.$$

We proceed to show that $\mu_1 = k-s$. Since $\alpha \in \mathfrak{S}^{lb}(\sigma, \pi|k)$, there exists $1 \leq j_1 \leq \text{sa}(\pi)$ (where $\text{sa}(\pi)$ is the smallest ascent of π) satisfying the conditions in Definition 4.10. Recall that $\alpha^{(i)}$ is the permutation obtained by removing $\pi_1, \ldots, \pi_i$ from α. From Definition 4.10, we see that $\text{des}(\alpha^{(j_1-1)}) = \text{des}(\alpha^{(j_1)})$, $d_{j_1}(\pi) = s - j_1 + 1$. Moreover, the elements before π_j in $\alpha^{(j_1-1)}$ are from σ, that is, $\sigma_1 < \cdots < \sigma_q$. Since j_1 is the smallest integer satisfying the conditions in Definition 4.10, it follows that for $a < j_1$, $\text{des}(\alpha^{(a-1)}) = \text{des}(\alpha^{(a)}) + 1$. Hence we conclude that $\text{des}(\alpha^{(j_1-1)}) = \text{des}(\alpha^{(0)}) - j_1 + 1 = k - j_1 + 1$, and by $\text{des}(\alpha^{(j_1-1)}) = \text{des}(\alpha^{(j_1)})$, we deduce that $\text{des}(\alpha^{(j_1)}) = k - j_1 + 1$. Under the condition that the elements before π_{j_1} in $\alpha^{(j_1-1)}$ are from σ, that is, $\sigma_1 < \cdots < \sigma_q$, we deduce that $\text{maj}(\alpha^{(j_1-1)}) - \text{maj}(\alpha^{(j_1)}) = k - j_1 + 1$. Moreover, by the definition of Φ, we see that

$$\mu_1 = t(j_1) = \text{maj}(\alpha^{(j_1-1)}) - \text{maj}(\alpha^{(j_1)}) - d_{j_1}(\pi) = k - j_1 + 1 - (s - j_1 + 1) = k - s.$$

It yields that $\mu \in \mathcal{P}_{n-k+r}(0, = k-s)$.

Conversely, let $\lambda \in \mathcal{P}_{k-r}(k-s, m)$ and $\mu \in \mathcal{P}_{n-k+r}(0, = k-s)$. We next show that the map Ψ given in Definition 3.6 is also a map from $\mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0, = k-s)$ to $\mathfrak{S}^{lb}(\sigma, \pi|k)$. Assume that $\Psi(\lambda, \mu) = \overline{\alpha}$. First, we can use Lemma 3.7 to derive that $\overline{\alpha} = \overline{\alpha}_1 \cdots \overline{\alpha}_{n+m}$ is a shuffle of σ and π with k descents. Assume that $\overline{\alpha}^{(i-1)}$ is the permutation obtained from $\overline{\alpha}^{(i)}$ by inserting π_i before $\overline{\alpha}^{(i)}_{k_i+1}$ and j_1 is the smallest integer such that $\mathrm{des}(\overline{\alpha}^{(j_1-1)}) = \mathrm{des}(\overline{\alpha}^{(j_1)})$. We aim to show that j_1 satisfies the conditions in Definition 4.10. Set $k_{j_1} = q$, it means that $\sigma_1, \ldots, \sigma_q$ should appear before π_{j_1} in $\overline{\alpha}^{(j_1-1)}$. Under the assumption that j_1 is the smallest integer such that $\mathrm{des}(\overline{\alpha}^{(j_1-1)}) = \mathrm{des}(\overline{\alpha}^{(j_1)})$, we derive that $\mathrm{des}(\overline{\alpha}^{(j_1-1)}) = k - j_1 + 1$. Consequently, $\mathrm{maj}(\overline{\alpha}^{(j_1-1)}) - \mathrm{maj}(\overline{\alpha}^{(j_1)}) \leq k - j_1 + 1$. In particular, we deduce that if $\mathrm{maj}(\overline{\alpha}^{(j_1-1)}) - \mathrm{maj}(\overline{\alpha}^{(j_1)}) = k - j_1 + 1$, then $\sigma_1 < \cdots < \sigma_q < \pi_{j_1}$ and either $\sigma_q > \overline{\alpha}^{(j_1)}_{q+1}$ or $\pi_{j_1} < \overline{\alpha}^{(j_1)}_{q+1}$. From the proofs of Lemma 3.5 and Lemma 3.7, we find that $\mu_1 = t(j_1) = \mathrm{maj}(\overline{\alpha}^{(j_1-1)}) - \mathrm{maj}(\overline{\alpha}^{(j_1)}) - d_{j_1}(\pi)$. Since $\mu_1 = k - s$, we see that

$$\mathrm{maj}(\overline{\alpha}^{(j_1-1)}) - \mathrm{maj}(\overline{\alpha}^{(j_1)}) - d_{j_1}(\pi) = k - s. \tag{4.15}$$

Since $\mathrm{maj}(\overline{\alpha}^{(j_1-1)}) - \mathrm{maj}(\overline{\alpha}^{(j_1)}) \leq k - j_1 + 1$ and $d_{j_1}(\pi) \geq s - j_1 + 1$, we find that (4.15) holds if and only if

$$\mathrm{maj}(\overline{\alpha}^{(j_1-1)}) - \mathrm{maj}(\overline{\alpha}^{(j_1)}) = k - j_1 + 1 \quad \text{and} \quad d_{j_1}(\pi) = s - j_1 + 1.$$

Hence we conclude that j_1 satisfies the condition in Definition 4.10, and so $\overline{\alpha} \in \mathfrak{S}^{lb}(\sigma, \pi)$. This completes the proof. ∎

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References

[1] R.M. Adin, I.M. Gessel, V. Reiner, and Y. Roichman, Cyclic quasi-symmetric functions, Israel J. Math. 243 (2021) 437–500.

[2] G.E. Andrews, The Theory of Partitions, Addison-Wesley Publishing Co., 1976.

[3] D. Baker-Jarvis, B.E. Sagan, Bijective proofs of shuffle compatibility results, Adv. Appl. Math. 113 (2020) 101973.

[4] R. Domagalski, J. Liang, Q. Minnich and B.E. Sagan, J. Schmidt, A. Sietsema, Cyclic shuffle compatibility, Sém. Lothar. Combin. 85 ([2020C2021]), Art. B85d, 11 pp.

[5] D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc. 19 (1968) 236–240.

[6] A. M. Garsia and I. M. Gessel, Permutation statistics and partitions, Adv. in Math. 31 (1979) 288–305.

[7] I.M. Gessel and Y. Zhuang, Shuffle-compatible permutation statistics, Adv. Math. 332 (2018) 85–141.

[8] D. Grinberg, Shuffle-compatible permutation statistics II: the exterior peak set, Electron. J. Combin. 25 (2018) #P4.17.

[9] I. P. Goulden, A bijective proof of Stanley's shuffling theorem, Trans. Amer. Math. Soc. 288 (1985) 147C-160.

[10] J. Haglund, N. Loehr and J.B. Remmel, Statistics on wreath products, perfect matchings, and signed words, European J. Combin. 26 (2005) 835–868.

[11] Kathy Q. Ji and Dax T.X. Zhang, A cyclic analogue of Stanley's shuffling theorem, Electron. J. Combin., 29 (2022) P4.20.

[12] P.A. MacMahon, Collected Papers. Vol. I, MIT Press, Cambridge, MA, 1978.

[13] M. Novick, A bijective proof of a major index theorem of Garsia and Gessel, Electron. J. Combin. 17 (2010), Research Paper 64, 12 pp.

[14] B.E. Sagan and C.D. Savage, Mahonian pairs, J. Combin. Theory Ser. A 119 (2012) 526–545.

[15] J. D. Stadler, Stanley's shuffling theorem revisited, J. Combin. Theory Ser. A 88 (1999) 176–187.

[16] R. P. Stanley, Ordered structures and partitions, Memoirs of the American Mathematical Society, No. 119. American Mathematical Society, Providence, R.I., 1972. iii+104 pp.

[17] R.P. Stanley, Eumerative Combinatorics, Vol. I, 2nd ed., Cambridge University Press, Cambridge, 2012.

[18] L. Yang and S.H.F. Yan, On a conjecture concerning shuffle-compatible permutation statistics, Electron. J. Combin. 29 (2022) #P3.3.