Nearly Equal Distributions of the Rank and the Crank of Partitions

William Y.C. Chen, Kathy Q. Ji and Wenston J.T. Zang

Dedicated to Professor Krishna Alladi on the occasion of his sixtieth birthday

Abstract Let $N(\leq m,n)$ denote the number of partitions of n with rank not greater than m, and let $M(\leq m,n)$ denote the number of partitions of n with crank not greater than m. Bringmann and Mahlburg observed that $N(\leq m,n) \leq M(\leq m,n) \leq N(\leq m+1,n)$ for $m<0$ and $1 \leq n \leq 100$. They also pointed out that these inequalities can be restated as the existence of a reordering τ_n on the set of partitions of n such that $|\ \mathrm{crank}(\lambda)\ | - |\ \mathrm{rank}(\tau_n(\lambda))\ | = 0$ or 1 for all partitions λ of n, that is, the rank and the crank are nearly equal distributions over partitions of n. In the study of the spt-function, Andrews, Dyson, and Rhoades proposed a conjecture on the unimodality of the spt-crank, and they showed that it is equivalent to the inequality $N(\leq m,n) \leq M(\leq m,n)$ for $m<0$ and $n \geq 1$. We proved this conjecture by combinatorial arguments. In this paper, we show that the inequality $M(\leq m,n) \leq N(\leq m+1,n)$ is true for $m<0$ and $n \geq 1$. Furthermore, we provide a description of such a reordering τ_n and show that it leads to nearly equal distributions of the rank and the crank. Using this reordering, we give an interpretation of the function $\mathrm{ospt}(n)$ defined by Andrews, Chan, and Kim, which yields an upper bound for $\mathrm{ospt}(n)$ due to Chan and Mao.

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Keywords: Partition · Rank · Crank · Andrews' spt-function · ospt-Function

2010 Mathematics Subject Classification. 11P81, 05A17, 05A20

1 Introduction

The objective of this paper is to confirm an observation of Bringmann and Mahlburg [9] on the nearly equal distributions of the rank and the crank of partitions. Recall that the rank of a partition was introduced by Dyson [12] as the largest part minus the number of parts. The crank of a partition was defined by Andrews and Garvan [5] as the largest part if the partition contains no ones, and otherwise as the number of parts larger than the number of ones minus the number of ones.

Let m be an integer. For $n \geq 1$, let $N(m,n)$ denote the number of partitions of n with rank m, and for $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m. For $n = 1$, set

$$M(0,1) = -1,\ M(1,1) = M(-1,1) = 1,$$

and for $n = 1$ and $m \neq -1, 0, 1$, set

$$M(m,1) = 0.$$

Define the rank and the crank cumulation functions by

$$N(\leq m,n) = \sum_{r \leq m} N(r,n), \tag{1.1}$$

and

$$M(\leq m,n) = \sum_{r \leq m} M(r,n). \tag{1.2}$$

Bringmann and Mahlburg [9] observed that for $m < 0$ and $1 \leq n \leq 100$,

$$N(\leq m,n) \leq M(\leq m,n) \leq N(\leq m+1,n). \tag{1.3}$$

For $m = -1$, an equivalent form of the inequality $N(\leq -1,n) \leq M(\leq -1,n)$ for $n \geq 1$ was conjectured by Kaavya [17]. Bringmann and Mahlburg [9] pointed out that this observation may also be stated as follows. For $1 \leq n \leq 100$, there must be some reordering τ_n of partitions λ of n such that

$$\mid \operatorname{crank}(\lambda) \mid - \mid \operatorname{rank}(\tau_n(\lambda)) \mid = 0 \text{ or } 1. \tag{1.4}$$

Moreover, they noticed that using (1.4), one can deduce the following inequality on the spt-function $\operatorname{spt}(n)$:

$$\mathrm{spt}(n) \leq \sqrt{2n}p(n), \tag{1.5}$$

where $\mathrm{spt}(n)$ is the spt-function defined by Andrews [2] as the total number of smallest parts in all partitions of n and $p(n)$ is the number of partitions of n.

In the study of the spt-crank, Andrews, Dyson, and Rhoades [4] conjectured that the sequence $\{N_S(m,n)\}_m$ is unimodal for $n \geq 1$, where $N_S(m,n)$ is the number of S-partitions of size n with spt-crank m, see Andrews, Garvan, and Liang [6]. They showed that this conjecture is equivalent to the inequality $N(\leq m,n) \leq M(\leq m,n)$ for $m < 0$ and $n \geq 1$. They obtained the following asymptotic formula for $M(\leq m,n) - N(\leq m,n)$, which implies that the inequality holds for fixed $m < 0$ and sufficiently large n.

Theorem 1.1 (Andrews, Dyson, and Rhoades). *For any given $m < 0$,*

$$M(\leq m,n) - N(\leq m,n) \sim -\frac{(1+2m)\pi^2}{96n}p(n) \quad as \quad n \to \infty. \tag{1.6}$$

By constructing a series of injections [11], we proved the conjecture of Andrews, Dyson, and Rhoades.

Theorem 1.2. *For $m < 0$ and $n \geq 1$,*

$$N(\leq m,n) \leq M(\leq m,n). \tag{1.7}$$

Mao [18] obtained an asymptotic formula for $N(\leq m+1,n) - M(\leq m,n)$, which implies that the inequality $M(\leq m,n) \leq N(\leq m+1,n)$ holds for any fixed $m < 0$ and sufficiently large n.

Theorem 1.3 (Mao). *For any given $m < 0$,*

$$N(\leq m+1,n) - M(\leq m,n) \sim \frac{\pi}{4\sqrt{6n}}p(n) \quad as \quad n \to \infty. \tag{1.8}$$

It turns out that our constructive approach in [11] can also be used to deduce the following assertion.

Theorem 1.4. *For $m < 0$ and $n \geq 1$,*

$$M(\leq m,n) \leq N(\leq m+1,n). \tag{1.9}$$

If we list the set of partitions of n in two ways, one by the ranks, and the other by the cranks, then we are led to a reordering τ_n of the partitions of n. Using the inequalities (1.3) for $m < 0$ and $n \geq 1$, we show that the rank and the crank are nearly equidistributed over partitions of n. Since there may be more than one partition with the same rank or crank, the aforementioned listings may not be unique. Nevertheless, this does not affect the required property of the reordering τ_n. It should be noted that the above description of τ_n relies on the two orderings of partitions of n, it would be interesting to find a definition of τ_n explicitly on a partition λ of n.

Theorem 1.5. *For $n \geq 1$, let τ_n be a reordering on the set of partitions of n as defined above. Then for any partition λ of n,*

$$\operatorname{crank}(\lambda) - \operatorname{rank}(\tau_{\mathrm{n}}(\lambda)) = \begin{cases} 0, & \text{if } \operatorname{crank}(\lambda) = 0, \\ 0 \ \text{or}\ 1, & \text{if } \operatorname{crank}(\lambda) > 0, \\ 0 \ \text{or}\ -1, & \text{if } \operatorname{crank}(\lambda) < 0. \end{cases} \tag{1.10}$$

Clearly, the above theorem implies relation (1.4). For example, for $n = 4$, the reordering τ_4 is illustrated in Table 1.

λ	crank(λ)	$\tau_4(\lambda)$	rank($\tau_4(\lambda)$)	crank(λ) − rank($\tau_4(\lambda)$)
(1, 1, 1, 1)	−4	(1,1,1,1)	−3	−1
(2,1,1)	−2	(2,1,1)	−1	−1
(3,1)	0	(2,2)	0	0
(2,2)	2	(3,1)	1	1
(4)	4	(4)	3	1

Table 1: The reordering τ_4

We find that the map τ_n is related to the function ospt(n) defined by Andrews, Chan, and Kim [3] as the difference between the first positive crank moment and the first positive rank moment, namely,

$$\operatorname{ospt}(n) = \sum_{m \geq 0} m M(m, n) - \sum_{m \geq 0} m N(m, n). \tag{1.11}$$

Andrews, Chan, and Kim [3] derived the following generating function of ospt(n).

Theorem 1.6 (Andrews, Chan, and Kim). *We have*

$$\sum_{n \geq 0} \operatorname{ospt}(n) q^n$$
$$= \frac{1}{(q;q)_\infty} \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+7i+5j+2}(1-q^{4i+2})(1-q^{4i+2j+3}) \right.$$

$$+ \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+1}(1-q^{2i+1})(1-q^{4i+2j+2}) \Bigg) .$$

Based on the above generating function, Andrews, Chan, and Kim [3] proved the positivity of $\mathrm{ospt}(n)$.

Theorem 1.7 (Andrews, Chan, and Kim). *For $n \geq 1$, $\mathrm{ospt}(n) > 0$.*

They also found a combinatorial interpretation of $\mathrm{ospt}(n)$ in terms of even strings and odd strings of a partition. The following theorem shows that the function $\mathrm{ospt}(n)$ is related to the reordering τ_n.

Theorem 1.8. *For $n > 1$, $\mathrm{ospt}(n)$ equals the number of partitions λ of n such that $\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1$.*

It can be seen that $\tau_n((n)) = (n)$ for $n > 1$, since the partition (n) has the largest rank and the largest crank among all partitions of n. It follows that $\mathrm{crank}((n)) - \mathrm{rank}(\tau_n((n))) = 1$ when $n > 1$. Thus Theorem 1.8 implies that $\mathrm{ospt}(n) > 0$ for $n > 1$.

The following upper bound for $\mathrm{ospt}(n)$ can be derived from Theorem 1.5 and Theorem 1.8.

Theorem 1.9. *For $n > 1$,*

$$\mathrm{ospt}(n) \leq \frac{p(n)}{2} - \frac{M(0,n)}{2}. \tag{1.12}$$

It is easily seen that $M(0,n) \geq 1$ for $n \geq 3$ since $\mathrm{crank}((n-1,1)) = 0$ when $n \geq 3$. Hence Theorem 1.9 implies the following inequality due to Chan and Mao [10]: For $n \geq 3$,

$$\mathrm{ospt}(n) < \frac{p(n)}{2}. \tag{1.13}$$

This paper is organized as follows. In Section 2, we give a combinatorial proof of Theorem 1.4 with the aid of m-Durfee rectangle symbols as introduced in [11]. In Section 3, we demonstrate that Theorem 1.5 follows from Theorem 1.4. Proofs of Theorem 1.8 and Theorem 1.9 are given in Section 4. For completeness, we include a derivation of inequality (1.5).

2 Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. To this end, we first reformulate the inequality $M(\leq m, n) \leq N(\leq m+1, n)$ for $m < 0$ and $n \geq 1$ in terms of the rank-set. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. Recall that the rank-set of λ introduced by Dyson [14] is the infinite sequence

$$[-\lambda_1, 1-\lambda_2, \ldots, j-\lambda_{j+1}, \ldots, \ell-1-\lambda_\ell, \ell, \ell+1, \ldots].$$

Let $q(m,n)$ denote the number of partitions λ of n such that m appears in the rank-set of λ. Dyson [14] established the following relation: For $n \geq 1$,

$$M(\leq m, n) = q(m, n), \tag{2.1}$$

see also Berkovich and Garvan [8, (3.5)].

Let $p(m,n)$ denote the number of partitions of n with rank at least m, namely,

$$p(m, n) = \sum_{r=m}^{\infty} N(r, n).$$

By establishing the relation

$$M(\leq m, n) - N(\leq m, n) = q(m, n) - p(-m, n), \tag{2.2}$$

for $m < 0$ and $n \geq 1$, we see that $M(\leq m,n) \geq N(\leq m,n)$ is equivalent to the inequality $q(m,n) \geq p(-m,n)$. This was justified by a number of injections in [11].

Similarly, to prove $N(\leq m+1,n) \geq M(\leq m,n)$ for $m < 0$ and $n \geq 1$, we need the following relation.

Theorem 2.1. *For $m < 0$ and $n \geq 1$,*

$$N(\leq m+1, n) - M(\leq m, n) = q(-m-1, n) - p(m+2, n). \tag{2.3}$$

Proof. Since

$$N(\leq m+1, n) = \sum_{r=-\infty}^{m+1} N(r, n)$$

and

$$p(m+2, n) = \sum_{r=m+2}^{\infty} N(r, n),$$

we get

$$N(\leq m+1, n) = \sum_{r=-\infty}^{\infty} N(r, n) - p(m+2, n). \tag{2.4}$$

In fact,

$$\sum_{r=-\infty}^{\infty} N(r, n) = p(n),$$

so that (2.4) takes the form

$$N(\leq m+1, n) = p(n) - p(m+2, n). \tag{2.5}$$

On the other hand, owing to the symmetry

$$M(m,n) = M(-m,n),$$

due to Dyson [14], (2.1) becomes

$$q(-m-1,n) = \sum_{r=m+1}^{\infty} M(r,n).$$

Hence

$$M(\leq m,n) = \sum_{r=-\infty}^{m} M(r,n) = \sum_{r=-\infty}^{\infty} M(r,n) - q(-m-1,n). \tag{2.6}$$

But

$$\sum_{r=-\infty}^{\infty} M(r,n) = p(n),$$

so we arrive at

$$M(\leq m,n) = p(n) - q(-m-1,n). \tag{2.7}$$

Subtracting (2.7) from (2.5) gives (2.3). This completes the proof. □

In view of Theorem 2.1, we see that Theorem 1.4 is equivalent to the following assertion.

Theorem 2.2. *For $m \geq 0$ and $n \geq 1$,*

$$q(m,n) \geq p(-m+1,n). \tag{2.8}$$

Let $P(-m+1,n)$ denote the set of partitions counted by $p(-m+1,n)$, that is, the set of partitions of n with rank at least $-m+1$, and let $Q(m,n)$ denote the set of partitions counted by $q(m,n)$, that is, the set of partitions λ of n such that m appears in the rank-set of λ. Then Theorem 2.2 can be interpreted as the existence of an injection Θ from the set $P(-m+1,n)$ to the set $Q(m,n)$ for $m \geq 0$ and $n \geq 1$.

In [11], we have constructed an injection Φ from the set $Q(m,n)$ to $P(-m,n)$ for $m \geq 0$ and $n \geq 1$. It turns out that the injection Θ in this paper is less involved than the injection Φ in [11]. More specifically, to construct the injection Φ, the set $Q(m,n)$ is divided into six disjoint subsets $Q_i(m,n)$ $(1 \leq i \leq 6)$ and the set $P(-m,n)$ is divided into eight disjoint subsets $P_i(-m,n)$ $(1 \leq i \leq 8)$. For $m \geq 1$, the injection Φ consists of six injections ϕ_i from the set $Q_i(m,n)$ to the set $P_i(-m,n)$, where $1 \leq i \leq 6$. When $m = 0$, the injection Φ requires considerations of more cases. For the purpose of this paper, the set $P(-m+1,n)$ will be divided into three disjoint subsets $P_i(-m+1,n)$ $(1 \leq i \leq 3)$ and the set $Q(m,n)$ will be divided into three disjoint subsets $Q_i(m,n)$ $(1 \leq i \leq 3)$. For $m \geq 0$, the injection Θ consists of three injections θ_1, θ_2 and θ_3, where θ_1 is the identity map, and for $i = 2, 3$, θ_i is an injection from $P_i(-m+1,n)$ to $Q_i(m,n)$.

To describe the injection Θ, we shall represent the partitions in $Q(m,n)$ and $P(-m+1,n)$ in terms of m-Durfee rectangle symbols. As a generalization of a Durfee symbol defined by Andrews [1], an m-Durfee rectangle symbol of a partition is defined in [11]. Let λ be a partition of n and let $\ell(\lambda)$ denote the number of parts of λ. The m-Durfee rectangle symbol of λ is defined as follows:

$$(\alpha,\beta)_{(m+j)\times j} = \begin{pmatrix} \alpha_1,\, \alpha_2,\, \ldots,\, \alpha_s \\ \beta_1,\, \beta_2,\, \ldots,\, \beta_t \end{pmatrix}_{(m+j)\times j}, \tag{2.9}$$

where $(m+j)\times j$ is the m-Durfee rectangle of the Ferrers diagram of λ and α consists of columns to the right of the m-Durfee rectangle and β consists of rows below the m-Durfee rectangle, see Fig. 1. For the partition $\lambda = (7,7,6,4,3,3,2,2,2)$, the 2-Durfee rectangle symbol of λ is

$$\begin{pmatrix} 4,\, 3,\, 3,\, 2 \\ 3,\, 2,\, 2,\, 2 \end{pmatrix}_{5\times 3}.$$

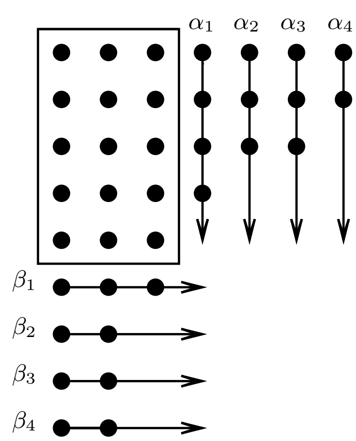

Fig. 1: The 2-Durfee rectangle representation of $(7,7,6,4,3,3,2,2,2)$.

Clearly, we have

$$m+j \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s, \quad j \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_t,$$

and

$$n = j(m+j) + \sum_{i=1}^{s} \alpha_i + \sum_{i=1}^{t} \beta_i.$$

When $m=0$, an m-Durfee rectangle symbol reduces to a Durfee symbol.

Notice that for a partition λ with $\ell(\lambda) \le m$, it has no m-Durfee rectangle. In this case, we adopt the convention that the m-Durfee rectangle has no columns, that is, $j = 0$, and so the m-Durfee rectangle symbol of λ is defined to be $(\lambda', \emptyset)_{m\times 0}$, where λ' is the conjugate of λ. For example, the 3-Durfee rectangle symbol of $\lambda = (5, 5, 1)$ is

$$\begin{pmatrix} 3,\ 2,\ 2,\ 2,\ 2 \\ \ \end{pmatrix}_{3\times 0}.$$

The partitions in $P(-m+1, n)$ can be characterized in terms of m-Durfee rectangle symbols.

Proposition 2.3. *Assume that $m \ge 0$ and $n \ge 1$. Let λ be a partition of n and let $(\alpha, \beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then the rank of λ is at least $-m+1$ if and only if either $j = 0$ or $j \ge 1$ and $\ell(\beta) + 1 \le \ell(\alpha)$.*

Proof. The proof is substantially the same as that of [11, Proposition 3.2]. Assume that the rank of λ is at least $-m + 1$. We are going to show that either $j = 0$ or $j \ge 1$ and $\ell(\beta) + 1 \le \ell(\alpha)$. There are two cases:
Case 1: $\ell(\lambda) \le m$. We have $j = 0$.
Case 2: $\ell(\lambda) \ge m+1$. We have $j \ge 1$, $\lambda_1 = j + \ell(\alpha)$ and $\ell(\lambda) = m + j + \ell(\beta)$. It follows that

$$\lambda_1 - \ell(\lambda) = (j + \ell(\alpha)) - (j + m + \ell(\beta)) = \ell(\alpha) - \ell(\beta) - m.$$

Under the assumption that $\lambda_1 - \ell(\lambda) \ge -m + 1$, we see that $\ell(\alpha) - \ell(\beta) \ge 1$, that is, $\ell(\beta) + 1 \le \ell(\alpha)$.

Conversely, we assume that $j = 0$ or $j \ge 1$ and $\ell(\beta)+1 \le \ell(\alpha)$. We proceed to show that the rank of λ is at least $-m + 1$. There are two cases:
Case 1: $j = 0$. Clearly, $\ell(\lambda) \le m$, which implies that the rank of λ is at least $-m + 1$.
Case 2: $j \ge 1$ and $\ell(\beta) + 1 \le \ell(\alpha)$. Note that $\lambda_1 = j + \ell(\alpha)$ and $\ell(\lambda) = j + m + \ell(\beta)$. It follows that

$$\lambda_1 - \ell(\lambda) = (j + \ell(\alpha)) - (j + m + \ell(\beta)) = -m + \ell(\alpha) - \ell(\beta). \qquad (2.10)$$

Under the assumption that $\ell(\alpha) - \ell(\beta) \ge 1$, (2.10) implies that $\lambda_1 - \ell(\lambda) \ge -m + 1$. This completes the proof. ☐

The following proposition will be used to describe the partitions in $Q(m, n)$ in terms of m-Durfee rectangle symbols.

Proposition 2.4. [11, Proposition 3.1] *Assume that $m \ge 0$ and $n \ge 1$. Let λ be a partition of n and let $(\alpha, \beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then m appears in the rank-set of λ if and only if either $j = 0$ or $j \ge 1$ and $\beta_1 = j$.*

If no confusion arises, we do not distinguish a partition λ and its m-Durfee rectangle symbol representation. We shall divide the set of the m-Durfee rectangle symbols $(\alpha,\beta)_{(m+j)\times j}$ in $P(-m+1,n)$ into three disjoint subsets $P_1(-m+1,n)$, $P_2(-m+1,n)$ and $P_3(-m+1,n)$. More precisely,

(1) $P_1(-m+1,n)$ is the set of m-Durfee rectangle symbols $(\alpha,\beta)_{(m+j)\times j}$ in $P(-m+1,n)$ for which either of the following conditions holds:
(i) $j=0$;
(ii) $j\geq 1$ and $\beta_1=j$;
(2) $P_2(-m+1,n)$ is the set of m-Durfee rectangle symbols $(\alpha,\beta)_{(m+j)\times j}$ in $P(-m+1,n)$ such that $j\geq 1$ and $\beta_1=j-1$;
(3) $P_3(-m+1,n)$ is the set of m-Durfee rectangle symbols $(\alpha,\beta)_{(m+j)\times j}$ in $P(-m+1,n)$ such that $j\geq 2$ and $\beta_1\leq j-2$.

The set $Q(m,n)$ will be divided into the following three subsets $Q_1(m,n)$, $Q_2(m,n)$ and $Q_3(m,n)$:

(1) $Q_1(m,n)$ is the set of m-Durfee rectangle symbols $(\gamma,\delta)_{(m+j')\times j'}$ in $Q(m,n)$ such that either of the following conditions holds:
(i) $j'=0$;
(ii) $j'\geq 1$ and $\ell(\delta)-\ell(\gamma)\leq -1$;
(2) $Q_2(m,n)$ is the set of m-Durfee rectangle symbols $(\gamma,\delta)_{(m+j')\times j'}$ in $Q(m,n)$ such that $j'\geq 1$, $\ell(\delta)-\ell(\gamma)\geq 0$ and $\gamma_1<m+j'$;
(3) $Q_3(m,n)$ is the set of m-Durfee rectangle symbols $(\gamma,\delta)_{(m+j')\times j'}$ in $Q(m,n)$ such that $j'\geq 1$, $\ell(\delta)-\ell(\gamma)\geq 0$ and $\gamma_1=m+j'$.

We are now ready to define the injections θ_i from the set $P_i(-m+1,n)$ to the set $Q_i(m,n)$, where $1\leq i\leq 3$. Since $P_1(-m+1,n)$ coincides with $Q_1(m,n)$, we set θ_1 to be the identity map. The following lemma gives an injection θ_2 from $P_2(-m+1,n)$ to $Q_2(m,n)$.

Lemma 2.5. *For $m\geq 0$ and $n>1$, there is an injection θ_2 from $P_2(-m+1,n)$ to $Q_2(m,n)$.*

Proof. To define the map θ_2, let

$$\lambda=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}_{(m+j)\times j}=\begin{pmatrix}\alpha_1,\,\alpha_2,\,\ldots,\,\alpha_s\\ \beta_1,\,\beta_2,\,\ldots,\,\beta_t\end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $P_2(-m+1,n)$. From the definition of $P_2(-m+1,n)$, we see that $s-t\geq 1$, $j\geq 1$, $\alpha_1\leq m+j$ and $\beta_1=j-1$.

Set

$$\theta_2(\lambda)=\begin{pmatrix}\gamma\\ \delta\end{pmatrix}_{(m+j')\times j'}=\begin{pmatrix}\alpha_1-1,\,\alpha_2-1,\,\ldots,\,\alpha_s-1\\ \beta_1+1,\,\beta_2+1,\,\ldots,\,\beta_t+1,\,1^{s-t}\end{pmatrix}_{(m+j)\times j}.$$

Clearly, $\theta_2(\lambda)$ is an m-Durfee rectangle symbol of n. Furthermore, $j'=j$, $\ell(\delta)-\ell(\gamma)\geq 0$. Since $\alpha_1\leq m+j$, we see that $\gamma_1=\alpha_1-1\leq m+j-1<m+j'$.

Noting that $\beta_1 = j-1$, we get $\delta_1 = \beta_1 + 1 = j = j'$. Moreover, $\delta_s = 1$ since $s - t \geq 1$. This proves that $\theta_2(\lambda)$ is in $Q_2(m,n)$.

To prove that θ_2 is an injection, define

$$H(m,n) = \{\theta_2(\lambda)\colon \lambda \in P_2(-m+1,n)\}.$$

Let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \gamma_1,\, \gamma_2,\, \ldots,\, \gamma_{s'} \\ \delta_1,\, \delta_2,\, \ldots,\, \delta_{t'} \end{pmatrix}_{(m+j')\times j'}$$

be an m-Durfee rectangle symbol in $H(m,n)$. Since $\mu \in Q_2(m,n)$, we have $t' \geq s'$, $\gamma_1 < m + j'$ and $\delta_1 = j'$. According to the construction of θ_2, $\delta_{t'} = 1$. Define

$$\sigma(\mu) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \gamma_1+1,\, \gamma_2+1,\, \ldots,\, \gamma_{s'}+1,\, 1^{t'-s'} \\ \delta_1-1,\, \delta_2-1,\, \ldots,\, \delta_{t'}-1 \end{pmatrix}_{(m+j')\times j'}.$$

Clearly, $\ell(\beta) < t'$ since $\delta_{t'} = 1$, so that $\ell(\alpha) - \ell(\beta) \geq 1$. Moreover, since $\delta_1 = j'$ and $j' = j$, we see that $\beta_1 = \delta_1 - 1 = j' - 1 = j - 1$. It is easily checked that $\sigma(\theta_2(\lambda)) = \lambda$ for any λ in $P_2(-m+1,n)$. Hence the map θ_2 is an injection from $P_2(-m+1,n)$ to $Q_2(m,n)$. This completes the proof. □

For example, for $m = 2$ and $n = 35$, consider the following 2-Durfee rectangle symbol in $P_2(-1,35)$:

$$\lambda = \begin{pmatrix} 5,\, 5,\, 3,\, 1,\, 1 \\ 2,\, 2,\, 1 \end{pmatrix}_{5\times 3}.$$

Applying the injection θ_2 to λ, we obtain

$$\mu = \theta_2(\lambda) = \begin{pmatrix} 4,\, 4,\, 2 \\ 3,\, 3,\, 2,\, 1,\, 1 \end{pmatrix}_{5\times 3},$$

which is in $Q_2(2,35)$. Applying σ to μ, we recover λ.

The following lemma gives an injection θ_3 from $P_3(-m+1,n)$ to $Q_3(m,n)$.

Lemma 2.6. *For $m \geq 0$ and $n > 1$, there is an injection θ_3 from $P_3(-m+1,n)$ to $Q_3(m,n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1,\, \alpha_2,\, \ldots,\, \alpha_s \\ \beta_1,\, \beta_2,\, \ldots,\, \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $P_3(-m+1,n)$. By definition, $s - t \geq 1$, $j \geq 2$ and $\beta_1 \leq j - 2$.

Define

$$\begin{aligned}\theta_3(\lambda) &= \begin{pmatrix}\gamma\\ \delta\end{pmatrix}_{(m+j')\times j'}\\ &= \begin{pmatrix} m+j-1,\, \alpha_1-1,\, \alpha_2-1,\, \ldots,\, \alpha_s-1 \\ j-1, \quad \beta_1+1,\, \beta_2+1,\, \ldots,\, \beta_t+1\ ,1^{s-t+1}\end{pmatrix}_{(m+j-1)\times(j-1)}.\end{aligned}$$

Evidently, $\ell(\delta) = s+2$ and $\ell(\gamma) \leq s+1$, and so $\ell(\delta) - \ell(\gamma) \geq 1$. Moreover, we have $\gamma_1 = m+j-1 = m+j'$, $\delta_1 = j-1 = j'$ and

$$\begin{aligned} j'(m+j') &+ \sum_{i=1}^{s+1}\gamma_i + \sum_{i=1}^{s+2}\delta_i \\ &= (m+j-1)(j-1) + \left(m+j-1+\sum_{i=1}^{s}(\alpha_i-1)\right)\\ &\quad + \left(j-1+s-t+1+\sum_{i=1}^{t}(\beta_i+1)\right)\\ &= j(m+j) + \sum_{i=1}^{s}\alpha_i + \sum_{i=1}^{t}\beta_i = n.\end{aligned}$$

This yields that $\theta_3(\lambda)$ is in $Q_3(m,n)$. In particular, since $s-t \geq 1$, we see that

$$\delta_{s+2} = \delta_{s+1} = 1. \tag{2.11}$$

To prove that the map θ_3 is an injection, define

$$I(m,n) = \{\theta_3(\lambda)\colon \lambda \in P_3(-m+1,n)\}.$$

Let

$$\mu = \begin{pmatrix}\gamma\\ \delta\end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix}\gamma_1,\, \gamma_2,\, \ldots,\, \gamma_{s'}\\ \delta_1,\, \delta_2,\, \ldots,\, \delta_{t'}\end{pmatrix}_{(m+j')\times j'}$$

be an m-Durfee rectangle symbol in $I(m,n)$. Since $\mu \in Q_3(m,n)$, we have $t' \geq s'$, $\gamma_1 = m+j'$ and $\delta_1 = j'$. By the construction of θ_3, $t'-s' \geq 1$. Define

$$\pi(\mu) = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}_{(m+j)\times j} = \begin{pmatrix}\gamma_2+1,\, \ldots,\, \gamma_{s'}+1,\, 1^{t'-s'-1}\\ \delta_2-1,\, \ldots,\, \delta_{t'}-1\end{pmatrix}_{(m+j'+1)\times(j'+1)}.$$

It follows from (2.11) that $\ell(\beta) \leq t'-3$ and $\ell(\alpha) = t'-2$. Therefore, $\ell(\alpha) \geq \ell(\beta)+1$ and $\beta_1 = \delta_2 - 1 \leq j'-1 = j-2$, so that $\pi(\mu)$ is in $P_3(-m+1,n)$. Moreover, it can be checked that $\pi(\theta_3(\lambda)) = \lambda$ for any λ in $P_3(-m+1,n)$.

This proves that the map θ_3 is an injection from $P_3(-m+1,n)$ to $Q_3(m,n)$. $\square$

For example, for $m = 3$ and $n = 63$, consider the following 3-Durfee rectangle symbol in $P_3(-2,63)$:

$$\lambda = \begin{pmatrix} 7,\ 7,\ 4,\ 3,\ 3,\ 2,\ 1 \\ 2,\ 2,\ 2,\ 1,\ 1 \end{pmatrix}_{7\times 4}.$$

Applying the injection θ_3 to λ, we obtain

$$\mu = \theta_3(\lambda) = \begin{pmatrix} 6,\ 6,\ 6,\ 3,\ 2,\ 2,\ 1 \\ 3,\ 3,\ 3,\ 3,\ 2,\ 2,\ 1,\ 1,\ 1 \end{pmatrix}_{6\times 3},$$

which is in $Q_3(3,63)$. Applying π to μ, we recover λ.

Combining the bijection θ_1 and the injections θ_2 and θ_3, we are led to an injection Θ from $P(-m+1,n)$ to $Q(m,n)$, and hence the proof of Theorem 2.2 is complete. More precisely, for a partition λ,

$$\Theta(\lambda) = \begin{cases} \theta_1(\lambda), & \text{if} \quad \lambda \in P_1(-m+1,n), \\ \theta_2(\lambda), & \text{if} \quad \lambda \in P_2(-m+1,n), \\ \theta_3(\lambda), & \text{if} \quad \lambda \in P_3(-m+1,n). \end{cases}$$

3 Proof of Theorem 1.5

In this section, we show that it is indeed the case that the reordering τ_n leads to the nearly equal distributions of the rank and the crank, with the aid of the inequalities in Theorem 1.2 and Theorem 1.4. For the sake of presentation, the inequalities in Theorem 1.2 and Theorem 1.4 for $m < 0$ can be recast for $m \geq 0$.

Theorem 3.1. *For $m \geq 0$ and $n \geq 1$,*

$$N(\leq m,n) \geq M(\leq m,n) \geq N(\leq m-1,n). \tag{3.1}$$

To see that the inequalities (3.1) for $m \geq 0$ can be derived from (1.7) in Theorem 1.2 and (1.9) in Theorem 1.4 for $m < 0$, we assume that $m \geq 0$, so that (1.7) and (1.9) can be stated as follows:

$$N(\leq -m-1,n) \leq M(\leq -m-1,n) \leq N(\leq -m,n), \tag{3.2}$$

and hence

$$p(n)-N(\leq -m-1,n) \geq p(n)-M(\leq -m-1,n) \geq p(n)-N(\leq -m,n). \quad (3.3)$$

It follows that

$$\sum_{r=-m}^{\infty} N(r,n) \geq \sum_{r=-m}^{\infty} M(r,n) \geq \sum_{r=-m+1}^{\infty} N(r,n). \quad (3.4)$$

Now, by the symmetry $N(m,n) = N(-m,n)$, see [13], we have

$$\sum_{r=-m}^{\infty} N(r,n) = N(\leq m,n) \quad \text{and} \quad \sum_{r=-m+1}^{\infty} N(r,n) = N(\leq m-1,n). \quad (3.5)$$

Similarly, the symmetry $M(m,n) = M(-m,n)$, see [14], leads to

$$\sum_{r=-m}^{\infty} M(r,n) = M(\leq m,n). \quad (3.6)$$

Substituting (3.5) and (3.6) into (3.4), we obtain (3.1). Conversely, one can reverse the above steps to derive (1.7) and (1.9) for $m < 0$ from (3.1) for $m \geq 0$. This means that the inequalities (3.1) for $m \geq 0$ are equivalent to the inequalities (1.7) and (1.9) for $m < 0$.

We can now prove Theorem 1.5.

Proof of Theorem 1.5. Let λ be a partition of n, and let $\tau_n(\lambda) = \mu$. Suppose that λ is the i-th partition of n when the partitions of n are listed in the increasing order of cranks used in the definition of τ_n. Meanwhile, μ is also the i-th partition in the list of partitions of n in the increasing order of ranks used in the definition of τ_n. Let $\text{crank}(\lambda) = a$ and $\text{rank}(\mu) = b$, so that

$$M(\leq a,n) \geq i > M(\leq a-1,n), \quad (3.7)$$

and

$$N(\leq b,n) \geq i > N(\leq b-1,n). \quad (3.8)$$

We now consider three cases:

Case 1: $a = 0$. We aim to show that $b = 0$. Assume to the contrary that $b \neq 0$. There are two subcases:

Subcase 1.1: $b < 0$. From (3.7) and (3.8), we have

$$N(\leq -1,n) \geq N(\leq b,n) \geq i > M(\leq -1,n),$$

which contradicts the inequality $N(\leq m,n) \leq M(\leq m,n)$ in Theorem 1.2 with $m = -1$.

Subcase 1.2: $b > 0$. From (3.7) and (3.8), we see that

$$M(\leq 0,n) \geq i > N(\leq b-1,n) \geq N(\leq 0,n),$$

which contradicts the inequality $M(\leq m, n) \leq N(\leq m, n)$ in (3.1) with $m = 0$. This completes the proof of Case 1.
Case 2: $a < 0$. We proceed to show that $b = a$ or $a + 1$. By (3.7) and the inequality $M(\leq m, n) \leq N(\leq m + 1, n)$ in Theorem 1.4 with $m = a$, we see that

$$N(\leq a + 1, n) \geq i. \tag{3.9}$$

Combining (3.8) and (3.9), we deduce that

$$N(\leq a + 1, n) > N(\leq b - 1, n),$$

and thus

$$a + 1 \geq b. \tag{3.10}$$

On the other hand, by (3.7) and the inequality $N(\leq m, n) \leq M(\leq m, n)$ in Theorem 1.2 with $m = a - 1$, we find that

$$N(\leq a - 1, n) < i.$$

Together with (3.8), this gives

$$N(\leq a - 1, n) < N(\leq b, n),$$

so that $a \leq b$. In view of (3.10), we obtain that $b = a$ or $a+1$. This completes the proof of Case 2.
Case 3: $a > 0$. We claim that $b = a$ or $a - 1$. Combining the inequality $M(\leq m, n) \geq N(\leq m - 1, n)$ in (3.1) with $m = a - 1$ and the inequality $M(\leq a - 1, n) < i$ in (3.7), we get

$$N(\leq a - 2, n) < i. \tag{3.11}$$

By means of (3.8) and (3.11), we find that

$$N(\leq b, n) > N(\leq a - 2, n),$$

whence

$$a - 1 \leq b. \tag{3.12}$$

On the other hand, combining the inequality $N(\leq m, n) \geq M(\leq m, n)$ in (3.1) with $m = a$ and the inequality $M(\leq a, n) \geq i$ in (3.7), we are led to

$$N(\leq a, n) \geq i, \tag{3.13}$$

which together with (3.8) yields that

$$N(\leq a, n) > N(\leq b - 1, n),$$

and hence $a \geq b$. But it has been shown that $b \geq a - 1$, whence $b = a - 1$ or a. This completes the proof of Case 3. $\square$

4 Proofs of Theorem 1.8 and Theorem 1.9

In this section, we give a proof of Theorem 1.8 concerning an interpretation of the ospt-function in terms of the reordering of τ_n. Then we use Theorem 1.8 to deduce Theorem 1.9, which gives an upper bound for the ospt-function. Finally, for completeness, we include a derivation of (1.5) from (1.4).

Proof of Theorem 1.8. Let $\mathcal{P}(n)$ denote the set of partitions of n. By the definition (1.11) of $\operatorname{ospt}(n)$, we see that

$$\operatorname{ospt}(n) = \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{crank}(\lambda)>0}} \operatorname{crank}(\lambda) - \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{rank}(\lambda)>0}} \operatorname{rank}(\lambda). \tag{4.1}$$

We claim that

$$\sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{rank}(\lambda)>0}} \operatorname{rank}(\lambda) = \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{crank}(\lambda)>0}} \operatorname{rank}(\tau_n(\lambda)). \tag{4.2}$$

From Theorem 1.5, we see that if $\operatorname{crank}(\lambda) > 0$, then $\operatorname{rank}(\tau_n(\lambda)) \geq 0$. This implies that

$$\{\lambda \in \mathcal{P}(n)\colon \operatorname{crank}(\lambda) > 0\} \subseteq \{\lambda \in \mathcal{P}(n)\colon \operatorname{rank}(\tau_n(\lambda)) \geq 0\}. \tag{4.3}$$

Therefore,

$$\sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{crank}(\lambda)>0}} \operatorname{rank}(\tau_n(\lambda)) \leq \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{rank}(\tau_n(\lambda))\geq 0}} \operatorname{rank}(\tau_n(\lambda)). \tag{4.4}$$

From Theorem 1.5, we also see that if $\operatorname{crank}(\lambda) = 0$, then $\operatorname{rank}(\tau_n(\lambda)) = 0$, and if $\operatorname{crank}(\lambda) < 0$, then $\operatorname{rank}(\tau_n(\lambda)) \leq 0$. Now,

$$\{\lambda \in \mathcal{P}(n)\colon \operatorname{rank}(\tau_n(\lambda)) > 0\} \subseteq \{\lambda \in \mathcal{P}(n)\colon \operatorname{crank}(\lambda) > 0\}. \tag{4.5}$$

Hence by (4.3),

$$\sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{rank}(\tau_n(\lambda))>0}} \operatorname{rank}(\tau_n(\lambda)) \leq \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{crank}(\lambda)>0}} \operatorname{rank}(\tau_n(\lambda)). \tag{4.6}$$

Since

$$\sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{rank}(\tau_n(\lambda))\geq 0}} \operatorname{rank}(\tau_n(\lambda)) = \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{rank}(\tau_n(\lambda))>0}} \operatorname{rank}(\tau_n(\lambda)),$$

from (4.4) and (4.6), we infer that

$$\sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{rank}(\tau_n(\lambda))>0}} \operatorname{rank}(\tau_n(\lambda)) = \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \operatorname{crank}(\lambda)>0}} \operatorname{rank}(\tau_n(\lambda)). \tag{4.7}$$

But

$$\sum_{\substack{\lambda\in\mathcal{P}(n)\\ \mathrm{rank}(\tau_n(\lambda))>0}} \mathrm{rank}(\tau_n(\lambda)) = \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \mathrm{rank}(\lambda)>0}} \mathrm{rank}(\lambda), \tag{4.8}$$

thus we arrive at (4.2), and so the claim is justified.

Substituting (4.2) into (4.1), we get

$$\begin{aligned}\mathrm{ospt}(n) &= \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \mathrm{crank}(\lambda)>0}} \mathrm{crank}(\lambda) - \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \mathrm{crank}(\lambda)>0}} \mathrm{rank}(\tau_n(\lambda))\\ &= \sum_{\substack{\lambda\in\mathcal{P}(n)\\ \mathrm{crank}(\lambda)>0}} (\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda))). \end{aligned}\tag{4.9}$$

Appealing to Theorem 1.5, we see that if $\mathrm{crank}(\lambda) > 0$, then

$$\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 0 \quad \text{or} \quad 1.$$

By (4.9),

$$\mathrm{ospt}(n) = \#\{\lambda \in \mathcal{P}(n) \colon \mathrm{crank}(\lambda) > 0 \text{ and } \mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1\}. \tag{4.10}$$

Also, by Theorem 1.5, we see that if $\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1$, then $\mathrm{crank}(\lambda) > 0$. Consequently,

$$\mathrm{ospt}(n) = \#\{\lambda \in \mathcal{P}(n) \colon \mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1\}, \tag{4.11}$$

as required. □

Theorem 1.9 can be easily deduced from Theorem 1.5 and Theorem 1.8.

Proof of Theorem 1.9. From the symmetry $M(m,n) = M(-m,n)$, we see that

$$p(n) = \sum_{m=-\infty}^{\infty} M(m,n) = M(0,n) + 2\sum_{m\geq 1} M(m,n). \tag{4.12}$$

Hence

$$\sum_{m\geq 1} M(m,n) = \frac{p(n)}{2} - \frac{M(0,n)}{2}. \tag{4.13}$$

In virtue of Theorem 1.5, if $\mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1$, then $\mathrm{crank}(\lambda) > 0$, and hence

$$\#\{\lambda \in \mathcal{P}(n) \colon \mathrm{crank}(\lambda) - \mathrm{rank}(\tau_n(\lambda)) = 1\} \leq \#\{\lambda \in \mathcal{P}(n) \colon \mathrm{crank}(\lambda) > 0\}. \tag{4.14}$$

This, combined with Theorem 1.8, leads to

$$\mathrm{ospt}(n) \leq \#\{\lambda \in \mathcal{P}(n) \colon \mathrm{crank}(\lambda) > 0\} = \sum_{m\geq 1} M(m,n). \tag{4.15}$$

Substituting (4.13) into (4.15), we obtain that

$$\mathrm{ospt}(n) \leq \frac{p(n)}{2} - \frac{M(0,n)}{2},$$

as required. □

We conclude this paper with a derivation of inequality (1.5), that is, $\mathrm{spt}(n) \leq \sqrt{2n}p(n)$. Recall that the k-th moment $N_k(n)$ of ranks and the k-th moment $M_k(n)$ of cranks were defined by Atkin and Garvan [7] as follows:

$$N_k(n) = \sum_{m=-\infty}^{\infty} m^k N(m,n), \tag{4.16}$$

$$M_k(n) = \sum_{m=-\infty}^{\infty} m^k M(m,n). \tag{4.17}$$

Andrews [2] showed that the spt-function can be expressed in terms of the second moment $N_2(n)$ of ranks,

$$\mathrm{spt}(n) = np(n) - \frac{1}{2}N_2(n). \tag{4.18}$$

Employing the following relation due to Dyson [14],

$$M_2(n) = 2np(n), \tag{4.19}$$

Garvan [15] observed that the following expression

$$\mathrm{spt}(n) = \frac{1}{2}M_2(n) - \frac{1}{2}N_2(n), \tag{4.20}$$

implies that $M_2(n) > N_2(n)$ for $n \geq 1$. In general, he conjectured and later proved that $M_{2k}(n) > N_{2k}(n)$ for $k \geq 1$ and $n \geq 1$, see [16].

Bringmann and Mahlburg [9] pointed out that inequality (1.5) can be derived by combining the reordering τ_n and the Cauchy-Schwarz inequality. By (4.20), we see that

$$\begin{aligned} 2\,\mathrm{spt}(n) &= \sum_{m=-\infty}^{\infty} m^2 M(m,n) - \sum_{m=-\infty}^{\infty} m^2 N(m,n) \\ &= \sum_{\lambda\in\mathcal{P}(n)} \mathrm{crank}^2(\lambda) - \sum_{\lambda\in\mathcal{P}(n)} \mathrm{rank}^2(\lambda). \end{aligned} \tag{4.21}$$

Since

$$\sum_{\lambda\in\mathcal{P}(n)} \mathrm{rank}^2(\lambda) = \sum_{\lambda\in\mathcal{P}(n)} \mathrm{rank}^2(\tau_n(\lambda)),$$

(4.21) can be rewritten as

$$
\begin{aligned}
2\,\mathrm{spt}(n) &= \sum_{\lambda\in\mathcal{P}(n)} \left(\mathrm{crank}^2(\lambda) - \mathrm{rank}^2(\tau_n(\lambda))\right) \\
&= \sum_{\lambda\in\mathcal{P}(n)} (\mid \mathrm{crank}(\lambda) \mid - \mid \mathrm{rank}(\tau_n(\lambda)) \mid)\cdot(\mid \mathrm{crank}(\lambda) \mid + \mid \mathrm{rank}(\tau_n(\lambda)) \mid).
\end{aligned}
\tag{4.22}
$$

By (1.4), we find that

$$\mid \mathrm{crank}(\lambda) \mid + \mid \mathrm{rank}(\tau_n(\lambda)) \mid \leq 2 \mid \mathrm{crank}(\lambda) \mid$$

and

$$0 \leq \mid \mathrm{crank}(\lambda) \mid - \mid \mathrm{rank}(\tau_n(\lambda)) \mid \leq 1.$$

Thus (4.22) gives

$$\mathrm{spt}(n) \leq \sum_{\lambda\in\mathcal{P}(n)} \mid \mathrm{crank}(\lambda) \mid . \tag{4.23}$$

Applying the inequality on the arithmetic and quadratic means

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \tag{4.24}$$

for nonnegative real numbers to the numbers $\mid \mathrm{crank}(\lambda) \mid$, where λ ranges over partitions of n, we are led to

$$
\begin{aligned}
\frac{\sum_{\lambda\in\mathcal{P}(n)} \mid \mathrm{crank}(\lambda) \mid}{p(n)} &\leq \sqrt{\frac{\sum_{\lambda\in\mathcal{P}(n)} \mid \mathrm{crank}(\lambda) \mid^2}{p(n)}}. \\
&= \sqrt{\frac{M_2(n)}{p(n)}}.
\end{aligned}
\tag{4.25}
$$

In light of Dyson's identity (4.19), this becomes

$$\sum_{\lambda\in\mathcal{P}(n)} \mid \mathrm{crank}(\lambda) \mid \leq \sqrt{2n}p(n). \tag{4.26}$$

Combining (4.23) and (4.26) completes the proof. □

Acknowledgments. This work was supported by the National Science Foundation of China. We wish to thank Kathrin Bringmann, Karl Mahlburg and the referee for their valuable comments and suggestions.

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