Ranks of overpartitions modulo 6 and 10

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Abstract. Lovejoy and Osburn proved formulas for the generating functions for the rank differences of overpartitions modulo 3 and 5. In this paper, we derive formulas for the generating functions for the rank differences of overpartitions modulo 6 and 10. With these generating functions, we obtain some equalities and inequalities on ranks of overpartitions modulo 6 and 10. We also relate these generating functions to the third order mock theta functions $\omega(q)$ and $\rho(q)$ and the tenth order mock theta functions $\phi(q)$ and $\psi(q)$.

Keywords: Overpartition, Dyson's rank, rank difference, generalized η-function, modular function, mock theta function.

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1 Introduction

The rank of a partition introduced by Dyson [11] as the largest part of the partition minus the number of parts. Dyson [11] conjectured that this partition statistic provided combinatorial interpretations of Ramanujan's congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, where $p(n)$ is the number of partitions of n. More precisely, let $N(m,n)$ denote the number of partitions of n with rank m and let $N(s,\ell,n)$ denote the number of partitions of n with rank congruent to s modulo ℓ. Dyson conjectured

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leq k \leq 4, \tag{1.1}$$

$$N(k,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \leq k \leq 6. \tag{1.2}$$

These two assertions were confirmed by Atkin and Swinnerton-Dyer [5]. In fact, they established generating functions for every rank difference $N(s,\ell,\ell n+d) - N(t,\ell,\ell n+d)$ with $\ell = 5$ or 7 and for $0 \leq d,s,t < \ell$, many of which are in terms of infinite products and generalized Lambert series. Although Dyson's rank fails to explain Ramanujan's congruence $p(11n+6) \equiv 0 \pmod{11}$ combinatorially, the generating functions for the rank differences $N(s,\ell,\ell n+d) - N(t,\ell,\ell n+d)$ with $\ell = 11$ have also been determined by Atkin and Hussain [4]. Since then, the rank differences of partitions modulo other numbers have been extensively studied, see, for example, Lewis established the rank differences of partitions modulo 2 in [20] and the rank differences of partitions modulo 9 in [19]. Santa-Gadea [29] obtained other rank differences of partitions modulo 9 and some rank differences of partitions modulo 12. Recently, Mao [25] established the generating functions for the rank differences of partitions modulo 10.

Dyson's rank can be extended to overpartitions in the obvious way. Recall that an overpartition [10] is a partition in which the first occurrence of a part may be overlined. The rank of an overpartition is defined to be the largest part of an overpartition minus its number of parts. Similarly, let $\overline{N}(m,n)$ denote the number of overpartitions of n with rank m, and let $\overline{N}(s,\ell,n)$ denote the number of overpartitions of n with rank congruent to s modulo ℓ. Lovejoy [21] obtained the following generating function for $\overline{N}(m,n)$,

$$\overline{R}(z;q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{N}(m,n) z^m q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}. \tag{1.3}$$

Analogous to the rank of a partition, Lovejoy and Osburn [22] studied the rank differences $\overline{N}(s,\ell,\ell n+d) - \overline{N}(t,\ell,\ell n+d)$ with $\ell = 3$ or 5 for $0 \le d,s,t < \ell$. The rank differences with $\ell = 7$ have been recently determined by Jennings-Shaffer [18]. It has been shown in [9] that there are no congruences of the form $\overline{p}(\ell n+d) \equiv 0 \pmod{\ell}$ for primes $\ell \ge 3$. The generating functions for these rank differences provide a measure of the extent to which the rank fails to produce a congruence $\overline{p}(\ell n+d) \equiv 0 \pmod{\ell}$. On the other hand, as remarked by Jennings-Shaffer in [18], determining these three difference formulas is equivalent to determining the 3-dissection of $\overline{R}(\exp(2i\pi/3);q)$, the 5-dissection of $\overline{R}(\exp(2i\pi/5);q)$ and the 7-dissection of $\overline{R}(\exp(2i\pi/7);q)$.

In this paper, we will establish the generating functions for the rank differences of overpartitions modulo 6 and 10. To do so, we will consider the 3-dissection of $\overline{R}(\exp(i\pi/3);q)$ and the 5-dissection of $\overline{R}(\exp(i\pi/5);q)$. The main results are summarized in Theorems 1.1, 1.2 and 1.3 below, which are stated in terms of the rank differences of overpartitions. Here and in the sequel, we use the notation

$$\begin{aligned}
&(x_1,x_2,\ldots,x_k;q)_\infty := \prod_{n=0}^{\infty}(1-x_1q^n)(1-x_2q^n)\cdots(1-x_kq^n),\\
&j(z;q) := (z,q/z,q;q)_\infty, \quad J_{a,m} := j(q^a;q^m),\\
&J_m := (q^m;q^m)_\infty, \ \overline{J}_{a,m} := j(-q^a;q^m).
\end{aligned}$$

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,n) + \overline{N}(1,6,n) - \overline{N}(2,6,n) - \overline{N}(3,6,n))q^n$$

$$=\frac{J_{18}^3 J_{9,18}}{J_6 J_{3,18}^2}+q\frac{2J_{18}^3}{J_6 J_{3,18}}+q^2\left\{\frac{4J_{18}^3}{J_6 J_{9,18}}-\frac{2}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n}}{1+q^{9n+3}}\right\}. \tag{1.4}$$

Theorem 1.2. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n\\ =&2A_0+2q\left(A_1+\frac{q^5}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+10}}\right)\\ &+2q^2A_2+2q^3A_3+2q^4A_4,\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}A_0 &:= \frac{J_{10,50}^2 J_{15,50}^2 J_{25,50}^4}{2J_{5,10}^3 J_{20,50} J_{50}^3}+4q^{10}\frac{J_{5,50}J_{50}^3}{J_{5,10}^2 J_{20,50}},\\ A_1 &:= \frac{J_{20,50}J_{25,50}^4 J_{50}^3}{J_{5,10}^4 J_{10,50}^2 J_{15,50}}-4q^5\frac{J_{5,50}^4 J_{20,50}^3 J_{25,50}^4}{J_{5,10}^5 J_{10,50}^2 J_{50}^3}-8q^{15}\frac{J_{10,50}^4 J_{15,50}^2 J_{25,50} J_{50}^3}{J_{5,10}^4 J_{20,50}^5},\\ A_2 &:= \frac{J_{5,50}^3 J_{20,50}^3 J_{25,50}^5}{J_{5,10}^5 J_{10,50}^2 J_{50}^3}+4q^{10}\frac{J_{10,50}^5 J_{25,50} J_{50}^7}{J_5^4 J_{5,10} J_{5,50}^3 J_{20,50}^4}-16q^{10}\frac{J_{5,50}J_{15,50}^2 J_{50}^3}{J_{5,10}^4 J_{20,50}},\\ A_3 &:= \frac{2J_{10,50}^4 J_{15,50}^5 J_{25,50}^4}{J_{5,10}^5 J_{5,50} J_{20,50}^3 J_{50}^3}+2q^5\frac{J_{10,50}J_{25,50}^3 J_{50}^3}{J_{5,10}^4 J_{20,50}^2}-16q^5\frac{J_{15,50}^2 J_{25,50} J_{50}^3}{J_{5,10}^4 J_{20,50}}\\ &\quad+8q^{10}\frac{J_{5,50}^3 J_{20,50} J_{25,50} J_{50}^3}{J_{5,10}^4 J_{10,50}^2 J_{15,50}},\\ A_4 &:= \frac{4J_{10,50}^4 J_{15,50}^6 J_{25,50}^3}{J_{5,10}^5 J_{5,50} J_{20,50}^3 J_{50}^3}-\frac{J_{10,50}J_{25,50}^4 J_{50}^3}{J_{5,10}^4 J_{5,50} J_{20,50}^2}-16q^5\frac{J_{15,50}^3 J_{50}^3}{J_{5,10}^4 J_{20,50}}\\ &\quad-8q^5\frac{J_{25}J_{50}^5}{J_5 J_{5,10}^3 J_{10,50}}.\end{aligned}$$

Theorem 1.3. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n\\ &=2B_0+2q\left(B_1-\frac{q^5}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+10}}\right)+2q^2B_2\\ &\quad+2q^3B_3+2q^4\left(B_4-\frac{1}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+20}}\right),\end{aligned} \tag{1.6}$$

where

$$B_0 := \frac{4q^5 J_5^5 J_{25}^5 J_{5,50}^4 J_{15,50}^2}{J_{5,10}^6 J_{50}^6 J_{10,50}^3} - \frac{q^5 J_{50}^3 J_{25,50}^2}{J_{5,10}^2 J_{15,50} J_{20,50}},$$

$$B_1 := \frac{4q^5 J_5^8 J_{50}^7 J_{25,50}}{J_{5,10}^6 J_{10,50}^4 J_{20,50}^5} - \frac{4q^{10} J_{50}^3 J_{5,50}^2 J_{25,50}^2}{J_{5,10}^4 J_{10,50} J_{15,50}} + \frac{8q^{15} J_{50}^6 J_{10,50}^6}{J_5^6 J_{5,50}^2 J_{20,50}^3},$$

$$B_2 := -\frac{J_{5,50}^7 J_{20,50}^7 J_{25,50}^6}{J_{5,10}^6 J_{10,50}^4 J_{50}^9} + \frac{2J_{50} J_{20,50}^3 J_{25,50}}{J_5^4} - \frac{4q^{10} J_{50}^6 J_{10,50}^6 J_{25,50}}{J_5^6 J_{5,50}^3 J_{20,50}^3}$$
$$+ \frac{16q^{10} J_{50}^8}{J_5^2 J_{5,10}^2 J_{10,50} J_{15,50}^2},$$

$$B_3 := -\frac{J_{50}^3 J_{25,50}^4}{J_{5,10}^4 J_{10,50} J_{15,50}} + \frac{2J_{50}^3 J_{15,50} J_{25,50}}{J_{5,10}^2 J_{5,50} J_{20,50}} + \frac{4q^5 J_{50}^6 J_{20,50}^3 J_{25,50}^2}{J_5^6 J_{15,50}^4}$$
$$+ \frac{16q^{15} J_{50}^3 J_{5,50}^3}{J_{5,10}^4 J_{20,50}},$$

$$B_4 := \frac{4J_{50}^3 J_{15,50}^2}{J_{5,10}^2 J_{5,50} J_{20,50}} - \frac{2J_{5,50}^3 J_{20,50}^3 J_{25,50}^5}{J_5^2 J_{5,10}^4 J_{50}^4} + \frac{8q^{10} J_{50}^8}{J_5^2 J_{5,10}^2 J_{15,50}^2 J_{20,50}}$$
$$+ \frac{16q^{10} J_{50}^6 J_{10,50}^3}{J_5^6 J_{5,50} J_{25,50}} - \frac{16q^{15} J_{50}^3 J_{5,50}^4}{J_{5,10}^4 J_{10,50} J_{15,50}}.$$

Besides the equalities on ranks of partitions, like (1.1) and (1.2), some inequalities have also been obtained by Andrews [2], Garvan [12], Mao [25], and so on. In particular, Bringmann and Kane [6] characterized the sign of the rank differences of partitions for all odd moduli. In this paper, we obtain the following equalities and inequalities between the ranks of overpartitions modulo 6 and 10. The proofs of these identities and inequalities are based on the generating functions in Theorems 1.1, 1.2 and 1.3, as well as identities (1.2), (1.3) and (1.4) of [22].

Theorem 1.4. *We have*

$$\overline{N}(1,6,3n) = \overline{N}(3,6,3n) \text{ for } n \geq 1, \tag{1.7}$$

$$\overline{N}(0,6,3n) > \overline{N}(2,6,3n) \text{ for } n \geq 1, \tag{1.8}$$

$$\overline{N}(1,6,3n+1) = \overline{N}(3,6,3n+1) \text{ for } n \geq 0, \tag{1.9}$$

$$\overline{N}(0,6,3n+1) > \overline{N}(2,6,3n+1) \text{ for } n \geq 0, \tag{1.10}$$

$$\overline{N}(0,6,3n+2) < \overline{N}(2,6,3n+2) \text{ for } n \geq 1, \tag{1.11}$$

$$\overline{N}(1,6,3n+2) > \overline{N}(3,6,3n+2) \text{ for } n \geq 0. \tag{1.12}$$

Theorem 1.5. *For $n \geq 0$,*

$$\overline{N}(0,10,5n) + \overline{N}(1,10,5n) > \overline{N}(4,10,5n) + \overline{N}(5,10,5n). \tag{1.13}$$

Computer evidence suggests that the following inequalities hold, but we fail to prove them and so we leave them in the following two conjectures.

Conjecture 1.6. *For $n \geq 0$ and $1 \leq i \leq 4$,*

$$\overline{N}(0,10,5n+i) + \overline{N}(1,10,5n+i) \geq \overline{N}(4,10,5n+i) + \overline{N}(5,10,5n+i). \tag{1.14}$$

Conjecture 1.7. *For $n \geq 0$,*

$$\overline{N}(1,10,n) + \overline{N}(2,10,n) \geq \overline{N}(3,10,n) + \overline{N}(4,10,n). \tag{1.15}$$

The rank differences of partitions and overpartitions are also related to mock theta functions. Many of the classical mock theta functions can be written in terms of the rank differences of partitions. For example, Andrews and Garvan [3] found that the fifth order mock theta functions $\chi_0(q)$ and $\chi_1(q)$ can be expressed in terms of the rank differences of partitions modulo 5, which were later proved by Hickerson [15]. Subsequently, Hickerson [16] showed that the seventh order mock theta functions $\mathscr{F}_0(q)$, $\mathscr{F}_1(q)$ and $\mathscr{F}_2(q)$ are related to the rank differences of partitions modulo 7. Recently, Lovejoy and Osburn [23] has proved that the tenth order mock theta functions $\phi(q)$ and $\psi(q)$ can be expressed in terms of the rank differences of overpartitions modulo 5. In this paper, we establish a relation between the third order mock theta functions $\omega(q)$ and $\rho(q)$ and the rank differences of overpartitions modulo 6, where $\omega(q)$ and $\rho(q)$ are defined by [30]:

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} \quad \text{and} \quad \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q;q^2)_{n+1}}{(q^3;q^6)_{n+1}}.$$

Theorem 1.8. *We have*

$$\begin{aligned} &\sum_{n=0}^{\infty} (\overline{N}(0,6,3n+2) + \overline{N}(1,6,3n+2) - \overline{N}(2,6,3n+2) - \overline{N}(3,6,3n+2))q^n \\ &= \frac{4}{3}\omega(q) + \frac{2}{3}\rho(q). \end{aligned} \tag{1.16}$$

In light of Theorem 1.2 and Theorem 1.3, we obtain the following relations between the tenth order mock theta functions $\phi(q)$ and $\psi(q)$ and the ranks of overpartitions modulo 10. The tenth order mock theta functions $\phi(q)$ and $\psi(q)$ are defined as [8]:

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}. \tag{1.17}$$

Theorem 1.9. *We have*

$$\sum_{n=0}^{\infty}(\overline{N}(0,10,5n+1)+\overline{N}(1,10,5n+1)-\overline{N}(4,10,5n+1)-\overline{N}(5,10,5n+1))q^n$$
$$=-\phi(q)+M_1(q), \tag{1.18}$$
$$\sum_{n=0}^{\infty}(\overline{N}(1,10,5n+1)+\overline{N}(2,10,5n+1)-\overline{N}(3,10,5n+1)-\overline{N}(4,10,5n+1))q^n$$
$$=\phi(q)+M_2(q), \tag{1.19}$$
$$\sum_{n=0}^{\infty}(\overline{N}(1,10,5n+4)+\overline{N}(2,10,5n+4)-\overline{N}(3,10,5n+4)-\overline{N}(4,10,5n+4))q^n$$
$$=q^{-1}\psi(q)+M_3(q), \tag{1.20}$$

where $M_1(q)$, $M_2(q)$ and $M_3(q)$ are (explicit) weakly holomorphic modular forms given by:

$$\begin{aligned} M_1(q)=&\frac{J_5J_{10}J_{4,10}}{J_{2,5}J_{2,10}}+2q\frac{J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{2,10}\overline{J}_{2,10}J_{3,10}\overline{J}_{3,10}}+\frac{2J_{4,10}J_{5,10}^4J_{10}^3}{J_{1,2}^4J_{2,10}^2J_{3,10}}\\ &-8q\frac{J_{1,10}^4J_{4,10}^3J_{5,10}^4}{J_{1,2}^5J_{2,10}^2J_{10}^3}-16q^3\frac{J_{2,10}^4J_{3,10}^2J_{5,10}J_{10}^3}{J_{1,2}^4J_{4,10}^5},\\ M_2(q)=&-\frac{J_5J_{10}J_{4,10}}{J_{2,5}J_{2,10}}-2q\frac{J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{2,10}\overline{J}_{2,10}J_{3,10}\overline{J}_{3,10}}+\frac{8qJ_1^8J_{10}^7J_{5,10}}{J_{1,2}^6J_{2,10}^4J_{4,10}^5}\\ &-\frac{8q^2J_{10}^3J_{1,10}^2J_{5,10}^2}{J_{1,2}^4J_{2,10}J_{3,10}}+\frac{16q^3J_{10}^6J_{2,10}^6}{J_1^6J_{1,10}^2J_{4,10}^3},\\ M_3(q)=&\frac{J_5J_{10}J_{2,10}}{J_{1,5}J_{4,10}}-\frac{2J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{1,10}\overline{J}_{1,10}J_{4,10}\overline{J}_{4,10}}+\frac{8J_{10}^3J_{3,10}^2}{J_{1,2}^2J_{1,10}J_{4,10}}-\frac{4J_{1,10}^3J_{4,10}^3J_{5,10}^5}{J_1^2J_{1,2}^4J_{10}^4}\\ &+\frac{16q^2J_{10}^8}{J_1^2J_{1,2}^2J_{3,10}^2J_{4,10}}+\frac{32q^2J_{10}^6J_{2,10}^3}{J_1^6J_{1,10}J_{5,10}}-\frac{32q^3J_{10}^3J_{1,10}^4}{J_{1,2}^4J_{2,10}J_{3,10}}. \end{aligned}$$

This paper is organized as follows. In Section 2, we prove Theorem 1.1 by establishing the 3-dissection of $\overline{R}(\exp(i\pi/3);q)$. In Section 3, we give the proofs of Theorem 1.2 and Theorem 1.3 by investigating the 5-dissection of $\overline{R}(\exp(i\pi/5);q)$. Section 4 is devoted to establishing some equalities and inequalities on ranks of overpartitions modulo 6 and 10 with the aid of Theorems 1.1, 1.2 and 1.3. In Section 5, we prove the relations between the rank differences of overpartitions and mock theta functions as stated in Theorem 1.8 and Theorem 1.9.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need to determine the 3-dissection of $\overline{R}(\exp(i\pi/3);q)$. First, we simplify the generating function $\overline{R}(z;q)$ defined in (1.3) when $z=\exp(i\pi/3)$.

Lemma 2.1. *We have*

$$\begin{aligned}\overline{R}(\exp(i\pi/3);q) &= \sum_{n=0}^{\infty}(\overline{N}(0,6,n)+\overline{N}(1,6,n)-\overline{N}(2,6,n)-\overline{N}(3,6,n))q^n \\ &= \frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}. \end{aligned} \tag{2.1}$$

Proof. Setting $z=\xi_6=\exp\left(\frac{\pi i}{3}\right)$ in (1.3), we have

$$\begin{aligned}\overline{R}(\exp(i\pi/3);q) &= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)\xi_6^m q^n \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-\xi_6)(1-\xi_6^{-1})(-1)^n q^{n^2+n}}{(1-\xi_6 q^n)(1-\xi_6^{-1}q^n)}. \end{aligned} \tag{2.2}$$

Using $\overline{N}(s,\ell,n)=\overline{N}(\ell-s,\ell,n)$ in [21], and noting that $1-\xi_6+\xi_6^2=0$ and $\xi_6^3=-1$, we find that the left-hand side of (2.2) can be simplified as:

$$\begin{aligned}\overline{R}(\exp(i\pi/3);q) &= \sum_{n=0}^{\infty}\sum_{s=0}^{5}\overline{N}(s,6,n)\xi_6^s q^n \\ &= \sum_{n=0}^{\infty}\left\{\overline{N}(0,6,n)+(\xi_6+\xi_6^5)\overline{N}(1,6,n)+(\xi_6^2+\xi_6^4)\overline{N}(2,6,n)+\xi_6^3\overline{N}(3,6,n)\right\}q^n \\ &= \sum_{n=0}^{\infty}(\overline{N}(0,6,n)+\overline{N}(1,6,n)-\overline{N}(2,6,n)-\overline{N}(3,6,n))q^n. \end{aligned}$$

We proceed to simplify the right-hand side of (2.2). In light of the fact that $1-\xi_6^{-1}-\xi_6=0$, we deduce that

$$\begin{aligned}\overline{R}(\exp(i\pi/3);q) &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(2-\xi_6^{-1}-\xi_6)(-1)^n q^{n^2+n}}{(1-\xi_6^{-1}q^n-\xi_6 q^n+q^{2n})} \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-q^n+q^{2n}} \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}(1+q^n)}{1+q^{3n}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2-2n}}{1+q^{-3n}}\right\}\\
&= \frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}},
\end{aligned}
$$

as desired. Thus, we complete the proof of Lemma 2.1. ∎

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.1, it suffices to show that

$$
\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}\\
&\quad= \frac{J_{18}^3 J_{9,18}}{2J_6 J_{3,18}^2}+q\frac{J_{18}^3}{J_6 J_{3,18}}+q^2\left\{\frac{2J_{18}^3}{J_6 J_{9,18}}-\frac{1}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n}}{1+q^{9n+3}}\right\}.
\end{aligned}
\tag{2.3}
$$

First, we split the sum on the left-hand side of (2.3) into three sums according to the summation index n modulo 3,

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}} &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+3n}}{1+q^{9n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n+2}}{1+q^{9n+3}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+15n+6}}{1+q^{9n+6}}\\
&:= S_0 - S_1 + S_2.
\end{aligned}
\tag{2.4}
$$

We claim that

$$
S_0+S_2=\frac{2qJ_{3,18}}{J_{9,18}}S_1+\frac{J_6^3 J_{3,18}^6 J_{9,18}^2}{2J_{18}^9}.
\tag{2.5}
$$

The identity (2.5) can be justified by using the following identity in [22, Lemma 4.1].

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+n}\left(\frac{\zeta^{-2n}}{1-z\zeta^{-1}q^n}+\frac{\zeta^{2n+2}}{1-z\zeta q^n}\right)\\
&= \frac{\zeta(\zeta^2,q\zeta^{-2},-1,-q;q)_\infty}{(\zeta,q\zeta^{-1},-\zeta,-q\zeta^{-1};q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-zq^n}\\
&+\frac{(\zeta,q\zeta^{-1},\zeta^2,q\zeta^{-2},-z,-qz^{-1};q)_\infty (q;q)_\infty^2}{(z,qz^{-1},z\zeta,qz^{-1}\zeta^{-1},z\zeta^{-1},q\zeta z^{-1},-\zeta,-q\zeta^{-1};q)_\infty}.
\end{aligned}
\tag{2.6}
$$

Replacing q, z and ζ in (2.6) by q^9, $-q^3$ and q^3, we find that

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2+9n}\left(\frac{q^{-6n}}{1+q^{9n}}+\frac{q^{6n+6}}{1+q^{9n+6}}\right)\\
=&\frac{q^3(-1,-q^9;q^9)_\infty}{(-q^3,-q^6;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n}}{1+q^{9n+3}}+\frac{(q^3,q^6;q^9)_\infty^3 (q^9;q^9)_\infty^2}{(-q^3,-q^6;q^9)_\infty^3(-1,-q^9;q^9)_\infty}
\end{aligned}
$$

$$=\frac{2qJ_{3,18}}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n+2}}{1+q^{9n+3}}+\frac{J_6^3J_{3,18}^6J_{9,18}^2}{2J_{18}^9},$$

which gives (2.5), and hence the claim is verified.

Substituting (2.5) into (2.4), we have

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}=-\left(1-\frac{2qJ_{3,18}}{J_{9,18}}\right)S_1+\frac{J_6^3J_{3,18}^6J_{9,18}^2}{2J_{18}^9}.$$

Using the identity in [22, Lemma 3.1]

$$\frac{(q;q)_\infty}{(-q;q)_\infty}=\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty}-2q(q^3,q^{15},q^{18};q^{18})_\infty=J_{9,18}-2qJ_{3,18}, \tag{2.7}$$

it follows that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}=-\frac{(q;q)_\infty}{(-q;q)_\infty}\cdot\frac{S_1}{J_{9,18}}+\frac{J_6^3J_{3,18}^6J_{9,18}^2}{2J_{18}^9}.$$

Hence the proof of (2.3) amounts to show the following identity:

$$\frac{J_6^3J_{3,18}^6J_{9,18}^2}{J_{18}^9}=\frac{(q;q)_\infty}{(-q;q)_\infty}\left\{\frac{J_{18}^3J_{9,18}}{J_6J_{3,18}^2}+q\frac{2J_{18}^3}{J_6J_{3,18}}+q^2\frac{4J_{18}^3}{J_6J_{9,18}}\right\}. \tag{2.8}$$

Substituting (2.7) into (2.8), we find that the right-hand side of (2.8) can be simplified as

$$\begin{aligned}&(J_{9,18}-2qJ_{3,18})\left\{\frac{J_{18}^3J_{9,18}}{J_6J_{3,18}^2}+q\frac{2J_{18}^3}{J_6J_{3,18}}+q^2\frac{4J_{18}^3}{J_6J_{9,18}}\right\}\\&=\left(\frac{J_{18}^3J_{9,18}^2}{J_{3,18}^2J_6}-8q^3\frac{J_{18}^3J_{3,18}}{J_6J_{9,18}}\right)+q\left(\frac{2J_{18}^3J_{9,18}}{J_{3,18}J_6}-\frac{2J_{18}^3J_{9,18}}{J_{3,18}J_6}\right)+q^2\left(\frac{4J_{18}^3}{J_6}-\frac{4J_{18}^3}{J_6}\right)\\&=\frac{J_{18}^3J_{9,18}^2}{J_{3,18}^2J_6}-8q^3\frac{J_{18}^3J_{3,18}}{J_6J_{9,18}}.\end{aligned}$$

Thus (2.8) becomes

$$\frac{J_{18}^3J_{9,18}^2}{J_{3,18}^2J_6}-8q^3\frac{J_{18}^3J_{3,18}}{J_6J_{9,18}}=\frac{J_6^3J_{3,18}^6J_{9,18}^2}{J_{18}^9}. \tag{2.9}$$

The derivation of (2.9) relies on the following identity in [5].

$$j(x;q)^2j(yz;q)j(yz^{-1};q)=j(y;q)^2j(xz;q)j(xz^{-1};q)-yz^{-1}j(z;q)^2j(xy;q)j(xy^{-1};q). \tag{2.10}$$

In (2.10), replacing q by q^9 and setting $x=-q^3$, $y=q^3$ and $z=-1$, we have

$$j(-q^3;q^9)^3-q^3j(-1;q^9)^3=\frac{j(q^3;q^9)^4}{j(-q^3;q^9)},$$

which is equivalent to

$$\frac{J_{18}^3}{J_{3,18}^3} - 8q^3\frac{J_{18}^3}{J_{9,18}^3} = \frac{J_{3,18}^5 J_6^4}{J_{18}^9}. \tag{2.11}$$

Then (2.9) is obtained by multiplying both sides of (2.11) by $J_{3,18}J_{9,18}^2/J_6$, and hence (2.8) is verified. Thus, we complete the proof of Theorem 1.1. ∎

3 Proofs of Theorem 1.2 and Theorem 1.3

To prove Theorem 1.2 and Theorem 1.3, we are required to consider the 5-dissection of $\overline{R}(\exp(i\pi/5); q)$.

Lemma 3.1. *We have*

$$\begin{aligned}\overline{R}(\exp(i\pi/5); q) &= \sum_{n=0}^{\infty}(\overline{N}(0,10,n) + \overline{N}(1,10,n) - \overline{N}(4,10,n) - \overline{N}(5,10,n))q^n \\ &\quad + (\xi_{10}^2 - \xi_{10}^3)\sum_{n=0}^{\infty}(\overline{N}(1,10,n) + \overline{N}(2,10,n) - \overline{N}(3,10,n) - \overline{N}(4,10,n))q^n \\ &= F_1(q) + (\xi_{10}^2 - \xi_{10}^3)F_2(q), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}F_1(q) &:= \frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{5n}}, \\ F_2(q) &:= \frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}(q^n-1)}{1+q^{5n}}.\end{aligned}$$

Proof. Plugging $z = \xi_{10} = \exp\left(\frac{\pi i}{5}\right)$ into (1.3), we have

$$\begin{aligned}\overline{R}(\exp(i\pi/5); q) &= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)\xi_{10}^m q^n \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-\xi_{10})(1-\xi_{10}^{-1})(-1)^n q^{n^2+n}}{(1-\xi_{10}q^n)(1-\xi_{10}^{-1}q^n)}. \end{aligned} \tag{3.2}$$

Using $\overline{N}(s,\ell,n) = \overline{N}(\ell - s,\ell,n)$ and noting that $\xi_{10}^5 = -1,\ 1 - \xi_{10} + \xi_{10}^2 - \xi_{10}^3 + \xi_{10}^4 = 0$, we find that the left-hand side of (3.2) can be simplified as

$$\begin{aligned}\overline{R}(\exp(i\pi/5); q) &= \sum_{n=0}^{\infty}\sum_{t=0}^{9}\overline{N}(t,10,n)\xi_{10}^t q^n \\ &= \sum_{n=0}^{\infty}\{\overline{N}(0,10,n) + (\xi_{10} - \xi_{10}^4)\overline{N}(1,10,n) + (\xi_{10}^2 - \xi_{10}^3)\overline{N}(2,10,n)\end{aligned}$$

$$+(\xi_{10}^3-\xi_{10}^2)\overline{N}(3,10,n)+(\xi_{10}^4-\xi_{10})\overline{N}(4,10,n)-\overline{N}(5,10,n)\Big\}q^n$$

$$=\sum_{n=0}^{\infty}\Big\{\overline{N}(0,10,n)+(1+\xi_{10}^2-\xi_{10}^3)\overline{N}(1,10,n)+(\xi_{10}^2-\xi_{10}^3)\overline{N}(2,10,n)$$

$$+(\xi_{10}^3-\xi_{10}^2)\overline{N}(3,10,n)-(1+\xi_{10}^2-\xi_{10}^3)\overline{N}(4,10,n)-\overline{N}(5,10,n)\Big\}q^n$$

$$=\sum_{n=0}^{\infty}(\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n$$

$$+(\xi_{10}^2-\xi_{10}^3)\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n.$$

We now turn to simplify the right-hand side of (3.2). Using the fact that $1-\xi_{10}-\xi_{10}^{-1}-\xi_{10}^{-3}-\xi_{10}^3=0$, we deduce that

$$(1-\xi_{10}q^n)(1-\xi_{10}^{-1}q^n)(1-\xi_{10}^3q^n)(1-\xi_{10}^{-3}q^n)(1+q^n)=1+q^{5n}$$

and

$$\begin{aligned}&(1-\xi_{10})(1-\xi_{10}^{-1})(1-\xi_{10}^3q^n)(1-\xi_{10}^{-3}q^n)(1+q^n)\\&=(1-\xi_{10}^2+\xi_{10}^3)+(-1+\xi_{10}^{-1}+\xi_{10})q^n+(\xi_{10}^{-1}-1+\xi_{10})q^{2n}+(1-\xi_{10}^2+\xi_{10}^3)q^{3n}\\&=(1-\xi_{10}^2+\xi_{10}^3)+(\xi_{10}^2-\xi_{10}^3)q^n+(\xi_{10}^2-\xi_{10}^3)q^{2n}+(1-\xi_{10}^2+\xi_{10}^3)q^{3n}.\end{aligned}$$

Hence the right-hand side of (3.2) can be simplified as

$$\begin{aligned}&\overline{R}(\exp(i\pi/5);q)\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-\xi_{10})(1-\xi_{10}^{-1})(1-\xi_{10}^3q^n)(1-\xi_{10}^{-3}q^n)(1+q^n)(-1)^nq^{n^2+n}}{(1-\xi_{10}q^n)(1-\xi_{10}^{-1}q^n)(1-\xi_{10}^3q^n)(1-\xi_{10}^{-3}q^n)(1+q^n)}\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}(1+q^{3n})}{1+q^{5n}}+(\xi_{10}^2-\xi_{10}^3)\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}(q^n+q^{2n}-1-q^{3n})}{1+q^{5n}}\right\}\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2-4n}}{1+q^{-5n}}\right\}+(\xi_{10}^2-\xi_{10}^3)\frac{(-q;q)_\infty}{(q;q)_\infty}\\&\quad\times\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{5n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2-3n}}{1+q^{-5n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2-4n}}{1+q^{-5n}}\right\}\\&=\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}-(\xi_{10}^2-\xi_{10}^3)\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}(1-q^n)}{1+q^{5n}},\end{aligned}$$

as desired. Thus, we complete the proof of Lemma 3.1. ∎

Since the coefficients of $F_1(q)$ and $F_2(q)$ are all integers and $[\mathbb{Q}(\xi_{10}):\mathbb{Q}]=4$, we equate the coefficients of ξ_{10}^k on both sides of (3.1) to obtain the following two corollaries.

Corollary 3.2. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n\\&=\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}.\end{aligned}\tag{3.3}$$

Corollary 3.3. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n\\&=\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}(q^n-1)}{1+q^{5n}}.\end{aligned}\tag{3.4}$$

By Corollary 3.2 and Corollary 3.3, we see that the proofs of Theorem 1.2 and Theorem 1.3 amount to the 5-dissections of right-hand sides of (3.3) and (3.4) respectively.

Lemma 3.4. *Let*

$$\begin{aligned}U_1&:=\frac{J_{5,50}^2J_{10,50}^3J_{15,50}^4J_{25,50}^2}{J_{50}^9},\\U_2&:=\frac{J_{5,50}^3J_{10,50}^2J_{15,50}^4J_{20,50}J_{25,50}}{J_{50}^9}.\end{aligned}$$

We have

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}=\frac{1}{2}U_1-q^2U_2+\frac{(q;q)_\infty}{(-q;q)_\infty}\cdot\frac{q^6}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+10}}.\tag{3.5}$$

Proof. First, we split the sum on the left-hand side of (3.5) into five sums according to the summation index n modulo 5,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}&=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+5n}}{1+q^{25n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+15n+2}}{1+q^{25n+5}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n+6}}{1+q^{25n+10}}\\&\quad-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+35n+12}}{1+q^{25n+15}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+45n+20}}{1+q^{25n+20}}\\&:=P_0-P_1+P_2-P_3+P_4.\end{aligned}\tag{3.6}$$

We next aim to establish the following two relations,

$$P_0 + P_4 = \frac{2q^4 J_{5,50}}{J_{25,50}} P_2 + \frac{1}{2} U_1, \tag{3.7}$$

$$P_1 + P_3 = \frac{2q J_{15,50}}{J_{25,50}} P_2 + q^2 U_2. \tag{3.8}$$

Replacing q, z and ζ in (2.6) by q^{25}, $-q^{10}$ and q^{10} respectively, we find that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (-1)^n q^{25n^2+25n} \left(\frac{q^{-20n}}{1+q^{25n}} + \frac{q^{20n+20}}{1+q^{25n+20}} \right) \\
&= \frac{(-1, q^5, q^{20}, -q^{25}; q^{25})_\infty}{(q^{10}, -q^{10}, q^{15}, -q^{15}; q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+10}}{1+q^{25n+10}} \\
&\quad + \frac{(q^5, q^{20}; q^{25})_\infty (q^{10}, q^{15}, q^{25}; q^{25})_\infty^2}{(-1, -q^5, -q^{20}, -q^{25}; q^{25})_\infty (-q^{10}, -q^{15}; q^{25})_\infty^2} \\
&= \frac{2q^4 J_{5,50}}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+6}}{1+q^{25n+10}} + \frac{J_{5,50}^2 J_{10,50}^3 J_{15,50}^4 J_{25,50}^2}{2 J_{50}^9},
\end{aligned}$$

which gives (3.7).

Replacing q, z and ζ in (2.6) by q^{25}, $-q^{10}$ and q^5 respectively, we have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (-1)^n q^{25n^2+25n} \left(\frac{q^{-10n}}{1+q^{25n+5}} + \frac{q^{10n+10}}{1+q^{25n+15}} \right) \\
&= \frac{(-1, q^{10}, q^{15}, -q^{25}; q^{25})_\infty}{(q^5, -q^5, q^{20}, -q^{20}; q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+5}}{1+q^{25n+10}} \\
&\quad + \frac{(q^5, q^{20}; q^{25})_\infty (q^{10}, q^{15}, q^{25}; q^{25})_\infty^2}{(-q^5, -q^{10}, -q^{15}, -q^{20}; q^{25})_\infty^2} \\
&= \frac{2 J_{15,50}}{q J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+6}}{1+q^{25n+10}} + \frac{J_{5,50}^3 J_{10,50}^2 J_{15,50}^4 J_{20,50} J_{25,50}}{J_{50}^9}.
\end{aligned}$$

This yields (3.8). Substituting (3.7) and (3.8) into (3.6), we have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1+q^{5n}} = \left(1 - \frac{2q J_{15,50}}{J_{25,50}} + \frac{2q^4 J_{5,50}}{J_{25,50}} \right) P_2 + \frac{1}{2} U_1 - q^2 U_2. \tag{3.9}$$

By [22, Lemma 3.1]

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = J_{25,50} - 2q J_{15,50} + 2q^4 J_{5,50},$$

we find that

$$1-\frac{2qJ_{15,50}}{J_{25,50}}+\frac{2q^4J_{5,50}}{J_{25,50}}=\frac{(q;q)_\infty}{(-q;q)_\infty}\frac{1}{J_{25,50}}. \tag{3.10}$$

Hence (3.9) becomes

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}=\frac{(q;q)_\infty}{(-q;q)_\infty}\cdot\frac{P_2}{J_{25,50}}+\frac{1}{2}U_1-q^2U_2,$$

which is (3.5). This completes the proof of Lemma 3.4. ∎

Lemma 3.5. *Let*

$$V_1:=\frac{J_{5,50}^4J_{15,50}^2J_{20,50}^3J_{25,50}^2}{J_{50}^9},$$
$$V_2:=\frac{J_{5,50}^4J_{10,50}J_{15,50}^3J_{20,50}^2J_{25,50}}{J_{50}^9}.$$

We have

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{5n}}=\frac{1}{2}V_1-q^3V_2-\frac{(q;q)_\infty}{(-q;q)_\infty}\cdot\frac{q^4}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+20}}. \tag{3.11}$$

Proof. First, split the sum on the left-hand side of (3.11) into five sums according to the summation index n modulo 5,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{5n}}&=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+10n}}{1+q^{25n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+20n+3}}{1+q^{25n+5}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+30n+8}}{1+q^{25n+10}}\\&\quad-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+40n+15}}{1+q^{25n+15}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+50n+24}}{1+q^{25n+20}}\\&:=T_0-T_1+T_2-T_3+T_4.\end{aligned} \tag{3.12}$$

Now, we proceed to derive the following two identities,

$$T_0-T_3=-\frac{2qJ_{15,50}}{J_{25,50}}T_4+\frac{1}{2}V_1, \tag{3.13}$$

$$-T_1+T_2=\frac{2q^4J_{5,50}}{J_{25,50}}T_4-q^3V_2. \tag{3.14}$$

The proofs of the above two identities rely on an identity in [7, Theorem 2.1]

$$\frac{(a_1,q/a_1,\ldots,a_r,q/a_r;q)_\infty(q;q)_\infty^2}{(b_1,q/b_1,\ldots,b_s,q/b_s;q)_\infty}$$

$$= \frac{(a_1/b_1, qb_1/a_1, \ldots, a_r/b_1, qb_1/a_r; q)_\infty}{(b_2/b_1, qb_1/b_2, \ldots, b_s/b_1, qb_1/b_s; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{(s-r)n} q^{(s-r)n(n+1)/2}}{1-b_1q^n} \left(\frac{a_1\cdots a_r b_1^{s-r-1}}{b_2\cdots b_s}\right)^n$$

$$+ \,\mathrm{idem}(b_1; b_2, \ldots, b_s). \tag{3.15}$$

Here we use the notation

$$\begin{aligned}&F(b_1, b_2, \cdots, b_m) + \mathrm{idem}(b_1; b_2, \cdots, b_m)\\&:= F(b_1, b_2, \cdots, b_m) + F(b_2, b_1, b_3, \cdots, b_m) + \cdots + F(b_m, b_2, \cdots, b_{m-1}, b_1).\end{aligned}$$

In (3.15), setting $r = 1, s = 3$, $a_1 = -b_3 = z$, $b_1 = -z\zeta^{-1}$, and $b_2 = -z\zeta q^{-1}$, we obtain the following identity, of which (3.13) and (3.14) are special cases.

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \left(\frac{\zeta^{-2n}q^{2n}}{1+z\zeta^{-1}q^n} - \frac{q^{-1}\zeta^{2n+2}}{1+z\zeta q^{n-1}}\right)\\&= -\frac{(-1, -q, \zeta^2q^{-1}, q^2\zeta^{-2}; q)_\infty}{(-\zeta, -q\zeta^{-1}, \zeta, q\zeta^{-1}; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n+1}}{1+zq^n}\\&\quad + \frac{(\zeta, q\zeta^{-1}, z, qz^{-1}, \zeta^2q^{-1}, q^2\zeta^{-2}; q)_\infty (q;q)_\infty^2}{(-z, -qz^{-1}, -\zeta, -q\zeta^{-1}, -z\zeta^{-1}, -q\zeta z^{-1}, -z\zeta q^{-1}, -q^2\zeta^{-1}z^{-1}; q)_\infty}.\end{aligned} \tag{3.16}$$

Replacing q, z and ζ in (3.16) by q^{25}, q^{20} and q^{20} respectively, we find that

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} (-1)^n q^{25n^2} \left(\frac{q^{10n}}{1+q^{25n}} - \frac{q^{40n+15}}{1+q^{25n+15}}\right)\\&= -\frac{(-1, q^{10}, q^{15}, -q^{25}; q^{25})_\infty}{(q^5, -q^5, q^{20}, -q^{20}; q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+25}}{1+q^{25n+20}}\\&\quad + \frac{(q^{10}, q^{15}; q^{25})_\infty (q^5, q^{20}, q^{25}; q^{25})_\infty^2}{(-1, -q^{10}, -q^{15}, -q^{25}; q^{25})_\infty (-q^5, -q^{20}; q^{25})_\infty^2}\\&= -\frac{2qJ_{15,50}}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+24}}{1+q^{25n+20}} + \frac{J_{5,50}^4 J_{15,50}^2 J_{20,50}^3 J_{25,50}^2}{2J_{50}^9},\end{aligned}$$

which gives (3.13).

Replacing q, z and ζ in (3.16) by q^{25}, q^{20} and q^{15} respectively, we have

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} (-1)^n q^{25n^2} \left(\frac{q^{20n}}{1+q^{25n+5}} - \frac{q^{30n+5}}{1+q^{25n+10}}\right)\\&= -\frac{(-1, q^5, q^{20}, -q^{25}; q^{25})_\infty}{(q^{10}, -q^{10}, q^{15}, -q^{15}; q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+25}}{1+q^{25n+20}}\\&\quad + \frac{(q^{10}, q^{15}; q^{25})_\infty (q^5, q^{20}, q^{25}; q^{25})_\infty^2}{(-q^5, -q^{10}, -q^{15}, -q^{20}; q^{25})_\infty^2}\end{aligned}$$

$$= -\frac{2qJ_{5,50}}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+24}}{1+q^{25n+20}} + \frac{J_{5,50}^4 J_{10,50} J_{15,50}^3 J_{20,50}^2 J_{25,50}}{J_{50}^9}.$$

This yields (3.14). Substituting (3.13) and (3.14) into (3.12), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{5n}} = \left(1 - \frac{2qJ_{15,50}}{J_{25,50}} + \frac{2q^4 J_{5,50}}{J_{25,50}}\right) T_4 + \frac{1}{2}V_1 - q^3 V_2. \tag{3.17}$$

Substituting (3.10) into (3.17), and noting that

$$\begin{aligned} T_4 &:= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+24}}{1+q^{25n+20}} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+4}(1+q^{25n+20}-1)}{1+q^{25n+20}} \\ &= -\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+4}}{1+q^{25n+20}}, \end{aligned}$$

we arrive at (3.11). Thus, we complete the proof of Lemma 3.5. ∎

To prove Theorem 1.2 and Theorem 1.3, we also need the following two lemmas.

Lemma 3.6. *Recall that U_1 and U_2 are defined in Lemma 3.4. The following identity holds.*

$$\frac{1}{2}U_1 - q^2 U_2 = \frac{(q;q)_\infty}{(-q;q)_\infty} \left\{A_0 + A_1 q + A_2 q^2 + A_3 q^3 + A_4 q^4\right\}, \tag{3.18}$$

where A_0, A_1, A_2, A_3, A_4 are defined in Theorem 1.2.

Lemma 3.7. *Recall that U_1 and U_2 are defined in Lemma 3.4 and V_1 and V_2 are defined in Lemma 3.5. The following identity holds.*

$$\frac{1}{2}V_1 - \frac{1}{2}U_1 + q^2 U_2 - q^3 V_2 = \frac{(q;q)_\infty}{(-q;q)_\infty} \left\{B_0 + B_1 q + B_2 q^2 + B_3 q^3 + B_4 q^4\right\}, \tag{3.19}$$

where B_0, B_1, B_2, B_3, B_4 are defined in Theorem 1.3.

Before verifying Lemma 3.6 and Lemma 3.7, we will give proofs of Theorem 1.2 and Theorem 1.3 based on Lemmas 3.4–3.7. We begin with Theorem 1.2.

Proof of Theorem 1.2. Substituting (3.5) in Lemma 3.4 into Corollary 3.2, we find that

$$\begin{aligned} &\sum_{n=0}^{\infty} (\overline{N}(0,10,n) + \overline{N}(1,10,n) - \overline{N}(4,10,n) - \overline{N}(5,10,n)) q^n \\ &\quad = \frac{2(-q;q)_\infty}{(q;q)_\infty} \left(\frac{1}{2}U_1 - q^2 U_2\right) + \frac{2q^6}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+10}}. \end{aligned} \tag{3.20}$$

Substituting (3.18) in Lemma 3.6 into (3.20), we obtain (1.5), and so Theorem 1.2 is verified. ∎

We now turn to prove Theorem 1.3 by using Lemmas 3.4, 3.5 and 3.7.

Proof of Theorem 1.3. Substituting (3.5) in Lemma 3.4 and (3.11) in Lemma 3.5 into Corollary 3.3, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n \\
&= \frac{2(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}V_1-q^3V_2\right)-\frac{2q^4}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+20}} \\
&\quad -\left\{\frac{2(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}U_1-q^2U_2\right)+\frac{2q^6}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+10}}\right\} \\
&= \frac{2(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}V_1-\frac{1}{2}U_1+q^2U_2-q^3V_2\right) \\
&\quad -\frac{2q^4}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+20}}-\frac{2q^6}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+10}}. \qquad (3.21)
\end{aligned}$$

Then we obtain (1.6) after substituting (3.19) in Lemma 3.7 into (3.21). Thus, we complete the proof of Theorem 1.3. ∎

We conclude this section by giving proofs of Lemma 3.6 and Lemma 3.7. The proofs require to use standard computational techniques from the theory of modular forms. Recall that the Dedekind η-function is defined by

$$\eta(\tau)=q^{\frac{1}{24}}(q;q)_\infty,$$

where $\tau\in\mathcal{H}:=\{\tau\in\mathcal{C}:Im\tau>0\}$ and $q=\exp(2\pi i\tau)$. The generalized Dedekind η-function is defined by

$$\eta_{\delta,g}(\tau)=q^{P_2(g/\delta)\delta/2}\prod_{\substack{n>0\\ n\equiv g \pmod{\delta}}}(1-q^n)\prod_{\substack{n>0\\ n\equiv -g \pmod{\delta}}}(1-q^n), \qquad (3.22)$$

where $P_2(t)=\{t\}^2-\{t\}+\frac{1}{6}$ is the second Bernoulli function, and $\{t\}:=t-[t]$ is the fractional part of t. Note that

$$\eta_{\delta,0}(\tau)=q^{\frac{\delta}{12}}(q^\delta;q^\delta)_\infty^2$$

and

$$\eta_{\delta,\frac{\delta}{2}}(\tau)=q^{-\frac{\delta}{24}}(q^{\frac{\delta}{2}};q^\delta)_\infty^2.$$

Let N be a fixed positive integer. A generalized Dedekind η-quotient of level N has the form

$$f(\tau)=\prod_{\substack{\delta\mid N\\ 0\le g<\delta}}\eta_{\delta,g}^{r_{\delta,g}}(\tau), \qquad (3.23)$$

where

$$r_{\delta,g} \in \begin{cases} \frac{1}{2}\mathbb{Z} & \text{if } g = 0 \text{ or } g = \delta/2, \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

Suppose f is a modular function with respect to the congruence subgroup Γ of $\Gamma_0(1)$. For any cusp $\zeta \in \Gamma$, there exists $A \in \Gamma_0(1)$ such that $A\zeta = \infty$. Denote the width of the cusp ζ with respect to Γ by M. If

$$f(A^{-1}\tau) = \sum_{m=m_0}^{\infty} b_m q^{m/M}$$

and $b_{m_0} \neq 0$, then we say m_0 is the order of f at ζ with respect to Γ and denote this value by $ORD(f, \zeta, \Gamma)$.

Robins [28] gave the sufficient conditions under which a generalized η-quotient is a modular function on $\Gamma_1(N)$.

Theorem 3.8 (Robins)**.** *Let $f(\tau)$ be a generalized η-quotient defined as* (3.23)*. If*

(1)

$$\sum_{\substack{\delta|N \\ 0\le g<\delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} \equiv 0 \pmod 2,$$

(2)

$$\sum_{\substack{\delta|N \\ 0\le g<\delta}} \frac{N}{\delta} P_2(0) r_{\delta,g} \equiv 0 \pmod 2.$$

Then $f(\tau)$ is a modular function on $\Gamma_1(N)$.

The following theorem due to Garvan and Liang [13] can be used to prove generalized η-quotient identities. This theorem is based on the valence formula for modular functions, along with the fact that a generalized η-quotient has no zeros nor poles in the upper-half plane $\mathcal{H}$.

Theorem 3.9 (Garvan and Liang)**.** *Let $f_1(\tau), f_2(\tau), \ldots, f_n(\tau)$ be generalized η-quotients that are modular functions on $\Gamma_1(N)$. Let $\mathcal{S}_N$ be a set of inequivalent cusps for $\Gamma_1(N)$. Define the constant*

$$B = \sum_{\substack{s\in\mathcal{S}_N \\ s\neq i\infty}} \min(\{ORD(f_j, s, \Gamma_1(N)) : 1 \le j \le n\} \cup \{0\}), \tag{3.24}$$

and consider

$$g(\tau) := \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau) + 1, \tag{3.25}$$

where each $\alpha_j \in \mathbb{C}$. Then

$$g(\tau) \equiv 0$$

if and only if

$$ORD(g(\tau), i\infty, \Gamma_1(N)) > -B. \tag{3.26}$$

We are now in a position to prove Lemma 3.6 and Lemma 3.7. We begin with Lemma 3.6.

Proof of Lemma 3.6. It is equivalent to show that

$$
\begin{aligned}
&\frac{1}{2}\frac{J_{5,50}^2J_{10,50}^3J_{15,50}^4J_{25,50}^2}{J_{50}^9}-q^2\frac{J_{5,50}^3J_{10,50}^2J_{15,50}^4J_{20,50}J_{25,50}}{J_{50}^9}\\
&=\frac{(q;q)_\infty}{(-q;q)_\infty}\left\{\frac{J_{10,50}^2J_{15,50}^2J_{25,50}^4}{2J_{5,10}^3J_{20,50}J_{50}^3}+4q^{10}\frac{J_{5,50}J_{50}^3}{J_{5,10}^2J_{20,50}}\right.\\
&+q\left(\frac{J_{20,50}J_{25,50}^4J_{50}^3}{J_{5,10}^4J_{10,50}^2J_{15,50}}-4q^5\frac{J_{5,50}^4J_{20,50}^3J_{25,50}^4}{J_{5,10}^5J_{10,50}^2J_{50}^3}-8q^{15}\frac{J_{10,50}^4J_{15,50}^2J_{25,50}J_{50}^3}{J_{5,10}^4J_{20,50}^5}\right)\\
&+q^2\left(\frac{J_{5,50}^3J_{20,50}^3J_{25,50}^5}{J_{5,10}^5J_{10,50}^2J_{50}^3}+4q^{10}\frac{J_{10,50}^5J_{25,50}J_{50}^7}{J_5^4J_{5,10}J_{5,50}^3J_{20,50}^4}-16q^{10}\frac{J_{5,50}J_{15,50}^2J_{50}^3}{J_{5,10}^4J_{20,50}}\right)\\
&+q^3\left(\frac{2J_{10,50}^4J_{15,50}^5J_{25,50}^4}{J_{5,10}^5J_{5,50}J_{20,50}^3J_{50}^3}+2q^5\frac{J_{10,50}J_{25,50}^3J_{50}^3}{J_{5,10}^4J_{20,50}^2}-16q^5\frac{J_{15,50}^2J_{25,50}J_{50}^3}{J_{5,10}^4J_{20,50}}\right.\\
&\qquad\left.+8q^{10}\frac{J_{5,50}^3J_{20,50}J_{25,50}J_{50}^3}{J_{5,10}^4J_{10,50}^2J_{15,50}}\right)\\
&+q^4\left(\frac{4J_{10,50}^4J_{15,50}^6J_{25,50}^3}{J_{5,10}^5J_{5,50}J_{20,50}^3J_{50}^3}-\frac{J_{10,50}J_{25,50}^4J_{50}^3}{J_{5,10}^4J_{5,50}J_{20,50}^2}-16q^5\frac{J_{15,50}^3J_{50}^3}{J_{5,10}^4J_{20,50}}\right.\\
&\qquad\left.\left.-\ 8q^5\frac{J_{25}J_{50}^5}{J_5J_{5,10}^3J_{10,50}}\right)\right\}.
\end{aligned}
$$

Multiplying both sides of the above identity by $2J_{50}^9J_{5,50}^{-2}J_{10,50}^{-3}J_{15,50}^{-4}J_{25,50}^{-2}$, we obtain the following identity, which is expressed in terms of the generalized η-quotient.

$$
\begin{aligned}
&1-\frac{2\eta_{50,5}(\tau)\eta_{50,20}(\tau)}{\eta_{50,10}(\tau)\eta_{50,25}(\tau)}=\frac{\eta_{1,0}(\tau)\eta_{50,0}(\tau)^{\frac{5}{2}}}{\eta_{2,0}(\tau)^{\frac{1}{2}}\eta_{5,0}(\tau)\eta_{10,0}(\tau)^2}\left\{\frac{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{50,10}(\tau)\eta_{50,20}(\tau)\eta_{50,25}(\tau)^3}{\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^3}\right.\\
&+\frac{8\eta_{10,0}(\tau)\eta_{50,5}(\tau)\eta_{50,20}(\tau)}{\eta_{50,0}(\tau)\eta_{10,5}(\tau)^2\eta_{50,10}(\tau)\eta_{50,15}(\tau)^2\eta_{50,25}(\tau)}+\frac{2\eta_{50,20}(\tau)^3\eta_{50,25}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^3}\\
&-\frac{8\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,5}(\tau)^4\eta_{50,20}(\tau)^5\eta_{50,25}(\tau)^3}{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^5\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^2}-\frac{16\eta_{50,10}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,20}(\tau)^3}\\
&+\frac{2\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,5}(\tau)^3\eta_{50,20}(\tau)^5\eta_{50,25}(\tau)^4}{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^5\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^2}+\frac{8\eta_{10,0}(\tau)^{\frac{3}{2}}\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,10}(\tau)^4}{\eta_{5,0}(\tau)^2\eta_{10,5}(\tau)\eta_{50,5}(\tau)^3\eta_{50,15}(\tau)^2\eta_{50,20}(\tau)^2}\\
&-\frac{32\eta_{50,5}(\tau)\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)\eta_{50,25}(\tau)}+\frac{4\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^3\eta_{50,25}(\tau)^3}{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^5\eta_{50,5}(\tau)\eta_{50,20}(\tau)}\\
&+\frac{4\eta_{50,25}(\tau)^2}{\eta_{10,5}(\tau)^4\eta_{50,15}(\tau)^2}-\frac{32\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)}+\frac{16\eta_{50,5}(\tau)^3\eta_{50,20}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^3}
\end{aligned}
$$

$$+\frac{8\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^4\eta_{50,25}(\tau)^2}{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^5\eta_{50,5}(\tau)\eta_{50,20}(\tau)}-\frac{2\eta_{50,25}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,5}(\tau)\eta_{50,15}(\tau)^2}$$
$$\left.-\frac{32\eta_{50,15}(\tau)\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)\eta_{50,25}(\tau)}-\frac{16\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{25,0}(\tau)^{\frac{1}{2}}\eta_{50,20}(\tau)^2}{\eta_{5,0}(\tau)^{\frac{1}{2}}\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^3\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^2\eta_{50,25}(\tau)}\right\}.$$

In light of Theorem 3.8, it can be shown that each term of the above identity is a modular function with respect to $\Gamma_1(50)$. Using the algorithm in [13], we could calculate the constant B in (3.24), which is equal to -145. Thus, by Theorem 3.9, it amounts to verify the identity in the q-expansion past q^{145}, as desired. Hence Lemma 3.6 is verified. ∎

Finally, we give a proof of Lemma 3.7.

Proof of Lemma 3.7. It is equivalent to show that

$$\frac{1}{2}\frac{J_{5,50}^4J_{15,50}^2J_{20,50}^3J_{25,50}^2}{J_{50}^9}-\frac{1}{2}\frac{J_{5,50}^2J_{10,50}^3J_{15,50}^4J_{25,50}^2}{J_{50}^9}$$
$$+q^2\frac{J_{5,50}^3J_{10,50}^2J_{15,50}^4J_{20,50}J_{25,50}}{J_{50}^9}-q^3\frac{J_{5,50}^4J_{10,50}J_{15,50}^3J_{20,50}^2J_{25,50}}{J_{50}^9}$$
$$=\frac{(q;q)_\infty}{(-q;q)_\infty}\left\{\frac{4q^5J_5^5J_{25}^5J_{5,50}^4J_{15,50}^2}{J_{5,10}^6J_{50}^6J_{10,50}^3}-\frac{q^5J_{50}^3J_{25,50}^2}{J_{5,10}^2J_{15,50}J_{20,50}}\right.$$
$$+q\left(\frac{4q^5J_5^8J_{50}^7J_{25,50}}{J_{5,10}^6J_{10,50}^4J_{20,50}^5}-\frac{4q^{10}J_{50}^3J_{5,50}^2J_{25,50}^2}{J_{5,10}^4J_{10,50}J_{15,50}}+\frac{8q^{15}J_{50}^6J_{10,50}^6}{J_5^6J_{5,50}^2J_{20,50}^3}\right)$$
$$+q^2\left(-\frac{J_{5,50}^7J_{20,50}^7J_{25,50}^6}{J_{5,10}^6J_{10,50}^4J_{50}^9}+\frac{2J_{50}J_{20,50}^3J_{25,50}}{J_5^4}-\frac{4q^{10}J_{50}^6J_{10,50}^6J_{25,50}}{J_5^6J_{5,50}^3J_{20,50}^3}+\frac{16q^{10}J_{50}^8}{J_5^2J_{5,10}^2J_{10,50}J_{15,50}^2}\right)$$
$$+q^3\left(-\frac{J_{50}^3J_{25,50}^4}{J_{5,10}^4J_{10,50}J_{15,50}}+\frac{2J_{50}^3J_{15,50}J_{25,50}}{J_{5,10}^2J_{5,50}J_{20,50}}+\frac{4q^5J_{50}^6J_{20,50}^3J_{25,50}^2}{J_5^6J_{15,50}^4}+\frac{16q^{15}J_{50}^3J_{5,50}^3}{J_{5,10}^4J_{20,50}}\right)$$
$$+q^4\left(\frac{4J_{50}^3J_{15,50}^2}{J_{5,10}^2J_{5,50}J_{20,50}}-\frac{2J_{5,50}^3J_{20,50}^3J_{25,50}^5}{J_5^2J_{5,10}^4J_{50}^4}+\frac{8q^{10}J_{50}^8}{J_5^2J_{5,10}^2J_{15,50}^2J_{20,50}}\right.$$
$$\left.\left.+\frac{16q^{10}J_{50}^6J_{10,50}^3}{J_5^6J_{5,50}J_{25,50}}-\frac{16q^{15}J_{50}^3J_{5,50}^4}{J_{5,10}^4J_{10,50}J_{15,50}}\right)\right\}.$$

Multiplying both sides of the above identity by $2J_{50}^9J_{5,50}^{-2}J_{10,50}^{-3}J_{15,50}^{-4}J_{25,50}^{-2}$ gives the following identity, which is expressed in terms of the generalized η-quotient.

$$-1+\frac{\eta_{50,5}(\tau)^2\eta_{50,20}(\tau)^3}{\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^2}+\frac{2\eta_{50,5}(\tau)\eta_{50,20}(\tau)}{\eta_{50,10}(\tau)\eta_{50,25}(\tau)}-\frac{2\eta_{50,5}(\tau)^2\eta_{50,20}(\tau)^2}{\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)\eta_{50,25}(\tau)}$$
$$=\frac{\eta_{1,0}(\tau)\eta_{50,0}(\tau)^{\frac{5}{2}}}{\eta_{2,0}(\tau)^{\frac{1}{2}}\eta_{5,0}(\tau)^2\eta_{10,0}(\tau)^2}\left\{\frac{8\eta_{5,0}(\tau)^{\frac{7}{2}}\eta_{25,0}(\tau)^{\frac{5}{2}}\eta_{50,5}(\tau)^4\eta_{50,20}(\tau)^2}{\eta_{10,0}(\tau)\eta_{50,0}(\tau)^4\eta_{10,5}(\tau)^6\eta_{50,10}(\tau)^4\eta_{50,25}(\tau)}\right.$$
$$-\frac{2\eta_{5,0}(\tau)\eta_{10,0}(\tau)\eta_{50,20}(\tau)\eta_{50,25}(\tau)}{\eta_{50,0}(\tau)\eta_{10,5}(\tau)^2\eta_{50,10}(\tau)\eta_{50,15}(\tau)^3}+\frac{8\eta_{5,0}(\tau)^4\eta_{50,5}(\tau)^2\eta_{50,25}(\tau)}{\eta_{10,0}(\tau)\eta_{10,5}(\tau)^6\eta_{50,0}(\tau)^2\eta_{50,10}(\tau)^3\eta_{50,20}(\tau)}$$

$$
\begin{aligned}
&- \frac{8\eta_{5,0}(\tau)\eta_{50,5}(\tau)^2\eta_{50,20}(\tau)^2\eta_{50,25}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^3} + \frac{16\eta_{10,0}(\tau)^2\eta_{50,0}(\tau)\eta_{50,10}(\tau)^5}{\eta_{5,0}(\tau)^2\eta_{50,5}(\tau)^2\eta_{50,15}(\tau)^2\eta_{50,20}(\tau)\eta_{50,25}(\tau)} \\
&- \frac{2\eta_{5,0}(\tau)\eta_{50,0}(\tau)\eta_{50,5}(\tau)^7\eta_{50,20}(\tau)^9\eta_{50,25}(\tau)^5}{\eta_{10,0}(\tau)\eta_{10,5}(\tau)^6\eta_{50,10}(\tau)^5\eta_{50,15}(\tau)^2} + \frac{4\eta_{10,0}(\tau)^2\eta_{50,20}(\tau)^5}{\eta_{5,0}(\tau)\eta_{50,10}(\tau)\eta_{50,15}(\tau)^2} \\
&- \frac{8\eta_{10,0}(\tau)^2\eta_{50,0}(\tau)\eta_{50,10}(\tau)^5}{\eta_{5,0}(\tau)^2\eta_{50,5}(\tau)^3\eta_{50,15}(\tau)^2\eta_{50,20}(\tau)} + \frac{32\eta_{10,0}(\tau)\eta_{50,20}(\tau)^2}{\eta_{10,5}(\tau)^2\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^4\eta_{50,25}(\tau)} \\
&- \frac{2\eta_{5,0}(\tau)\eta_{50,20}(\tau)^2\eta_{50,25}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^3} + \frac{4\eta_{5,0}(\tau)\eta_{10,0}(\tau)\eta_{50,20}(\tau)}{\eta_{50,0}(\tau)\eta_{10,5}(\tau)^2\eta_{50,5}(\tau)\eta_{50,10}(\tau)\eta_{50,15}(\tau)} \\
&+ \frac{8\eta_{10,0}(\tau)^2\eta_{50,0}(\tau)\eta_{50,20}(\tau)^5\eta_{50,25}(\tau)}{\eta_{5,0}(\tau)^2\eta_{50,10}(\tau)\eta_{50,15}(\tau)^6} + \frac{32\eta_{5,0}(\tau)\eta_{50,5}(\tau)^3\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)\eta_{50,15}(\tau)^2\eta_{50,25}(\tau)} \\
&+ \frac{8\eta_{5,0}(\tau)\eta_{10,0}(\tau)\eta_{50,20}(\tau)}{\eta_{50,0}(\tau)\eta_{10,5}(\tau)^2\eta_{50,5}(\tau)\eta_{50,10}(\tau)\eta_{50,25}(\tau)} - \frac{4\eta_{50,0}(\tau)\eta_{50,5}(\tau)^3\eta_{50,20}(\tau)^5\eta_{50,25}(\tau)^4}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)\eta_{50,15}(\tau)^2} \\
&+ \frac{16\eta_{10,0}(\tau)\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^2\eta_{50,10}(\tau)\eta_{50,15}(\tau)^4\eta_{50,25}(\tau)} + \frac{32\eta_{10,0}(\tau)^2\eta_{50,0}(\tau)\eta_{50,10}(\tau)^2\eta_{50,20}(\tau)^2}{\eta_{5,0}(\tau)^2\eta_{50,5}(\tau)\eta_{50,15}(\tau)^2\eta_{50,25}(\tau)^2} \\
&- \left.\frac{32\eta_{5,0}(\tau)\eta_{50,5}(\tau)^4\eta_{50,20}(\tau)^2}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^3\eta_{50,25}(\tau)}\right\}.
\end{aligned}
$$

We then use Theorem 3.8 to show that each term of the above identity is a modular function with respect to $\Gamma_1(50)$. Employing the algorithm in [13], we find that the constant B in (3.24) is equal to -155. By Theorem 3.9, it suffices to verify the identity in the q-expansion past q^{155}, as desired. Thus, we complete the proof of Lemma 3.7. ∎

4 Proofs of Theorem 1.4 and Theorem 1.5

To prove Theorem 1.4 and Theorem 1.5, we are required to recall the following result due to Liaw [24].

Theorem 4.1 (Liaw). *If p and r are positive integers with $p \geq 2$ and $r < p$, define*

$$\sum_{n=0}^{\infty} b_{p,r}(n)q^n := \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty (q^{p-r};q^p)_\infty},$$

then $b_{p,r}(n) \geq 0$ for all n.

We now give a proof of Theorem 1.4.

Proof of Theorem 1.4. (1) We first show (1.7) and (1.8). Comparing the coefficients of q^{3n} of (1.4) in Theorem 1.1, we find that

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n)+\overline{N}(1,6,3n)-\overline{N}(2,6,3n)-\overline{N}(3,6,3n))q^n = \frac{J_6^3 J_{3,6}}{J_{1,6}^2 J_2}. \tag{4.1}$$

Together with the identity (1.2) in [22, Theorem 1.1],

$$\sum_{n=0}^{\infty}(\overline{N}(0,3,3n)-\overline{N}(1,3,3n))q^n = -1+\frac{(q^3;q^3)_\infty^2(-q;q)_\infty}{(q;q)_\infty(-q^3;q^3)_\infty^2} = -1+\frac{J_6^3J_{3,6}}{J_{1,6}^2J_2},$$

we deduce that for $n\geq 1$,

$$\begin{aligned}&\overline{N}(0,6,3n)+\overline{N}(1,6,3n)-\overline{N}(2,6,3n)-\overline{N}(3,6,3n)\\&=\overline{N}(0,3,3n)-\overline{N}(1,3,3n).\end{aligned} \tag{4.2}$$

Observe that

$$\overline{N}(s,\ell,n)=\overline{N}(s,2\ell,n)+\overline{N}(\ell+s,2\ell,n)=\overline{N}(s,2\ell,n)+\overline{N}(\ell-s,2\ell,n), \tag{4.3}$$

so (4.2) is equivalent to

$$\begin{aligned}&\overline{N}(0,6,3n)+\overline{N}(1,6,3n)-\overline{N}(2,6,3n)-\overline{N}(3,6,3n)\\&=\overline{N}(0,6,3n)+\overline{N}(3,6,3n)-\overline{N}(1,6,3n)-\overline{N}(2,6,3n).\end{aligned}$$

This implies that for $n\geq 1$

$$\overline{N}(1,6,3n)=\overline{N}(3,6,3n),$$

which is (1.7).

We turn to show (1.8). By (1.7), it suffices to show that the coefficients of q^n on the right-hand side of (4.1) are positive. First, observe that

$$\frac{J_6^3J_{3,6}}{J_{1,6}^2J_2}=\frac{(q^3;q^6)_\infty^2(q^6;q^6)_\infty}{(q,q^5;q^6)_\infty^2(q^2,q^4;q^6)_\infty}=\frac{(q^3;q^3)_\infty}{(q,q^2;q^3)_\infty}\frac{(q^3;q^6)_\infty}{(q,q^5;q^6)_\infty}=\frac{(q^3;q^3)_\infty}{(q,q^2;q^3)_\infty}\frac{(q^3;q^6)_\infty^2}{(q;q^2)_\infty},$$

and

$$\frac{(q^3;q^6)_\infty^2}{(q;q^2)_\infty}=\frac{(-q;q)_\infty}{(-q^3;q^3)_\infty^2}=\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(-q^3;q^3)_\infty}=\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty}{(q^6;q^6)_\infty}.$$

By Jacobi's triple product identity [1, p. 18, Theorem 2.8], we see that

$$(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty=\sum_{n=-\infty}^{\infty}q^{(3n^2-n)/2},$$

and using Cauchy's q-binomial theorem [1, p. 17, Theorem 2.1], we derive that

$$\frac{(q^3;q^3)_\infty}{(q,q^2;q^3)_\infty}=\sum_{m=0}^{\infty}\frac{q^{2m}}{(q^{3m+1};q^3)_\infty(q^3;q^3)_m}.$$

Hence (4.1) can be written as:

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n)+\overline{N}(1,6,3n)-\overline{N}(2,6,3n)-\overline{N}(3,6,3n))q^n$$

$$= \frac{1}{(q^6;q^6)_\infty} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{q^{2m+(3n^2-n)/2}}{(q^{3m+1};q^3)_\infty (q^3;q^3)_m}. \tag{4.4}$$

It is easy to see that the coefficients of q^n in each term of (4.4) are nonnegative for $n \geq 0$. In particular, the term corresponding to $m = n = 0$ of (4.4)

$$\frac{1}{(q;q^3)_\infty (q^6;q^6)_\infty}$$

has positive coefficients. Hence we conclude that the coefficients of q^n in (4.4) are positive for $n \geq 0$, and so, for $n \geq 0$

$$\overline{N}(0,6,3n) + \overline{N}(1,6,3n) - \overline{N}(2,6,3n) - \overline{N}(3,6,3n) > 0.$$

Together with (1.7), we obtain (1.8).

(2) We next show (1.9) and (1.10). Comparing the coefficients of q^{3n+1} of (1.4) in Theorem 1.1, we find that

$$\sum_{n=0}^{\infty} (\overline{N}(0,6,3n+1) + \overline{N}(1,6,3n+1) - \overline{N}(2,6,3n+1) - \overline{N}(3,6,3n+1))q^n$$
$$= \frac{2J_6^3}{J_{1,6}J_2}. \tag{4.5}$$

Combining the identity (1.3) in [22, Theorem 1.1],

$$\sum_{n=0}^{\infty} (\overline{N}(0,3,3n+1) - \overline{N}(1,3,3n+1))q^n = \frac{2(q^3;q^3)_\infty (q^6;q^6)_\infty}{(q;q)_\infty} = \frac{2J_6^3}{J_{1,6}J_2},$$

we derive that for $n \geq 0$,

$$\overline{N}(0,6,3n+1) + \overline{N}(1,6,3n+1) - \overline{N}(2,6,3n+1) - \overline{N}(3,6,3n+1)$$
$$= \overline{N}(0,3,3n+1) - \overline{N}(1,3,3n+1). \tag{4.6}$$

Using (4.3), we see that (4.6) can be written as

$$\overline{N}(0,6,3n+1) + \overline{N}(1,6,3n+1) - \overline{N}(2,6,3n+1) - \overline{N}(3,6,3n+1)$$
$$= \overline{N}(0,6,3n+1) + \overline{N}(3,6,3n+1) - \overline{N}(1,6,3n+1) - \overline{N}(2,6,3n+1),$$

which implies that for $n \geq 0$

$$\overline{N}(1,6,3n+1) = \overline{N}(3,6,3n+1),$$

so (1.9) is verified.

To show (1.10), by (1.9), we find that (4.5) can be written as

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+1)-\overline{N}(2,6,3n+1))q^n=\frac{2J_6^3}{J_{1,6}J_2}=\frac{(q^6;q^6)_\infty}{(q^2,q^4;q^6)_\infty}\frac{2}{(q,q^5;q^6)_\infty}. \tag{4.7}$$

It is easy to see that the coefficients of q^n in

$$\frac{2}{(q,q^5;q^6)_\infty}$$

are positive for $n\geq 0$. From Theorem 4.1, we see that the coefficients of q^n in

$$\frac{(q^6;q^6)_\infty}{(q^2,q^4;q^6)_\infty}$$

are nonnegative for $n\geq 0$. Hence, from (4.7), we deduce that $\overline{N}(0,6,3n+1)>\overline{N}(2,6,3n+1)$ for $n\geq 0$, and so (1.10) is verified.

(3) To show (1.11), we first establish the generating function of $\overline{N}(0,6,3n+2)-\overline{N}(2,6,3n+2)$. We aim to show that

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)-\overline{N}(2,6,3n+2))q^n\\&=-\frac{2(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+6n+1}}{1-q^{6n+2}}.\end{aligned} \tag{4.8}$$

By Theorem 1.1, we derive

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)-\overline{N}(2,6,3n+2)-\overline{N}(3,6,3n+2))q^n\\&=\frac{4J_6^3}{J_2J_{3,6}}-\frac{2}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}}.\end{aligned} \tag{4.9}$$

Together with the identity (1.4) in [22]

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,3,3n+2)-\overline{N}(1,3,3n+2))q^n\\&=\frac{4(-q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty}-\frac{6(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{3n+1}}\\&=\frac{4J_6^3}{J_2J_{3,6}}-\frac{6}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{3n+1}},\end{aligned} \tag{4.10}$$

and by (4.3), we obtain

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)-\overline{N}(2,6,3n+2))q^n$$
$$=\frac{4J_6^3}{J_2J_{3,6}}-\frac{3}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{3n+1}}-\frac{1}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}}. \tag{4.11}$$

Then the identity (4.8) can be derived from (4.11) by using the following identity in [27, p.1],

$$\frac{J_1^3}{j(z;q)}=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\binom{n+1}{2}}}{1-zq^n}. \tag{4.12}$$

Replacing $q\to q^6$ and setting $z=q^2$ in (4.12), we have

$$\frac{J_6^3}{J_2}=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{6n+2}}$$
$$=\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}}+\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{3n+1}}. \tag{4.13}$$

Plug (4.13) into (4.11) to get

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)-\overline{N}(2,6,3n+2))q^n$$
$$=\frac{1}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}}-\frac{1}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{3n+1}}$$
$$=-\frac{2(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+6n+1}}{1-q^{6n+2}},$$

which is the desired generating function (4.8).

We now consider the negativity of (4.8). Note that

$$\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+6n+1}}{1-q^{6n+2}}$$
$$=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^nq^{3n^2+6n+1}}{1-q^{6n+2}}+\sum_{n=0}^{\infty}\frac{(-1)^nq^{3n^2+6n+2}}{1-q^{6n+4}}\right)$$
$$=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+1}}{1-q^{12n+2}}-\sum_{n=0}^{\infty}\frac{q^{12n^2+24n+10}}{1-q^{12n+8}}+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{12n+4}}-\sum_{n=0}^{\infty}\frac{q^{12n^2+24n+11}}{1-q^{12n+10}}\right)$$
$$=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+1}}{(1-q^{12n+2})(1-q^{12n+8})}\left((1-q^{12n+8})(1-q^{12n+9})+q^{24n+11}(1-q^6)\right)$$

$$+\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{(1-q^{12n+4})(1-q^{12n+10})}\left((1-q^{12n+9})(1-q^{12n+10})+q^{24n+13}(1-q^6)\right)$$

$$=\frac{(-q^3;q^3)_\infty}{(q^6;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+1}}{1-q^{12n+2}}\frac{1-q^{12n+9}}{1-q^3}+\sum_{n=0}^{\infty}\frac{q^{12n^2+36n+12}(1+q^3)}{(1-q^{12n+2})(1-q^{12n+8})}\right.$$
$$\left.+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{12n+4}}\frac{1-q^{12n+9}}{1-q^3}+\sum_{n=0}^{\infty}\frac{q^{12n^2+36n+15}(1+q^3)}{(1-q^{12n+4})(1-q^{12n+10})}\right)$$

$$=\frac{(-q^3;q^3)_\infty}{(q^6;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+1}}{1-q^{12n+2}}\sum_{m=0}^{4n+2}q^{3m}+\sum_{n=0}^{\infty}\frac{q^{12n^2+36n+12}(1+q^3)}{(1-q^{12n+2})(1-q^{12n+8})}\right.$$
$$\left.+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{12n+4}}\sum_{m=0}^{4n+2}q^{3m}+\sum_{n=0}^{\infty}\frac{q^{12n^2+36n+15}(1+q^3)}{(1-q^{12n+4})(1-q^{12n+10})}\right).$$

It is easy to check that each term in the above identity has nonnegative coefficients. In particular, the terms corresponding to $n=0$ and $m=0,1$ in the first series of the above identity

$$\frac{q}{1-q^2}+\frac{q^4}{1-q^2}$$

have positive coefficients. Hence the coefficients of q^n in (4.8) are negative for $n\geq 1$, and hence $\overline{N}(0,6,3n+2)<\overline{N}(2,6,3n+2)$ for $n\geq 1$. This completes the proof of (1.11).

(4) We finish the proof of Theorem 1.4 by showing (1.12). Instead, we aim to show that for $n\geq 0$

$$\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)>\overline{N}(2,6,3n+2)+\overline{N}(3,6,3n+2). \tag{4.14}$$

Inequality (1.12) can be derived by subtracting (1.11) from (4.14).

Combining (4.9) and (4.13), we find that

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)-\overline{N}(2,6,3n+2)-\overline{N}(3,6,3n+2))q^n$$
$$=\frac{2(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{3n+1}}. \tag{4.15}$$

We now investigate the positivity of (4.15). Observe that

$$\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{3n+1}}$$
$$=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{q^{12n^2+6n}}{1-q^{6n+1}}-\sum_{n=-\infty}^{\infty}\frac{q^{12n^2+18n+6}}{1-q^{6n+4}}\right)$$

$$
\begin{aligned}
&= \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+6n}}{1-q^{6n+1}} - \sum_{n=0}^{\infty}\frac{q^{12n^2+24n+11}}{1-q^{6n+5}} - \sum_{n=0}^{\infty}\frac{q^{12n^2+18n+6}}{1-q^{6n+4}} + \sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{6n+2}}\right)\\
&= \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+6n}}{(1-q^{6n+1})(1-q^{6n+4})}((1-q^{6n+4})(1-q^{12n+6})+q^{18n+7}(1-q^3))\right.\\
&\qquad\left.+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{(1-q^{6n+2})(1-q^{6n+5})}((1-q^{6n+5})(1-q^{12n+9})+q^{18n+11}(1-q^3))\right)\\
&= \frac{(-q^3;q^3)_\infty}{(q^6;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+6n}}{1-q^{6n+1}}\sum_{m=0}^{4n+1}q^{3m} + \sum_{n=0}^{\infty}\frac{q^{12n^2+24n+7}}{(1-q^{6n+1})(1-q^{6n+4})}\right)\\
&\quad + \frac{(-q^3;q^3)_\infty}{(q^6;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{6n+2}}\sum_{m=0}^{4n+2}q^{3m} + \sum_{n=0}^{\infty}\frac{q^{12n^2+30n+13}}{(1-q^{6n+2})(1-q^{6n+5})}\right).
\end{aligned}
$$

It is easy to see that each term of the above identity has nonnegative coefficients. Especially, from $n=0$ and $m=0$ in the first series of the above identity, we get the term $1/(1-q)$ which gives strictly positive coefficients of q^n for $n \geq 1$. Hence (4.14) holds and (1.12) is proved. Thus we complete the proof of Theorem 1.4. ∎

We conclude this section by showing Theorem 1.5.

Proof of Theorem 1.5. From Theorem 1.2, we derive that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(\overline{N}(0,10,5n)+\overline{N}(1,10,5n)-\overline{N}(4,10,5n)-\overline{N}(5,10,5n))q^n\\
&= \frac{J_{2,10}^2 J_{3,10}^2 J_{5,10}^4}{J_{1,2}^3 J_{4,10} J_{10}^3} + \frac{8q^2 J_{1,10} J_{10}^3}{J_{1,2}^2 J_{4,10}}\\
&= \frac{(q^5;q^5)_\infty^2}{(q,q^4;q^5)_\infty^2}\frac{1}{(q^{10};q^{10})_\infty(q,q^9;q^{10})_\infty^2(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty(q^3,q^7;q^{10})_\infty^3}\\
&\quad + \frac{(q^{10};q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}\frac{8q^2}{(q,q^9;q^{10})_\infty^3(q^2,q^8;q^{10})_\infty^2(q^3,q^7;q^{10})_\infty^3(q^4,q^6;q^{10})_\infty^3(q^5;q^{10})_\infty^4}.
\end{aligned}
$$

By Theorem 4.1, we see that the coefficients of q^n in

$$
\frac{(q^5;q^5)_\infty^2}{(q,q^4;q^5)_\infty^2} \quad \text{and} \quad \frac{(q^{10};q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}
$$

are nonnegative for $n \geq 0$ respectively. The series $1/(q;q)_\infty$ gives strictly positive coefficients of q^n for $n \geq 0$. This leads to the inequality (1.13). Thus, we completes the proof of Theorem 1.5. ∎

5 Proofs of Theorem 1.8 and Theorem 1.9

This section is devoted to showing the relations between the rank differences of overpartitions and mock theta functions stated in Theorem 1.8 and Theorem 1.9. It is known that mock theta functions can be expressed in terms of the Appell-Lerch sum $m(x, q, z)$. Recall that the Appell-Lerch sum is defined as

$$m(x, q, z) := \frac{1}{j(z; q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1 - q^{r-1} xz}, \tag{5.1}$$

where $x, z \in \mathbb{C}^*$ such that neither z nor xz is an integral power of q.

The third order mock theta functions $\omega(q)$ and $\rho(q)$ and the tenth order mock theta functions $\phi(q)$ and $\psi(q)$ can be expressed in term of $m(x, q, z)$ as follows, see [17]

$$\omega(q) = -2q^{-1} m(q, q^6, q^2) + \frac{J_6^3}{J_2 J_{3,6}}, \tag{5.2}$$

$$\rho(q) = q^{-1} m(q, q^6, -q), \tag{5.3}$$

$$\phi(q) = -2q^{-1} m(q, q^{10}, q^2) + \frac{J_5 J_{10} J_{4,10}}{J_{2,5} J_{2,10}}, \tag{5.4}$$

$$\psi(q) = -2m(q^3, q^{10}, q) - \frac{q J_5 J_{10} J_{2,10}}{J_{1,5} J_{4,10}}. \tag{5.5}$$

In order to prove Theorem 1.8 and Theorem 1.9, we also need to recall the universal mock theta function $g_2(x, q)$ defined by Gordon and McIntosh [14]

$$g_2(x, q) := \frac{1}{J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1 - xq^n}.$$

Hickerson and Mortenson [17, Proposition 4.4] showed that $g_2(x, q)$ and $m(x, q, z)$ have the following relation,

$$g_2(x, q) = -x^{-1} m(x^{-2} q, q^2, x). \tag{5.6}$$

It should be noted that Hickerson and Mortenson use the notation $h(x, q)$ instead of $g_2(x, q)$ in [17]. It also should be noted that $g_2(x, q)$ has the following relation with the generating function $\overline{R}(z; q)$ for $\overline{N}(m, n)$, see [26, (3.2)].

$$(1 + z)\overline{R}(z; q) = (1 - z) + 2z(1 - z) g_2(z, q). \tag{5.7}$$

In [26], the author use the notation $K_2(x, q)$ to express $\overline{R}(x; q)$ and $H(x, q)$ to express $g_2(x, q)$.

The following two identities on $m(x, q, z)$ are also required in the proof of Theorem 1.9.

$$m(x, q, z) = x^{-1} m(x^{-1}, q, z^{-1}), \tag{5.8}$$

$$m(x,q,z_1)-m(x,q,z_0)=\frac{z_0J_1^3j(z_1/z_0;q)j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}. \tag{5.9}$$

see [17, Proposition 3.1, Theorem 3.3].

We are now in a position to give a proof of Theorem 1.8.

Proof of Theorem 1.8. From Theorem 1.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)-\overline{N}(2,6,3n+2)-\overline{N}(3,6,3n+2))q^n\\
&=\frac{4J_6^3}{J_2J_{3,6}}-\frac{2}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}}=\frac{4J_6^3}{J_2J_{3,6}}-2g_2(-q,q^3).
\end{aligned} \tag{5.10}$$

Replacing q by q^3 in (5.6) and setting $x=-q$, we have

$$g_2(-q,q^3)=q^{-1}m(q,q^6,-q), \tag{5.11}$$

and by (5.3), we deduce that

$$\rho(q)=g_2(-q,q^3). \tag{5.12}$$

Together with the following identity in [30, p. 63]

$$\omega(q)+2\rho(q)=\frac{3J_6^3}{J_2J_{3,6}}, \tag{5.13}$$

we find that (5.10) can be transformed as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)-\overline{N}(2,6,3n+2)-\overline{N}(3,6,3n+2))q^n\\
&=\frac{4}{3}(\omega(q)+2\rho(q))-2\rho(q)\\
&=\frac{4}{3}\omega(q)+\frac{2}{3}\rho(q),
\end{aligned}$$

which is (1.16). Thus we complete the proof of Theorem 1.8. ∎

We finish this paper with the proof of Theorem 1.9.

Proof of Theorem 1.9. (1) We first show (1.18). By Theorem 1.2, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\overline{N}(0,10,5n+1)+\overline{N}(1,10,5n+1)-\overline{N}(4,10,5n+1)-\overline{N}(5,10,5n+1))q^n\\
&=\frac{2J_{4,10}J_{5,10}^4J_{10}^3}{J_{1,2}^4J_{2,10}^2J_{3,10}}-8q\frac{J_{1,10}^4J_{4,10}^3J_{5,10}^4}{J_{1,2}^5J_{2,10}^2J_{10}^3}-16q^3\frac{J_{2,10}^4J_{3,10}^2J_{5,10}J_{10}^3}{J_{1,2}^4J_{4,10}^5}
\end{aligned}$$

$$+\frac{2q}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+5n}}{1+q^{5n+2}}. \tag{5.14}$$

We claim that the following identity holds.

$$\frac{1}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+5n}}{1+q^{5n+2}}=-\frac{1}{2}q^{-1}\phi(q)+\frac{J_5J_{10}J_{4,10}}{2qJ_{2,5}J_{2,10}}+\frac{J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{2,10}\overline{J}_{2,10}J_{3,10}\overline{J}_{3,10}}. \tag{5.15}$$

Identity (1.18) can be derived by plugging (5.15) into (5.14).

From the definition of $g_2(x,q)$, we find that

$$\frac{1}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+5n}}{1+q^{5n+2}}=g_2(-q^2,q^5). \tag{5.16}$$

Letting $q\to q^5$ and setting $x=-q^2$ in (5.6) yields

$$g_2(-q^2,q^5)=q^{-2}m(q,q^{10},-q^2). \tag{5.17}$$

Replacing q by q^{10}, putting $z_1=-q^2$, $z_0=q^2$ and $x=q$ in (5.9), it follows that

$$m(q,q^{10},-q^2)-m(q,q^{10},q^2)=\frac{q^2J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{2,10}\overline{J}_{2,10}J_{3,10}\overline{J}_{3,10}}. \tag{5.18}$$

Substituting (5.18) into (5.17) and by (5.4), we derive that

$$g_2(-q^2,q^5)=-\frac{1}{2}q^{-1}\phi(q)+\frac{J_5J_{10}J_{4,10}}{2qJ_{2,5}J_{2,10}}+\frac{J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{2,10}\overline{J}_{2,10}J_{3,10}\overline{J}_{3,10}}. \tag{5.19}$$

Hence, combining (5.19) and (5.16), we obtain (5.15). This completes the proof of (1.18).

(2) Analogue to the above process, we proceed to show (1.19). By Theorem 1.3, we see that

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(1,10,5n+1)+\overline{N}(2,10,5n+1)-\overline{N}(3,10,5n+1)-\overline{N}(4,10,5n+1))q^n\\&=\frac{8qJ_1^8J_{10}^7J_{5,10}}{J_{1,2}^6J_{2,10}^4J_{4,10}^5}-\frac{8q^2J_{10}^3J_{1,10}^2J_{5,10}^2}{J_{1,2}^4J_{2,10}J_{3,10}}+\frac{16q^3J_{10}^6J_{2,10}^6}{J_1^6J_{1,10}^2J_{4,10}^3}-\frac{2q}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+5n}}{1+q^{5n+2}}.\end{aligned} \tag{5.20}$$

Then the desired identity (1.19) can be immediately obtained when substituting (5.15) into (5.20).

(3) Finally, we show (1.20). Similarly, using Theorem 1.3, we derive that

$$\sum_{n=0}^{\infty}(\overline{N}(1,10,5n+4)+\overline{N}(2,10,5n+4)-\overline{N}(3,10,5n+4)-\overline{N}(4,10,5n+4))q^n$$

$$= \frac{8J_{10}^3 J_{3,10}^2}{J_{1,2}^2 J_{1,10} J_{4,10}} - \frac{4J_{1,10}^3 J_{4,10}^3 J_{5,10}^5}{J_1^2 J_{1,2}^4 J_{10}^4} + \frac{16q^2 J_{10}^8}{J_1^2 J_{1,2}^2 J_{3,10}^2 J_{4,10}} + \frac{32q^2 J_{10}^6 J_{2,10}^3}{J_1^6 J_{1,10} J_{5,10}} - \frac{32q^3 J_{10}^3 J_{1,10}^4}{J_{1,2}^4 J_{2,10} J_{3,10}}$$
$$- \frac{2}{J_{5,10}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+5n}}{1+q^{5n+4}}. \tag{5.21}$$

To prove (1.20), it is necessary to show that

$$\frac{1}{J_{5,10}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+5n}}{1+q^{5n+4}} = -\frac{1}{2} q^{-1}\psi(q) - \frac{J_5 J_{10} J_{2,10}}{2J_{1,5} J_{4,10}} + \frac{J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{1,10} \overline{J}_{1,10} J_{4,10} \overline{J}_{4,10}}. \tag{5.22}$$

We then obtain (1.20) upon substituting (5.22) into (5.21).

From the definition of $g_2(x,q)$, we note that

$$\frac{1}{J_{5,10}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+5n}}{1+q^{5n+4}} = g_2(-q^4, q^5). \tag{5.23}$$

Letting $q \to q^5$ and setting $x = -q^4$ in (5.6) yields

$$g_2(-q^4, q^5) = q^{-4} m(q^{-3}, q^{10}, -q^4). \tag{5.24}$$

On the other hand, by letting $q \to q^{10}$, $x = q^3$ and $z = -q^{-4}$ in (5.8), we obtain

$$m(q^3, q^{10}, -q^{-4}) = q^{-3} m(q^{-3}, q^{10}, -q^4). \tag{5.25}$$

Combining (5.24) and (5.25) yields

$$g_2(-q^4, q^5) = q^{-1} m(q^3, q^{10}, -q^{-4}). \tag{5.26}$$

Replacing q by q^{10}, putting $z_1 = -q^{-4}$, $z_0 = q$ and $x = q^3$ in (5.9), it follows that

$$m(q^3, q^{10}, -q^{-4}) - m(q^3, q^{10}, q) = \frac{q J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{1,10} \overline{J}_{1,10} J_{4,10} \overline{J}_{4,10}}. \tag{5.27}$$

Substituting (5.27) into (5.26) and by (5.5), we derive that

$$g_2(-q^4, q^5) = -\frac{1}{2} q^{-1}\psi(q) - \frac{J_5 J_{10} J_{2,10}}{2J_{1,5} J_{4,10}} + \frac{J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{1,10} \overline{J}_{1,10} J_{4,10} \overline{J}_{4,10}}. \tag{5.28}$$

Thus (5.22) follows from (5.23) and (5.28). This completes the proof of Theorem 1.9. ∎

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