The crank moments weighted by the parity of cranks

Kathy Q. Ji[1] and Alice X.H. Zhao[2]

[1]Center for Applied Mathematics
Tianjin University, Tianjin 300072, P.R. China
kathyji@tju.edu.cn

[2]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
zhaoxiaohua@mail.nankai.edu.cn

Abstract. In this note, we introduce the $2k$-th crank moment $\mu_{2k}(-1,n)$ weighted by the parity of cranks and show that $(-1)^n\mu_{2k}(-1,n)>0$ for $n\geq k\geq 0$. When $k=0$, the inequality $(-1)^n\mu_{2k}(-1,n)>0$ reduces to Andrews and Lewis's inequality $(-1)^n(M_e(n)-M_o(n))>0$ for $n\geq 0$, where $M_e(n)$ (resp. $M_o(n)$) denotes the number of partitions of n with even (resp. odd) crank. Several generating functions of $\mu_{2k}(-1,n)$ are also studied in order to show the positivity of $(-1)^n\mu_{2k}(-1,n)$.

Keywords: Partition, crank moments, positivity, parity.

2000 AMS Classification Numbers: Primary, 11P81; Secondary, 05A17

1 Introduction

Dyson's rank [8] and the Andrews-Garvan-Dyson crank [4] are two fundamental statistics in the theory of partitions. They provide combinatorial explanations for Ramanujan's famous congruences of the partition function $p(n)$, where $p(n)$ counts the number of partitions of n. A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1,\lambda_2,\ldots,\lambda_r)$ such that $\sum_{i=1}^r\lambda_i=n$. For a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$, the rank of λ, denoted by $r(\lambda)$, is the largest part of λ minus the number of parts. The crank [4] is defined by

$$c(\lambda)=\begin{cases}\lambda_1, & \text{if } n_1(\lambda)=0,\\ \mu(\lambda)-n_1(\lambda), & \text{if } n_1(\lambda)>0,\end{cases}$$

where $n_1(\lambda)$ is the number of ones in λ and $\mu(\lambda)$ is the number of parts larger than $n_1(\lambda)$. For $n>1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for

$n \leq 1$, we set $M(0,1) = -1$, $M(0,0) = M(-1,1) = M(1,1) = 1$, $M(m,n) = 0$ otherwise. The generating function for $M(m,n)$ is given in [9, 10].

$$\begin{aligned}
C(z,q) &:= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n) z^m q^n \\
&= \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty} \\
&= \frac{1-z}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{1-zq^n}.
\end{aligned}$$

Here and throughout the paper, we adopt the standard notation on q-series [1].

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n), \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

We let $M_e(n)$ (resp. $M_o(n)$) denote the number of partitions of n with even (resp. odd) crank. The first study of $M_e(n) - M_o(n)$ was done by Andrews and Lewis [5]. By setting $z = -1$ in (1.1), we get

$$\sum_{n=0}^{\infty} (M_e(n) - M_o(n)) q^n = \frac{(q;q)_\infty}{(-q;q)_\infty^2}.$$

Andrews and Lewis proved that

Theorem 1.1 (Andrews-Lewis)**.** *For $n \geq 0$, $(-1)^n (M_e(n) - M_o(n)) > 0$.*

In [7], Choi, Kang and Lovejoy established congruences and asymptotic properties satisfied by $M_e(n) - M_o(n)$.

Analogous to the symmetrized rank moments defined by Andrews [3], Garvan [11] introduced the symmetrized crank moments in the study of the higher order spt-function. To be more specific, the k-th symmetrized crank moment $\mu_k(n)$ is defined as follows.

$$\mu_k(n) = \sum_{m=-\infty}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m,n). \tag{1.1}$$

From the symmetry $M(m,n) = M(-m,n)$, it is clear that $\mu_{2k+1}(n) = 0$. As for an even symmetrized moment $\mu_{2k}(n)$, Chen, Ji and Shen [6] gave a combinatorial interpretation of $\mu_{2k}(n)$ by introducing k-marked Dyson symbols.

In this paper, we introduce the crank moments weighted by the parity of cranks.

$$\mu_{2k}(-1,n) = \sum_{m=-\infty}^{\infty} \binom{m+k-1}{2k} (-1)^m M(m,n). \tag{1.2}$$

When $k=0$, it reduces to

$$\mu_0(-1,n)=M_e(n)-M_o(n).$$

Furthermore, $\mu_{2k}(-1,n)$ is a linear combination of the twisted crank moments $M_k(-1,n)$ introduced by Rhoades [12] as given by

$$M_k(-1,n)=\sum_{m=-\infty}^{\infty}(-1)^m m^k M(m,n). \tag{1.3}$$

For example,

$$\mu_4(-1,n)=\frac{1}{24}M_4(-1,n)-\frac{1}{24}M_2(-1,n).$$

In [12], Rhoades established the congruences and asymptotic properties satisfied by $M_k(-1,n)$.

The main objective of this note is to show the following positivity property of $(-1)^n\mu_{2k}(-1,n)$.

Theorem 1.2. *For $n\geq k\geq 0$, $(-1)^n\mu_{2k}(-1,n)>0$.*

It should be noted that Theorem 1.1 is the case $k=0$ of Theorem 1.2. To prove Theorem 1.2, we first establish the following explicit generating function for $\mu_{2k}(-1,n)$ with the aid of Andrews's k-fold generalization of q-Whipple's theorem [2, p.199, Theorem 4].

Theorem 1.3. *For $k\geq 0$, we have*

$$\sum_{n=0}^{\infty}\mu_{2k}(-1,n)q^n=\frac{(q;q)_\infty}{(-q;q)_\infty^2}\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 1}\frac{(-1)^k q^{n_1+n_2+\cdots+n_k}}{(1+q^{n_1})^2(1+q^{n_2})^2\cdots(1+q^{n_k})^2}. \tag{1.4}$$

We next show that the above generating function (1.6) is equivalent to the following form which plays crucial role in the proof of Theorem 1.2.

Theorem 1.4. *For $k\geq 0$, we have*

$$\sum_{n=0}^{\infty}\mu_{2k}(-1,n)q^n=\frac{(q;q)_\infty}{(-q;q)_\infty^2}\sum_{m_k>m_{k-1}>\cdots>m_1\geq 1}\frac{(-1)^{m_k}m_1(m_2-m_1)\cdots(m_k-m_{k-1})q^{m_k}}{(1-q^{m_1})(1-q^{m_2})\cdots(1-q^{m_k})}. \tag{1.5}$$

The paper is organized as follows. In Section 2, we first introduce the generalized crank moments $\mu_{2k}(z,n)$. When $z=1$, $\mu_{2k}(z,n)$ corresponds to the crank moment $\mu_{2k}(n)$ and when $z=-1$, $\mu_{2k}(z,n)$ is the crank moment $\mu_{2k}(-1,n)$ weighted by the parity of cranks. Then we establish the generating function of $\mu_{2k}(z,n)$, and hence we derive a generating function of $\mu_{2k}(-1,n)$ stated in Corollary 2.2. In Section 3, we first derive the generating function of $\mu_{2k}(-1,n)$ in Theorem 1.3 by applying Andrews's k-fold generalization of q-Whipple's theorem. Then we show the generating function in Theorem 1.4 is equivalent to the generating function in Theorem 1.3. In Section 4, we show Theorem 1.2 follows from Theorem 1.4 and Jacobi's triple product identity.

2 The generating function of $\mu_{2k}(z,n)$

We define $\mu_{2k}(z,n)$ as

$$\mu_{2k}(z,n)=\sum_{m=-\infty}^{\infty}\binom{m+k-1}{2k}z^m M(m,n). \tag{2.1}$$

When $z=1$, $\mu_{2k}(z,n)$ corresponds to $\mu_{2k}(n)$. When $z=-1$, $\mu_{2k}(z,n)$ coincides with $\mu_{2k}(-1,n)$. We have the following generating function for $\mu_{2k}(z,n)$.

Theorem 2.1. *We have*

$$\sum_{n=0}^{\infty}\mu_{2k}(z,n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{\binom{n}{2}}(1-q^n)\Big(\frac{z^{k+1}q^{n(k+1)}}{(1-zq^n)^{2k+1}}+\frac{z^{-k}q^{nk}}{(1-z^{-1}q^n)^{2k+1}}\Big). \tag{2.2}$$

Proof. As in the proof of [11, Theorem 2.2], we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mu_{2k}(z,n)q^n&=\frac{z^{k+1}}{(2k)!}\left(\frac{\partial}{\partial z}\right)^{2k}z^{k-1}C(z,q)\\
&=\frac{z^{k+1}}{(2k)!}\sum_{j=0}^{k-1}\binom{2k}{j}(k-1)(k-2)\cdots(k-j)z^{k-1-j}C^{(2k-j)}(z,q)\\
&=\frac{1}{(q;q)_\infty}\sum_{j=0}^{k-1}\binom{k-1}{j}z^{2k-j}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^{n-1}q^{\binom{n}{2}+(2k-j)n}(1-q^n)}{(1-zq^n)^{2k-j+1}}\\
&=\frac{1}{(q;q)_\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^{n-1}q^{\binom{n}{2}+2kn}(1-q^n)z^{2k}}{(1-zq^n)^{2k+1}}\left(1+\frac{z^{-1}q^{-n}}{(1-zq^n)^{-1}}\right)^{k-1}\\
&=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{\binom{n}{2}}(1-q^n)\Big(\frac{z^{k+1}q^{n(k+1)}}{(1-zq^n)^{2k+1}}+\frac{z^{-k}q^{nk}}{(1-z^{-1}q^n)^{2k+1}}\Big).
\end{aligned}$$

Thus we have completed the proof of Theorem 2.1. ∎

Set $z=-1$ in (2.2), we get

Corollary 2.2. *For $k\geq 0$*

$$\sum_{n=0}^{\infty}\mu_{2k}(-1,n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n+k-1}q^{\binom{n}{2}+nk}\frac{(1-q^n)^2}{(1+q^n)^{2k+1}}. \tag{2.3}$$

3 Proof of Theorems 1.3 and 1.4

In this section, we first prove that Theorem 1.3 follows from Corollary 2.2 and Andrews's k-fold generalization of q-Whipple's theorem and then show that Theorem 1.3 is equivalent to Theorem 1.4. Recall that Andrews's k-fold generalization of q-Whipple's theorem [2, p.199, Theorem 4] is stated as follows. For $k \geq 1$,

$$
\begin{aligned}
&{}_{2k+4}\phi_{2k+3}\left(\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b_1, & c_1, & \cdots, & b_k, & c_k, & q^{-n} \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b_1, & aq/c_1, & \cdots, & aq/b_k, & aq/c_k, & aq^{n+1} \end{matrix} ; q, \frac{a^k q^{n+k}}{b_1\cdots b_k c_1 \cdots c_k}\right) \\
&= \frac{(aq;q)_n (aq/b_k c_k;q)_n}{(aq/b_k;q)_n (aq/c_k;q)_n} \\
&\quad \times \sum_{m_1,\ldots,m_{k-1}\geq 0} \frac{(aq/b_1c_1;q)_{m_1}}{(q;q)_{m_1}} \frac{(aq/b_2c_2;q)_{m_2}}{(q;q)_{m_2}} \cdots \frac{(aq/b_{k-1}c_{k-1};q)_{m_{k-1}}}{(q;q)_{m_{k-1}}} \\
&\quad \times \frac{(b_2;q)_{m_1}}{(aq/b_1;q)_{m_1}} \frac{(c_2;q)_{m_1}}{(aq/c_1;q)_{m_1}} \frac{(b_3;q)_{m_1+m_2}}{(aq/b_2;q)_{m_1+m_2}} \frac{(c_3;q)_{m_1+m_2}}{(aq/c_2;q)_{m_1+m_2}} \\
&\quad \times \frac{\cdots (b_k;q)_{m_1+\cdots+m_{k-1}}}{\cdots (aq/b_{k-1};q)_{m_1+\cdots+m_{k-1}}} \frac{(c_k;q)_{m_1+\cdots+m_{k-1}}}{(aq/c_{k-1};q)_{m_1+\cdots+m_{k-1}}} \frac{(q^{-n};q)_{m_1+\cdots+m_{k-1}}}{(b_kc_kq^{-n}/a;q)_{m_1+\cdots+m_{k-1}}} \\
&\quad \times \frac{(aq)^{m_{k-2}+2m_{k-3}+\cdots+(k-2)m_1} q^{m_1+m_2+\cdots+m_{k-1}}}{(b_2c_2)^{m_1}(b_3c_3)^{m_1+m_2}\cdots(b_{k-1}c_{k-1})^{m_1+m_2+\cdots+m_{k-2}}}.
\end{aligned} \tag{3.1}
$$

We are now in position to give a proof of Theorem 1.3.

Proof of Theorem 1.3: In (3.1), replacing k by $k+1$, setting $b_i = c_i = -1$ for $1 \leq i \leq k+1$, $a = 1$ and putting $n \to \infty$, after simplification, we obtain

$$
\begin{aligned}
&1 + \sum_{n=1}^{\infty} \frac{2^{2k+2}(-1)^n q^{\binom{n+1}{2}+kn}}{(1+q^n)^{2k+1}} \\
&= \frac{(q;q)_\infty^2}{(-q;q)_\infty^2} \sum_{m_1,\ldots,m_k\geq 0} \frac{(-1;q)_{m_1}^2 (-1;q)_{m_1+m_2}^2 \cdots (-1;q)_{m_1+m_2+\cdots+m_k}^2 q^{km_1+(k-1)m_2+\cdots+m_k}}{(-q;q)_{m_1}^2 (-q;q)_{m_1+m_2}^2 \cdots (-q;q)_{m_1+m_2+\cdots+m_k}^2}.
\end{aligned} \tag{3.2}
$$

If we make the substitution $n_1 = m_1,\ n_2 = m_1 + m_2,\ \ldots,\ n_k = m_1 + m_2 + \cdots + m_k$, then the right hand side of (3.2) becomes

$$
\frac{(q;q)_\infty^2}{(-q;q)_\infty^2} \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(-1;q)_{n_1}^2 (-1;q)_{n_2}^2 \cdots (-1;q)_{n_k}^2 q^{n_1+n_2+\cdots+n_k}}{(-q;q)_{n_1}^2 (-q;q)_{n_2}^2 \cdots (-q;q)_{n_k}^2}.
$$

Thus (3.2) can be written as

$$1+\sum_{n=1}^{\infty}\frac{2^{2k+2}(-1)^n q^{\binom{n+1}{2}+kn}}{(1+q^n)^{2k+1}} \tag{3.3}$$
$$=\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{(-1;q)_{n_1}^2(-1;q)_{n_2}^2\cdots(-1;q)_{n_k}^2 q^{n_1+n_2+\cdots+n_k}}{(-q;q)_{n_1}^2(-q;q)_{n_2}^2\cdots(-q;q)_{n_k}^2}.$$

Now let us examine the term $n_1=0$ on the right hand of (3.3), that is

$$\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_2\geq 0}\frac{(-1;q)_{n_2}^2(-1;q)_{n_3}^2\cdots(-1;q)_{n_k}^2 q^{n_2+n_3+\cdots+n_k}}{(-q;q)_{n_2}^2(-q;q)_{n_3}^2\cdots(-q;q)_{n_k}^2}. \tag{3.4}$$

Using (3.3) for k replaced by $k-1$, we see that (3.4) is equal to

$$1+\sum_{n=1}^{\infty}\frac{2^{2k}(-1)^n q^{\binom{n}{2}+kn}}{(1+q^n)^{2k-1}}.$$

Hence

$$\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 1}\frac{(-1;q)_{n_1}^2(-1;q)_{n_2}^2\cdots(-1;q)_{n_k}^2 q^{n_1+n_2+\cdots+n_k}}{(-q;q)_{n_1}^2(-q;q)_{n_2}^2\cdots(-q;q)_{n_k}^2}$$
$$=\left(1+\sum_{n=1}^{\infty}\frac{2^{2k+2}(-1)^n q^{\binom{n+1}{2}+kn}}{(1+q^n)^{2k+1}}\right)-\left(1+\sum_{n=1}^{\infty}\frac{2^{2k}(-1)^n q^{\binom{n}{2}+kn}}{(1+q^n)^{2k-1}}\right) \tag{3.5}$$
$$=\sum_{n=1}^{\infty}\frac{2^{2k}(-1)^{n-1}q^{\binom{n}{2}+kn}(1-q^n)^2}{(1+q^n)^{2k+1}}.$$

Multiplying both sides of (3.5) by $(-1)^k/(q;q)_\infty$, by Corollary 2.2 and the standard algebraic manipulation, we obtain (1.6). This is complete the proof of Theorem 1.3. ∎

To estimate the sign of $\mu_{2k}(-1,n)$, we need to reform (1.6) in Theorem 1.3 to (1.7) in Theorem 1.4.

Proof of Theorem 1.4: By Theorem 1.3, it suffices to show that for $k\geq 0$,

$$\sum_{n_k\geq n_{k-1}\geq\cdots\geq n_1\geq 1}\frac{(-1)^k q^{n_1+n_2+\cdots+n_k}}{(1+q^{n_1})^2(1+q^{n_2})^2\cdots(1+q^{n_k})^2}$$
$$=\sum_{m_k>m_{k-1}>\cdots>m_1\geq 1}\frac{(-1)^{m_k}m_1(m_2-m_1)\cdots(m_k-m_{k-1})q^{m_k}}{(1-q^{m_1})(1-q^{m_2})\cdots(1-q^{m_k})}. \tag{3.6}$$

In general, we wish to show that for $m \geq 0$ and $k \geq 0$,

$$\begin{aligned} &\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-1)^k q^{(m+1)n_k + n_{k-1} + \cdots + n_1}}{(1+q^{n_1})^2 (1+q^{n_2})^2 \cdots (1+q^{n_k})^2} \\ &= \sum_{m_k > m_{k-1} > \cdots > m_1 > m} \frac{(-1)^{m_k - m}(m_1 - m)(m_2 - m_1) \cdots (m_k - m_{k-1}) q^{m_k}}{(1-q^{m_1})(1-q^{m_2}) \cdots (1-q^{m_k})}, \end{aligned} \tag{3.7}$$

which turns to (3.6) when $m = 0$.

We proceed by induction on k. It is trivial for $k = 0$. For $k = 1$, it follows that

$$\begin{aligned} &\sum_{n_1 \geq 1} \frac{-q^{(m+1)n_1}}{(1+q^{n_1})^2} \\ &= \sum_{n_1 \geq 1} q^{mn_1} \sum_{m_1 \geq 1} (-1)^{m_1} m_1 q^{m_1 n_1} \\ &= \sum_{m_1 \geq 1} \frac{(-1)^{m_1} m_1 q^{m+m_1}}{1 - q^{m+m_1}} \\ &= \sum_{m_1 > m} \frac{(-1)^{m_1 - m}(m_1 - m) q^{m_1}}{1 - q^{m_1}}. \end{aligned}$$

Thus (3.7) holds when $k = 1$.

Following the same argument as above, we could derive the following identity.

$$\sum_{n_2 \geq n_1} \frac{-q^{(m+1)n_2}}{(1+q^{n_2})^2} = \sum_{m_1 > m} \frac{(-1)^{m_1 - m}(m_1 - m) q^{m_1 n_1}}{1 - q^{m_1}}. \tag{3.8}$$

We assume that (3.7) holds for any positive integer $k-1$. It will be shown that it also holds for k. Exchanging the order of summation, we get

$$\begin{aligned} &\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-1)^k q^{(m+1)n_k + n_{k-1} + \cdots + n_1}}{(1+q^{n_1})^2 (1+q^{n_2})^2 \cdots (1+q^{n_k})^2} \\ &= \sum_{n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-1)^{k-1} q^{n_1 + \cdots + n_{k-1}}}{(1+q^{n_1})^2 \cdots (1+q^{n_{k-1}})^2} \sum_{n_k \geq n_{k-1}} \frac{-q^{(m+1)n_k}}{(1+q^{n_k})^2} \\ &= \sum_{n_{k-1} \geq \cdots \geq n_1 \geq 1} \frac{(-1)^{k-1} q^{n_1 + \cdots + n_{k-1}}}{(1+q^{n_1})^2 \cdots (1+q^{n_{k-1}})^2} \sum_{m_1 > m} \frac{(-1)^{m_1 - m}(m_1 - m) q^{m_1 n_{k-1}}}{1 - q^{m_1}} \qquad \text{by (3.8)} \\ &= \sum_{m_1 > m} \frac{(-1)^{m_1 - m}(m_1 - m)}{1 - q^{m_1}} \sum_{n_{k-1} \geq n_{k-2} \geq \cdots \geq n_1 \geq 1} \frac{(-1)^{k-1} q^{(m_1+1)n_{k-1} + n_{k-2} + \cdots + n_1}}{(1+q^{n_1})^2 \cdots (1+q^{n_{k-1}})^2} \end{aligned}$$

$$= \sum_{m_1>m} \frac{(-1)^{m_1-m}(m_1-m)}{1-q^{m_1}} \sum_{m_k>m_{k-1}>\cdots>m_2>m_1} \frac{(-1)^{m_k-m_1}(m_2-m_1)\cdots(m_k-m_{k-1})q^{m_k}}{(1-q^{m_2})\cdots(1-q^{m_k})}$$

$$= \sum_{m_k>m_{k-1}>\cdots>m_1>m} \frac{(-1)^{m_k-m}(m_1-m)(m_2-m_1)\cdots(m_k-m_{k-1})q^{m_k}}{(1-q^{m_1})(1-q^{m_2})\cdots(1-q^{m_k})},$$

where the penultimate step follows by the induction hypothesis. Thus (3.7) holds for k, which implies (3.6) holds for k. Therefore we have completed the proof of Theorem 1.4. ∎

4 Proof of Theorem 1.2

In this section, we aim to prove Theorem 1.2 with the aid of Theorem 1.4.

Proof of Theorem 1.2. Replacing q by $-q$ in (1.7), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n\mu_{2k}(-1,n)q^n\\&=\frac{(-q;-q)_\infty}{(q;-q)_\infty^2}\sum_{m_k>m_{k-1}>\cdots>m_1\geq 1}\frac{m_1(m_2-m_1)\cdots(m_k-m_{k-1})q^{m_k}}{(1-(-q)^{m_1})(1-(-q)^{m_2})\cdots(1-(-q)^{m_k})}.\end{aligned} \tag{4.1}$$

By the standard algebraic manipulations of the infinite products, we see that

$$\begin{aligned}\frac{(-q;-q)_\infty}{(q;-q)_\infty^2}&=(-q;q^2)_\infty(q^2;q^2)_\infty(-q;q^2)_\infty^2 \quad \text{(by Euler's partition theorem)}\\&=(-q;q^2)_\infty\sum_{n=-\infty}^{\infty}q^{n^2} \quad \text{(by Jacobi's triple product identity)}\\&=(-q;q^2)_\infty\left(1+2\sum_{n=1}^{\infty}q^{n^2}\right).\end{aligned}$$

Hence (4.1) becomes

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n\mu_{2k}(-1,n)q^n\\&=\left(1+2\sum_{n=1}^{\infty}q^{n^2}\right)\sum_{m_k>m_{k-1}>\cdots>m_1\geq 1}\frac{m_1(m_2-m_1)\cdots(m_k-m_{k-1})q^{m_k}(-q;q^2)_\infty}{(1-(-q)^{m_1})(1-(-q)^{m_2})\cdots(1-(-q)^{m_k})}.\end{aligned}$$

Given $m_k > m_{k-1} > \cdots > m_1 \geq 1$, define

$$\sum_{m\geq 0} f_{m_1,m_2,\ldots,m_k}(m)q^m := \frac{(-q;q^2)_\infty}{(1-(-q)^{m_1})(1-(-q)^{m_2})\cdots(1-(-q)^{m_k})}.$$

In the above sum, if m_i is odd, then the term in the denominator can be canceled by $(-q;q^2)_\infty$, since the m_i are all distinct. Thus, we deduce that $f_{m_1,m_2,\ldots,m_k}(m) \geq 0$ and $f_{m_1,m_2,\ldots,m_k}(0) = 1$, which impies $(-1)^n \mu_{2k}(-1,n) > 0$ for any nonnegative integer $n \geq k$. Thus, we have completed the proof of Theorem 1.2. ∎

Acknowledgments. We wish to thank the referee for helpful suggestions. This work was supported by the 973 Project, the PCSIRT Project and the National Science Foundation of China.

References

[1] G.E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications, Vol. 2, Addison-Wesley, Reading, 1976 (Reissued: Cambridge University Press, Cambridge, 1985).

[2] G.E. Andrews, Problems and prospects for basic hypergeometric functions, In: Askey, R. (ed.) Theory and Appl. of Special Functions, pp. 191–224. Academic Press, New York (1975).

[3] G.E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.

[4] G.E. Andrews and F.G. Garvan, Dyson's crank of a partition, Bull. Amer. Math. Soc. 18 (1988) 167–171.

[5] G.E. Andrews and R. Lewis, The ranks and cranks of partitions moduli 2, 3 and 4, J. Number Theory 85 (2000) 74–84.

[6] W.Y.C. Chen, K.Q. Ji and E.Y.Y. Shen, k-marked Dyson symbols and congruences for moments of cranks, submitted. arXiv:1312.2080.

[7] D. Choi, S.-Y. Kang and J. Lovejoy, Partitions weighted by the parity of the crank, J. Combin. Theory Ser. A 116 (2009) 1034–1046.

[8] F.J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

[9] F.J. Dyson, Mappings and symmetries of partitions, J. Combin. Theory Ser. A 51 (1989) 169–180.

[10] F.G. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7, 11, Trans. Amer. Math. Soc. 305 (1988) 47–77.

[11] F.G. Garvan, Higher order spt-functions, Adv. Math. 228 (2011) 241–265.

[12] R.C. Rhoades, Families of quasimodular forms and Jacobi forms: the crank statistic for partitions, Proc. Amer. Math. Soc. 141 (2013) 29–39.