COMBINATORIAL PROOF OF A PARTIAL THETA FUNCTION IDENTITY OF WARNAAR

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Abstract. By constructing a sign-reversing involution, we prove Warnaar's identity involving a partial theta function, which plays many important roles in the study of asymptotic behaviors and quantum modularities in number theory. We also obtain a Euler-like theorem for a certain kind of unimodal sequences from Warnaar's identity.

1. Introduction

A partial theta function is a sum of the form

$$\sum_{n=0}^{\infty} q^{An^2+Bn} z^n.$$

We can find many identities involving partial theta functions in Ramanujan's lost notebook [2]. Though recent studies on quantum modular forms [9, 13, 18, 20] and asymptotics [8, 16, 17, 19] shed lights on the role of partial theta functions in number theory and combinatorics, it is still far from the complete understanding of their roles. In particular, in combinatorics, identities containing a partial theta function are very interesting since they indicate what remains after numerous cancellations of certain kinds of partitions. There have been extensive studies on finding combinatorial proofs for these identities [1, 4, 5, 12]. However, as a slight change in an identity makes dramatic changes in its combinatorial nature, finding a combinatorial proof of a new partial theta function identity often gives new insights.

By employing Bailey's Lemma, S.O. Warnaar [15] generalized partial theta function identities of Ramanujan and obtained several new identities. In this paper, we will discuss the following identity of Warnaar.

Theorem 1.1. [15, p. 380] *We have*

$$\sum_{n=0}^{\infty} (-1)^n a^n q^{n^2+n} = \sum_{n=0}^{\infty} \frac{(q;q^2)_n (aq;q^2)_n (aq)^n}{(-aq;q)_{2n+1}}, \tag{1.1}$$

where

$$(a;q)_n := \prod_{j=0}^{n-1} (1-aq^j).$$

Date: April 14, 2015.

2010 *Mathematics Subject Classification.* Primary: 11P81, 05A17.

Key words and phrases. partial theta function, Warnaar's identity, combinatorial proof, partition, overpartition, unimodal sequence.

As mentioned in [15], when $a = 1$, Theorem 1.1 becomes an identity in Ramanujan's lost notebook [19, p. 13] proved by Andrews [2, Eq. (6.2)]. B.C. Berndt and the second author [3] used (1.1) to show that the asymptotic expansion of a partial theta function has integral coefficients. More precisely, via (1.1), one can show that

$$\left(\frac{1-t}{1+t}\right)^{\frac{2b-1}{4}} \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!2^{2n}} \log^n\left(\frac{1+t}{1-t}\right) H_{2n}\left(\frac{b-1}{2}\log^{1/2}\left(\frac{1+t}{1-t}\right)\right)$$

has integral coefficients in t, for all positive integers b, where E_{2n} denotes the $2n$-th Euler number, and $H_n(x)$ is the n-th Hermite polynomial. Moreover, at the same paper, they proved that (1.1) can be used to obtain special values of L-series associated with a polynomial. More recently, K. Brinmann, T. Creutzig, and L. Rolen [6] used (1.1) to get the quantum modularity of the Fourier coefficients of a special kind of Jacobi form of negative index.

The combinatorial nature of (1.1) is also very interesting. In the literature, most partial theta function identities have been proved using a Franklin-type involution. We use a successive chain of involutions to prove (1.1), which can be regarded as a combinatorial telescoping method [10, 11]. In some sense, this is also a combinatorial analogue of Bailey lemma or chains in q-series. We expect this new idea could be applied to a wide variety of identities involving partial theta functions.

2. Combinatorial Proof for Theorem 1.1

By replacing a by b/q in (1.1), we obtain the following equivalent form with Theorem 1.1.

Theorem 2.1. *We have*

$$\sum_{n\ge 0} b^n q^{n^2} = \sum_{n\ge 0} \frac{(-b)^n (q;q^2)_n (-b;q^2)_n}{(b;q)_{2n+1}}.$$

In order to prove Theorem 2.1 we need several definitions.

For a sequence $\mu = (\mu_1, \dots, \mu_n)$, not necessarily a partition, we define $\ell(\mu) = n$ and $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n$.

A *good pair* is a pair (λ, μ) of an overpartition λ and a sequence μ such that $\mu_i \in \{0, 2i-1\}$ for $i = 1, 2, \dots, \ell(\mu)$, the largest part of λ is at most $2\ell(\mu)$, and only $0, 2, 4, \dots, 2\ell(\mu)-2$ can be overlined in λ. By overpartitions, we mean that we may overline the first occurrence of each part. We denote by $\mathcal{G}$ the set of good pairs.

It is easy to see that

$$\frac{(-b)^n (q;q^2)_n (-b;q^2)_n}{(b;q)_{2n+1}} = \sum_{(\lambda,\mu)\in\mathcal{G}, \ell(\mu)=n} b^{\ell(\lambda)+\ell(\mu)} q^{|\lambda|+|\mu|} (-1)^{\mathrm{zero}(\mu)},$$

where $\mathrm{zero}(\mu)$ is the number of 0's in μ. Thus in order to prove Theorem 2.1 it suffices to show the following identity:

$$\sum_{(\lambda,\mu)\in\mathcal{G}} b^{\ell(\lambda)+\ell(\mu)} q^{|\lambda|+|\mu|} (-1)^{\mathrm{zero}(\mu)} = \sum_{n\ge 0} b^n q^{n^2}. \tag{2.1}$$

Let $\mathcal{G}_k = \{(\lambda, \mu) \in \mathcal{G} : \ell(\lambda) + \ell(\mu) = k\}$ and $\mathrm{wt}(\lambda, \mu) = q^{|\lambda|+|\mu|}(-1)^{\mathrm{zero}(\mu)}$. Then (2.1) is equivalent to the following proposition.

Proposition 2.2. *For a nonnegative integer k,*

$$\sum_{(\lambda,\mu)\in\mathcal{G}_k} \mathrm{wt}(\lambda, \mu) = q^{k^2}.$$

Let $\mathcal{G}_{k,r}$ be the set of $(\lambda, \mu) \in \mathcal{G}_k$ such that the last r entries of μ are nonzero and λ has no parts smaller than $2r$. By definition we have $\mathcal{G}_k = \mathcal{G}_{k,0}$, and $\mathcal{G}_{k,k}$ has only one element $(\emptyset, (1, 3, \ldots, 2k - 1))$. Thus

$$\sum_{(\lambda,\mu)\in\mathcal{G}_{k,0}} \mathrm{wt}(\lambda, \mu) = \sum_{(\lambda,\mu)\in\mathcal{G}_k} \mathrm{wt}(\lambda, \mu), \qquad \sum_{(\lambda,\mu)\in\mathcal{G}_{k,k}} \mathrm{wt}(\lambda, \mu) = q^{k^2}.$$

Proposition 2.2 follows from the following lemma.

Lemma 2.3. *Let $0 \le r < k$. Then*

$$\sum_{(\lambda,\mu)\in\mathcal{G}_{k,r}} \mathrm{wt}(\lambda, \mu) = \sum_{(\lambda,\mu)\in\mathcal{G}_{k,r+1}} \mathrm{wt}(\lambda, \mu).$$

Proof. Note that $\mathcal{G}_{k,r+1} \subseteq \mathcal{G}_{k,r}$. We will find a sign-reversing involution ϕ on $\mathcal{G}_{k,r}$ with fixed point set $\mathcal{G}_{k,r+1}$. In other words, for $(\lambda, \mu) \in \mathcal{G}_{k,r}$, if $\phi(\lambda, \mu) = (\lambda', \mu') \neq (\lambda, \mu)$, then $\mathrm{wt}(\lambda', \mu') = -\mathrm{wt}(\lambda, \mu)$, and $\phi(\lambda, \mu) = (\lambda, \mu)$ if and only if $(\lambda, \mu) \in \mathcal{G}_{k,r+1}$. The existence of such an involution clearly implies the desired identity.

Let $(\lambda, \mu) \in \mathcal{G}_{k,r}$ and $n = \ell(\mu)$. Then $\lambda_1 \le 2n$ and $\mu_n = 2n - 1, \mu_{n-1} = 2n - 3, \ldots, \mu_{n-r+1} = 2n - 2r + 1$.

We first divide the set $\mathcal{G}_{k,r}$ into following disjoint subsets according to whether $\overline{2r}$ is a part of λ and the size of μ_{n-r}. Here, for a partition ν and an integer i, $i \in \nu$ means that ν has a part i, and $i \notin \nu$ means that ν does not have a part i.

$$\begin{aligned}
\mathcal{A} &:= \{(\lambda, \mu) \in \mathcal{G}_{k,r} : \overline{2r} \in \lambda\},\\
\mathcal{B} &:= \{(\lambda, \mu) \in \mathcal{G}_{k,r} : \overline{2r} \notin \lambda, r = n\},\\
\mathcal{C} &:= \{(\lambda, \mu) \in \mathcal{G}_{k,r} : \overline{2r} \notin \lambda, r \neq n, \mu_{n-r} = 0\},\\
\mathcal{D} &:= \{(\lambda, \mu) \in \mathcal{G}_{k,r} : \overline{2r} \notin \lambda, r \neq n, \mu_{n-r} = 2n - 2r - 1\}.
\end{aligned}$$

We note that, for $(\lambda, \mu) \in \mathcal{A}$, it must be $r \neq n$ since $\overline{2n} \notin \lambda$. Next we further divide the set $\mathcal{C}$ into seven disjoint sets.

$$\begin{aligned}
\mathcal{C}_1 &:= \{(\lambda, \mu) \in \mathcal{C} : \lambda_1 = 2n, 2r \in \lambda\},\\
\mathcal{C}_2 &:= \{(\lambda, \mu) \in \mathcal{C} : \lambda_1 = 2n, 2r \notin \lambda\},\\
\mathcal{C}_3 &:= \{(\lambda, \mu) \in \mathcal{C} : \lambda_1 = 2n - 1\},\\
\mathcal{C}_4 &:= \{(\lambda, \mu) \in \mathcal{C} : \lambda_1 \le 2n - 2, \overline{2n-2} \in \lambda, \mu_{n-r-1} = 0\},\\
\mathcal{C}_5 &:= \{(\lambda, \mu) \in \mathcal{C} : \lambda_1 \le 2n - 2, \overline{2n-2} \in \lambda, \mu_{n-r-1} \neq 0\},\\
\mathcal{C}_6 &:= \{(\lambda, \mu) \in \mathcal{C} : \lambda_1 \le 2n - 2, \overline{2n-2} \notin \lambda, r = n - 1\},\\
\mathcal{C}_7 &:= \{(\lambda, \mu) \in \mathcal{C} : \lambda_1 \le 2n - 2, \overline{2n-2} \notin \lambda, r \neq n - 1\}.
\end{aligned}$$

We note that for the sets $\mathcal{C}_4$ and $\mathcal{C}_5$, we do not need to consider the case $r = n-1$ because this gives a contradiction $\overline{2n-2} = \overline{2r} \notin \lambda$. Finally, we divide the set $\mathcal{D}$ into four disjoint sets.

$$\begin{aligned}
\mathcal{D}_1 &:= \{(\lambda,\mu) \in \mathcal{D} : 2r \in \lambda, \lambda_1 = 2n\},\\
\mathcal{D}_2 &:= \{(\lambda,\mu) \in \mathcal{D} : 2r \in \lambda, \lambda_1 < 2n\},\\
\mathcal{D}_3 &:= \{(\lambda,\mu) \in \mathcal{D} : 2r \notin \lambda, 2r+1 \in \lambda\},\\
\mathcal{D}_4 &:= \{(\lambda,\mu) \in \mathcal{D} : 2r \notin \lambda, 2r+1 \notin \lambda\} = \mathcal{G}_{k,r+1}.
\end{aligned}$$

We define $\phi(\lambda,\mu) = (\lambda',\mu')$ as follows. Here, for a partition ν and an integer i, we denote by $\nu + (i)$ (respectively $\nu - (i)$) the partition obtained from ν by adding a part i (respectively removing a part i).

Now we begin to construct the map $\phi(\lambda,\mu) = (\lambda',\mu')$ on the thirteen disjoint subsets of $\mathcal{G}_{k,r}$.

The case $(\lambda,\mu) \in \mathcal{A}$. We define

$$\lambda' = \lambda - (\overline{2r}), \qquad \mu' = (\mu_1,\dots,\mu_{n-r},0,\mu_{n-r+1}+2,\dots,\mu_n+2).$$

Then we have $(\lambda',\mu') \in \mathcal{C}_7$.

The case $(\lambda,\mu) \in \mathcal{C}_7$. In this case we define

$$\lambda' = \lambda + (\overline{2r}), \qquad \mu' = (\mu_1,\dots,\mu_{n-r-1},\mu_{n-r+1}-2,\dots,\mu_n-2).$$

Then we have $(\lambda',\mu') \in \mathcal{A}$.

The case $(\lambda,\mu) \in \mathcal{B}$. In this case, $\mu = (1,3,\dots,2r-1)$ and $\lambda = (\underbrace{2r,\dots,2r}_{k-r})$. We define

$$\lambda' = \lambda - (2r), \qquad \mu' = (0,3,5,\dots,2n+1).$$

Then we have $(\lambda',\mu') \in \mathcal{C}_6$.

The case $(\lambda,\mu) \in \mathcal{C}_6$. In this case we have $\lambda = (\overbrace{2r,\dots,2r}^{k-n})$ and $\mu = (0,3,5,\dots,2n-1)$. We define

$$\lambda' = \lambda + (2r), \qquad \mu' = (1,3,\dots,2n-3).$$

Then we have $(\lambda',\mu') \in \mathcal{B}$.

The case $(\lambda,\mu) \in \mathcal{C}_1$. In this case we define

$$\begin{aligned}
\lambda' &= \lambda - (2r) - (2n) + (\overline{2n}),\\
\mu' &= (\mu_1,\dots,\mu_{n-r-1},\mu_{n-r},0,\mu_{n-r+1}+2,\dots,\mu_n+2).
\end{aligned}$$

Then we have $(\lambda',\mu') \in \mathcal{C}_4$.

The case $(\lambda,\mu) \in \mathcal{C}_4$. In this case, we define

$$\begin{aligned}
\lambda' &= \lambda + (2r) - (\overline{2n-2}) + (2n-2),\\
\mu' &= (\mu_1,\dots,\mu_{n-r-1},\mu_{n-r+1}-2,\mu_{n-r+2}-2,\dots,\mu_n-2).
\end{aligned}$$

Then we have $(\lambda',\mu') \in \mathcal{C}_1$.

The case $(\lambda,\mu)\in\mathcal{C}_2$. In this case we define

$$\lambda' = \lambda - (2n) + (2r+1),$$
$$\mu' = (\mu_1,\dots,\mu_{n-r-1}, 2n-2r-1, \mu_{n-r+1},\dots,\mu_n).$$

Then we have $(\lambda',\mu')\in\mathcal{D}_3$.
The case $(\lambda,\mu)\in\mathcal{D}_3$. In this case we define

$$\lambda' = \lambda - (2r+1) + (2n),$$
$$\mu' = (\mu_1,\dots,\mu_{n-r-1}, 0, \mu_{n-r+1},\dots,\mu_n).$$

Then we have $(\lambda',\mu')\in\mathcal{C}_2$.
The case $(\lambda,\mu)\in\mathcal{C}_3$. In this case we define

$$\lambda' = \lambda - (2n-1) + (2r),$$
$$\mu' = (\mu_1,\dots,\mu_{n-r-1}, 2n-2r-1, \mu_{n-r+1},\dots,\mu_n).$$

Then we have $(\lambda',\mu')\in\mathcal{D}_2$.
The case $(\lambda,\mu)\in\mathcal{D}_2$. In this case we define

$$\lambda' = \lambda - (2r) + (2n-1),$$
$$\mu' = (\mu_1,\dots,\mu_{n-r-1}, 0, \mu_{n-r+1},\dots,\mu_n).$$

Then we have $(\lambda',\mu')\in\mathcal{C}_3$.
The case $(\lambda,\mu)\in\mathcal{C}_5$. In this case we define

$$\lambda' = \lambda + (2r) - (\overline{2n-2}) + (2n-2),$$
$$\mu' = (\mu_1,\dots,\mu_{n-r-1}, \mu_{n-r+1}-2,\dots,\mu_n-2).$$

Then we have $(\lambda',\mu')\in\mathcal{D}_1$.
The case $(\lambda,\mu)\in\mathcal{D}_1$. In this case we define

$$\lambda' = \lambda - (2r) - (2n) + (\overline{2n}),$$
$$\mu' = (\mu_1,\dots,\mu_{n-r}, 0, \mu_{n-r+1}+2,\dots,\mu_n+2).$$

Then we have $(\lambda',\mu')\in\mathcal{C}_5$.
The case $(\lambda,\mu)\in\mathcal{D}_4$. In this case we define $(\lambda',\mu')=(\lambda,\mu)$.

It is straightforward to check that ϕ is a sign-reversing involution ϕ on $\mathcal{G}_{k,r}$ with fixed point set $\mathcal{G}_{k,r+1}=\mathcal{D}_4$. More precisely, ϕ fixes only the set $\mathcal{D}_4$ and maps bijectively $\mathcal{A}$ to $\mathcal{C}_7$, $\mathcal{B}$ to $\mathcal{C}_6$, $\mathcal{C}_1$ to $\mathcal{C}_4$, $\mathcal{C}_2$ to $\mathcal{D}_3$, $\mathcal{C}_3$ to $\mathcal{D}_2$, and $\mathcal{C}_5$ to $\mathcal{D}_1$. $\square$

For example, for $k=4$ and $r=1$, there are eight elements (λ,μ) in the set $\mathcal{G}_{4,1}\setminus\mathcal{G}_{4,2}$ such that $|\lambda|+|\mu|=9$. Applying the above involution ϕ, we get the following correspondence.

$$\mathcal{A}\ni((4,\overline{2}),(0,3))\rightleftharpoons((4),\ (0,0,5))\in\mathcal{C}_7,$$
$$\mathcal{A}\ni((3,\overline{2}),(1,3))\rightleftharpoons((3),\ (1,0,5))\in\mathcal{C}_7,$$
$$\mathcal{C}_3\ni((3,3),(0,3))\rightleftharpoons((3,2),\ (1,3))\in\mathcal{D}_2,$$
$$\mathcal{C}_1\ni((4,2),(0,3))\rightleftharpoons((\overline{4}),\ (0,0,5))\in\mathcal{C}_4.$$

3. A connection with unimodal sequences

A *unimodal sequence* is a sequence $w = (a_1, a_2, \dots, a_r, \overline{c}, b_1, b_2, \dots, b_s)$ of nonnegative integers with a marked integer c called the *peak* satisfying

$$a_1 \le a_2 \le \cdots \le a_r \le c \ge b_1 \ge b_2 \ge \cdots \ge b_s. \tag{3.1}$$

In this case we say that the sequence has *weight* $|w| = c + \sum_{i=1}^{r} a_i + \sum_{i=1}^{s} b_i$ and *rank* $\mathrm{rank}(w) = r - s$. Various unimodal sequences and their ranks have attracted many researchers [7, 8, 9, 13]. By replacing a by q in Warnaar's identity, we see that

$$\sum_{n=0}^{\infty} \frac{(q;q)_{2n} q^{2n+2}}{(-q^2;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{(n+1)^2+1}. \tag{3.2}$$

Let $W(n)$ be the set of unimodal sequences $(a_1, a_2, \dots, a_r, \overline{c}, b_1, b_2, \dots, b_s)$ of weight n such that the peak c is an even integer at least 2, $0 \le a_1 < a_2 < \cdots < a_r \le c-2$, and $c \ge b_1 \ge b_2 \ge \cdots \ge b_s \ge 2$. Then, it is easy to check that the left hand side of (3.2) is equal to the generating function

$$\sum_{n=0}^{\infty} \sum_{w \in W(n)} (-1)^{\mathrm{rank}(w)} q^{|w|}.$$

Therefore, we get the following analog of Euler's pentagonal number theorem.

Theorem 3.1. *Let $W_e(n)$ (respectively $W_o(n)$) be the set of unimodal sequences in $W(n)$ with even rank (respectively odd rank). Then,*

$$|W_e(n)| - |W_o(n)| = \begin{cases} (-1)^{k-1}, & \text{if } n-1 = k^2 \text{ for a positive integer } k, \\ 0, & \text{otherwise.} \end{cases}$$

It is natural to consider the following generating function

$$\sum_{n=0}^{\infty} \sum_{w \in W(n)} a^{\mathrm{rank}(w)} q^{|w|} = \sum_{n=0}^{\infty} \frac{(-aq;q)_{2n} q^{2n+2}}{(q^2/a;q)_{2n+1}}.$$

However, since every term in the right hand side has a positive coefficient, there is no cancelation. Hence the above generating function is not a partial theta function.

4. Concluding Remarks

In this paper we gave a combinatorial proof of Warnaar's partial theta function identity (1.1). It would be interesting to find a class of partial theta function identities, which can be proven through the arguments given in this paper.

Acknowledgement

Kathy Q. Ji was supported by the 973 Project, the PCSIRT Project of the Ministry of Education and the National Science Foundation of China.

Byungchan Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2013R1A1A2061326).

Jang Soo Kim was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2013R1A1A2061006). Jang Soo Kim was partially supported by the TJ Park Science Fellowship of POSCO TJ Park Foundation.

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