Bilateral truncated Jacobi's identity

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Abstract. Recently, Andrews and Merca considered the truncated version of Euler's pentagonal number theorem and obtained a non-negative result on the coefficients of this truncated series. Guo and Zeng showed the coefficients of two truncated Gauss' identities are non-negative and they conjectured that the truncated Jacobi's identity also has non-negative coefficients. Mao provided a proof of this conjecture by using an algebraic method. In this paper, we consider the bilateral truncated Jacobi's identity and show that when the upper and lower bounds of the summation satisfy some certain restrictions, then this bilateral truncated identity has non-negative coefficients. As a corollary, we show the conjecture of Guo and Zeng holds. Our proof is purely combinatorial and mainly based on a bijection for Jacobi's identity.

Keywords: Partitions, Jacobi's identity.

AMS Classifications: 05A17, 11P82, 11P83.

1 Introduction

This paper is concerned with bilateral truncated Jacobi's identity. Recall that Jacobi's identity [9, p.257, Eq.(5)] (see also [19, Theorem 357]) is given by

$$(q;q)_\infty^3 = \sum_{j=0}^{+\infty} (-1)^j (2j+1) q^{\binom{j+1}{2}}, \qquad (1.1)$$

which is equivalent to

$$(q;q)_\infty^3 = \sum_{j=-\infty}^{+\infty} (-1)^j j q^{\binom{j+1}{2}}. \qquad (1.2)$$

It plays an important role in the study of partition congruences and representations of integers as sums of squares, see, for example, Andrews [1, 2], Ewell [8], Hirschhorn [12], Hirschhorn and Sellers [13] and Ramanujan [15, 16].

The truncated theta series were recently studied by several authors, see, for example, Andrews and Merca [4], Chapman [7], Guo and Zeng [11], Mao [14], Shanks [17,18], and Yee [20]. In [4], Andrews and Merca considered the truncated Euler's pentagonal number theorem. Euler's pentagonal number theorem is one of the most well-known theorem in partition theory.

$$(q;q)_\infty = \sum_{j=-\infty}^{+\infty} (-1)^j q^{j(3j+1)/2}. \tag{1.3}$$

This gives

$$\frac{1}{(q;q)_\infty} \sum_{j=0}^{+\infty} (-1)^j q^{j(3j+1)/2}(1-q^{2j+1}) = 1. \tag{1.4}$$

Here and throughout this paper, we have adopted the standard notation on partitions and q-series [3]

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

Let $p(n)$ be the number of partitions of n. From (1.4), it is easy to obtain the following recursive formula for $p(n)$:

$$\sum_{j=0}^{\infty} (-1)^j \left(p\left(n - \frac{3j^2+j}{2}\right) - p\left(n - \frac{3j^2+5j}{2} - 1\right) \right) = 0.$$

Andrews and Merca [4] showed that the following truncated Euler's pentagonal number theorem

$$(-1)^{k-1} \frac{1}{(q;q)_\infty} \sum_{j=0}^{k-1} (-1)^j q^{j(3j+1)/2}(1-q^{2j+1})$$

has non-negative coefficients. This leads to the following inequality on $p(n)$,

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \left(p\left(n - \frac{3j^2+j}{2}\right) - p\left(n - \frac{3j^2+5j}{2} - 1\right) \right) \geq 0,$$

where $n \geq 1$ and $k \geq 1$. They also conjectured that for positive integers k, R, S with $k \geq 1$ and $1 \leq S < R/2$, the following truncated Jacobi's triple product identity

$$\frac{(-1)^{k-1}}{(q^S;q^R)_\infty (q^{R-S};q^R)_\infty (q^R;q^R)_\infty} \sum_{j=0}^{k-1} (-1)^j q^{Rj(j+1)/2-Sj}(1-q^{(2j+1)S}) \tag{1.5}$$

has non-negative coefficients. When $R = 3$ and $S = 1$, this truncated identity is the truncated Euler's pentagonal number theorem and has been proved by Andrews and

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Merca [4]. Guo and Zeng [11] showed this conjecture holds when $R = 4$ and $S = 1$. Mao [14] showed this conjecture is valid by exploiting an algebraic method. Yee [20] gave a combinatorial proof of this conjecture with the aid of Wright's bijection [19] for the Jacobi's triple product identity.

In [11], Guo and Zeng considered the truncated Jacobi's identity. It is easy to see that Jacobi's identity (1.1) can also be written as

$$\frac{1}{(q;q)_\infty^3} \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{\binom{j+1}{2}} = 1. \tag{1.6}$$

Define $t(n)$ to be the number of partitions of n into three kinds of parts, then

$$\sum_{n \geq 1} t(n) q^n = \frac{1}{(q;q)_\infty^3}.$$

Thus, from (1.6), it is easy to see that for $n \geq 1$,

$$\sum_{j=0}^{\infty} (-1)^j (2j+1) t(n - j(j+1)/2) = 0.$$

Guo and Zeng [11] conjectured that for $n, k \geq 1$,

$$(-1)^k \sum_{j=0}^{k} (-1)^j (2j+1) t(n - j(j+1)/2) \geq 0. \tag{1.7}$$

In other words, the following truncated Jacobi's identity

$$\frac{(-1)^k}{(q;q)_\infty^3} \sum_{j=0}^{k} (-1)^j q^{\binom{j+1}{2}} (2j+1)$$

has non-negative coefficients. Mao [14] gave an algebraic proof of this conjecture.

In this paper, we consider the following bilateral truncated Jacobi's identity

$$\frac{1}{(q;q)_\infty^3} \sum_{j=a}^{b} (-1)^j j q^{\binom{j+1}{2}}$$

and give a sufficient condition so that this bilateral truncated series has non-negative coefficients. More specifically, we have

Theorem 1.1. *Given any two integers $a \leq b$, define $sg(a) := 1$ if $a \geq 0$ and $sg(a) := -1$ if $a < 0$. If $(-1)^{a+b} sg(a-1) sg(b) = 1$, then the coefficient of q^n in*

$$\frac{(-1)^a}{(q;q)_\infty^3} sg(a-1) \sum_{j=a}^{b} (-1)^j j q^{\binom{j+1}{2}} \tag{1.8}$$

is non-negative for $n \geq 1$ and is positive for $a \neq 0$ or $b \neq 0$ and $n \geq \max\left\{1, \min\left\{\binom{a + \frac{sg(a)+1}{2}}{2}, \binom{b + \frac{sg(b)+3}{2}}{2}\right\}\right\}$.

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Observe that the coefficient of q^n in (1.8) is equal to

$$(-1)^a sg(a-1) \sum_{j=a}^{b} (-1)^j jt(n-j(j+1)/2).$$

Hence, Theorem 1.1 is equivalent to the following corollary.

Corollary 1.2. *When $(-1)^{a+b} sg(a-1)sg(b) = 1$ and for $n \geq 1$, there holds*

$$(-1)^a sg(a-1) \sum_{j=a}^{b} (-1)^j jt(n-j(j+1)/2) \geq 0$$

with strict inequality if $a \neq 0$ or $b \neq 0$ and $n \geq \max \left\{ 1, \min \left\{ \binom{a+\frac{sg(a)+1}{2}}{2}, \binom{b+\frac{sg(b)+3}{2}}{2} \right\} \right\}$.

Note that Guo and Zeng's conjecture (1.7) is the special case of Corollary 1.2 with $a = -k - 1$ and $b = k$.

As an example of Corollary 1.2, let $a = -2$ and $b = 3$, we obtain

$$t(n) - 3t(n-1) + 2t(n-3) - 3t(n-6) \leq 0.$$

Another example, let $a = -3$ and $b = 4$, we get

$$t(n) - 3t(n-1) + 5t(n-3) - 3t(n-6) + 4t(n-10) \geq 0.$$

This paper is organized as follows. In Section 2, we first recall the definitions of synchronized F-partitions and rooted synchronized F-partitions and give the generating functions for rooted synchronized F-partitions. Then we state a bijection between the set of degenerate rooted synchronized F-partitions and the set of synchronized F-partitions without the zero part and an involution on the set of non-degenerate rooted synchronized F-partitions. Finally, we restrict these two bijections to the set of rooted synchronized F-partitions whose discrepancy lies in the interval $[a, b]$. Thus, we obtain a partition identity, which plays an important role in the proof of Theorem 1.1. In Section 3, we give a combinatorial proof of Theorem 1.1 based on the partition identity in Section 2.

2 Synchronized F-partitions

The notions of synchronized F-partitions and rooted synchronized F-partitions were first introduced in [5] which are used to give a combinatorial proof of finite form of Jacobi's identity. In [5], synchronized F-partitions and rooted synchronized F-partitions are named as synchronized partitions and rooted synchronized partitions. In this paper, to illustrate the connection between synchronized partitions and Frobenius partitions, we will call synchronized partitions as synchronized F-partitions. Assume that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a

partition with distinct parts and $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ is also a partition with distinct parts under the assumption that the last part β_s may be zero. Then a synchronized F-partition is a representation of (α, β) as a two-row array such that some $*$ symbols may be added at the end of α or β so that they are of the same length depending on which is of smaller length. We may denote a synchronized F-partition with underlying partitions α and β by $S(\alpha, \beta)$. The difference $\ell(\alpha) - \ell(\beta)$ is called the *discrepancy* of the synchronized F-partition. A synchronized F-partition with a positive discrepancy k can be represented as follows:

$$S(\alpha,\ \beta) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_s & \alpha_{s+1} & \cdots & \alpha_{s+k} \\ \beta_1 & \beta_2 & \cdots & \beta_s & * & * & * \end{pmatrix}$$

and a synchronized F-partition with a negative discrepancy $-k$ ($k > 0$) can be represented as follows:

$$S(\alpha,\ \beta) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r & * & * & * \\ \beta_1 & \beta_2 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_{r+k} \end{pmatrix}.$$

A synchronized F-partition with zero discrepancy can be simply represented as a two-row array without any star added, which is referred to as a Frobenius partition, see Andrews [1], Corteel and Lovejoy [6]. A *rooted synchronized F-partition* is defined as a synchronized F-partition with a distinguished star symbol, which we denote by $\bar{*}$. Clearly, a rooted synchronized F-partition has an underlying synchronized F-partition with nonzero discrepancy. To distinguish a synchronized F-partition, we may denote a rooted synchronized F-partition with underlying partitions α and β by $\bar{S}(\alpha, \beta)$.

For example, there are five rooted synchronized F-partitions of 2 :

$$\begin{pmatrix} 2 \\ \bar{*} \end{pmatrix} \begin{pmatrix} 1 & \bar{*} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{*} \\ 2 \end{pmatrix} \begin{pmatrix} \bar{*} & * \\ 2 & 0 \end{pmatrix} \begin{pmatrix} * & \bar{*} \\ 2 & 0 \end{pmatrix}.$$

It is easy to see that the generating function of synchronized F-partitions equals

$$(-q; q)_\infty (-1; q)_\infty, \tag{2.1}$$

and the generating function of synchronized F-partitions without the zero part equals

$$(-q; q)_\infty (-q; q)_\infty. \tag{2.2}$$

On the other hand, the generating function of synchronized F-partitions with a non-negative discrepancy k equals

$$\frac{q^{\binom{k+1}{2}}}{(q; q)_\infty}, \tag{2.3}$$

and the generating function of synchronized F-partitions with a negative discrepancy $-k$ equals

$$\frac{q^{\binom{-k+1}{2}}}{(q; q)_\infty} = \frac{q^{\binom{k}{2}}}{(q; q)_\infty}. \tag{2.4}$$

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The generating functions (2.3) and (2.4) can be deduced from the Wright's bijection [19] for the Jacobi's triple product identity. Let us recall that a synchronized F-partition $S(\alpha, \beta)$ with a discrepancy k can be represented as follows: put r solid circles on the diagonal, where r is equal to the number of parts in α. Then for $j = 1, 2, \ldots, r$, put $\alpha_j - 1$ circles in row j to the right of the diagonal and β_j circles in column $k + j$ below the diagonal. For example, Figure 2.1 gives the representations of the synchronized F-partitions $\begin{pmatrix} 7 & 5 & 4 & 3 & 2 & 1 \\ 6 & 5 & 2 & 0 & * & * \end{pmatrix}$ and $\begin{pmatrix} 6 & 5 & 4 & * & * & * \\ 7 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$.

Figure 2.1: The Wright's Representations.

Let us consider the case discrepancy $k > 0$. Note that $q^{\binom{k+1}{2}}$ is the generating function of partition $(k, k-1, \cdots, 1)$. Hence $q^{\binom{k+1}{2}}/(q; q)_\infty$ is the generating function for a pair of partitions (T_k, λ), where $T_k = (k, k-1, \cdots, 1)$ and λ is a partition with no restriction. Then Wright's bijection [19] as illustrated by Figure 2.2 yields the generating function (2.3).
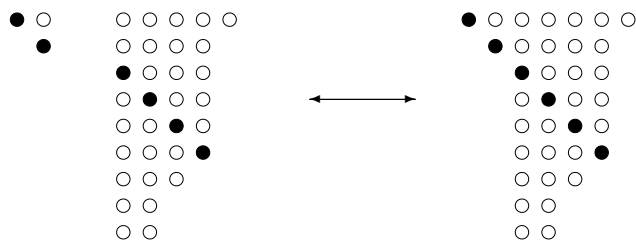

Figure 2.2: The Wright's Bijection for $k \geq 0$.

The case $-k < 0$ is similar to the previous situation. Notice that $q^{\binom{-k+1}{2}} = q^{\binom{k}{2}}$ is the generating function of partition $(k-1, \ldots, 1)$. Thus $q^{\binom{-k+1}{2}}/(q; q)_\infty$ is the generating
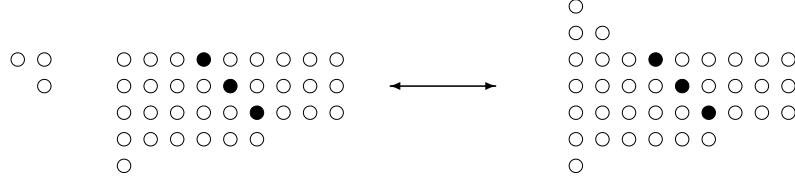
Figure 2.3: The Wright's Bijection for $-k < 0$.

function for a pair of partitions (T_{k-1}, λ), where $T_{k-1} = (k-1, \cdots, 1)$ and λ is a partition with no restriction. Then Wright's bijection yields the generating function (2.4), as shown in Figure 2.3.

Let us define the sign of a synchronized F-partition as $(-1)^{\delta(S)}$, where $\delta(S)$ is the number of stars in $S(\alpha, \beta)$ and define a sign of a rooted synchronized F-partition $\bar{S}(\alpha, \beta)$ as $(-1)^{\delta(\bar{S})}$, where $\delta(\bar{S})$ is the number of stars in $\bar{S}(\alpha, \beta)$ under the assumption that a star with the bar in the top row is not counted. In other words, the sign of a rooted synchronized F-partition equals $(-1)^k sg(k)$, where k is the discrepancy.

From (2.3) and (2.4), it is easy to show that the generating function of synchronized F-partitions $S(\alpha, \beta)$ with sign $(-1)^{\delta(S)}$ equals

$$\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k+1}{2}}, \tag{2.5}$$

and the generating function of rooted synchronized F-partitions $\bar{S}(\alpha, \beta)$ with sign $(-1)^{\delta(\bar{S})}$ equals

$$\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k k q^{\binom{k+1}{2}}. \tag{2.6}$$

For any two integers $a \le b$, let $\mathbb{S}_{a,b}$ denote the set of synchronized F-partitions $S(\alpha, \beta)$ whose discrepancy lies in the interval $[a, b]$, and let $\mathbb{R}_{a,b}$ be the set of rooted synchronized F-partitions $\bar{S}(\alpha, \beta)$ whose discrepancy lies in the interval $[a, b]$. Again, from (2.3) and (2.4), we have

Lemma 2.1. *The generating function of synchronized F-partitions $S(\alpha, \beta)$ in $\mathbb{S}_{a,b}$ with sign $(-1)^{\delta(S)}$ equals*

$$\sum_{S(\alpha,\beta)\in\mathbb{S}_{a,b}} (-1)^{\delta(S)} q^{|\alpha|+|\beta|} = \frac{1}{(q;q)_\infty} \sum_{k=a}^{b} (-1)^k q^{\binom{k+1}{2}}, \tag{2.7}$$

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and the generating function of rooted synchronized F-partitions $\bar{S}(\alpha, \beta)$ *in* $\mathbb{R}_{a,b}$*with sign* $(-1)^{\delta(\bar{S})}$ *equals*

$$\sum_{\bar{S}(\alpha,\beta)\in\mathbb{R}_{a,b}} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} = \frac{1}{(q;q)_\infty} \sum_{k=a}^{b} (-1)^k k q^{\binom{k+1}{2}}. \tag{2.8}$$

A rooted synchronized F-partition $\bar{S}(\alpha, \beta)$ is called *degenerate* if

$$\bar{S}(\alpha, \beta) = \begin{pmatrix} \alpha_1 & \cdots & \alpha_s & \alpha_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & \beta_s & \bar{\ast} & \cdots & \ast \end{pmatrix}$$

or

$$\bar{S}(\alpha, \beta) = \begin{pmatrix} \alpha_1 & \cdots & \alpha_r & \ast & \cdots & \ast & \bar{\ast} \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_s & 0 \end{pmatrix},$$

where $\alpha_1 > \alpha_2 > \ldots > \alpha_r \geq 1$ and $\beta_1 > \beta_2 > \ldots > \beta_s \geq 1$; otherwise $\bar{S}(\alpha, \beta)$ is called *non-degenerate.*

Next, we give a bijection between the set of degenerate rooted synchronized F-partitions and the set of synchronized F-partitions without the zero part, as stated in [5].

Theorem 2.2. *There is a sign preserving bijection ϕ between the set of degenerate rooted synchronized F-partitions of n and the set of synchronized F-partitions of n that do not contain the zero part.*

Proof. For a degenerated rooted synchronized F-partition $\bar{S}(\alpha, \beta)$, we can construct a synchronized F-partition $S(\alpha', \beta')$ that do not contain the zero part. Let $\ell(\lambda)$ denote the number of parts in an ordinary partition λ. We consider the following two cases.

Case 1: If $\ell(\alpha) > \ell(\beta)$, then delete the bar to the first '$\ast$' on the bottom row.

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_s & a_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & \beta_s & \bar{\ast} & \cdots & \ast \end{pmatrix} \longleftrightarrow \begin{pmatrix} \alpha_1 & \cdots & \alpha_s & \alpha_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & \beta_s & \ast & \cdots & \ast \end{pmatrix}.$$

Case 2: If $\ell(\alpha) < \ell(\beta)$, then delete a zero part on the bottom row along with a barred star on the top row.

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_r & \ast & \cdots & \ast & \bar{\ast} \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_s & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} \alpha_1 & \cdots & \alpha_r & \ast & \cdots & \ast \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_s \end{pmatrix}.$$

Clearly, the procedure is reversible and it preserves the signs. ∎

We now state a sign reversing involution on the set of non-degenerate rooted synchronized F-partitions (see [5, Theorem 3.1]).

Theorem 2.3. *There is a sign reversing involution τ on the set of non-degenerate rooted synchronized F-partitions of n.*

Proof. For a non-degenerate rooted synchronized F-partition $\bar{S}(\alpha, \beta)$, we proceed to construct a non-degenerate rooted synchronized F-partition $\bar{S}(\alpha', \beta')$. We consider the following two cases.

Case 1: The partition β has a zero part.

- If $\ell(\alpha) > \ell(\beta)$, then replace the zero part by a star $*$.
- If $\ell(\alpha) < \ell(\beta)$, then delete the whole column of the zero part.

Case 2: The partition β has no zero part.

- If $\ell(\alpha) > \ell(\beta)$, then replace the first '$*$' on the bottom row by a zero part.
- If $\ell(\alpha) < \ell(\beta)$, then add a zero part along with a star on the top as a column.

The above bijection can be illustrated as follows:

$$
\begin{pmatrix} \alpha_1 & \cdots & \alpha_s & a_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & 0 & * & \bar{*} & * \end{pmatrix} \xleftrightarrow{\ell(\alpha)>\ell(\beta)} \begin{pmatrix} \alpha_1 & \cdots & \alpha_s & \alpha_{s+1} & \cdots & \alpha_r \\ \beta_1 & \cdots & * & * & \bar{*} & * \end{pmatrix},
$$

$$
\begin{pmatrix} \alpha_1 & \cdots & \alpha_r & * & \bar{*} & * & * \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_{s-1} & 0 \end{pmatrix} \xleftrightarrow{\ell(\alpha)<\ell(\beta)} \begin{pmatrix} \alpha_1 & \cdots & \alpha_r & * & \bar{*} & * \\ \beta_1 & \cdots & \beta_r & \beta_{r+1} & \cdots & \beta_{s-1} \end{pmatrix}.
$$

It is easy to check that the above construction is a sign reversing involution. ∎

Using (2.2) and (2.6) and combining Theorems 2.2 and 2.3, we are led to a combinatorial proof of Jacobi's identity

$$
\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k k q^{\binom{k+1}{2}} = (q;q)_\infty^2.
$$

In [10], Joichi and Stanton also provided two combinatorial proofs of Jacobi's identity.

In the remainder part of this section, we will restrict the bijection ϕ in Theorem 2.2 and the involution τ in Theorem 2.3 to the set $\mathbb{R}_{a,b}$. Let $\mathbb{D}_{a,b}$ denote the set of degenerate rooted synchronized F-partitions in $\mathbb{R}_{a,b}$ and let $\mathbb{N}_{a,b}$ denote the set of non-degenerate rooted synchronized F-partitions in $\mathbb{R}_{a,b}$. Obviously,

$$
\mathbb{R}_{a,b} = \mathbb{D}_{a,b} \cup \mathbb{N}_{a,b}. \tag{2.9}
$$

We first restrict the bijection ϕ in Theorem 2.2 to the set $\mathbb{D}_{a,b}$. Let $\bar{S}(\alpha, \beta)$ be a degenerate rooted synchronized F-partition with the discrepancy k and let $S(\alpha', \beta') = \phi(\bar{S}(\alpha, \beta))$ and its discrepancy is equal to k'. From the definition of ϕ, we see that $S(\alpha', \beta')$ is a synchronized F-partition without the zero part. Furthermore, when $k > 0$, $k' = k$; when $k < 0$, $k' = k + 1$. Hence, if $\bar{S}(\alpha, \beta)$ is in $\mathbb{D}_{a,b}$, that is, $a \leq k \leq b$, then $a + (1 + sg(-a))/2 \leq k' \leq b + (1 - sg(b))/2$ which implies that $S(\alpha', \beta')$ is in $\mathbb{S}_{a+(1+sg(-a))/2, b+(1-sg(b))/2}$. Thus, we have the following conclusion.

Corollary 2.4. *The bijection ϕ in Theorem 2.2 is also a sign preserving bijection between the set of degenerate rooted synchronized F-partitions of n in $\mathbb{R}_{a,b}$ and the set of synchronized F-partitions of n in $\mathbb{S}_{a+(1+sg(-a))/2,b+(1-sg(b))/2}$ that do not contain the zero part.*

Let $\mathbb{Z}_{a,b}$ denote the set of synchronized F-partitions of n in $\mathbb{S}_{a,b}$ that do not contain the zero part, from Corollary 2.4, we see that

$$\sum_{\bar{S}(\alpha,\beta)\in\mathbb{D}_{a,b}}(-1)^{\delta(\bar{S})}q^{|\alpha|+|\beta|} = \sum_{S(\alpha,\beta)\in\mathbb{Z}_{a+(1+sg(-a))/2,b+(1-sg(b))/2}}(-1)^{\delta(S)}q^{|\alpha|+|\beta|}. \tag{2.10}$$

We next restrict the bijection τ in Theorem 2.3 to the set $\mathbb{N}_{a,b}$. It should be noted that when we apply the involution τ into non-degenerate rooted synchronized F-partitions $\bar{S}(\alpha,\beta)$ with the zero part, whose discrepancy is b, then we delete the zero part of β to obtain $\bar{S}(\alpha',\beta')$ which has $b+1$ discrepancy. This new rooted synchronized F-partition $\bar{S}(\alpha',\beta')$ is not in $\mathbb{N}_{a,b}$. Besides, if we apply the involution τ into non-degenerate rooted synchronized F-partitions $\bar{S}(\alpha,\beta)$ without the zero part, whose discrepancy is a, we then add a part 0 to β to get $\bar{S}(\alpha',\beta')$ which has $a-1$ discrepancy. This rooted synchronized F-partition $\bar{S}(\alpha',\beta')$ is also not in $\mathbb{N}_{a,b}$.

Define $\mathbb{T}_a$ to be the set of non-degenerate rooted synchronized F-partitions $\bar{S}(\alpha,\beta)$ without the zero part, whose discrepancy is a and $\mathbb{T}_b$ to be the set of non-degenerate rooted synchronized F-partitions $\bar{S}(\alpha,\beta)$ with the zero part, whose discrepancy is b. Let $\mathbb{M}_{a,b}$ denote the set of non-degenerate rooted synchronized F-partitions in $\mathbb{N}_{a,b}$ and not in $\mathbb{T}_a$ and $\mathbb{T}_b$. Obviously, we have

$$\mathbb{N}_{a,b} = \mathbb{M}_{a,b} \cup \mathbb{T}_a \cup \mathbb{T}_b. \tag{2.11}$$

It is not difficult to verify the following consequence.

Corollary 2.5. *The involution τ in Theorem 2.3 is also a sign reversing involution τ on the set $\mathbb{M}_{a,b}$.*

As a consequence of Corollary 2.5, we obtain the following identity.

$$\sum_{\bar{S}(\alpha,\beta)\in\mathbb{M}_{a,b}}(-1)^{\delta(\bar{S})}q^{|\alpha|+|\beta|} = 0. \tag{2.12}$$

Hence, from the relation (2.9) and (2.11), we have

$$\sum_{\bar{S}(\alpha,\beta)\in\mathbb{R}_{a,b}} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}$$

$$= \sum_{\bar{S}(\alpha,\beta)\in\mathbb{D}_{a,b}} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} + \sum_{\bar{S}(\alpha,\beta)\in\mathbb{N}_{a,b}} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}$$

$$= \sum_{\bar{S}(\alpha,\beta)\in\mathbb{D}_{a,b}} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} + \sum_{\bar{S}(\alpha,\beta)\in\mathbb{M}_{a,b}} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}$$

$$+ \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_a} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} + \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_b} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}. \tag{2.13}$$

Submitting (2.10) and (2.12) into (2.13), we obtain the following identity, which is useful in the proof of our main theorem.

$$\sum_{\bar{S}(\alpha,\beta)\in\mathbb{R}_{a,b}} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}$$

$$= \sum_{S(\alpha,\beta)\in\mathbb{Z}_{a+(1+sg(-a))/2, b+(1-sg(b))/2}} (-1)^{\delta(S)} q^{|\alpha|+|\beta|}$$

$$+ \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_a} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} + \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_b} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}. \tag{2.14}$$

3   Proof of Theorem 1.1

In this section, we give a combinatorial proof of Theorem 1.1. To this end, we first interpret (1.8) in terms of rooted synchronized F-partitions, and then show our theorem holds based on (2.14).

From (2.8) in Lemma 2.1, it is known that (1.8) can be interpreted as follows.

$$\frac{(-1)^a}{(q;q)_\infty^3} sg(a-1) \sum_{j=a}^{b} (-1)^j j q^{\binom{j+1}{2}} = \frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{\bar{S}(\alpha,\beta)\in\mathbb{R}_{a,b}} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}. \tag{3.1}$$

Using (2.14) in the above identity, we find that

$$\frac{(-1)^a}{(q;q)_\infty^3} sg(a-1) \sum_{j=a}^{b} (-1)^j j q^{\binom{j+1}{2}}$$

$$= \frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_a} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}$$

$$+ \frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_b} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|}$$

$$+ \frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{S(\alpha,\beta)\in\mathbb{Z}_{a+(1+sg(-a))/2,b+(1-sg(b))/2}} (-1)^{\delta(S)} q^{|\alpha|+|\beta|}. \qquad (3.2)$$

Obviously, (3.2) can be seen as the sum of three terms. We proceed to show that each sum has non-negative coefficients under the assumption $(-1)^{a+b}sg(a-1)sg(b) = 1$. At first, we have the following conclusion.

Lemma 3.1. *Given any integer a, the coefficient of q^n in*

$$\frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_a} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} \qquad (3.3)$$

is non-negative for $n \geq 1$. Moreover, when $a \neq 0$ and $a \neq 1$, the coefficient of q^n in (3.3) is positive for $n \geq \binom{a+\frac{sg(a)+1}{2}}{2}$.

Proof. Let $\bar{S}(\alpha,\beta)$ be a non-degenerate rooted synchronized F-partition in $\mathbb{T}_a$. From the definition of $\mathbb{T}_a$, we see that $(-1)^{\delta(\bar{S})} = (-1)^a sg(a)$. Furthermore, when $a \neq 0$, it is clear to see that $sg(a-1) = sg(a)$. So

$$\frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_a} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} = \frac{1}{(q;q)_\infty^2} \sum_{\bar{S}(\alpha,\beta)\in\mathbb{T}_a} q^{|\alpha|+|\beta|}, \qquad (3.4)$$

which clearly has non-negative coefficients. When $a = 0$, by the definition of $\mathbb{T}_a$, we see that there is no element in $\mathbb{T}_a$, so (3.3) is equal to zero.

We next show that when $a \neq 0$ and $a \neq 1$, the coefficient of q^n in (3.4) is positive for $n \geq \binom{a+\frac{sg(a)+1}{2}}{2}$. It is easy to see that the coefficient of q^n in (3.4) is equal to the number of triplets $(\gamma, \delta, \bar{S}(\alpha,\beta))$ of n, where γ and δ are ordinary partitions, $\bar{S}(\alpha,\beta) \in \mathbb{T}_a$ and

$$|\gamma| + |\delta| + |\alpha| + |\beta| = n.$$

So it suffices to show that when $a \neq 0$ and $a \neq 1$ and for any $n \geq \binom{a+\frac{sg(a)+1}{2}}{2}$, there exists at least one such triplet $(\gamma, \delta, \bar{S}(\alpha,\beta))$. We consider the following two cases.

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When $a > 1$, we set $\gamma = \delta = \emptyset$ and

$$\bar{S}(\alpha, \beta) = \begin{pmatrix} n - \binom{a}{2} & a - 1 & a - 2 & \cdots & 1 \\ * & * & * & \cdots & \bar{*} \end{pmatrix}.$$

When $a < 0$, we set $\gamma = \delta = \emptyset$ and

$$\bar{S}(\alpha, \beta) = \begin{pmatrix} \bar{*} & * & * & \cdots & * \\ n - \binom{-a}{2} & -a - 1 & -a - 2 & \cdots & 1 \end{pmatrix}.$$

In either case, it is clear that $|\gamma| + |\delta| + |\alpha| + |\beta| = n$ and note that $n \geq \binom{a + \frac{sg(a)+1}{2}}{2} = \binom{|a|+1}{2}$ which implies $n - \binom{|a|}{2} \geq |a|$, so $\bar{S}(\alpha, \beta) \in \mathbb{T}_a$. Thus we complete the proof. ∎

It should be noted that when $a = 0$ or $a = 1$, there is no element in $\mathbb{T}_a$ which implies the number of such triplet is equal to zero. So in these two cases, the coefficient of q^n in (3.3) is equal to zero.

The next lemma determines when the second sum of (3.2) has the non-negative coefficients.

Lemma 3.2. *For any integers $a \leq b$, when $(-1)^{a+b} sg(a-1) sg(b) = 1$, the coefficient of q^n in*

$$\frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{S(\alpha,\beta) \in \mathbb{T}_b} (-1)^{\delta(S)} q^{|\alpha|+|\beta|} \tag{3.5}$$

is non-negative for $n \geq 1$. Moreover, when $(-1)^{a+b} sg(a-1) sg(b) = 1$, $b \neq -1$ and $b \neq 0$, the coefficient of q^n in (3.5) is positive for $n \geq \binom{b + \frac{sg(b)+3}{2}}{2}$.

Proof. Let $\bar{S}(\alpha, \beta)$ be a non-degenerate rooted synchronized F-partition in $\mathbb{T}_b$. From the definition of $\mathbb{T}_b$, it is known that $(-1)^{\delta(\bar{S})} = (-1)^b sg(b)$. Since $(-1)^{a+b} sg(a-1) sg(b) = 1$, we have

$$\frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{\bar{S}(\alpha,\beta) \in \mathbb{T}_b} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} = \frac{1}{(q;q)_\infty^2} \sum_{\bar{S}(\alpha,\beta) \in \mathbb{T}_b} q^{|\alpha|+|\beta|}. \tag{3.6}$$

Clearly, the above identity has non-negative coefficients.

We proceed to show that when $b \neq -1$ and $b \neq 0$, the coefficient of q^n in (3.6) is positive for $n \geq \binom{b + \frac{sg(b)+3}{2}}{2}$. It is clear to see that the coefficient of q^n in (3.6) can be combinatorially interpreted as the number of triplets $(\gamma, \delta, \bar{S}(\alpha, \beta))$ of n, where γ and δ are ordinary partitions, $\bar{S}(\alpha, \beta) \in \mathbb{T}_b$ and

$$|\gamma| + |\delta| + |\alpha| + |\beta| = n.$$

So it suffices to show that when $b \neq -1$, $b \neq 0$ and for any positive integer $n \geq \binom{b + \frac{sg(b)+3}{2}}{2}$, there exists at least one such triplet $(\gamma, \delta, \bar{S}(\alpha, \beta))$. There are two following cases. When

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$b > 0$ and $n \geq \binom{b+2}{2}$, we set $\gamma = \delta = \emptyset$ and

$$\bar{S}(\alpha, \beta) = \begin{pmatrix} n - \binom{b+1}{2} & b & b-1 & \cdots & 1 \\ 0 & * & * & \cdots & \bar{*} \end{pmatrix}.$$

When $b < -1$ and $n \geq \binom{b+1}{2}$, we set $\gamma = \delta = \emptyset$ and

$$\bar{S}(\alpha, \beta) = \begin{pmatrix} \bar{*} & * & * & \cdots & * & * \\ n - \binom{-b-1}{2} & -b-2 & -b-1 & \cdots & 1 & 0 \end{pmatrix}.$$

In either case, we have $|\gamma| + |\delta| + |\alpha| + |\beta| = n$ and $\bar{S}(\alpha, \beta) \in \mathbb{T}_b$. ∎

It should be remarked that when $b = -1$ or $b = 0$, there is no element in $\mathbb{T}_b$ which implies the number of such triplet is equal to zero. So the coefficient of q^n in (3.6) is equal to zero.

Finally, we aim to show that the third sum of (3.2) has non-negative coefficients when $(-1)^{a+b} sg(a-1) sg(b) = 1$.

Lemma 3.3. *For any integers $a \leq b$, when $(-1)^{a+b} sg(a-1) sg(b) = 1$, the coefficient of q^n in*

$$\frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{S(\alpha,\beta) \in \mathbb{Z}_{a + \frac{1+sg(-a)}{2}, b + \frac{1-sg(b)}{2}}} (-1)^{\delta(S)} q^{|\alpha|+|\beta|} \tag{3.7}$$

is non-negative for $n \geq 1$. In particular, when $a = 1$ or $b = -1$ or $ab = 0$ but $a^2 + b^2 > 0$, the coefficient of q^n in (3.7) is positive for all $n \geq 1$.

Proof. Let $\mathbb{O}_n$ denote the set of triplets $(\gamma, \delta, S(\alpha, \beta))$ of n, where

- γ is an ordinary partition;

- δ is an ordinary partition;

- $S(\alpha, \beta)$ is a synchronized F-partition without zero parts such that $a + \frac{1+sg(-a)}{2} \leq \delta(S) \leq b + \frac{1-sg(b)}{2}$;

- $|\gamma| + |\delta| + |\alpha| + |\beta| = n$.

Let $\pi = (\gamma, \delta, S(\alpha, \beta)) \in \mathbb{O}_n$, we associate π with a sign

$$\omega(\pi) = (-1)^a sg(a-1)(-1)^{\delta(S)}. \tag{3.8}$$

It is easy to see that the coefficient of q^n in (3.7) is equal to

$$\sum_{\pi \in \mathbb{O}_n} \omega(\pi).$$

Hence, it suffices to show that the number of triplets in $\mathbb{O}_n$ with positive sign is not less than the number of triplets in $\mathbb{O}_n$ with negative sign. To this end, we construct a sign-reversing involution ψ defined on the set $\mathbb{O}_n$. Let $\pi = (\gamma, \delta, S(\alpha, \beta)) \in \mathbb{O}_n$, we proceed to construct another triplet $\pi' = (\gamma', \delta', S(\alpha', \beta'))$. We consider the following four cases.

Case 1. $\gamma = \alpha = \emptyset$ and $\beta_1 \geq \delta_1$. We remove β_1 from β to get β' and add β_1 to δ to generate δ'.

Case 2. $\gamma = \alpha = \emptyset$ and $\beta_1 < \delta_1$. We remove δ_1 from δ to get δ' and add δ_1 to β to generate β'.

Case 3: $\gamma \neq \emptyset$ or $\alpha \neq \emptyset$ and $\alpha_1 \geq \gamma_1$. Remove α_1 from α to get α' and add α_1 to γ to generate γ'.

Case 4: $\gamma \neq \emptyset$ or $\alpha \neq \emptyset$ and $\alpha_1 < \gamma_1$. Remove γ_1 from γ to generate γ' and add γ_1 to α to get α'.

From the definition of ψ, it is easy to see that the map ψ changes the parity of $\ell(\alpha) - \ell(\beta)$. Note that $(-1)^{\delta(S)} = (-1)^{\ell(\alpha) - \ell(\beta)}$, so ψ changes the parity of the sign $\omega(\pi)$.

Furthermore, it is not difficult to check that if we apply the map ψ into the following four kinds of triplets $(\gamma, \delta, S(\alpha, \beta))$ in $\mathbb{O}_n$, then their images are not in the set $\mathbb{O}_n$.

(1). $\alpha = \gamma = \emptyset$, $\ell(\beta) = -\left(a + \frac{1+sg(-a)}{2}\right)$ and $\delta_1 > \beta_1$.

(2). $\alpha = \gamma = \emptyset$, $\ell(\beta) = -\left(b + \frac{1-sg(b)}{2}\right)$ and $\delta_1 \leq \beta_1$.

(3). $\gamma \neq \emptyset$ or $\alpha \neq \emptyset$, $\ell(\alpha) - \ell(\beta) = a + \frac{1+sg(-a)}{2}$ and $\gamma_1 \leq \alpha_1$.

(4). $\gamma \neq \emptyset$ or $\alpha \neq \emptyset$, $\ell(\alpha) - \ell(\beta) = b + \frac{1-sg(b)}{2}$ and $\gamma_1 > \alpha_1$.

We denote the set of these triplets by $\mathbb{E}_n$. For $\pi = (\gamma, \delta, S(\alpha, \beta)) \in \mathbb{E}_n$ and note that $(-1)^{a+b}sg(a-1)sg(b) = 1$, an easy calculation deduces that the sign of π is positive. Let $\overline{\mathbb{O}}_n$ denote the set of triplets in $\mathbb{O}_n$ and not in $\mathbb{E}_n$. Obviously, the map ψ is a sign-reversing involution on the set $\overline{\mathbb{O}}_n$, which implies that

$$\sum_{\pi \in \overline{\mathbb{O}}_n} \omega(\pi) = 0.$$

Hence, we have

$$\sum_{\pi \in \mathbb{O}_n} \omega(\pi) = \sum_{\pi \in \mathbb{E}_n} \omega(\pi), \tag{3.9}$$

which is non-negative since $w(\pi) = 1$ for any $\pi \in \mathbb{E}_n$.

To show that (3.7) has positive coefficients when $a = 1$ or $b = -1$, or $ab = 0$ but $a^2 + b^2 > 0$, we will show that in these three cases, for any positive integer n, there exists at least one triple $\pi = (\gamma, \delta, S(\alpha, \beta))$ in the set $\mathbb{E}_n$.

When $a = 1$ or $a = 0$, but $b \neq 0$, let

$$\pi = (\emptyset, \emptyset, S((n), \emptyset)).$$

Clearly, π satisfies that $\gamma_1 \leq \alpha_1$ and $\ell(\alpha) - \ell(\beta) = 1 = a + \frac{1+sg(-a)}{2}$. So $\pi \in \mathbb{E}_n$.

When $b = -1$ or $b = 0$, but $a \neq 0$, let

$$\pi = ((n), \emptyset, S(\emptyset, \emptyset)).$$

Obviously π satisfies that $\gamma_1 > \alpha_1$ and $\ell(\alpha) - \ell(\beta) = 0 = b + \frac{1-sg(b)}{2}$. So $\pi \in \mathbb{E}_n$. Thus, we completes the proof of this lemma. ∎

It should be noted that when $a = b = 0$, we have $a + \frac{1+sg(-a)}{2} = 1$, but $b + \frac{1-sg(b)}{2} = 0$, which gives that $a + \frac{1+sg(-a)}{2} > b + \frac{1-sg(b)}{2}$. So there is no element in $\mathbb{Z}_{a+\frac{1+sg(-a)}{2}, b+\frac{1-sg(b)}{2}}$. Thus, the coefficient of q^n in (3.7) is equal to zero.

Proof of Theorem 1.1. Recall that the bilateral truncated Jacobi's identity has the following combinatorial interpretation

$$\frac{(-1)^a}{(q;q)_\infty^3} sg(a-1) \sum_{j=a}^{b} (-1)^j j q^{\binom{j+1}{2}} \tag{3.10}$$

$$= \frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{\bar{S}(\alpha,\beta) \in \mathbb{T}_a} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} \tag{3.11}$$

$$+ \frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{\bar{S}(\alpha,\beta) \in \mathbb{T}_b} (-1)^{\delta(\bar{S})} q^{|\alpha|+|\beta|} \tag{3.12}$$

$$+ \frac{(-1)^a}{(q;q)_\infty^2} sg(a-1) \sum_{S(\alpha,\beta) \in \mathbb{Z}_{a+\frac{1+sg(-a)}{2}, b+\frac{1-sg(b)}{2}}} (-1)^{\ell(\alpha)-\ell(\beta)} q^{|\alpha|+|\beta|}. \tag{3.13}$$

From Lemmas 3.1, 3.2, and 3.3, we see that when $(-1)^{a+b} sg(a-1) sg(b) = 1$, the coefficient of q^n in (3.10) is non-negative for $n \geq 1$.

To study the positivity of the coefficient of q^n in (3.10), we first let

$$L(a,b) = \max\left\{ 1, \min\left\{ \binom{a+\frac{sg(a)+1}{2}}{2}, \binom{b+\frac{sg(b)+3}{2}}{2} \right\} \right\}.$$

We consider the following three cases:

Case 1: $a \neq 0$, $a \neq 1$, $b \neq 0$ and $b \neq -1$. From Lemma 3.1, we see that the coefficient of q^n in (3.11) is positive for all $n \geq \binom{a+\frac{sg(a)+1}{2}}{2}$. By Lemma 3.2, it is known that the coefficient of q^n in (3.12) is positive for all $n \geq \binom{b+\frac{sg(b)+3}{2}}{2}$. Note that the coefficient of q^n in (3.13) is non-negative, so the coefficient of q^n in (3.10) is positive for all $n \geq L(a,b)$.

Case 2: $a \neq 0$, $a \neq 1$, and $b = 0$ or $b = -1$. In this case, by Lemma 3.3, we know that the coefficient of q^n in (3.13) is positive for all $n \geq 1$. Since the coefficients of q^n in (3.11) and (3.12) are non-negative, then for all $n \geq L(a, b)$, the coefficient of q^n in (3.10) is positive.

Case 3: $a = 1$ or $a = 0$ but $b \neq 0$. In this case, it follows from Lemma 3.3 that the coefficient of q^n in (3.13) is positive for all $n \geq 1$. Hence, for all $n \geq L(a, b)$, the coefficient of q^n in (3.10) is positive since the coefficients of q^n in (3.11) and (3.12) are non-negative.

Therefore, when $a \neq 0$ or $b \neq 0$, the coefficient of q^n in (3.10) is positive for $n \geq L(a, b)$. This completes the proof. ∎

References

[1] G. E. Andrews, Generalized Frobenius partitions, Mem. Amer. Math. Soc. 49 (1984), No. 301, iv+, 44 pp.

[2] G. E. Andrews and R. Roy, Ramanujan's method in q-series congruences, Electron. J. Combin. 4(2) (1997) #R2.

[3] G.E. Andrews, The Theory of Partitions, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998.

[4] G. E. Andrews and M. Merca, The truncated pentagonal number theorem, J. Combin. Theory Ser. A 119 (2012) 1639–1643.

[5] W. Y.C. Chen and K. Q. Ji, Jacobi's identity and Synchronized Partitions, arXiv:math/0601309.

[6] S. Corteel and J. K. Lovejoy, Frobenius partitions and the combinatorics of Ramanujan's ${}_1\psi_1$ summation, J. Combin. Theory Ser. A 97 (2002) 177–183.

[7] R. Chapman, Partition identities arising from involutions, Australas J. Combin. 27(2003) 285–291.

[8] J. A. Ewell, Completion of a Gaussian derivation, Proc. Amer. Math. Soc. 84(2) (1982) 311–314.

[9] C. Jacobi, Fundamenta nova theoriae functionum ellipticarum, Mathematische Werke, Vol. 1, (1829) 49–239.

[10] J. T. Joichi and D. Stanton, An involution for Jacobi's identity, Discrete Math. 73 (1989) 261–271.

[11] V. J. W. Guo and J. Zeng, Two truncated identities of Gauss, J. Combin. Theory Ser. A 120 (2013) 700–707.

[12] M. D. Hirschhorn, Jacobi's two-square theorem and related identities, Ramanujan J. 3 (1999) 153–158.

[13] M. D. Hirschhorn and J. A. Sellers, Two congruences involving 4-cores, The Foata Festschrift. Electron. J. Combin. 3 (1996) #R10.

[14] R. Mao, Proofs of two conjectures on truncated series, J. Combin. Theory Ser. A 130 (2015) 15–25.

[15] S. Ramanujan, Some propertities of $p(n)$, the number of partitions of n, Proc. Cambridge Philos. Soc. 19 (1919) 207–210.

[16] S. Ramanujan, Congruence properties of partitions, Math. Zeitschr. 9 (1921) 147–153.

[17] D. Shanks, A short proof of an identity of Euler, Proc. Amer. Math. Soc. 2 (1951) 747–749.

[18] D. Shanks, Two theorems of Gauss, Pacific J. Math. 8 (1958) 609–612.

[19] E. M. Wright, An enumerative proof of an identity of Jacobi, J. London Math. Soc. 40 (1965) 55–57.

[20] A. J. Yee, The truncated Jacobi triple product theorem, J. Combin. Theory Ser. A 130 (2015) 1–14.