k-Marked Dyson Symbols and Congruences for Moments of Cranks

William Y.C. Chen[1], Kathy Q. Ji[2] and Erin Y.Y. Shen[3]

[1,2,3]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

[1]Center for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

[1]chen@nankai.edu.cn, [2]ji@nankai.edu.cn, [3]shenyiying@mail.nankai.edu.cn

Abstract. By introducing k-marked Durfee symbols, Andrews found a combinatorial interpretation of $2k$-th symmetrized moment $\eta_{2k}(n)$ of ranks of partitions of n. Recently, Garvan introduced the $2k$-th symmetrized moment $\mu_{2k}(n)$ of cranks of partitions of n in the study of the higher-order spt-function $spt_k(n)$. In this paper, we give a combinatorial interpretation of $\mu_{2k}(n)$. We introduce k-marked Dyson symbols based on a representation of ordinary partitions given by Dyson, and we show that $\mu_{2k}(n)$ equals the number of $(k+1)$-marked Dyson symbols of n. We then introduce the full crank of a k-marked Dyson symbol and show that there exist an infinite family of congruences for the full crank function of k-marked Dyson symbols which implies that for fixed prime $p \geq 5$ and positive integers r and $k \leq (p-1)/2$, there exist infinitely many non-nested arithmetic progressions $An+B$ such that $\mu_{2k}(An+B) \equiv 0 \pmod{p^r}$.

1 Introduction

Dyson's rank [9] and the Andrews-Garvan-Dyson crank [2] are two fundamental statistics in the theory of partitions. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, the rank of λ, denoted $r(\lambda)$, is the largest part of λ minus the number of parts. The crank $c(\lambda)$ is defined by

$$c(\lambda) = \begin{cases} \lambda_1, & \text{if } n_1(\lambda) = 0, \\ \mu(\lambda) - n_1(\lambda), & \text{if } n_1(\lambda) > 0, \end{cases}$$

where $n_1(\lambda)$ is the number of ones in λ and $\mu(\lambda)$ is the number of parts larger than $n_1(\lambda)$.

Andrews [3] introduced the symmetrized moments $\eta_{2k}(n)$ of ranks of partitions of n given by

$$\eta_k(n) = \sum_{m=-\infty}^{+\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m,n), \tag{1.1}$$

where $N(m, n)$ is the number of partitions of n with rank m.

In view of the symmetry $N(-m, n) = N(m, n)$, we have $\eta_{2k+1}(n) = 0$. As for the even symmetrized moments $\eta_{2k}(n)$, Andrews [3] showed that for fixed $k \geq 1$, $\eta_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of n. Kursungoz [15] and Ji [13] provided the alternative proof of this result respectively. Bringmann, Lovejoy and Osburn [7] defined two-parameter generalization of $\eta_{2k}(n)$ and k-marked Durfee symbols. In [3], Andrews also introduced the full rank of a k-marked Durfee symbol and defined the full rank function $NF_k(r, t; n)$ to be the number of k-marked Durfee symbols of n with full rank congruent to r modulo t.

The full rank function $NF_k(r, t; n)$ have been extensively studied and they posses many congruence properties, see for example, [5–8,14]. Recently, Bringmann, Garvan and Mahlburg [6] used the automorphic properties of the generating functions of $NF_k(r, t; n)$ to prove the existence of infinitely many congruences for $NF_k(r, t; n)$. More precisely, for given positive integers j, $k \geq 3$, odd positive integer t, and prime Q not divisible by $6t$, there exist infinitely many arithmetic progressions $An + B$ such that for every $0 \leq r < t$, we have

$$NF_k(r, t; An + B) \equiv 0 \pmod{Q^j}. \tag{1.2}$$

Since

$$\eta_{2k}(n) = \sum_{r=0}^{t-1} NF_{k+1}(r, t; n),$$

by (1.2), we see that there exist an infinite family of congruences for $\eta_{2k}(n)$, namely, for given positive integers k and j, prime $Q > 3$, there exist infinitely many non-nested arithmetic progressions $An + B$ such that

$$\eta_{2k}(An + B) \equiv 0 \pmod{Q^j}.$$

Analogous to the symmetrized moments $\eta_k(n)$ of ranks, Garvan [12] introduced the k-th symmetrized moments $\mu_k(n)$ of cranks of partitions of n in the study of the higher-order spt-function $spt_k(n)$. To be more specific,

$$\mu_k(n) = \sum_{m=-\infty}^{+\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m, n), \tag{1.3}$$

where $M(m, n)$ denotes the number of partitions of n with crank m for $n > 1$. For $n = 1$ and $m \neq -1, 0, 1$, we set $M(m, 1) = 0$; otherwise, we define

$$M(-1, 1) = 1, \ M(0, 1) = -1, \ M(1, 1) = 1.$$

It is clear that $\mu_{2k+1}(n) = 0$, since $M(m, n) = M(-m, n)$.

In this paper, we give a combinatorial interpretation of $\mu_{2k}(n)$. We first introduce the notion of k-marked Dyson symbols based on a representation for ordinary partitions given by Dyson [9]. We show that for fixed $k \geq 1$, $\mu_{2k}(n)$ equals the number of $(k+1)$-marked Dyson symbols of n. Moreover, we define the full crank of a k-marked Dyson symbol and define full crank function $NC_k(r, t; n)$ to be the number of k-marked Dyson symbols of n with full crank congruent to r modulo t. We prove that for fixed prime $p \geq 5$ and positive integers r and $k \leq (p+1)/2$, there exists infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq i \leq p^r - 1$,

$$NC_k(i, p^r; An + B) \equiv 0 \pmod{p^r}. \tag{1.4}$$

Note that

$$\mu_{2k}(n) = \sum_{i=0}^{p^r-1} NC_{k+1}(i, p^r; n),$$

so that from (1.4) we can deduce that there exist an infinite family of congruences for $\mu_{2k}(n)$, that is, for fixed prime $p \geq 5$, positive integers r and $k \leq (p-1)/2$, there exist infinitely many non-nested arithmetic progressions $An + B$ such that

$$\mu_{2k}(An + B) \equiv 0 \pmod{p^r}.$$

2 Dyson symbols and k-marked Dyson symbols

In this section, we introduce the notion of k-marked Dyson symbols. A 1-marked Dyson symbol is called a Dyson symbol, which is a representation of a partition introduced by Dyson [10]. For $1 \leq i \leq k$, we define the i-th crank of a k-marked Dyson symbol. Moreover, we define the function $F_k(m_1, m_2, \ldots, m_k; n)$ to be the number of k-marked Dyson symbol of n with the i-th crank equal to m_i for $1 \leq i \leq k$. The following theorem shows that the number of k-marked Dyson symbols of n can be expressed in terms of the number of Dyson symbols of n.

Theorem 2.1. *For fixed integers $m_1, m_2, \ldots, m_k$, we have*

$$F_k(m_1, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1}=0}^{+\infty} F_1\left(\sum_{i=1}^{k} |m_i| + 2\sum_{i=1}^{k-1} t_i + k - 1; n\right). \tag{2.1}$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $\ell(\lambda)$ denote the number of parts of λ and $|\lambda|$ denote the sum of parts of λ. A Dyson symbol of n is a pair of restricted partitions (α, β) satisfying the following conditions:

(1) If $\ell(\alpha) = 0$, then $\beta_1 = \beta_2$;

(2) If $\ell(\alpha) = 1$, then $\alpha_1 = 1$;

(3) If $\ell(\alpha) > 1$, then $\alpha_1 = \alpha_2$;

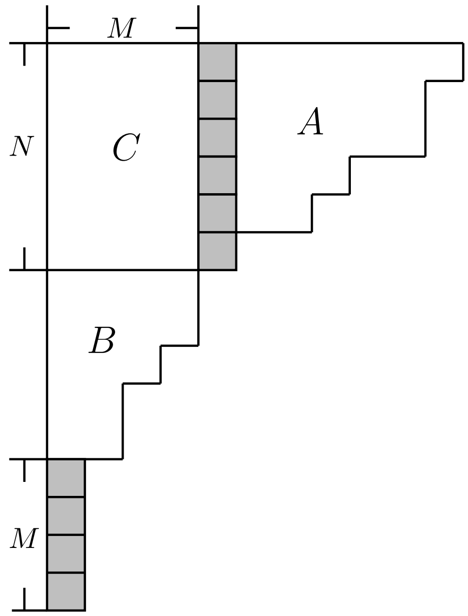

Figure 2.1: The decomposition of λ.

(4) $n = |\alpha| + |\beta| + \ell(\alpha)\ell(\beta)$.

When we display a Dyson symbol, we shall put α on the top of β in the form of a Durfee symbol [3] or a Frobenius partition [1].

For example, there are 5 Dyson symbols of 4:

$$\begin{pmatrix} & \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} & & & \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & \end{pmatrix}.$$

Theorem 2.2 (Dyson)**.** *There is a bijection Ω between the set of partitions of n and the set of Dyson symbols of n.*

For completeness, we give a proof of the above theorem.

Proof of Theorem 2.2: Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of n. A Dyson symbol (α, β) of n can be constructed via the following procedure. There are two cases.

Case 1: One is not a part of λ. We set $\alpha = \emptyset$ and $\beta = \lambda'$.

Case 2: One is a part of λ. Assume that one occurs M times in λ. We decompose the Ferrers diagram of λ into three blocks as illustrated in Figure 2.1, where N is the number of parts of λ that are greater than M. In this case, we see that $\lambda = (\lambda_1, \ldots, \lambda_N, \lambda_{N+1}, \ldots, \lambda_s, 1^M)$, where $\lambda_N > M$, $\lambda_{N+1} \leq M$ and 1^M means M occurrences of 1. Then remove all parts equal to one from λ and insert a new part M, so that we get a partition $\mu = (\lambda_1, \ldots, \lambda_N, M, \lambda_{N+1}, \ldots, \lambda_s)$ as shown in Figure 2.2.

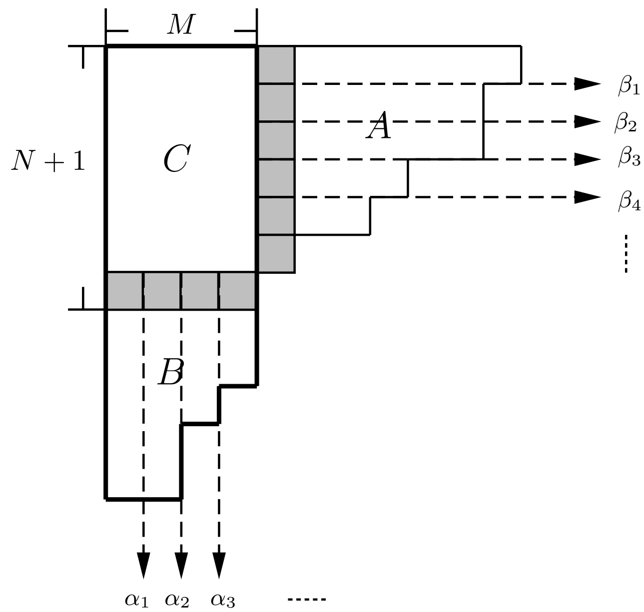

Figure 2.2: The Dyson symbol (α, β).

Now the partitions α and β can be obtained from μ. First, let $\beta = (\lambda_1 - M, \lambda_2 - M, \ldots, \lambda_N - M)$, and let $\nu = (M, \lambda_{N+1}, \ldots, \lambda_s)$. Then we get $\alpha = (\nu'_1, \nu'_2, \ldots, \nu'_M)$, where ν' the conjugate of ν, see Figure 2.2.

It is easy to verify that (α, β) is a Dyson symbol of n and the above procedure is reversible, and hence the proof is complete. ∎

For a Dyson symbol (α, β), Dyson [10] considered the difference between the number of parts of α and β, which we call the crank of (α, β). Let $F_1(m; n)$ denote the number of Dyson symbols of n with crank m. Dyson [10] observed the following relation based on the construction in Theorem 2.2.

Corollary 2.3 (Dyson)**.** *For $n \geq 2$ and integer m,*

$$M(-m, n) = F_1(m; n). \tag{2.2}$$

A k-marked Dyson symbol is defined as the following array

$$\eta = \left(\begin{array}{ccccccc} \alpha^{(k)}, & & \alpha^{(k-1)}, & & \ldots, & & \alpha^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_1, & \\ \beta^{(k)}, & & \beta^{(k-1)}, & & \ldots, & & \beta^{(1)} \end{array} \right),$$

consisting of k pairs of partitions $(\alpha^{(i)}, \beta^{(i)})$ and a partition $p = (p_{k-1}, p_{k-2}, \ldots, p_0)$ subject to the following conditions:

(1) The smallest part of p equals 1, that is, $p_{k-1} \geq \cdots \geq p_1 \geq p_0 = 1$.

(2) For $1 \le i \le k-1$, each part of $\alpha^{(i)}$ and $\beta^{(i)}$ is between p_{i-1} and p_i, namely,

$$p_i \ge \alpha_1^{(i)} \ge \alpha_2^{(i)} \ge \cdots \ge \alpha_\ell^{(i)} \ge p_{i-1} \quad \text{and} \quad p_i \ge \beta_1^{(i)} \ge \beta_2^{(i)} \ge \cdots \ge \beta_\ell^{(i)} \ge p_{i-1}.$$

(3) Each part of $\alpha^{(k)}$ and $\beta^{(k)}$ is no less than p_{k-1}, namely,

$$\alpha_1^{(k)} \ge \alpha_2^{(k)} \ge \cdots \ge \alpha_\ell^{(k)} \ge p_{k-1} \quad \text{and} \quad \beta_1^{(k)} \ge \beta_2^{(k)} \ge \cdots \ge \beta_\ell^{(k)} \ge p_{k-1}.$$

(4) If $\ell(\alpha^{(k)}) = 1$, then $\alpha_1^{(k)} = p_{k-1}$;
If $\ell(\alpha^{(k)}) > 1$, then $\alpha_1^{(k)} = \alpha_2^{(k)}$;
If $\ell(\alpha^{(k)}) = 0$ and $\ell(\beta^{(k)}) = 1$, then $\beta_1^{(k)} = p_{k-1}$;
If $\ell(\alpha^{(k)}) = 0$ and $\ell(\beta^{(k)}) \ge 2$, then $\beta_1^{(k)} = \beta_2^{(k)}$;
If $\ell(\alpha^{(k)}) = 0$ and $\ell(\beta^{(k)}) = 0$, then $p_{k-1} = \max\{\alpha_1^{(k-1)}, \beta_1^{(k-1)}\}$.

For example, the array below

$$\eta = \begin{pmatrix} (5,5,4) & & (3,3,2) & & (1,1) \\ & 4 & & 2 & \\ (4) & & (3,2,2) & & (2,1,1) \end{pmatrix} \tag{2.3}$$

is a 3-marked Dyson symbol.

We next define the weight of a k-marked Dyson symbol. Recall that for a pair of partitions (α, β) with $\ell(\alpha) \ge \ell(\beta)$, a balanced part β_i of β is defined recursively as follow. If the number of parts greater than β_i in α *is equal to* the number of unbalanced parts before β_i in β, that is, the number of unbalanced parts β_j with $1 \le j < i$; otherwise, we call β_i is an unbalanced part, see [13, p.992]. We use $b(\alpha, \beta)$ to denote the number of balanced parts of (α, β).

For example, for the pair of partitions

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 & 1 \\ 3 & 2 & 2 & \end{pmatrix},$$

the first part 3 of β is balanced, and the second part 2 and the third part 2 are unbalanced. Therefore, $b(\alpha, \beta) = 1$.

We now define the i-th crank and the i-th balanced number of a k-marked Dsyon symbol. Let

$$\eta = \begin{pmatrix} \alpha^{(k)}, & & \alpha^{(k-1)}, & & \ldots, & & \alpha^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_1 & \\ \beta^{(k)}, & & \beta^{(k-1)}, & & \ldots, & & \beta^{(1)} \end{pmatrix}$$

be a k-marked Dyson symbol. The pair of partitions $(\alpha^{(i)}, \beta^{(i)})$ is called the i-th vector of η. For $1 \le i \le k$, we define $c_i(\eta)$, the i-th *crank* of η, to be the difference between the number of parts of $\alpha^{(i)}$ and $\beta^{(i)}$, that is, $c_i(\eta) = \ell(\alpha^{(i)}) - \ell(\beta^{(i)})$.

For $1 \le i < k$, we define $b_i(\eta)$, the i-th balanced number of η by

$$b_i(\eta) = \begin{cases} b(\alpha^{(i)}, \beta^{(i)}), & \text{if } \ell(\alpha^{(i)}) \ge \ell(\beta^{(i)}), \\ b(\beta^{(i)}, \alpha^{(i)}), & \text{if } \ell(\alpha^{(i)}) < \ell(\beta^{(i)}). \end{cases}$$

For $i = k$, we set $b_k(\eta) = 0$.

For the 3-marked Dyson symbol η in (2.3), we have $c_1(\eta) = -1$, $c_2(\eta) = 0$, $c_3(\eta) = 2$ and $b_1(\eta) = 1$, $b_2(\eta) = 1$, $b_3(\eta) = 0$.

For $1 \le i \le k$, we define $l_i(\eta)$, the i-th large length of η by

$$l_i(\eta) = \begin{cases} \ell(\alpha^{(i)}), & \text{if } \ell(\alpha^{(i)}) \ge \ell(\beta^{(i)}), \\ \ell(\beta^{(i)}), & \text{if } \ell(\alpha^{(i)}) < \ell(\beta^{(i)}). \end{cases}$$

Similarly, we define the i-th small length $s_i(\eta)$ of η by

$$s_i(\eta) = \begin{cases} \ell(\beta^{(i)}), & \text{if } \ell(\alpha^{(i)}) \ge \ell(\beta^{(i)}), \\ \ell(\alpha^{(i)}), & \text{if } \ell(\alpha^{(i)}) < \ell(\beta^{(i)}). \end{cases}$$

The weight of k-marked Dyson symbol is defined by

$$|\eta| = \sum_{i=1}^{k} (|\alpha^{(i)}| + |\beta^{(i)}|) + \sum_{i=1}^{k-1} p_i + (l(\eta) + D + k - 1)(s(\eta) - D), \tag{2.4}$$

where

$$l(\eta) = \sum_{i=1}^{k} l_i(\eta), \quad s(\eta) = \sum_{i=1}^{k} s_i(\eta), \quad \text{and} \quad D = \sum_{i=1}^{k} b_i(\eta). \tag{2.5}$$

For example, the weight of the 3-marked Dyson symbol η in (2.3) equals 97.

For a k-marked Dyson symbol η, if the weight of η equals n, we call η a k-marked Dyson symbol of n. We can now define the function $F_k(m_1, \ldots, m_k; n)$ as the number of k-marked Dyson symbols of n with the i-th crank equal to m_i for $1 \le i \le k$. Note that a 1-marked Dyson symbol is a Dyson symbol and $F_1(m; n) = M(-m, n)$. The following theorem shows the function $F_k(m_1, \ldots, m_k; n)$ has the mirror symmetry with respect to each m_j.

Theorem 2.4. *For $n \ge 2$, $k \ge 1$ and $1 \le j \le k$, we have*

$$F_k(m_1, \ldots, m_j, \ldots, m_k; n) = F_k(m_1, \ldots, -m_j, \ldots, m_k; n). \tag{2.6}$$

Proof. The above identity is trivial for $m_j = 0$. We now assume that $m_j > 0$. Let $H_k(m_1, \ldots, m_k; n)$ denote the set of k-marked Dyson symbols of n counted by $F_k(m_1, \ldots,$

$m_k; n)$. We aim to build a bijection Λ between the set $H_k(m_1, \ldots, m_j, \ldots, m_k; n)$ and the set $H_k(m_1, \ldots, -m_j, \ldots, m_k; n)$.

Let

$$\eta = \begin{pmatrix} \alpha^{(k)}, & & \alpha^{(k-1)}, & & \ldots, & & \alpha^{(j)}, & \ldots, & & & \alpha^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_j & & & \cdots & p_1 & \\ \beta^{(k)}, & & \beta^{(k-1)}, & & \ldots, & & \beta^{(j)}, & \ldots, & & & \beta^{(1)} \end{pmatrix}$$

be a k-marked Dyson symbol in $H_k(m_1, \ldots, m_j, \ldots, m_k; n)$. To define the map Λ, we need to construct a new j-th vector $(\bar{\alpha}^{(j)}, \bar{\beta}^{(j)})$ from $(\alpha^{(j)}, \beta^{(j)})$. There are four cases.

Case 1: $1 \leq j \leq k-1$. Set $\bar{\alpha}^{(j)} = \beta^{(j)}$ and $\bar{\beta}^{(j)} = \alpha^{(j)}$.

Case 2: $j = k$ and $\ell(\alpha^{(k)}) = 1$. In this case, we have $\alpha_1^{(k)} = p_{k-1}$ and $\beta^{(k)} = \emptyset$. Set $\bar{\alpha}^{(k)} = \emptyset$ and $\bar{\beta}^{(k)} = \alpha^{(k)}$.

Case 3: $j = k$, $\ell(\alpha^{(k)}) \geq 2$ and $\ell(\beta^{(k)}) \neq 1$. Let $t = \beta_1^{(k)} - \beta_2^{(k)}$. Set

$$\bar{\alpha}^{(k)} = (\beta_1^{(k)} - t,\ \beta_2^{(k)},\ \ldots,\ \beta_\ell^{(k)}) \quad \text{and} \quad \bar{\beta}^{(k)} = (\alpha_1^{(k)} + t,\ \alpha_2^{(k)},\ \ldots,\ \alpha_\ell^{(k)}).$$

Case 4: $j = k$, $\ell(\alpha^{(k)}) \geq 2$ and $\ell(\beta^{(k)}) = 1$. Let $t = \beta_1^{(k)} - p_{k-1}$. Set

$$\bar{\alpha}^{(k)} = (\beta_1^{(k)} - t) \quad \text{and} \quad \bar{\beta}^{(k)} = (\alpha_1^{(k)} + t,\ \alpha_2^{(k)},\ \ldots,\ \alpha_\ell^{(k)}).$$

From the above construction, it can be checked that

$$\ell(\bar{\alpha}^{(j)}) - \ell(\bar{\beta}^{(j)}) = -(\ell(\alpha^{(j)}) - \ell(\beta^{(j)})).$$

Then $\Lambda(\eta)$ is defined as

$$\begin{pmatrix} \alpha^{(k)}, & & \alpha^{(k-1)}, & & \ldots, & & \bar{\alpha}^{(j)}, & \ldots, & & & \alpha^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_j & & & \cdots & p_1 & \\ \beta^{(k)}, & & \beta^{(k-1)}, & & \ldots, & & \bar{\beta}^{(j)}, & \ldots, & & & \beta^{(1)} \end{pmatrix}.$$

Hence $\Lambda(\eta)$ is a k-marked Dyson symbol in $H_k(m_1, \ldots, -m_j, \ldots, m_k; n)$. Furthermore, it can be seen that the above process is reversible. Thus Λ is a bijection. ∎

We are now ready to prove Theorem 2.1, which says that the number of k-marked Dyson symbols of n can be expressed in terms of the number of Dyson symbols of n. This theorem is needed in the combinatorial interpretation of $\mu_{2k}(n)$ given in Theorem 3.1. By Theorem 2.4, we see that Theorem 2.1 can be deduced from the following formula.

Theorem 2.5. *For $n \geq 2$ and $m_1, m_2, \ldots, m_k \geq 0$, we have*

$$F_k(m_1, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1} = 0}^{+\infty} F_1\left(\sum_{i=1}^{k} m_i + 2\sum_{i=1}^{k-1} t_i + k - 1; n\right). \tag{2.7}$$

To prove the above theorem, we introduce the structure of strict k-marked Dyson symbols. Recall that a strict bipartition of n is a pair of partitions (α, β) such that $\alpha_i > \beta_i$ for $i = 1, 2, \ldots, \ell(\beta)$ and $|\alpha| + |\beta| = n$. Note that for a strick bipartition (α, β) we have $\ell(\alpha) \geq \ell(\beta)$. For example,

$$\begin{pmatrix} 3 & 3 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 & \end{pmatrix}$$

is a strict bipartition.

Strict bipartitions are the building blocks of strict k-marked Dyson symbols. For $k \geq 2$, let

$$\eta = \begin{pmatrix} \alpha^{(k)}, & & \alpha^{(k-1)}, & & \ldots, & & \alpha^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_1 & \\ \beta^{(k)}, & & \beta^{(k-1)}, & & \ldots, & & \beta^{(1)} \end{pmatrix}$$

be a k-marked Dyson symbols of n. If $(\alpha^{(i)}, \beta^{(i)})$ is a strict bipartition for any $1 \leq i < k$, we say that η a strict k-marked Dyson symbol of n.

Notice that there is no balanced part in a strict bipartition. Consequently, if η is a strict k-marked Dyson symbol, then the i-th balanced number $b_i(\eta)$ of η equals zero for $1 \leq i < k$. To prove Theorem 2.5, we define a function $F_k^s(m_1, \ldots, m_k; n)$ as the number of strict k-marked Dyson symbols of n with the i-th crank equal to m_i for $1 \leq i \leq k$ and define a function $F_k(m_1, \ldots, m_k, t_1, \ldots, t_{k-1}; n)$ as the number of k-marked Dyson symbols of n with the i-th crank equal to m_i for $1 \leq i \leq k$ and the i-th balance number equal to t_i for $1 \leq i \leq k-1$. The relation stated in Theorem 2.5 can be established via two steps as stated in the following two theorems.

Theorem 2.6. *For $n \geq 2$, $k \geq 2$, $m_1, m_2, \ldots, m_k \geq 0$ and $t_1, t_2, \ldots, t_{k-1} \geq 0$, we have*

$$F_k(m_1, \ldots, m_k, t_1, \ldots, t_{k-1}; n) = F_k^s(m_1 + 2t_1, \ldots, m_{k-1} + 2t_{k-1}, m_k; n). \tag{2.8}$$

Theorem 2.7. *For $n \geq 2$, $k \geq 2$ and $m_1, m_2, \ldots, m_k \geq 0$, we have*

$$F_k^s(m_1, \ldots, m_k; n) = F_1\left(\sum_{i=1}^{k} m_i + k - 1; n\right). \tag{2.9}$$

To prove Theorem 2.6, we need a bijection in [13, Theorem 2.4]. Let $P(r; n)$ denote the set of pairs of partitions (α, β) of n where there are r balanced parts and $\ell(\alpha) - \ell(\beta) \geq 0$, and let $Q(r; n)$ denote the set of strict bipartitions $(\bar{\alpha}, \bar{\beta})$ of n with $\ell(\bar{\alpha}) - \ell(\bar{\beta}) \geq r$. Given two positive integers n and r, there is a bijection ψ between $P(r; n)$ and $Q(2r; n)$. Furthermore, the bijection ψ possesses the following properties. For $(\alpha, \beta) \in P(r; n)$, let $(\bar{\alpha}, \bar{\beta}) = \psi(\alpha, \beta)$. Then we have

$$\bar{\alpha}_1 = \max\{\alpha_1, \beta_1\}, \quad \bar{\alpha}_\ell = \alpha_\ell, \quad \text{and} \quad \bar{\beta}_\ell \geq \beta_\ell. \tag{2.10}$$

$$\ell(\bar{\alpha}) = \ell(\alpha) + r \quad \text{and} \quad \ell(\bar{\beta}) = \ell(\beta) - r. \tag{2.11}$$

We next give a proof of Theorem 2.6 by using the bijection ψ.

Proof of Theorem 2.6. Let $P_k(m_1, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n)$ denote the set of k-marked Dyson symbols of n with the i-th crank equal to m_i and the i-th balanced number equal to t_i, and let $Q_k(m_1, \ldots, m_k; n)$ denote the set of strict k-marked Dyson symbols of n with the i-th crank equal to m_i. We proceed to define a bijection Ω between $P_k(m_1, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n)$ and $Q_k(m_1 + 2t_1, \ldots, m_{k-1} + 2t_{k-1}, m_k; n)$.

Let

$$\eta = \begin{pmatrix} \alpha^{(k)}, & & \alpha^{(k-1)}, & & \ldots, & & \alpha^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_1 & \\ \beta^{(k)}, & & \beta^{(k-1)}, & & \ldots, & & \beta^{(1)} \end{pmatrix}$$

be a k-marked Dyson symbol in $P_k(m_1, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n)$. For $1 \leq i < k$, we apply the bijection ψ described above to $(\alpha^{(i)}, \beta^{(i)})$ to get a pair of partitions $(\bar{\alpha}^{(i)}, \bar{\beta}^{(i)})$. From the properties of the bijection ψ, we see that$(\bar{\alpha}^{(i)}, \bar{\beta}^{(i)})$ is a strict bipartition and

$$\bar{\alpha}_1^{(i)} = \max\{\alpha_1^{(i)}, \beta_1^{(i)}\}, \quad \bar{\alpha}_\ell^{(i)} = \alpha_\ell^{(i)}, \quad \bar{\beta}_\ell^{(i)} \geq \beta_\ell^{(i)} \tag{2.12}$$

and

$$\ell(\bar{\alpha}^{(i)}) = \ell(\alpha^{(i)}) + t_i, \quad \ell(\bar{\beta}^{(i)}) = \ell(\beta^{(i)}) - t_i. \tag{2.13}$$

Then $\Omega(\eta)$ is defined to be

$$\begin{pmatrix} \alpha^{(k)}, & & \bar{\alpha}^{(k-1)}, & & \ldots, & & \bar{\alpha}^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_1 & \\ \beta^{(k)}, & & \bar{\beta}^{(k-1)}, & & \ldots, & & \bar{\beta}^{(1)} \end{pmatrix}.$$

By (2.12), we see that that for $1 \leq i < k-1$, each part of $\bar{\alpha}^{(i)}$ and $\bar{\beta}^{(i)}$ is between p_{i-1} and p_i, namely,

$$p_i \geq \bar{\alpha}_1^{(i)} \geq \bar{\alpha}_2^{(i)} \geq \cdots \geq \bar{\alpha}_\ell^{(i)} \geq p_{i-1} \quad \text{and} \quad p_i \geq \bar{\beta}_1^{(i)} \geq \bar{\beta}_2^{(i)} \geq \cdots \geq \bar{\beta}_\ell^{(i)} \geq p_{i-1}.$$

It is also clear from (2.13) that the i-th crank of $\Omega(\eta)$ is equal to $m_i + 2t_i$ for $1 \leq i < k$ and the k-th crank of $\Omega(\eta)$ is equal to m_k. Using (2.13) again, we get

$$l(\Omega(\eta)) = \sum_{i=1}^{k-1} \ell(\bar{\alpha}^{(i)}) + \ell(\alpha^k) = \sum_{i=1}^{k} (\ell(\alpha^{(i)}) + t_i) = \sum_{i=1}^{k} \ell(\alpha^{(i)}) + D = l(\eta) + D$$

and

$$s(\Omega(\eta)) = \sum_{i=1}^{k-1} \ell(\bar{\beta}^{(i)}) + \ell(\beta^k) = \sum_{i=1}^{k} (\ell(\beta^{(i)}) - t_i) = \sum_{i=1}^{k} \ell(\alpha^{(i)}) - D = s(\eta) - D.$$

Thus the weight of $\Omega(\eta)$ is equal to

$$\begin{aligned} &\sum_{i=1}^{k} (|\bar{\alpha}^{(i)}| + |\bar{\beta}^{(i)}|) + \sum_{i=1}^{k-1} p_i + (l(\Omega(\eta)) + k - 1) \cdot s(\Omega(\eta)) \\ &= \sum_{i=1}^{k} (|\alpha^{(i)}| + |\beta^{(i)}|) + \sum_{i=1}^{k-1} p_i + (l(\eta) + k - 1 + D) \cdot (s(\eta) - D), \end{aligned}$$

which is in accordance with the definition of $|\eta|$. So $\Omega(\eta)$ is in $Q_k(m_1+2t_1,\ldots,m_{k-1}+2t_{k-1},m_k;n)$. Since ψ is a bijection, it is readily verified that Ω is also a bijection, and hence the proof is complete. ∎

We now turn to the proof of Theorem 2.7.

Proof of Theorem 2.7. Recall that $Q_k(m_1,\ldots,m_k;n)$ denotes the set of strict k-marked Dyson symbols of n with the i-th crank equal to m_i and $H_1(m;n)$ denotes the set of Dyson symbols of n with crank m. To establish a bijection Φ between $Q_k(m_1,\ldots,m_k;n)$ and $H_1(m_1+\cdots+m_k+k-1;n)$, let

$$\eta=\begin{pmatrix} \alpha^{(k)}, & & \alpha^{(k-1)}, & & \ldots, & & \alpha^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_1 & \\ \beta^{(k)}, & & \beta^{(k-1)}, & & \ldots, & & \beta^{(1)} \end{pmatrix}$$

be a strict k-marked Dyson symbol in $Q_k(m_1,\ldots,m_k;n)$. Let α be the partition consisting of all parts of $\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(k)}$ together with $p_1,\ldots,p_{k-1}$, and let β be the partition consisting of all parts of $\beta^{(1)},\beta^{(2)},\ldots,\beta^{(k)}$. Then $\Phi(\eta)$ is defined to be (α,β). From the definition of k-marked Dyson symbols, we see that (α,β) is a Dyson symbol. It is also easily seen that

$$\ell(\alpha)=l(\eta)+k-1,\quad \ell(\beta)=s(\eta) \tag{2.14}$$

and

$$|\alpha|=\sum_{i=1}^{k}|\alpha^{(i)}|+\sum_{i=1}^{k-1}p_i,\quad |\beta|=\sum_{i=1}^{k}|\beta^{(i)}|. \tag{2.15}$$

It follows from (2.14) that

$$\ell(\alpha)-\ell(\beta)=\sum_{i=1}^{k}m_i+k-1.$$

Combining (2.14) and (2.15), we deduce that the weight of (α,β) equals

$$|\alpha|+|\beta|+\ell(\alpha)\ell(\beta)=\sum_{i=1}^{k}|\alpha^{(i)}|+\sum_{i=1}^{k-1}p_i+\sum_{i=1}^{k}|\beta^{(i)}|+(l(\eta)+k-1)s(\eta)=|\eta|.$$

This proves that (α,β) is a Dyson symbol in $H_1(m_1+\cdots+m_k+k-1;n)$.

We next describe the reverse map of Φ. Let

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\alpha_1 & \alpha_2 & \ldots & \alpha_\ell\\ \beta_1 & \beta_2 & \ldots & \beta_\ell\end{pmatrix}$$

be a Dyson symbol in $H_1(m_1+\cdots+m_k+k-1;n)$. We proceed to show that a strict k-marked Dyson symbol η can be recovered from the Dyson symbol (α,β).

First, we see that the k-th vector $(\alpha^{(k)}, \beta^{(k)})$ of η and p_{k-1} can be recovered from (α, β). Let j_k be largest nonnegative integer such that $\beta_{j_k} \geq \alpha_{m_k+j_k+1}$, that is, for any $i \geq j_k + 1$, we have $\beta_i < \alpha_{m_k+i+1}$. Define

$$\begin{pmatrix} \alpha^{(k)} \\ \beta^{(k)} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{m_k+j_k} \\ \beta_1 & \beta_2 & \dots & \beta_{j_k} \end{pmatrix} \quad \text{and} \quad p_{k-1} = \alpha_{m_k+j_k+1}.$$

Obviously, $\ell(\alpha^{(k)}) - \ell(\beta^{(k)}) = m_k$.

To recover $(\alpha^{(k-1)}, \beta^{(k-1)})$ and p_{k-1}, we let

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} \alpha_{m_k+j_k+2} & \alpha_{m_k+j_k+3} & \dots & \alpha_\ell \\ \beta_{j_k+1} & \beta_{j_k+2} & \dots & \beta_\ell \end{pmatrix}.$$

By the choice of j_k, we find that $\alpha_{m_k+j_k+i+1} > \beta_{j_k+i}$ for any i, in other words, $\alpha'_i > \beta'_i$. Consequently, (α', β') is a strict bipartition. Then $(\alpha^{(k-1)}, \beta^{(k-1)})$ and p_{k-1} can be constructed from (α', β'). Let j_{k-1} be the largest nonnegative integer such that $\beta'_{j_{k-1}} \geq \alpha'_{m_{k-1}+j_{k-1}+1}$. Define

$$\begin{pmatrix} \alpha^{(k-1)} \\ \beta^{(k-1)} \end{pmatrix} = \begin{pmatrix} \alpha'_1 & \alpha'_2 & \dots & \alpha'_{m_{k-1}+j_{k-1}} \\ \beta'_1 & \beta'_2 & \dots & \beta'_{j_{k-1}} \end{pmatrix} \quad \text{and} \quad p_{k-2} = \alpha'_{m_{k-1}+j_{k-1}+1}.$$

Now we have $\ell(\alpha^{(k-1)}) - \ell(\beta^{(k-1)}) = m_{k-1}$. Since (α', β') is a strict bipartition, we deduce that $(\alpha^{(k-1)}, \beta^{(k-1)})$ is a strict bipartition.

The above procedure can be repeatedly used to determine $(\alpha^{(k-2)}, \beta^{(k-2)}), p_{k-3}, \dots, (\alpha^{(2)}, \beta^{(2)}), p_1, (\alpha^{(1)}, \beta^{(1)})$. The k-marked Dyson symbol η can be defined as

$$\begin{pmatrix} \alpha^{(k)}, & & \alpha^{(k-1)}, & & \dots, & & \alpha^{(1)} \\ & p_{k-1}, & & p_{k-2}, & \cdots & p_1 & \\ \beta^{(k)}, & & \beta^{(k-1)}, & & \dots, & & \beta^{(1)} \end{pmatrix}.$$

It can be checked that η is a strict k-marked Dyson symbol in $Q_k(m_1, \dots, m_k; n)$. Moreover, it can be seen that $\Phi(\eta) = (\alpha, \beta)$, that is, Φ is indeed a bijection. This completes the proof. ∎

Here is an example to illustrate the reverse map Φ^{-1}. Assume that $m_1 = 1, m_2 = 1, m_3 = 0$, and

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 6 & 6 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 \\ 5 & 5 & 4 & 2 & 1 & 1 & 1 & & & & \end{pmatrix},$$

which a Dyson symbol of 127, that is, $(\alpha, \beta) \in H_1(4; 127)$. From (α, β), we get

$$\begin{pmatrix} \alpha^{(3)} \\ \beta^{(3)} \end{pmatrix} = \begin{pmatrix} 6 & 6 & 3 \\ 5 & 5 & 4 \end{pmatrix}, \quad p_2 = 3, \quad \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 3 & 3 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & & & \end{pmatrix}.$$

Based on (α', β'), we get

$$\begin{pmatrix} \alpha^{(2)} \\ \beta^{(2)} \end{pmatrix} = \begin{pmatrix} 3 & 3 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 & \end{pmatrix}, \quad p_2 = 1, \quad \begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ \ \end{pmatrix}.$$

Finally, we obtain

$$\eta = \begin{pmatrix} (6\ 6\ 3) & & (3\ 3\ 2\ 2\ 1) & & (1) \\ & 3 & & 1 & \\ (5\ 5\ 4) & & (2\ 1\ 1\ 1) & & \end{pmatrix}.$$

It can be checked that $\eta \in Q_3(1, 1, 0; 127)$.

3 A combinatorial interpretation of $\mu_{2k}(n)$

In this section, we use Theorem 2.1 to give a combinatorial interpretation of $\mu_{2k}(n)$ in terms of k-marked Dyson symbols.

Theorem 3.1. *For $k \geq 1$ and $n \geq 2$, $\mu_{2k}(n)$ is equal to the number of $(k+1)$-marked Dyson symbols of n.*

Proof. By definition of $F_k(m_1, \ldots, m_k; n)$, the assertion of the theorem can be stated as follows

$$\sum_{m_1,\ldots,m_{k+1}=-\infty}^{\infty} F_{k+1}(m_1, \ldots, m_{k+1}; n) = \mu_{2k}(n). \tag{3.1}$$

Using Theorem 2.1, we see that the left-hand side of (3.1) equals

$$\begin{aligned} &\sum_{m_1,m_2,\ldots,m_{k+1}=-\infty}^{\infty} F_{k+1}(m_1, \ldots, m_{k+1}; n) \\ &\qquad = \sum_{m_1,m_2,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} F_1\left(\sum_{i=1}^{k+1} |m_i| + 2\sum_{i=1}^{k} t_i + k; n\right). \end{aligned} \tag{3.2}$$

Given k and n, let $c_k(j)$ denote the number of integer solutions to the equation

$$|m_1| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = j$$

in $m_1, m_2, \ldots, m_{k+1}$ and $t_1, t_2, \ldots, t_k$ subject to the further condition that $t_1, t_2, \ldots, t_k$ are nonnegative. It can be shown that generating function of $c_k(j)$ is equal to

$$\sum_{j=0}^{\infty} c_k(j) q^j = \frac{1+q}{(1-q)^{2k+1}},$$

so that

$$c_k(j) = \binom{2k+j}{2k} + \binom{2k+j-1}{2k}.$$

Substituting j by $m-k$, we get

$$c_k(m-k) = \binom{m+k-1}{2k} + \binom{m+k}{2k}.$$

Thus (3.2) simplifies to

$$\sum_{m_1,m_2,\dots,m_{k+1}=-\infty}^{\infty} F_{k+1}(m_1,\dots,m_{k+1};n) \\ = \sum_{m=1}^{\infty} \left[\binom{m+k-1}{2k} + \binom{m+k}{2k} \right] F_1(m;n).$$

Using Corollary 2.3 and noting that $M(-m,n) = M(m,n)$, we conclude that

$$\sum_{m_1,m_2,\dots,m_{k+1}=-\infty}^{\infty} F_{k+1}(m_1,\dots,m_{k+1};n) \\ = \sum_{m=1}^{\infty} \left[\binom{m+k-1}{2k} + \binom{m+k}{2k} \right] M(m,n),$$

which equals $\mu_{2k}(n)$, as claimed. ∎

For example, for $n = 5$ and $k = 1$, we have $\mu_2(5) = 35$, and there are 35 2-marked Dyson symbols of 5 as listed in the following table.

$$\begin{pmatrix} & & & & \\ & & & & 1 \\ (1 & 1 & 1 & 1) & \end{pmatrix} \qquad \begin{pmatrix} (1) & \\ & 1 \\ (1) & \end{pmatrix} \qquad \begin{pmatrix} (2 & 2) & \\ & & 1 \\ & & \end{pmatrix}$$

$$\begin{pmatrix} & (1) \\ 2 & \\ & (2) \end{pmatrix} \qquad \begin{pmatrix} & & & (1) \\ & & 1 & \\ (1 & 1) & & (1) \end{pmatrix} \qquad \begin{pmatrix} & (1 & 1) \\ 1 & & \\ & (1 & 1) \end{pmatrix}$$

$$\begin{pmatrix} (1 & 1 & 1 & 1) & \\ & & & & 1 \\ & & & & \end{pmatrix} \qquad \begin{pmatrix} & & \\ & & 1 \\ (2 & 2) & \end{pmatrix} \qquad \begin{pmatrix} (1 & 1) & & (1) \\ & & 1 & \\ & & & (1) \end{pmatrix}$$

$$\begin{pmatrix} & (1 & 1 & 1 & 1) \\ 1 & & & & \\ & & & & \end{pmatrix} \qquad \begin{pmatrix} (1 & 1) & & (1 & 1) \\ & & 1 & & \\ & & & & \end{pmatrix} \qquad \begin{pmatrix} & & & & (1) \\ & & & 1 & \\ (1 & 1 & 1) & & \end{pmatrix}$$

$$
\begin{pmatrix} (2) & & (1) \\ & 2 & \\ & & \end{pmatrix}
\quad
\begin{pmatrix} & & (1\ \ 1\ \ 1) \\ & 1 & \\ (1) & & \end{pmatrix}
\quad
\begin{pmatrix} (1\ \ 1\ \ 1) & & (1) \\ & 1 & \\ & & \end{pmatrix}
$$

$$
\begin{pmatrix} (1) & & (1\ \ 1) \\ & 1 & \\ & & (1) \end{pmatrix}
\quad
\begin{pmatrix} (1) & & (1\ \ 1\ \ 1) \\ & 1 & \\ & & \end{pmatrix}
\quad
\begin{pmatrix} & & (1\ \ 1) \\ & 1 & \\ (1\ \ 1) & & \end{pmatrix}
$$

$$
\begin{pmatrix} & & (2\ \ 1) \\ & 2 & \\ & & \end{pmatrix}
\quad
\begin{pmatrix} & & (1\ \ 1\ \ 1) \\ & 1 & \\ & & (1) \end{pmatrix}
\quad
\begin{pmatrix} & & (1\ \ 1) \\ & 1 & \\ (1) & & (1) \end{pmatrix}
$$

$$
\begin{pmatrix} & & (1) \\ & 2 & \\ (2) & & \end{pmatrix}
\quad
\begin{pmatrix} (1) & & (1) \\ & 1 & \\ & & (1\ \ 1) \end{pmatrix}
\quad
\begin{pmatrix} & & (1) \\ & 1 & \\ & & (1\ \ 1\ \ 1) \end{pmatrix}
$$

$$
\begin{pmatrix} & & \\ & 1 & \\ & & (1\ \ 1\ \ 1\ \ 1) \end{pmatrix}
\quad
\begin{pmatrix} (1) & & \\ & 1 & \\ & & (1\ \ 1\ \ 1) \end{pmatrix}
\quad
\begin{pmatrix} & & \\ & 1 & \\ (1\ \ 1) & & (1\ \ 1) \end{pmatrix}
$$

$$
\begin{pmatrix} & & \\ & 2 & \\ & & (2\ \ 1) \end{pmatrix}
\quad
\begin{pmatrix} (1\ \ 1) & & \\ & 1 & \\ & & (1\ \ 1) \end{pmatrix}
\quad
\begin{pmatrix} & & \\ & 1 & \\ (1\ \ 1\ \ 1) & & (1) \end{pmatrix}
$$

$$
\begin{pmatrix} (2) & & \\ & 2 & \\ & & (1) \end{pmatrix}
\quad
\begin{pmatrix} & & \\ & 1 & \\ (1) & & (1\ \ 1\ \ 1) \end{pmatrix}
\quad
\begin{pmatrix} (1\ \ 1\ \ 1) & & \\ & 1 & \\ & & (1) \end{pmatrix}
$$

$$
\begin{pmatrix} & & (1) \\ & 1 & \\ (1) & & (1\ \ 1) \end{pmatrix}
\quad
\begin{pmatrix} & & \\ & 2 & \\ (2) & & (1) \end{pmatrix}
$$

4 Congruences for $\mu_{2k}(n)$

In this section, we introduce the full crank of a k-marked Dyson symbol. We show that there exist an infinite family of congruences for the full crank function of k-marked Dyson symbols.

To define the full crank of a k-marked Dyson symbol η, denoted $FC(\eta)$, we recall that $c_k(\eta)$ denotes the k-th crank of η, $l(\eta)$ denotes the large length of η and $s(\eta)$ denotes the small length of η and D denotes the balanced number of η. Then $FC(\eta)$ is given by

$$FC(\eta) = \begin{cases} l(\eta) - s(\eta) + 2D + k - 1, & \text{if} \quad c_k(\eta) > 0, \\ -(l(\eta) - s(\eta) + 2D + k - 1), & \text{if} \quad c_k(\eta) \leq 0. \end{cases}$$

It is clear that for $k = 1$, the full crank of a 1-marked Dyson symbol reduces to the crank of a Dyson symbol.

Analogous to the full rank function for a k-marked Durfee symbol defined by Andrews [3], we define the full crank function $NC_k(i, t; n)$ as the number of k-marked Dyson symbols of n with the full crank congruent to i modulo t. The following theorem gives an infinite family of congruences of the full crank function.

Theorem 4.1. *For fixed prime $p \geq 5$ and positive integers r and $k \leq (p+1)/2$. Then there exist infinitely many non-nested arithmetic progressions $An + B$ such that for each $0 \leq i \leq p^r - 1$,*

$$NC_k(i, p^r; An + B) \equiv 0 \pmod{p^r}.$$

Since

$$\mu_{2k}(n) = \sum_{i=0}^{p^r-1} NC_{k+1}(i, p^r; n),$$

Theorem 4.1 implies the following congruences for $\mu_{2k}(n)$.

Theorem 4.2. *For fixed prime $p \geq 5$, positive integers r and $k \leq (p-1)/2$. Then there exists infinitely many non-nested arithmetic progressions $An + B$ such that*

$$\mu_{2k}(An + B) \equiv 0 \pmod{p^r}.$$

To prove Theorem 4.1, let $NC_k(m; n)$ denote the number of k-marked Dyson symbols of n with the full crank equal to m. In this notation, we have the following relation.

Theorem 4.3. *For $n \geq 2$, $k \geq 1$ and integer m,*

$$NC_k(m; n) = \binom{m + k - 2}{2k - 2} M(m, n). \tag{4.1}$$

Proof. Recall that $F_k(m_1, \ldots, m_k, t_1, \ldots, t_{k-1}; n)$ is the number of k-marked Dyson symbols of n such that for $1 \leq i \leq k$, the i-th crank equal to m_i and the i-th balance number equal to t_i. By the definition of $NC_k(m, n)$, we see that if $m \geq 1$, then we have

$$NC_k(m; n) = \sum F_k(m_1, m_2, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n), \tag{4.2}$$

where the summation ranges over all integer solutions to the equation

$$|m_1| + \cdots + |m_k| + 2t_1 + \cdots + 2t_{k-1} = m - k + 1 \tag{4.3}$$

in $m_1, m_2, \ldots, m_k$ and $t_1, t_2, \ldots, t_{k-1}$ subject to the further condition that m_k is positive and $t_1, t_2, \ldots, t_{k-1}$ are nonnegative.

Combining Theorem 2.6 and Theorem 2.7, we find that

$$F_k(m_1, m_2, \ldots, m_k, t_1, t_2, \ldots, t_{k-1}; n) = F_1\left(\sum_{i=1}^{k} |m_i| + 2\sum_{i=1}^{k-1} t_i + k - 1; n\right). \tag{4.4}$$

Substituting (4.4) into (4.2), we get

$$NC_k(m; n) = \sum F_1\left(\sum_{i=1}^{k} |m_i| + 2\sum_{i=1}^{k-1} t_i + k - 1; n\right), \tag{4.5}$$

where the summation ranges over all solutions to the equation (4.3). Let $\bar{c}_k(m-k+1)$ denote the number of integer solutions to the equation (4.3). It is not difficult to verify that

$$\bar{c}_k(m-k+1) = \binom{m+k-2}{2k-2}.$$

Thus, (4.5) simplifies to

$$NC_k(m; n) = \binom{m+k-2}{2k-2} F_1(m; n).$$

Using Corollary 2.3 and noting that $M(-m, n) = M(m, n)$, we conclude that

$$NC_k(m; n) = \binom{m+k-2}{2k-2} M(m, n),$$

as required. Similarly, it can be shown that relation (4.1) also holds for $m \leq 0$. ∎

Let $M(i, t; n)$ denote the number of partitions of n with the crank congruent to i modulo t. The following congruences for $M(i, t; n)$ given by Mahlburg [16] will be used in the proof of Theorem 4.1.

Theorem 4.4 (Mahlburg)**.** *For fixed prime $p \geq 5$ and positive integers τ and r, there are infinitely many non-nested arithmetic progressions $An + B$ such that for each $0 \leq m \leq p^r - 1$,*

$$M(m, p^r; An + B) \equiv 0 \pmod{p^\tau}.$$

We are now ready to complete the proof of Theorem 4.1 by using Theorems 4.3 and 4.4.

Proof of Theorem 4.1. For $0 \leq i \leq p^r - 1$, by the definition of $NC_k(i, p^r; n)$, we have

$$NC_k(i, p^r; n) = \sum_{t=-\infty}^{+\infty} NC_k(p^r t + i; n). \tag{4.6}$$

Replacing m by $p^r t + i$ in (4.1), we get

$$NC_k(p^r t + i; n) = \binom{p^r t + i + k - 2}{2k - 2} M(p^r t + i, n). \tag{4.7}$$

Substituting (4.7) into (4.6), we find that

$$NC_k(i, p^r; n) = \sum_{t=-\infty}^{+\infty} \binom{p^r t + i + k - 2}{2k - 2} M(p^r t + i, n). \tag{4.8}$$

Since p is a prime and $k \leq (p+1)/2$, we see that $(2k-2)!$ is not divisible by p. It follows that

$$\binom{p^r t + i + k - 2}{2k - 2} \equiv \binom{i + k - 2}{2k - 2} \pmod{p^r}.$$

Thus (4.8) implies that

$$\begin{aligned} NC_k(i, p^r; n) &\equiv \sum_{t=-\infty}^{+\infty} \binom{i + k - 2}{2k - 2} M(p^r t + i, n) \pmod{p^r} \\ &= \binom{i + k - 2}{2k - 2} M(i, p^r; n). \end{aligned}$$

Setting $\tau = r$ in Theorem 4.4, we deduce that there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq i \leq p^r - 1$

$$M(i, p^r; An + B) \equiv 0 \pmod{p^r}.$$

Consequently, there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq m \leq p^r - 1$

$$NC_k(i, p^r; An + B) \equiv 0 \pmod{p^r},$$

and hence the proof is complete. ∎

Acknowledgments. This work was supported by the 973 Project and the National Science Foundation of China.

References

[1] G. E. Andrews, Generalized Frobenius partitions, Mem. Amer. Math. Soc. 49 (1984) No. 301 iv+ 44 pp.

[2] G. E. Andrews and F. G. Garvan, Dyson's crank of a partition, Bull. Amer. Math. Soc. 18 (1988) 167–171.

[3] G. E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.

[4] A. O. L. Atkin and F. G. Garvan, Relations between the ranks and cranks of partitions, Ramanujan J. **7** (2003) 343–366.

[5] K. Bringmann, On the explicit construction of higher deformations of partition statistics, Duke Math. J. 114 (2008) 195–233.

[6] K. Bringmann, F. G. Garvan and K. Mahlburg, Partition statistics and quasiweak Maass forms, Int. Math. Res. Not. IMRN (2009) 63–97.

[7] K. Bringmann, J. Lovejoy and R. Osburn, Automorphic properties of generating functions for generalized rank moments and Durfee symbols, Int. Math. Res. Not. IMRN (2010) 238–260.

[8] K. Bringmann and B. Kane, Inequalities for full rank differences of 2-marked Durfee symbols, J. Combin. Theory A 119 (2012) 483–501.

[9] F. J. Dyson, Some guesses in the theorey of partitions, Eureka (Cambridge) 8 (1944) 10–15.

[10] F. J. Dyson, Mappings and symmetries of partitions, J. Combin. Theory A 51 (1989) 169–180.

[11] F. G. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7, 11, Trans. Amer. Math. Soc. 305 (1988) 47–77.

[12] F. G. Garvan, Higher order spt-functions, Adv. Math. 228 (2011) 241–265.

[13] K. Q. Ji, The combinatorics of k-marked Durfee symbols, Trans. Amer. Math. Soc. 363 (2011) 987–1005.

[14] W. J. Keith, Distribution of the full rank in residue classes for odd moduli, Discrete Math. 309 (2009) 4960–4968.

[15] K. Kursungoz, Counting k-marked Durfee symbols, Electron. J. Combin. 18 (2011) #P41.

[16] K. Mahlburg, Partition congruences and the Andrews-Garvan-Dyson crank, Proc. Natl. Acad. Sci. USA 102 (2005) 15373–15376.