On the Positive Moments of Ranks of Partitions

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Abstract. By introducing k-marked Durfee symbols, Andrews found a combinatorial interpretation of $2k$-th symmetrized moment $\eta_{2k}(n)$ of ranks of partitions of n in terms of $(k+1)$-marked Durfee symbols of n. In this paper, we consider the k-th symmetrized positive moment $\bar{\eta}_k(n)$ of ranks of partitions of n which is defined as the truncated sum over positive ranks of partitions of n. As combintorial interpretations of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$, we show that for fixed k and i with $1 \leq i \leq k+1$, $\bar{\eta}_{2k-1}(n)$ equals the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being zero and $\bar{\eta}_{2k}(n)$ equals the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being positive. The interpretations of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ also imply the interpretation of $\eta_{2k}(n)$ given by Andrews since $\eta_{2k}(n)$ equals $\bar{\eta}_{2k-1}(n)$ plus twice of $\bar{\eta}_{2k}(n)$. Moreover, we obtain the generating functions of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$.

Keywords: rank of a partition, k-marked Durfee symbol, moment of ranks

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1 Introduction

This paper is concerned with a combinatorial study of the symmetrized positive moments of ranks of partitions. The notion of symmetrized moments was introduced by Andrews [1]. The odd symmetrized moments are zero due to the symmetry of ranks. For even symmetrized moments, Andrews found a combinatorial interpretation by introducing k-marked Durfee symbols. It is natural to investigate the combinatorial interpretation of the odd symmetrized moments which are truncated sum over positive ranks of partitions of n. We give combinatorial interpretations of the even and odd positive moments in terms of k-marked Durfee symbols, which also lead to the combinatorial interpretation of the even symmetrized moments of ranks given by Andrews.

The rank of a partition λ introduced by Dyson [6] is defined as the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m. The generating function of $N(m,n)$ is given by

Theorem 1.1 (Dyson-Atkin-Swinnerton-Dyer [3]). *For fixed integer m, we have*

$$\sum_{n=0}^{+\infty} N(m,n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{+\infty} (-1)^{n-1} q^{n(3n-1)/2+|m|n}(1-q^n). \tag{1.1}$$

Recently, Andrews [1] introduced the k-th symmetrized moment $\eta_k(n)$ of ranks of partitions of n as given by

$$\eta_k(n) = \sum_{m=-\infty}^{+\infty} \binom{m+\lfloor \frac{k-1}{2} \rfloor}{k} N(m,n). \tag{1.2}$$

It can be easily seen that for given k, $\eta_k(n)$ is a linear combination of the moments $N_j(n)$ of ranks given by Atkin and Garvan [4]

$$N_j(n) = \sum_{m=-\infty}^{\infty} m^j N(m,n).$$

For example,

$$\eta_6(n) = \frac{1}{720} N_6(n) - \frac{1}{144} N_4(n) + \frac{1}{180} N_2(n).$$

In view of the symmetry $N(-m,n) = N(m,n)$, we have $\eta_{2k+1}(n) = 0$. As for the even symmetrized moments $\eta_{2k}(n)$, Andrews gave the following combinatorial interpretation by introducing k-marked Durfee symbols. For the definition of k-marked Durfee symbols, see Section 2.

Theorem 1.2 (Andrews [1]). *For fixed $k \geq 1$, $\eta_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of n.*

Andrews [1] proved the above theorem by using the k-fold generalization of Watson's q-analog of Whipple's theorem. Ji [8] gave a combinatorial proof of Theorem 1.2 by establishing a map from k-marked Durfee symbols to ordinary partitions. Kursungoz [9] provided another proof of Theorem 1.2 by using an alternative representation of k-marked Durfee symbols.

In this paper, we introduce the k-th symmetrized positive moment $\bar{\eta}_k(n)$ of ranks as given by

$$\bar{\eta}_k(n) = \sum_{m=1}^{\infty} \binom{m+\lfloor \frac{k-1}{2} \rfloor}{k} N(m,n),$$

or equivalently,

$$\bar{\eta}_{2k-1}(n) = \sum_{m=1}^{\infty} \binom{m+k-1}{2k-1} N(m,n) \tag{1.3}$$

and

$$\bar{\eta}_{2k}(n) = \sum_{m=1}^{\infty} \binom{m+k-1}{2k} N(m,n). \tag{1.4}$$

Furthermore, it is easy to see that for given k, $\bar{\eta}_k(n)$ is a linear combination of the positive moments $\overline{N}_j(n)$ of ranks introduced by Andrews, Chan and Kim [2] as given by

$$\overline{N}_j(n) = \sum_{m=1}^{\infty} m^j N(m,n).$$

For example,

$$\begin{aligned}
\bar{\eta}_4(n) &= \frac{1}{24}\overline{N}_4(n) - \frac{1}{12}\overline{N}_3(n) - \frac{1}{24}\overline{N}_2(n) + \frac{1}{12}\overline{N}_1(n), \\
\bar{\eta}_5(n) &= \frac{1}{120}\overline{N}_5(n) - \frac{1}{24}\overline{N}_3(n) + \frac{1}{30}\overline{N}_1(n).
\end{aligned}$$

By the symmetry $N(-m,n) = N(m,n)$, it is readily seen that

$$\eta_{2k}(n) = 2\bar{\eta}_{2k}(n) + \bar{\eta}_{2k-1}(n). \tag{1.5}$$

The main objective of this paper is to give combintorial interpretations of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$. We show that for given k and i with $1 \le i \le k+1$, $\bar{\eta}_{2k-1}(n)$ equals the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being zero and $\bar{\eta}_{2k}(n)$ equals the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being positive. It should be noted that $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ are independent of i since the ranks of k-marked Durfee symbols are symmetric, see Andrews [1, Corollary 12].

With the aid of Theorem 2.1 and Theorem 2.2 together with the generating function (1.1) of $N(m,n)$, we obtain the generating functions of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$.

2 Combinatorial interpretations

In this section, we give combinatorial interpretations of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ in terms of the k-marked Durfee symbols. For a partition λ, we write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$, so that λ_1 is the largest part and λ_s is the smallest part of λ. Recall that a k-marked Durfee symbol of n introduced by Andrews [1] is a two-line array composed of k pairs (α^i, β^i) of partitions along with a positive integer D which is represented in the following form:

$$\tau = \begin{pmatrix} \alpha^k, & \alpha^{k-1}, & \ldots, & \alpha^1 \\ \beta^k, & \beta^{k-1}, & \ldots, & \beta^1 \end{pmatrix}_D,$$

where the partitions α^i and β^i satisfy the following four conditions:

(1) The partitions α^i $(1 \le i < k)$ are nonempty, while α^k and β^i $(1 \le i \le k)$ are allowed to be empty;

(2) $\beta_1^{i-1} \le \alpha_1^{i-1} \le \min\{\alpha_s^i, \beta_s^i\}$ for $2 \le i \le k$;

(3) $\alpha_1^k,\ \beta_1^k \le D$;

(4) $\sum_{i=1}^k (|\alpha^i| + |\beta^i|) + D^2 = n$.

Let

$$\tau = \begin{pmatrix} \alpha^k, & \alpha^{k-1}, & \ldots, & \alpha^1 \\ \beta^k, & \beta^{k-1}, & \ldots, & \beta^1 \end{pmatrix}_D$$

be a k-marked Durfee symbol. The pair (α^i, β^i) of partitions is called the i-th vector of τ. Andrews defined the i-th rank $\rho_i(\tau)$ of τ as follows

$$\rho_i(\tau) = \begin{cases} \ell(\alpha^i) - \ell(\beta^i) - 1, & \text{for } 1 \le i < k, \\ \ell(\alpha^k) - \ell(\beta^k). & \text{for } i = k. \end{cases}$$

For example, consider the following 3-marked Durfee symbol τ.

$$\tau = \left(\begin{array}{ccc} \overbrace{5_3, 4_3,}^{\alpha^3} & \overbrace{4_2, 3_2, 3_2, 2_2,}^{\alpha^2} & \overbrace{2_1}^{\alpha^1} \\ \underbrace{4_3}_{\beta^3}, & \underbrace{3_2, 2_2, 2_2}_{\beta^2}, & \underbrace{2_1, 2_1}_{\beta^1} \end{array} \right)_5 .$$

We have $\rho_1(\tau) = -2$, $\rho_2(\tau) = 0$, and $\rho_3(\tau) = 1$.

For odd symmetrized moments $\bar{\eta}_{2k-1}(n)$, we have the following combinatorial interpretation.

Theorem 2.1. *For fixed positive integers k and i with $1 \le i \le k+1$, $\bar{\eta}_{2k-1}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of n with the i-th rank equal to zero.*

For the even case, we have the following interpretation.

Theorem 2.2. *For fixed positive integers k and i with $1 \le i \le k+1$, $\bar{\eta}_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being positive.*

The proofs of the above two interpretations are based on the following partition identity given by Ji [8]. We shall adopt the notation $D_k(m_1, m_2, \ldots, m_k; n)$ as used by Andrews [1] to denote the number of k-marked Durfee symbols of n with i-th rank equal to m_i.

Theorem 2.3. *Given $k \ge 2$ and $n \ge 1$, we have*

$$D_k(m_1, m_2, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1}=0}^{\infty} N\left(\sum_{i=1}^k |m_i| + 2\sum_{i=1}^{k-1} t_i + k - 1, n \right). \tag{2.1}$$

To prove the above two interpretations, we also need the following symmetric property given by Andrews [1]. Boulet and Kursungoz [5] found a combinatorial proof of this fact.

Theorem 2.4. *For $k \geq 2$ and $n \geq 1$, $D_k(m_1, \ldots, m_k; n)$ is symmetric in $m_1,\ m_2, \ldots, m_k$.*

We are now in a position to prove Theorem 2.1 and Theorem 2.2 with the aid of Theorem 2.3 and Theorem 2.4.

Proof of Theorem 2.1. By Theorem 2.4, it suffices to show that

$$\sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n) = \bar{\eta}_{2k-1}(n). \tag{2.2}$$

Using Theorem 2.3, we get

$$\begin{aligned}\sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty} & D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n) \\ &= \sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} N\left(\sum_{i=2}^{k+1} |m_i| + 2\sum_{i=1}^{k} t_i + k, n\right).\end{aligned} \tag{2.3}$$

Given k and n, let $c_k(n)$ denote the number of integer solutions to the equation

$$|m_2| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = n,$$

where the variables m_i are integers and the variables t_i are nonnegative integers. It is easy to see that the generating function of $c_k(n)$ is equal to

$$\begin{aligned}\sum_{n=0}^{\infty} c_k(n)q^n &= (1 + 2q + 2q^2 + 2q^3 + \cdots)^k (1 + q^2 + q^4 + q^6 + \cdots)^k \\ &= \left(\frac{1+q}{1-q}\right)^k \left(\frac{1}{1-q^2}\right)^k \\ &= \frac{1}{(1-q)^{2k}} \\ &= \sum_{n=0}^{\infty} \binom{n+2k-1}{2k-1} q^n.\end{aligned} \tag{2.4}$$

Equating the coefficients of q^n on the both sides of (2.4), we get

$$c_k(n) = \binom{n+2k-1}{2k-1},$$

that is,

$$c_k(m-k) = \binom{m+k-1}{2k-1}.$$

Thus, (2.3) can be written as

$$\sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty} D_{k+1}(0,m_2,m_3,\ldots,m_{k+1};n)$$
$$= \sum_{m=1}^{\infty} \binom{m+k-1}{2k-1} N(m,n)$$

which is equal to $\bar{\eta}_{2k-1}(n)$. This completes the proof. ∎

Proof of Theorem 2.2. Similarly, by Theorem 2.4, it is enough to show that

$$\sum_{\substack{m_1>0\\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty} D_{k+1}(m_1,m_2,\ldots,m_{k+1};n) = \bar{\eta}_{2k}(n). \tag{2.5}$$

Using Theorem 2.3, we get

$$\sum_{\substack{m_1>0\\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty} D_{k+1}(m_1,m_2,\ldots,m_{k+1};n)$$
$$= \sum_{\substack{m_1>0\\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} N\left(m_1+\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k,n\right). \tag{2.6}$$

Given k and n, let $\bar{c}_k(n)$ denote the number of integer solutions to the equation

$$m_1+|m_2|+\cdots+|m_{k+1}|+2t_1+\cdots+2t_k=n,$$

where the variable m_1 is a positive integer, the variables m_i $(2\le i\le k+1)$ are integers and the variables t_i are nonnegative integers. An easy computation shows that

$$\sum_{n=0}^{\infty} \bar{c}_k(n)q^n = \frac{q}{(1-q)^{2k+1}}, \tag{2.7}$$

so that

$$\bar{c}_k(n) = \binom{n+2k-1}{2k}.$$

We write

$$\bar{c}_k(m-k) = \binom{m+k-1}{2k}.$$

It follows that

$$\sum_{\substack{m_1>0\\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty} D_{k+1}(m_1,m_2,\ldots,m_{k+1};n) = \sum_{m=1}^{\infty}\binom{m+k-1}{2k}N(m,n),$$

which equals $\bar{\eta}_{2k}(n)$, as required. ∎

Note that the number $D_k(m_1,\ldots,m_k;n)$ has the mirror symmetry with respect to each m_i, that is, for $1\leq i\leq k$, we have

$$D_k(m_1,\ldots,m_i,\ldots,m_k;n) = D_k(m_1,\ldots,-m_i,\ldots,m_k;n).$$

Using this mirror symmetry, Theorem 2.2 can be restated as follows.

Theorem 2.5. *For fixed positive integers k and i with $1\leq i\leq k+1$, $\bar{\eta}_{2k}(n)$ is also equal to the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being negative.*

$\overline{\eta}_1(5)$	$\overline{\eta}_2(5)$	$\overline{\eta}_2(5)$
$\begin{pmatrix} 1_2 & 1_2 & 1_2 & 1_1 \\ & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & 1_1 & 1_1 \\ & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & & \\ 1_1 & 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & 1_1 & 1_1 \\ 1_1 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 & 1_1 & 1_1 \\ & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 \\ 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & 1_2 & 1_1 \\ 1_2 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_2 & 1_1 & 1_1 \\ & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_2 & 1_1 \\ 1_1 & & \end{pmatrix}_1$
$\begin{pmatrix} 1_1 & & \\ 1_2 & 1_2 & 1_2 \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & 1_1 \\ 1_1 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 \\ 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_1 & 1_1 \\ 1_2 & 1_1 \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & 1_1 \\ 1_2 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & & \\ 1_2 & 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & 1_1 \\ 1_2 & 1_2 \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 & 1_1 \\ 1_2 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 \\ 1_2 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_1 \\ \end{pmatrix}_2$	$\begin{pmatrix} 1_1 & 1_1 \\ 1_2 & 1_2 \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & & \\ 1_2 & 1_2 & 1_1 \end{pmatrix}_1$

Table 2.1: 2-Marked Durfee Symbols of 5.

For example, for $n=5$, $k=1$ and $i=1$, there are twenty-one 2-marked Durfee symbols of 5 as listed in Table 2.1. The first column in Table 2.1 gives seven 2-marked Durfee symbols τ with $\rho_1(\tau)=0$, the second column contains seven 2-marked Durfee symbols τ with $\rho_1(\tau)>0$ and the third column contains seven 2-marked Durfee symbols τ with $\rho_1(\tau)<0$. It can be verified that $\overline{\eta}_1(5)=7$, $\overline{\eta}_2(5)=7$ and $\eta_2(5)=\overline{\eta}_1(5)+2\overline{\eta}_2(5)=21$.

3 The generating functions of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$

In this section, we obtain the generating functions of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ with the aid of Theorem 2.1 and Theorem 2.2. In doing so, we use the generating function of $N(m,n)$ to derive the generating functions of $D_{k+1}(0,m_2,\ldots,m_{k+1};n)$ and $D_{k+1}(m_1,m_2,\ldots,m_{k+1};n)$ $(m_1>0)$.

Theorem 3.1. *For $k\geq 1$, we have*

$$\sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty}\sum_{n=0}^{\infty}D_{k+1}(0,m_2,\ldots,m_{k+1};n)x_1^{m_2}\cdots x_k^{m_{k+1}}q^n$$
$$=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+kn}\frac{(1-q^n)}{\prod_{j=1}^{k}(1-x_jq^n)(1-x_j^{-1}q^n)}. \tag{3.1}$$

Proof. Let

$$G_k(x_1,\ldots,x_k;q)=\sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty}\sum_{n=0}^{\infty}D_{k+1}(0,m_2,\ldots,m_{k+1};n)x_1^{m_2}\cdots x_k^{m_{k+1}}q^n.$$

By Theorem 2.3, we have

$$G_k(x_1,\ldots,x_k;q)$$
$$=\sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty}\sum_{t_1,\ldots,t_k=0}^{\infty}x_1^{m_2}\cdots x_k^{m_{k+1}}\sum_{n=0}^{\infty}N\left(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k,n\right)q^n. \tag{3.2}$$

Using (1.1) with m replaced by $\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k$, we get

$$\sum_{n=0}^{\infty}N\left(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k,n\right)q^n$$
$$=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+n(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k)}(1-q^n).$$

Therefore (3.2) becomes

$$G_k(x_1,\ldots,x_k;q)=\sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty}\sum_{t_1,\ldots,t_k=0}^{\infty}x_1^{m_2}\cdots x_k^{m_{k+1}}$$
$$\times\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+n(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k)}(1-q^n). \tag{3.3}$$

Write (3.3) in the following form

$$G_k(x_1,\ldots,x_k;q)=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+kn}(1-q^n)$$
$$\times\sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty}\sum_{t_1,\ldots,t_k=0}^{\infty}x_1^{m_2}\cdots x_k^{m_{k+1}}q^{n(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i)}. \tag{3.4}$$

Notice that

$$\sum_{a=-\infty}^{+\infty}\sum_{b=0}^{+\infty}x^aq^{n(|a|+2b)}=\frac{1}{(1-xq^n)(1-x^{-1}q^n)}. \tag{3.5}$$

Applying the above formula repeatedly to (3.4), we deduce that

$$G_k(x_1,\ldots,x_k;q)=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+kn}\frac{(1-q^n)}{\prod_{j=1}^{k}(1-x_jq^n)(1-x_j^{-1}q^n)},$$

as required. ∎

Setting $x_j=1$ for $1\leq j\leq k$ in Theorem 3.1 and using Theorem 2.1, we obtain the following generating function of $\bar{\eta}_{2k-1}(n)$.

Corollary 3.2. *For $k\geq 1$, we have*

$$\sum_{n=1}^{\infty}\bar{\eta}_{2k-1}(n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+kn}\frac{1}{(1-q^n)^{2k-1}}. \tag{3.6}$$

Taking $k=1$ in (3.6) and observing that $\bar{\eta}_1(n)=\overline{N}_1(n)$, we are led to the generating function for $\overline{N}_1(n)$ as given by Andrews, Chan and Kim in [2, Theorem 1].

The following generating function can be shown by using the same reasoning as in the proof of Theorem 3.1.

Theorem 3.3. *For $k \geq 1$, we have*

$$\sum_{\substack{m_1>0 \\ m_2,\ldots,m_{k+1}=-\infty}}^{\infty} \sum_{n=1}^{\infty} D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n) x_1^{m_1} \cdots x_{k+1}^{m_{k+1}} q^n$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n+1)/2+kn} \frac{x_1(1-q^n)}{(1-x_1q^n)\prod_{j=2}^{k+1}(1-x_jq^n)(1-x_j^{-1}q^n)}. \tag{3.7}$$

Setting $x_j = 1$ for $1 \leq j \leq k+1$ in Theorem 3.3 and using Theorem 2.2, we arrive at the following generating function of $\bar{\eta}_{2k}(n)$.

Corollary 3.4. *For $k \geq 1$, we have*

$$\sum_{n=1}^{\infty} \bar{\eta}_{2k}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n+1)/2+kn} \frac{1}{(1-q^n)^{2k}}. \tag{3.8}$$

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