The spt-Crank for Ordinary Partitions

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Dedicated to Professor George E. Andrews on the Occasion of His 75th Birthday

Abstract. The spt-function $spt(n)$ was introduced by Andrews as the weighted counting of partitions of n with respect to the number of occurrences of the smallest part. Andrews, Garvan and Liang defined the spt-crank of an S-partition which leads to combinatorial interpretations of the congruences of $spt(n)$ mod 5 and 7. Let $N_S(m,n)$ denote the net number of S-partitions of n with spt-crank m. Andrews, Garvan and Liang showed that $N_S(m,n)$ is nonnegative for all integers m and positive integers n, and they asked the question of finding a combinatorial interpretation of $N_S(m,n)$. In this paper, we introduce the structure of doubly marked partitions and define the spt-crank of a doubly marked partition. We show that $N_S(m,n)$ can be interpreted as the number of doubly marked partitions of n with spt-crank m. Moreover, we establish a bijection between marked partitions of n and doubly marked partitions of n. A marked partition is defined by Andrews, Dyson and Rhoades as a partition with exactly one of the smallest parts marked. They consider it a challenge to find a definition of the spt-crank of a marked partition so that the set of marked partitions of $5n+4$ and $7n+5$ can be divided into five and seven equinumerous classes. The definition of spt-crank for doubly marked partitions and the bijection between the marked partitions and doubly marked partitions leads to a solution to the problem of Andrews, Dyson and Rhoades.

Keywords: spt-function, spt-crank, congruence, marked partition, doubly marked partition.

AMS Classifications: 05A17, 05A19, 11P81, 11P83.

1 Introduction

Andrews [4] introduced the spt-function $spt(n)$ as the weighted counting of partitions with respect to the number of occurrences of the smallest part and he discovered that the spt-function bears striking resemblance to the classical partition function $p(n)$. Much attention has been drawn to the investigation of the spt-function, in particular, the spt-crank of an S-partition, see, for example, Andrews, Dyson and Rhoades [5], Andrews, Garvan and Liang [6,7], Folsom and Ono [13], Garvan [14] and Ono [19].

In this paper, we introduce the structure of doubly marked partitions and define the spt-crank of a doubly marked partition. This gives a solution to a problem posed by Andrews, Garvan and Liang [6] on the spt-crank of an S-partition. Moreover, we find a bijection between marked partitions and doubly marked partitions, which leads to a solution to a problem of finding the definition of the spt-crank for ordinary partitions posed by Andrews, Dyson and Rhoades [5].

Let us give an overview of notation and known results on the spt-function $spt(n)$. For a partition λ of n, we use $n_s(\lambda)$ to denote the number of occurrences of the smallest part in λ. Let $P(n)$ denote the set of ordinary partitions of n, then we have

$$spt(n) = \sum_{\lambda \in P(n)} n_s(\lambda). \tag{1.1}$$

For example, for $n = 4$, we have $spt(4) = 10$. Partitions in $P(4)$ and the values of $n_s(\lambda)$ are listed below:

λ	$n_s(\lambda)$
(4)	1
$(3,1)$	1
$(2,2)$	2
$(2,1,1)$	2
$(1,1,1,1)$	4

The spt-function $spt(n)$ can also be interpreted by marked partitions, see Andrews, Dyson and Rhoades [5]. A marked partition of n means a pair (λ, k) where λ is an ordinary partition of n and k is an integer identifying one of its smallest parts. If there are s smallest parts in λ, then $k = 1, 2, \ldots, s$. However, for the purpose of this paper, we shall mark the unique smallest part by its index. More precisely, if λ_k is the marked smallest part, then we use (λ, k) to denote this marked partition. For example, there are ten marked partitions of 4.

$$\begin{array}{lllll} ((4),1), & ((3,1),2), & ((2,2),1), & ((2,2),2), & ((2,1,1),2) \\ ((2,1,1),3), & ((1,1,1,1),1), & ((1,1,1,1),2), & ((1,1,1,1),3), & ((1,1,1,1),4). \end{array}$$

From the definition (1.1) of spt-function, one can derive the following generating function

$$\sum_{n\geq 1} spt(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty}, \tag{1.2}$$

see Andrews [4]. Based on the above formula, Andrews, Garvan and Liang [6] noticed that the generating function of $spt(n)$ can expressed in the following form

$$\sum_{n\geq 0} spt(n)q^n = \sum_{n=1}^{+\infty} \frac{q^n(q^{n+1};q)_\infty}{(q^n;q)_\infty(q^n;q)_\infty}, \tag{1.3}$$

and they introduced the structure of S-partitions and interpreted the right-hand side of (1.3) as the generating function of the net number of S-partitions of n, that is, the sum of signs of S-partitions of n. In other words, Andrews, Garvan and Liang established the following relation

$$spt(n) = \sum_{\pi} \omega(\pi),$$

where π ranges over S-partitions of n and $\omega(\pi)$ is the sign of π.

To be precise, let $\mathcal{D}$ denote the set of partitions into distinct parts and $\mathcal{P}$ denote the set of partitions. For $\lambda \in \mathcal{P}$, we use $s(\lambda)$ to denote the smallest part of λ with the convention that $s(\emptyset) = +\infty$. Let $\ell(\lambda)$ denote the number of parts of λ. The set of S-partitions is defined by

$$S = \{(\pi_1, \pi_2, \pi_3) \in \mathcal{D} \times \mathcal{P} \times \mathcal{P} \mid \pi_1 \neq \emptyset \text{ and } s(\pi_1) \leq \min\{s(\pi_2), s(\pi_3)\}\}.$$

For $\pi = (\pi_1,\, \pi_2,\, \pi_3) \in S$, we define the weight of π to be $|\pi_1|+|\pi_2|+|\pi_3|$ and we associate with π a sign

$$\omega(\pi) = (-1)^{\ell(\pi_1)-1}.$$

Using the generating function (1.2) and Watson's q-analog of Whipple's theorem [16, p.43, eq. (2.5.1)], Andrews showed that the spt-function can be expressed in terms of the second moment $N_2(n)$ of ranks, namely,

$$spt(n) = np(n) - \frac{1}{2}N_2(n). \tag{1.4}$$

In general, the kth moment $N_k(n)$ of ranks was introduced by Atkin and Garvan [9] as given by

$$N_k(n) \;=\; \sum_{m=-\infty}^{+\infty} m^k N(m,n),$$

where $N(m,n)$ is the number of partitions of n with rank m, and the rank of a partition is defined as the largest part minus the number of parts.

In view of relation (1.4) and identities on the refinements of $N(m,n)$, Andrews proved that $spt(n)$ satisfies congruences mod 5, 7 and 13 reminiscent to Ramanujan's congruences for $p(n)$. To be more specific, let $N(i,t,n)$ denote the number of partitions of n with rank congruent $i \mod t$. Dyson [10] conjectured

$$N(i,5,5n+4) \;=\; \frac{p(5n+4)}{5} \quad \text{for} \quad 0 \leq i \leq 4, \tag{1.5}$$

$$N(i,7,7n+5) \;=\; \frac{p(7n+5)}{7} \quad \text{for} \quad 0 \leq i \leq 6. \tag{1.6}$$

These relations were confirmed by Atkin and Swinnerton-Dyer [8] which imply Ramanujan's congruences mod 5 and 7 for $p(n)$ [20]:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, && (1.7)\\ p(7n+5) &\equiv 0 \pmod 7, && (1.8)\\ p(11n+6) &\equiv 0 \pmod{11}. && (1.9) \end{aligned}$$

Using relation (1.4) along with (1.5) and (1.6), Andrews [4] showed that

$$\begin{aligned} spt(5n+4) &\equiv 0 \pmod 5, && (1.10)\\ spt(7n+5) &\equiv 0 \pmod 7. && (1.11) \end{aligned}$$

Andrews also proved the following congruence

$$spt(13n+6) \equiv 0 \pmod{13} \tag{1.12}$$

by using identities on $N(i, 13, 13n+6)$ due to O'Brien [18]. Let

$$r_{a,b}(d) = \sum_{n=0}^{\infty} (N(a, 13, 13n+d) - N(b, 13, 13n+d))q^{13n},$$

and for $1 \le i \le 5$, let

$$S_i(d) = r_{(i-1),i}(d) - (7-i)r_{5,6}(d).$$

O' Brien proved that

$$S_1(6) + 2S_2(6) - 5S_5(6) \equiv 0 \pmod{13} \tag{1.13}$$

and

$$S_2(6) + 5S_3(6) + 3S_4(6) + 3S_5(6) \equiv 0 \pmod{13}. \tag{1.14}$$

By relation (1.4), Andrews derived an expression of $spt(13n+6)$ in terms of the numbers $N(i, 13, 13n+6)$ mod 13. Then the congruence (1.12) follows from (1.13) and (1.14).

To give combinatorial interpretations of the spt-congruences (1.10) and (1.11), Andrews, Garvan and Liang [6] defined the spt-crank for S-partitions, which takes the same form as the crank for vector partitions. Recall that the crank for vector partitions has been used to interpret Ramanujan's congruences for $p(n)$ mod 5, 7 and 11, see Andrews and Garvan [3], Dyson [11] and Garvan [15]. Let π be an S-partition, the spt-crank of π, denoted $r(\pi)$, is defined to be the number of parts of π_2 minus the number of parts of π_3, that is,

$$r(\pi) = \ell(\pi_2) - \ell(\pi_3).$$

Let $N_S(m, n)$ denote the net number of S-partitions of n with spt-crank m, that is,

$$N_S(m, n) = \sum_{\substack{|\pi|=n \\ r(\pi)=m}} \omega(\pi) \tag{1.15}$$

and let $N_S(k,t,n)$ denote the net number of S-partitions of n with spt-crank congruent $k \pmod t$, namely,

$$N_S(k,t,n) = \sum_{m \equiv k \pmod t} N_S(m,n).$$

Andrews, Garvan and Liang [6] established the following relations.

Theorem 1.1. *For $0 \le k \le 4$, we have*

$$N_S(k,5,5n+4) \quad = \quad \frac{spt(5n+4)}{5},$$

and for $0 \le k \le 6$, we have

$$N_S(k,7,7n+5) \quad = \quad \frac{spt(7n+5)}{7}.$$

By using generating functions, Andrews, Garvan and Liang [6] obtained the following positivity result for $N_S(m,n)$.

Theorem 1.2. *For all integers m and positive integers n, we have*

$$N_S(m,n) \ge 0. \tag{1.16}$$

Dyson [12] gave an alternative proof of this fact by using the following recurrence relation.

$$N_S(m,n) = \sum_{k=1}^{\infty} (-1)^{k-1} \sum_{j=0}^{k-1} p(n-k(m+j)-(k(k+1)/2)).$$

Andrews, Garvan and Liang [6] asked the question of finding a combinatorial interpretation of $N_S(m,n)$. Using the generating function for $N_S(m,n)$ given by Andrews, Garvan and Liang, we show that $N_S(m,n)$ can be interpreted as the number of doubly marked partitions of n with spt-crank m. This gives a solution to the problem of Andrews, Garvan and Liang.

Andrews, Dyson and Rhoades [5] proposed the problem of finding a definition of the spt-crank for marked partitions so that the set of marked partitions of $5n+4$ and $7n+5$ can be divided into five and seven equinumerous classes. We establish a bijection Δ between the set of marked partitions of n and the set of doubly marked partitions of n. While the spt-crank of a doubly marked partition does not lead to an explicit formula in terms of the corresponding marked partition, there is a way to define the spt-crank of a marked partition based on the bijection Δ between marked partitions and doubly marked partitions. Hence, in principle, the spt-crank of a doubly marked partition can be considered as a solution to the problem of Andrews, Dyson and Rhoades. It would be interesting to find an spt-crank that can be directly defined on marked partitions.

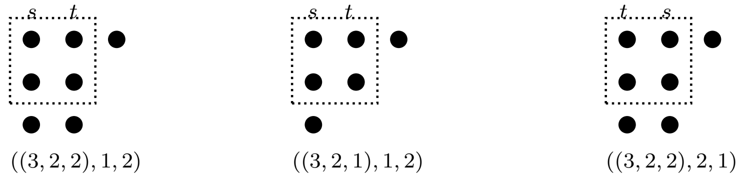

Figure 2.1: Illustration for doubly marked partitions

2 Combinatorial interpretation of $N_S(m,n)$

In this section, we first define the doubly marked partitions and the spt-crank of a doubly marked partition. Then we show that $N_S(m,n)$ equals the number of doubly marked partitions of n with spt-crank m. In the following definition, we assume that a partition λ of n is represented by its Ferrers diagram, and we use $D(\lambda)$ to denote size of the Durfee square of λ, see [2, p. 28].

We now give the definition of doubly marked partitions.

Definition 2.1. *A doubly marked partition of n is an ordinary partition λ of n along with two distinguished columns indexed by s and t, denoted (λ, s, t), where*

(1) $1 \leq s \leq D(\lambda)$;

(2) $s \leq t \leq \lambda_1$;

(3) $\lambda'_s = \lambda'_t$.

For example, $((3,2,2),1,2)$ is a doubly marked partition, whereas $((3,2,1),1,2)$ and $((3,2,2),2,1)$ are not doubly marked partitions, see Figure 2.1.

To define the spt-crank of a doubly marked partition (λ, s, t), let

$$g(\lambda, s, t) = \lambda'_s - s + 1, \tag{2.1}$$

where λ' denotes the conjugate of λ, in other words, λ'_s is the number of parts in λ that are not less than s. Since $s \leq D(\lambda)$, we see that $\lambda'_s \geq s$, which implies that $g(\lambda, s, t) \geq 1$.

Definition 2.2. *Let (λ, s, t) be a doubly marked partition, and let $g = g(\lambda, s, t)$. The spt-crank of (λ, s, t) is defined by*

$$c(\lambda, s, t) = g - \lambda_g + t - s. \tag{2.2}$$

For example, for the doubly marked partition $((4,4,1,1),2,3)$, we have $g = 2 - 1 = 1$ and the spt-crank equals $1 - \lambda_1 + 3 - 2 = -2$.

The following theorem gives a combinatorial interpretation of $N_S(m,n)$.

Theorem 2.3. *For any integer m and any positive integer n, $N_S(m,n)$ equals the number of doubly marked partitions of n with spt-crank m.*

For example, for $n = 4$, the sixteen S-partitions of 4, their spt-cranks and the ten doubly marked partitions of 4 and their spt-cranks are listed in Table 2.1. It can be checked that

$$N_S(3,4) = N_S(-3,4) = N_S(2,4) = N_S(-2,4) = 1,$$

and

$$N_S(1,4) = N_S(-1,4) = N_S(0,4) = 2.$$

S-partition	weight	spt-crank	doubly marked partition	spt-crank
$((1),(1,1,1),\emptyset)$	$+1$	3	$((1,1,1,1),1,1)$	3
$((1),(2,1),\emptyset)$	$+1$	2	$((2,1,1),1,1)$	2
$((1),(1,1),(1))$	$+1$	1	$((3,1),1,1)$	1
$((1),(3),\emptyset)$	$+1$	1	$((2,2),1,2)$	1
$((2,1),(1),\emptyset)$	-1	1		
$((2),(2),\emptyset)$	$+1$	1		
$((1),(2),(1))$	$+1$	0	$((2,2),1,1)$	0
$((1),(1),(2))$	$+1$	0	$((4),1,4)$	0
$((3,1),\emptyset,\emptyset)$	-1	0		
$((4),\emptyset,\emptyset)$	$+1$	0		
$((1),(1),(1,1))$	$+1$	-1	$((2,2),2,2)$	-1
$((1),\emptyset,(3))$	$+1$	-1	$((4),1,3)$	-1
$((2,1),\emptyset,(1))$	-1	-1		
$((2),\emptyset,(2))$	$+1$	-1		
$((1),\emptyset,(2,1))$	$+1$	-2	$((4),1,2)$	-2
$((1),\emptyset,(1,1,1))$	$+1$	-3	$((4),1,1)$	-3

Table 2.1: S-partitions and doubly marked partitions.

The proof of Theorem 2.3 relies on the following generating function of $N_S(m,n)$ given by Andrews, Garvan and Liang [6].

Theorem 2.4.

$$\sum_{m=-\infty}^{+\infty}\sum_{n\geq 0} N_S(m,n)z^m q^n$$
$$= 1+\sum_{m=0}^{+\infty} z^m \sum_{j=0}^{+\infty} \frac{q^{j^2+mj+2j+m+1}}{(q;q)_{j+m}} \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{m+1+j+h})}$$
$$+\sum_{m=1}^{+\infty} z^{-m} \sum_{j=m}^{+\infty} \frac{q^{j^2-mj+2j-m+1}}{(q;q)_{j-m}} \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{j-m+1+h})}. \tag{2.3}$$

Let $Q_{m,n}$ denote the set of doubly marked partitions of n with spt-crank m. We aim to show that for $m \geq 0$

$$\sum_{n\geq 1}\sum_{(\lambda,s,t)\in Q_{m,n}} q^{|\lambda|} = \sum_{j=0}^{+\infty} \frac{q^{j^2+mj+2j+m+1}}{(q;q)_{j+m}} \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{m+1+j+h})}, \tag{2.4}$$

and

$$\sum_{n\geq 1}\sum_{(\lambda,s,t)\in Q_{-m,n}} q^{|\lambda|} = \sum_{j=m}^{+\infty} \frac{q^{j^2-mj+2j-m+1}}{(q;q)_{j-m}} \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{j+h+1-m})}. \tag{2.5}$$

To do this, we first represent a doubly marked partition (λ,s,t) as a pair of partitions (α,β). Let (λ,s,t) be a doubly marked partition of n with spt-crank m, we define a pair of partitions $(\alpha,\beta)=\psi(\lambda,s,t)$ as follows.

$$\alpha = (\lambda_1 - t + s - 1, \ldots, \lambda_{\lambda'_s} - t + s - 1, \lambda_{\lambda'_s+1}, \ldots, \lambda_\ell) \tag{2.6}$$

and

$$\beta = (\lambda'_s, \lambda'_{s+1}, \ldots, \lambda'_t) \tag{2.7}$$

as illustrated in Figure 2.2. By the definition of a doubly marked partition, β is a partition with equal parts. Let P_n denote the set of pairs of partitions (α,β) of n where β is a partition with equal parts. We first claim that this representation is unique for the doubly marked partitions in $Q_{m,n}$. We then characterize the image set of ψ. Thus we can use the image set of ψ to compute the generating function of $Q_{m,n}$.

Lemma 2.5. *Given integer m and positive integer n, then the map ψ is an injection from the set $Q_{m,n}$ to the set P_n. In other words, any doubly marked partition (λ,s,t) in $Q_{m,n}$ can be uniquely represented by (α,β) in P_n.*

Proof. Let (λ,s,t) and $(\bar{\lambda},\bar{s},\bar{t})$ be two doubly marked partitions of n with spt-crank m such that

$$\psi(\lambda,s,t) = \psi(\bar{\lambda},\bar{s},\bar{t}) = (\alpha,\beta). \tag{2.8}$$

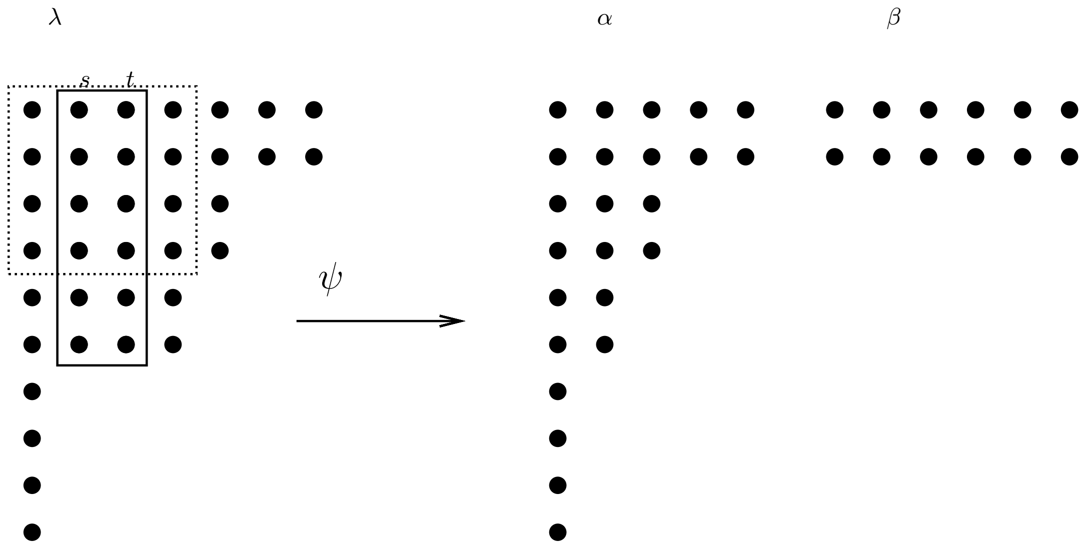

Figure 2.2: The map ψ: $((7,7,5,5,4,4,1,1,1,1),2,3) \mapsto ((5,5,3,3,2,2,1,1,1,1),(6,6))$.

We proceed to show that $(\lambda, s, t) = (\bar{\lambda}, \bar{s}, \bar{t})$.

By the construction of ψ and relation (2.8), we have

$$\lambda = \bar{\lambda} = (\alpha_1 + \ell(\beta), \alpha_2 + \ell(\beta), \ldots, \alpha_{\beta_1} + \ell(\beta), \alpha_{\beta_1+1}, \ldots, \alpha_\ell).$$

It remains to show that $s = \bar{s}$ and $t = \bar{t}$.

By the definition (2.2) of the spt-crank, we have

$$g - \lambda_g + \ell(\beta) - 1 = m$$

and

$$\bar{g} - \bar{\lambda}_{\bar{g}} + \ell(\beta) - 1 = m,$$

where

$$g = \lambda'_s - s + 1 \tag{2.9}$$

and

$$\bar{g} = \bar{\lambda}'_{\bar{s}} - \bar{s} + 1. \tag{2.10}$$

It follows that

$$g - \lambda_g = \bar{g} - \bar{\lambda}_{\bar{g}}. \tag{2.11}$$

We claim that $g = \bar{g}$. Assume to the contrary that $g \neq \bar{g}$. Without loss of generality, we may assume that $g > \bar{g}$. By (2.11), we see that $\lambda_g > \bar{\lambda}_{\bar{g}}$. Since $\lambda = \bar{\lambda}$, we get $\lambda_g > \lambda_{\bar{g}}$ which implies $g < \bar{g}$, contradicting the assumption that $g > \bar{g}$. So we have verified that $g = \bar{g}$.

Now that $g = \bar{g}$, by (2.9) and (2.10), we see that $\lambda'_s - s + 1 = \bar{\lambda}'_{\bar{s}} - \bar{s} + 1$, that is,

$$\lambda'_s - s = \bar{\lambda}'_{\bar{s}} - \bar{s}.$$

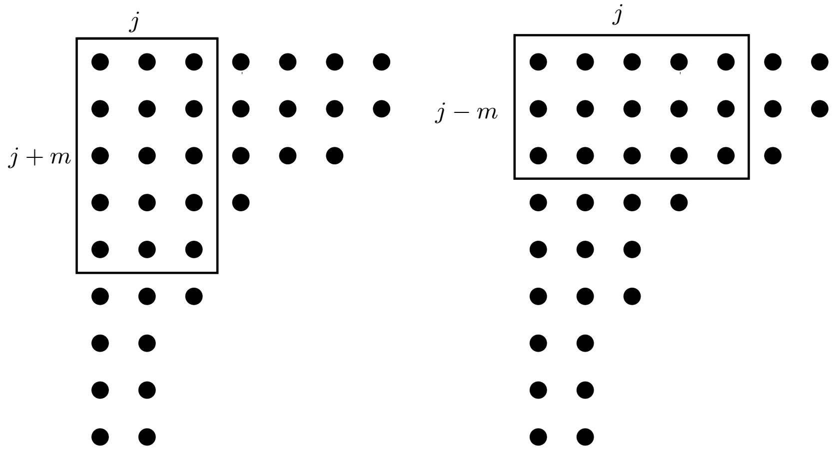

Figure 2.3: The 2-Durfee rectangle and −2-Durfee rectangle of $(7, 7, 6, 4, 3, 3, 2, 2)$.

Using the same argument as in the derivation of $g = \bar{g}$ from relation (2.11), we deduce that $s = \bar{s}$. By the construction of ψ, we see that

$$\ell(\beta) = t - s + 1 = \bar{t} - \bar{s} + 1.$$

But we have shown that $s = \bar{s}$, it follows that $t = \bar{t}$. Thus we reach the conclusion that $(\lambda, s, t) = (\bar{\lambda}, \bar{s}, \bar{t})$. This completes the proof. ∎

The following lemma will be used to characterize the image set of the map ψ. To present this lemma, we need to recall the definitions of the rank-set and the m-Durfee rectangle of a partition. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an ordinary partition. The rank-set of λ was introduced by Dyson [11] as an infinite sequence

$$[-\lambda_1, 1 - \lambda_2, \ldots, j - \lambda_{j+1}, \ldots, \ell - 1 - \lambda_\ell, \ell, \ell + 1, \ldots].$$

For example, the rank-set of $\lambda = (5, 5, 4, 3, 1)$ is $[-5, -4, -2, 0, 3, 5, 6, 7, 8, \ldots]$.

When m is an integer, the m-Durfee rectangle of a partition λ was introduced by Gordon and Houten [17], see also, Andrews [1], as the largest $(m + j) \times j$ rectangle contained in the Ferrers diagram of λ, see Figure 2.3, where j is called the width of the m-Durfee rectangle of λ. An m-Durfee rectangle reduces to a Durfee square when $m = 0$. It should be noticed that m may be negative.

It is worth mentioning that for a partition λ with $\ell(\lambda) \leq m$, there is no m-Durfee rectangle. In this case, we adopt a convention that the m-Durfee rectangle has no columns, that is, we set $j = 0$.

Lemma 2.6. *Let (λ, s, t) be a doubly marked partition of n with spt-crank m, and let $(\alpha, \beta) = \psi(\lambda, s, t)$. Then we have the following properties.*

(1) *The rank-set of α contains m;*

(2) *Let j be the width of the m-Durfee rectangle of α and h be the maximum integer such that $\alpha_{j+m+1+h} \geq h$, then*

$$\beta_1 = j + m + 1 + h.$$

Proof. We first show that m appears in the rank-set of α. Since (λ, s, t) has spt-crank m, by definition, we have

$$g - \lambda_g + t - s = m, \tag{2.12}$$

where $g = \lambda'_s - s + 1$. Since $s \geq 1$, we have $g \leq \lambda'_s$. Noting that $s \leq D(\lambda)$, we get $\lambda'_s \geq s$. It follows that $g \geq 1$. So we have $1 \leq g \leq \lambda'_s$. By the construction of ψ, we see that

$$\lambda_g - (t - s + 1) = \alpha_g. \tag{2.13}$$

Substituting (2.13) into (2.12), we obtain

$$g - 1 - \alpha_g = m, \tag{2.14}$$

which implies m appears the rank-set of α.

We continue to show that $\beta_1 = j+m+h+1$. Recall that j is the width of the m-Durfee rectangle of α. Since m appears in the rank-set of α, we find that

$$j + m - \alpha_{j+m+1} = m. \tag{2.15}$$

From (2.14) and (2.15), we deduce that

$$g = j + m + 1. \tag{2.16}$$

Since $g = \lambda'_s - s + 1$, it follows from (2.16) that

$$\lambda'_s - s = j + m. \tag{2.17}$$

From the construction of ψ, we see that $\beta_1 = \lambda'_s$. Hence

$$\beta_1 = s + j + m.$$

We claim that $s = h + 1$. By the choice of h, it suffices to show that $\alpha_{j+m+s} \geq s - 1$ and $\alpha_{j+m+s+1} < s$. By (2.17), we have

$$\alpha_{j+m+1+s} = \alpha_{\lambda'_s+1},$$

and

$$\alpha_{j+m+s} = \alpha_{\lambda'_s}.$$

From the construction of ψ, we see that

$$\alpha_{\lambda'_s+1} = \lambda_{\lambda'_s+1},$$

and

$$\alpha_{\lambda'_s} = \lambda_{\lambda'_s} - t + s - 1.$$

Examining the Ferrers diagram of λ, we see that $\lambda_{\lambda'_s+1} < s$. Consequently,

$$\alpha_{j+m+s+1} = \alpha_{\lambda'_s+1} = \lambda_{\lambda'_s+1} < s. \tag{2.18}$$

Since $\lambda'_t = \lambda'_s$, it can be seen that $\lambda_{\lambda'_s} \geq t$. So we deduce that

$$\alpha_{j+m+s} = \alpha_{\lambda'_s} = \lambda_{\lambda'_s} - t + s - 1 \geq t - t + s - 1 = s - 1. \tag{2.19}$$

Combining (2.18) and (2.19), we deduce that $h = s - 1$. So we conclude that $\beta_1 = j + m + h + 1$. This completes the proof. ∎

It turns out that the properties in Lemma 2.6 are sufficient to characterize the image of the map ψ.

Theorem 2.7. *Given an integer m and a positive integer n, let $V_{m,n}$ denote the set of pairs of partitions (α, β) of n satisfying the conditions in Lemma 2.6. The map ψ is a bijection between the set $Q_{m,n}$ and the set $V_{m,n}$.*

Proof. By Lemma 2.5 and Lemma 2.6, it suffices to construct the inverse map φ such that for all (α, β) in $V_{m,n}$, we have $\psi(\varphi(\alpha, \beta)) = (\alpha, \beta)$.

Given a pair of partitions (α, β) in $V_{m,n}$, we construct a doubly marked partition (λ, s, t). Recall that h is the maximum integer such that $\alpha_{j+m+1+h} \geq h$. Let $s = h + 1$, $t = h + \ell(\beta)$ and $\lambda = (\alpha_1 + \ell(\beta), \alpha_2 + \ell(\beta), \ldots, \alpha_{\beta_1} + \ell(\beta), \alpha_{\beta_1+1}, \ldots, \alpha_\ell)$. Then it can be seen that (λ, s, t) is a doubly marked partition. Set $\varphi(\alpha, \beta) = (\lambda, s, t)$. By the definitions of ψ and φ, it can be verified that $\psi(\varphi(\alpha, \beta)) = (\alpha, \beta)$ for all (α, β) in $V_{m,n}$. The detailed steps are omitted. This completes the proof. ∎

We are now in a position to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. By the bijection ψ in Theorem 2.7 between the set $Q_{m,n}$ and the set $V_{m,n}$, we see that for all integers m and positive integers n,

$$\sum_{(\lambda,s,t)\in Q_{m,n}} q^{|\lambda|} = \sum_{(\alpha,\beta)\in V_{m,n}} q^{|\alpha|+|\beta|}.$$

To prove (2.4) and (2.5), it suffices to show that for $m \geq 0$, we have

$$\sum_{n\geq 1} \sum_{(\alpha,\beta)\in V_{m,n}} q^{|\alpha|+|\beta|} = \sum_{j=0}^{+\infty} \frac{q^{j^2+mj+2j+m+1}}{(q;q)_{j+m}} \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{m+1+j+h})}, \tag{2.20}$$

and

$$\sum_{n\geq 1} \sum_{(\alpha,\beta)\in V_{-m,n}} q^{|\alpha|+|\beta|} = \sum_{j=m}^{+\infty} \frac{q^{j^2-mj+2j-m+1}}{(q;q)_{j-m}} \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{j+h+1-m})}. \tag{2.21}$$

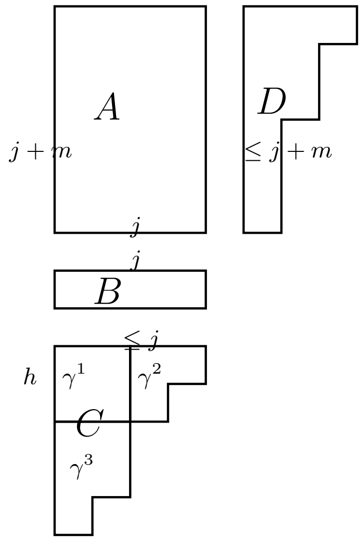

Figure 2.4: The decomposition of α

We first consider (2.20). Let $V_{m,n}^{j,h}$ denote the set of pairs of partitions (α, β) in $V_{m,n}$ such that the width of the m-Durfee rectangle of α is j and h is the maximum number satisfying $\alpha_{j+m+1+h} \geq h$. Note that $\alpha_{j+m+1} = j$. Hence we have $0 \leq h \leq j$. Now, we have the following relation for the sum on the left hand side of (2.20)

$$\sum_{n\geq 1} \sum_{(\alpha,\beta)\in V_{m,n}} q^{|\alpha|+|\beta|} = \sum_{j=0}^{+\infty} \sum_{h=0}^{j} \sum_{n\geq 1} \sum_{(\alpha,\beta)\in V_{m,n}^{j,h}} q^{|\alpha|+|\beta|}. \tag{2.22}$$

We shall show that

$$\sum_{n\geq 1} \sum_{(\alpha,\beta)\in V_{m,n}^{j,h}} q^{|\alpha|+|\beta|} = \frac{q^{j^2+mj+2j+m+1}}{(q;q)_{j+m}} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{m+1+j+h})}. \tag{2.23}$$

For any (α, β) in $V_{m,n}^{j,h}$, by Lemma 2.6, we see that β is a partition with each part equal to $j + m + 1 + h$. So the generating function for β is given by

$$\frac{q^{h+j+m+1}}{1 - q^{m+1+j+h}}. \tag{2.24}$$

To derive the generating function of α, we decompose the Ferrers diagram of α into four regions as shown in Figure 2.4. The generating function for α can be determined by computing the generating function of each region.

Region A forms to a partition into $j + m$ parts for which each part equals j. The generating function for this region is $q^{(m+j)\times j}$. Region B is a partition with only one part j, whose generating function is q^j.

Region C forms to a partition with each part not exceeding j. Let γ be the partition in this region. Since h is the maximum number such that $\alpha_{j+m+1+h} \geq h$, we see that the size of the Durfee square of γ is h. Therefore, we may divide γ into three partitions γ^1, γ^2 and γ^3:

(1) γ^1 is the Durfee square of γ, which is of size $h \times h$.

(2) γ^2 is the partition formed by the parts to the right of γ^1, which is a partition into at most h parts, each part not exceeding $j - h$;

(3) γ^3 is the partition formed by the parts below γ^1, which is a partition with each part not exceeding h.

Hence the generating function for all possible partitions γ in this region is given by

$$q^{h^2} \begin{bmatrix} j \\ h \end{bmatrix} \frac{1}{(q;q)_h},$$

where

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \frac{(q;q)_{n+m}}{(q;q)_n (q;q)_m}$$

is the generating function for partitions with at most m parts, each part not exceeding n, see [2, p.35].

Region D forms a partition into at most $m + j$ parts. So the generating function for possible partitions in this region equals

$$\frac{1}{(q;q)_{m+j}}.$$

Taking all the regions into consideration, we obtain the generating function of all possible partitions α satisfying the constraints of the set $V_{m,n}^{j,h}$

$$q^{(m+j)\times j} \cdot q^j \cdot q^{h^2} \begin{bmatrix} j \\ h \end{bmatrix} \frac{1}{(q;q)_h} \cdot \frac{1}{(q;q)_{m+j}}. \tag{2.25}$$

It should be noticed that for given m, n, j and h, to form a pair (α, β) in $V_{m,n}^{j,h}$, the choices α and β are independent. Hence the generating function for (α, β) in $V_{m,n}^{j,h}$ is the product of the generating functions of α and β subject to their individual constraints. Thus (2.23) follows from (2.24) and (2.25). So we arrive at (2.20).

We now turn to the proof of (2.21). The generating function for pairs of partitions in $V_{-m,n}$ can be computed in the same way as the derivation of (2.20). Let $V_{-m,n}^{j,h}$ denote the set of pairs of partitions (α, β) in $V_{-m,n}$ such that the width of the $-m$-Durfee rectangle of α is j, or the size of the $-m$-Durfee rectangle is $(j-m)\times j$. Assume that h is the maximum number such that $\alpha_{j+m+1+h}\geq h$. Since $j\geq m$, we have

$$\sum_{n\geq 1}\sum_{(\alpha,\beta)\in V_{-m,n}} q^{|\alpha|+|\beta|}=\sum_{j=m}^{+\infty}\sum_{h=0}^{j}\sum_{n\geq 1}\sum_{(\alpha,\beta)\in V_{-m,n}^{j,h}} q^{|\alpha|+|\beta|}. \tag{2.26}$$

Using the same argument as in the proof of (2.23), we can deduce that

$$\sum_{n\geq 1}\sum_{(\alpha,\beta)\in V_{-m,n}^{j,h}} q^{|\alpha|+|\beta|}=\frac{q^{j^2-mj+2j-m+1}}{(q;q)_{j-m}}\begin{bmatrix} j\\ h\end{bmatrix}\frac{q^{h^2+h}}{(q;q)_h(1-q^{j-m+1+h})}. \tag{2.27}$$

Combining (2.26) and (2.27), we obtain (2.21). This completes the proof. ∎

3 Marked partition and doubly marked partitions

In this section, we establish a correspondence between marked partitions and doubly marked partitions, so that one can divide the set of marked partitions of $5n+4$ and $7n+5$ into five and seven equinumerous classes by employing the spt-crank of doubly marked partitions.

Theorem 3.1. *There is a bijection Δ between the set of marked partitions (μ,k) of n and the set of doubly marked partitions (λ,s,t) of n.*

To prove the above theorem, we need to use the notation (λ,s,t) to mean a partition λ with two distinguished columns in the Ferrers diagram. In other words, we no longer assume that (λ,s,t) is a doubly marked partition unless it is explicitly stated. Let Q_n denote the set of doubly marked partitions of n, and let

$$U_n=\{(\lambda,s,t)\mid |\lambda|=n,\ 1\leq s\leq D(\lambda),\ 1\leq t\leq \lambda_1\}.$$

Obviously, $Q_n\subseteq U_n$.

Before we give a description of the bijection Δ, we introduce a transformation τ from $U_n\setminus Q_n$ to U_n.

The transformation τ: Assume that $(\lambda,s,t)\in U_n\setminus Q_n$, that is, λ is an ordinary partition of n with two distinguished columns s and t such that $1\leq s\leq D(\lambda)$ and either $1\leq t<s$ or $\lambda'_s>\lambda'_t$. We wish to construct a partition μ with two distinguished columns a and b. Let p be the maximum integer such that $\lambda'_p=\lambda'_s$. Define

$$\delta=(\lambda_1-p+s-1,\lambda_2-p+s-1,\ldots,\lambda_{\lambda'_s}-p+s-1,\lambda_{\lambda'_s+1},\ldots,\lambda_\ell). \tag{3.1}$$

Set a to be the minimum integer such that $\delta_a<\lambda'_s$ and

$$\mu=(\delta_1,\ldots,\delta_{a-1},\lambda'_s,\ldots,\lambda'_p,\delta_a,\ldots,\delta_\ell). \tag{3.2}$$

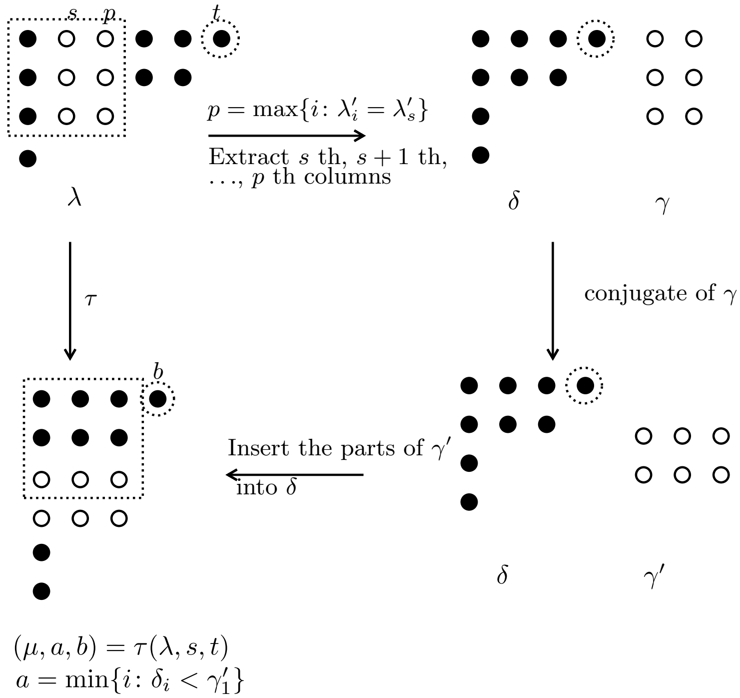

Figure 3.1: Illustration for the map $\tau\colon ((6,5,3,1),2,6) \mapsto ((4,3,3,3,1,1),3,4)$.

If $t < s$, then set $b = t$ and if $\lambda'_s > \lambda'_t$, then set $b = t-p+s-1$. Define $\tau(\lambda, s, t) = (\mu, a, b)$. Figure 3.1 gives an illustration of the map τ.

The conditions in the following lemma will be used to characterize the image set of the map τ.

Lemma 3.2. *Assume that $(\lambda, s, t) \in U_n \setminus Q_n$, and denote $\tau(\lambda, s, t)$ by (μ, a, b). We have $(\mu, a, b) \in U_n$. Furthermore, if $a = 1$, then we have $\mu'_b > s(\mu')$.*

Proof. We first show that $(\mu, a, b) \in U_n$, that is, we need to verify that $1 \leq a \leq D(\mu)$ and $1 \leq b \leq \mu_1$.

It is clear that $a \geq 1$. We proceed to show that $a \leq D(\mu)$. To this end, we first prove that $\delta_{\lambda'_s} < \lambda'_s$. Then we show that $a \leq D(\mu)$ can be deduced from the fact that $\delta_{\lambda'_s} < \lambda'_s$.

By the definition of δ in (3.1), we see that

$$\delta_{\lambda'_s} = \lambda_{\lambda'_s} - p + s - 1 < \lambda_{\lambda'_s}. \tag{3.3}$$

Since $1 \leq s \leq D(\lambda)$, that is, $\lambda'_s \geq D(\lambda)$, we deduce that

$$\lambda_{\lambda'_s} \leq \lambda'_s. \tag{3.4}$$

Combining (3.3) and (3.4) yields $\delta_{\lambda'_s} < \lambda'_s$.

Recall that a is the minimum integer such that $\delta_a < \lambda'_s$. But we have shown that $\delta_{\lambda'_s} < \lambda'_s$, this implies that $a \le \lambda'_s$. On the other hand, by the construction of τ, we find that $\lambda'_s = \mu_a$. So we deduce that $a \le \mu_a$, that is, $a \le D(\mu)$. This completes the proof of the assertion that $1 \le a \ \le D(\mu)$.

Next, we continue to prove that $1 \le b \le \mu_1$. There are two cases.

Case 1: $1 \le t < s$. By the construction of τ, we have $b = t$ and

$$\mu_1 \ge \lambda'_s. \tag{3.5}$$

Since $s \le D(\lambda)$, we get

$$\lambda'_s \ge s > t. \tag{3.6}$$

Combining (3.5) and (3.6), we deduce that $t < \mu_1$. Since $b = t$ and $t \ge 1$, we conclude that $1 \le b < \mu_1$.

Case 2: $\lambda'_s > \lambda'_t$. By the construction of τ, we have $b = t - p + s - 1$ and

$$\mu_1 \ge \delta_1 = \lambda_1 - p + s - 1. \tag{3.7}$$

Since $\lambda'_p = \lambda'_s$ and $\lambda'_s > \lambda'_t$, we have $t > p$, and so $b = t - p + s - 1 \ge s \ge 1$. Using (3.7) and the fact that $t \le \lambda_1$, we obtain $\mu_1 \ge t - p + s - 1 = b$. It follows that $1 \le b \le \mu_1$.

Up to now, we have shown that $(\mu, a, b) \in U_n$. Finally, we prove that if $a = 1$, then $\mu'_b > s(\mu')$. We now assume that $a = 1$, and we claim that in this case

$$b \le \lambda_1 - p + s - 1. \tag{3.8}$$

By the choice of a, if $a = 1$, then we have

$$\delta_1 < \lambda'_s \tag{3.9}$$

and

$$\mu = (\lambda'_s, \ldots, \lambda'_p, \delta_1, \ldots, \delta_\ell). \tag{3.10}$$

To prove the claim, we consider the following two cases.

Case 1: $1 \le t < s$. By the construction of τ, we have $b = t \le s - 1$. Since $p \le \lambda_1$, we see that

$$b = t \le s - 1 \le \lambda_1 - p + s - 1.$$

Case 2: $\lambda'_s > \lambda'_t$. By the construction of τ, we find that $b = t - p + s - 1$. Using the fact that $t \le \lambda_1$, we get

$$b = t - p + s - 1 \le \lambda_1 - p + s - 1.$$

So the claim is proved.

Combining (3.9) and (3.10), we get $\delta_1 < \mu_1$, or equivalently, $\mu'_{\delta_1} > \mu'_{\mu_1}$. Note that $\mu'_{\mu_1} = s(\mu')$, so we have $\mu'_{\delta_1} > s(\mu')$. On the other hand, from the definition (3.1) of δ, we have $\delta_1 = \lambda_1 - p + s - 1$. By the claim that $b \le \lambda_1 - p + s - 1$, we obtain that $b \le \delta_1$, this yields $\mu'_b \ge \mu'_{\delta_1}$. So we reach the conclusion that $\mu'_b > s(\mu')$. This completes the proof. ∎

The following theorem gives the image set W_n of the transformation τ, and it shows that τ is bijection between $U_n \setminus Q_n$ and W_n.

Theorem 3.3. *Given a positive integer n, let*

$$W_n = \{(\mu, a, b) \mid (\mu, a, b) \in U_n \text{ and } \mu'_b > s(\mu') \text{ whenever } a = 1\}. \tag{3.11}$$

Then the transformation τ is a bijection between $U_n \setminus Q_n$ and W_n.

Proof. By Lemma 3.2, it suffices to construct a map σ defined on W_n such that for all $(\lambda, s, t) \in U_n \setminus Q_n$, we have $\sigma(\tau(\lambda, s, t)) = (\lambda, s, t)$ and for all $(\mu, a, b) \in W_n$, we have $\tau(\sigma(\mu, a, b)) = (\mu, a, b)$.

Let $(\mu, a, b) \in W_n$, we wish to construct a partition λ with two distinguished columns s and t. Let r be the maximum integer such that $\mu_r = \mu_a$. Define

$$\gamma = (\mu_1, \ldots, \mu_{a-1}, \mu_{r+1}, \ldots, \mu_\ell).$$

Set s to be the minimum integer such that $\gamma'_s < \mu_a$, and

$$\lambda = (\gamma_1 + r - a + 1, \ldots, \gamma_{\mu_a} + r - a + 1, \gamma_{\mu_a+1}, \ldots, \gamma_\ell).$$

If $b < s$, then we set $t = b$. Otherwise, we set $t = b + r - a + 1$. Define $\sigma(\mu, a, b) = (\lambda, s, t)$. Using the same argument as in the proof of Lemma 3.2, we deduce that $\sigma(\mu, a, b) \in U_n \setminus Q_n$.

By the constructions of τ and σ, it is straightforward to check that $\sigma(\tau(\lambda, s, t)) = (\lambda, s, t)$ for all $(\lambda, s, t) \in U_n \setminus Q_n$ and $\tau(\sigma(\mu, a, b)) = (\mu, a, b)$ for all $(\mu, a, b) \in W_n$. The details are omitted. This completes the proof. ∎

We now describe the bijection Δ in Theorem 3.1 based on the bijection τ.

The definition of Δ: Let (μ, k) be a marked partition of n, we wish to construct a doubly marked partition (λ, s, t) of n.

We first consider $(\mu', 1, k)$. If $(\mu', 1, k)$ is already a doubly marked partition, then there is nothing to be done and we just set $(\lambda, s, t) = (\mu', 1, k)$. Otherwise, we iteratively apply the map τ to $(\mu', 1, k)$ until we get a doubly marked partition (λ, s, t). We shall show that this process terminates and it is reversible.

For example, let $n = 6$, $\mu = (2, 1, 1, 1, 1)$ and $k = 5$. We have $\mu' = (5, 1)$. Note that $(\mu', 1, k) = ((5, 1), 1, 5)$, which is not a doubly marked partition. It can be checked that $\tau(\mu', 1, k) = ((4, 2), 2, 4)$, which is not a doubly marked partition. Repeating this process, we get $\tau((4, 2), 2, 4) = ((3, 2, 1), 2, 3)$, and $\tau((3, 2, 1), 2, 3) = ((2, 2, 1, 1), 2, 2)$, which is eventually a doubly marked partition. See Figure 3.2. Thus, we obtain

$$\Delta((2, 1, 1, 1, 1), 5) = ((2, 2, 1, 1), 2, 2).$$

The following lemma shows that the map Δ is well-defined.

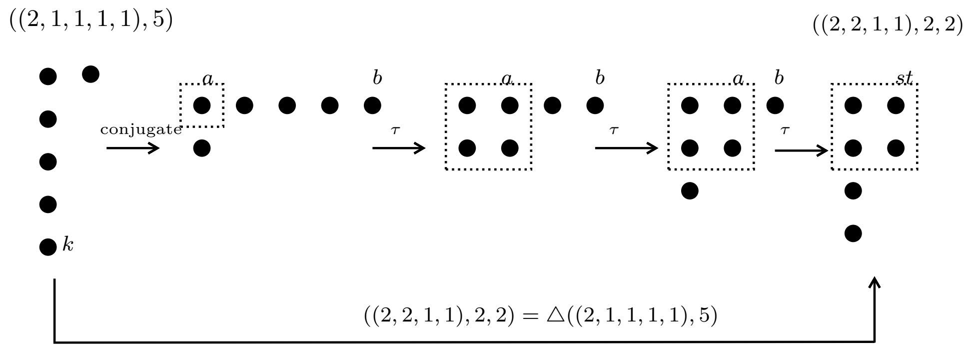

Figure 3.2: The bijection $\triangle$: $((2,1,1,1,1),5) \mapsto ((2,2,1,1),2,2)$.

Lemma 3.4. *The map Δ is well-defined, that is, for each marked partition (μ, k), there exists i such that $\tau^i(\mu', 1, k)$ is a doubly marked partition.*

Proof. Assume to the contrary that there exists a marked partition (μ, k) of n such that for any $i \geq 0$, $\tau^i(\mu', 1, k)$ is not a doubly marked partition of n. Let $(\lambda^{(i)}, s^{(i)}, t^{(i)}) = \tau^i(\mu', 1, k)$. By Lemma 3.2, we see that $\lambda^{(i)}$ is an ordinary partition of n, $s^{(i)}$ and $t^{(i)}$ are both bounded by n. Thus the set

$$\{(\lambda^{(i)}, s^{(i)}, t^{(i)})|\ i \geq 0\}$$

is finite. So there exist integers ℓ and m such that $\ell < m$ and $(\lambda^{(\ell)}, s^{(\ell)}, t^{(\ell)}) = (\lambda^{(m)}, s^{(m)}, t^{(m)})$, that is,

$$\tau^\ell(\mu', 1, k) = \tau^m(\mu', 1, k). \tag{3.12}$$

We may choose ℓ to the minimum integer such that $\tau^\ell(\mu', 1, k) = \tau^m(\mu', 1, k)$ for some $m > \ell$. We claim that $\ell \geq 1$, that is, there does not exist $m \geq 1$ such that

$$(\mu', 1, k) = \tau^m(\mu', 1, k). \tag{3.13}$$

Denote $(\mu', 1, k)$ by (λ, a, b), so that we have $\lambda' = \mu$. Since (μ, k) is a marked partition, that is, $\mu_k = s(\mu)$, we see that $\lambda'_b = \mu_k = s(\lambda')$. Since $a = 1$, by the definition (3.11) of W_n, we see that (λ, a, b) is in W_n if any only if $\lambda'_b > s(\lambda')$. So we deduce that $(\mu', 1, k)$ is not in W_n. On the other hand, by Theorem 3.3, we see that W_n is the image-set of τ. Since $\tau^m(\mu', 1, k)$ lies in the image-set of τ, it follows that $(\mu', 1, k) \neq \tau^m(\mu', 1, k)$ for any $m \geq 1$. This proves that $\ell \geq 1$.

By the choice of ℓ, we see that for any $m > \ell$,

$$\tau^{\ell-1}(\mu', 1, k) \neq \tau^{m-1}(\mu', 1, k). \tag{3.14}$$

By the assumption that $\tau^i(\mu', 1, k)$ is not a doubly marked partition for any $i \geq 0$, $\tau^{\ell-1}(\mu', 1, k)$ and $\tau^{m-1}(\mu', 1, k)$ are not doubly marked partitions. Since τ is a bijection, we obtain that

$$\tau^\ell(\mu', 1, k) \neq \tau^m(\mu', 1, k),$$

for any $m > \ell$, contradicting the choice of ℓ. Hence we conclude that Δ is well-defined. This completes the proof. ∎

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1: We have given the description of the map Δ and have shown that Δ is well-defined. It remains to show that Δ is reversible. To this end, we construct a map Λ defined on the set of doubly marked partitions of n and we shall show that it is the inverse map of Δ.

The map Λ can be described as follows. Let (λ, s, t) be a doubly marked partition of n. We aim to construct a marked partition (μ, k). If $s = 1$ and $\lambda'_t = s(\lambda')$, we set $(\mu, k) = (\lambda', t)$. If $s > 1$ or $\lambda'_t > s(\lambda')$, we can iteratively use the inverse map τ^{-1} to transform (λ, s, t) into an ordinary partition δ with two distinguished columns a and b such that $a = 1$ and $\delta'_b = s(\delta')$. Set $(\mu, k) = (\delta', b)$. Finally, define $\Lambda(\lambda, s, t) = (\mu, k)$.

Parallel to the proof of the fact that Δ is well-defined, it can be shown that the map Λ is well-defined. The details are omitted. Furthermore, since τ is a bijection, it is routine to check that Λ is the inverse map of Δ. This completes the proof. ∎

Employing the bijection Δ and the spt-crank for doubly marked partitions, one can divide the set of marked partitions of $5n+4$ and $7n+5$ into five and seven equinumerous classes.

For example, for $n = 4$, we have $spt(4) = 10$. The ten marked partitions of 4, the corresponding doubly marked partitions, and the spt-crank modulo 5 are listed in Table 3.1.

(μ, k)	$(\lambda, s, t) = \Delta(\mu, k)$	$c(\lambda, s, t)$	$c(\lambda, s, t) \mod 5$
$((4), 1)$	$((1,1,1,1), 1, 1)$	3	3
$((3,1), 2)$	$((3,1), 1, 1)$	1	1
$((2,2), 1)$	$((2,2), 1, 1)$	0	0
$((2,2), 2)$	$((2,2), 1, 2)$	1	1
$((2,1,1), 2)$	$((2,1,1), 1, 1)$	2	2
$((2,1,1), 3)$	$((2,2), 2, 2)$	-1	4
$((1,1,1,1), 1)$	$((4), 1, 1)$	-3	2
$((1,1,1,1), 2)$	$((4), 1, 2)$	-2	3
$((1,1,1,1), 3)$	$((4), 1, 3)$	-1	4
$((1,1,1,1), 4)$	$((4), 1, 4)$	0	0

Table 3.1: The case for $n = 4$.

For $n = 5$, we have $spt(5) = 14$. The fourteen marked partitions of 5, the corresponding doubly marked partitions, and the spt-crank modulo 7 are listed in Table 3.2.

(μ, k)	$(\lambda, s, t) = \Delta(\mu, k)$	$c(\lambda, s, t)$	$c(\lambda, s, t) \mod 7$
$((5), 1)$	$((1,1,1,1,1),1,1)$	4	4
$((4,1), 2)$	$((4,1),1,1)$	1	1
$((3,2), 2)$	$((3,1,1),1,1)$	2	2
$((3,1,1), 2)$	$((3,2),1,1)$	0	0
$((3,1,1), 3)$	$((3,2),1,2)$	1	1
$((2,2,1), 3)$	$((2,2,1),1,1)$	2	2
$((2,1,1,1), 2)$	$((2,1,1,1),1,1)$	3	3
$((2,1,1,1), 3)$	$((3,2),2,2)$	-2	5
$((2,1,1,1), 4)$	$((2,2,1),2,2)$	-1	6
$((1,1,1,1,1), 1)$	$((5),1,1)$	-4	3
$((1,1,1,1,1), 2)$	$((5),1,2)$	-3	4
$((1,1,1,1,1), 3)$	$((5),1,3)$	-2	5
$((1,1,1,1,1), 4)$	$((5),1,4)$	-1	6
$((1,1,1,1,1), 5)$	$((5),1,5)$	0	0

Table 3.2: The case for $n = 5$.

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education and the National Science Foundation of China.

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