A Cyclic Analogue of Stanley’s Shuffling Theorem

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Abstract. We introduce the cyclic major index of a cyclic permutation and give a bivariate analogue of the enumerative formula for the cyclic shuffles with a given cyclic descent number due to Adin, Gessel, Reiner and Roichman, which can be viewed as a cyclic analogue of Stanley’s shuffling theorem. This gives an answer to a question of Adin, Gessel, Reiner and Roichman, which has been posed by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema again.

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1 Introduction

The main theme of this note is to establish a cyclic analogue of Stanley’s shuffling theorem. Recall that Stanley’s shuffling theorem establishes an explicit expression for the generating function of the number of shuffles of two disjoint permutations $\sigma$ and $\pi$ with a given cyclic descent number and a given major index. Here we adopt some common notation and terminology on permutations as used in [13, Chapter 1]. We say that $\pi = \pi_1 \pi_2 \cdots \pi_n$ is a permutation of length $n$ if it is a sequence of $n$ distinct letters (not necessarily from 1 to $n$). For example, $\pi = 9 2 8 10 12 3 7$ is a permutation of length 7. Let $\mathfrak{S}_n$ denote the set of all permutations of length $n$.

Let $\pi \in \mathfrak{S}_n$. We say that $1 \leq i \leq n - 1$ is a descent of $\pi$ if $\pi_i > \pi_{i+1}$. The set of descents of $\pi$ is called the descent set of $\pi$, denoted $\text{Des} (\pi)$, viz.,

$$\text{Des} (\pi) := \{1 \leq i \leq n - 1 : \pi_i > \pi_{i+1}\}.$$ 

The number of its descents is called the descent number, denoted $\text{des} (\pi)$, namely,

$$\text{des} (\pi) := \# \text{Des} (\pi),$$ 

where the hash symbol $\# \mathcal{T}$ stands for the cardinality of a set $\mathcal{T}$. The major index of $\pi$, denoted $\text{maj} (\pi)$, is defined to be the sum of its descents. To wit,

$$\text{maj} (\pi) := \sum_{k \in \text{Des} (\pi)} k.$$ 

Let $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$ be disjoint permutations, that is, permutations with no letters in common. We say that $\alpha \in \mathfrak{S}_{n+m}$ is a shuffle of $\sigma$ and $\pi$ if both $\sigma$ and $\pi$ are subsequences of $\alpha$. The set of shuffles of $\sigma$
and $\pi$ is denoted $S(\sigma, \pi)$. For example,

$$S(63, 14) = \{6314, 6134, 6143, 1463, 1634, 1643\}.$$ 

Clearly, the number of permutations in $S(\sigma, \pi)$ is $\binom{m+n}{n}$ for two disjoint permutations $\sigma \in S_n$ and $\pi \in S_m$.

Stanley’s shuffling theorem states that

**Theorem 1.1.** Let $\sigma \in S_n$ and $\pi \in S_m$ be disjoint permutations, where $\text{des}(\sigma) = r$ and $\text{des}(\pi) = s$. Then

$$\sum_{\alpha \in \mathcal{S}(\sigma, \pi)} q^{\text{maj}(\alpha)} = \binom{n-r+s}{k-r} \binom{n-s+r}{k-s} q^{\text{maj}(\sigma)+\text{maj}(\pi)+(k-s)(k-r)}. \quad (1.1)$$

Here

$$\binom{n}{m} = \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n-m+1})}{(1-q^m)(1-q^{m-1}) \cdots (1-q)}$$

is the Gaussian polynomial (also called the $q$-binomial coefficient), see Andrews [2, Chapter 1].

Stanley [12] obtained the above expression in light of the $q$-Pfaff-Saalschütz identity in his setting of $P$-partitions. Bijective proofs of Stanley’s shuffling theorem have been given by Goulden [6], Stadler [11], Ji and Zhang [10].

Recently, Adin, Gessel, Reiner and Roichman [1] introduced a cyclic version of quasisymmetric functions with a corresponding cyclic shuffle operation. A cyclic permutation $[\pi]$ of length $n$ is the set of all rotations of a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, i.e,

$$[\pi] = \{\pi_1 \pi_2 \cdots \pi_n, \pi_2 \pi_3 \cdots \pi_1, \ldots, \pi_n \pi_1 \cdots \pi_{n-1}\}.$$ 

For example,

$$[4231] = \{4231, 2314, 3142, 1423\} \quad (1.2)$$

is a cyclic permutation of length 4, where

$$[4231] = [2314] = [3142] = [1423].$$

Let $\pi_l$ be the largest element in $[\pi]$. The linear permutation $\hat{\pi} = \pi_l \pi_{l+1} \cdots \pi_n \pi_1 \cdots \pi_{l-1}$ corresponding to the cyclic permutation $[\pi]$ is called the representative of the cyclic permutation $[\pi]$. For the example above, $4231$ is the representative of the cyclic permutation $[4231]$. Here and in the sequel, we use the representative to represent each cyclic permutation $[\pi]$. For example, we use $[4231]$ to represent the cyclic permutation in (1.2). In this way, all cyclic permutations of $\{1, 2, 3, 4\}$ are listed as follows:

$$[4123], [4312], [4132], [4213], [4231], [4321].$$

Let $S_n^c$ denote the set of all cyclic permutations of length $n$ and let $[\sigma] \in S_n^c$ and $[\pi] \in S_m^c$ be disjoint cyclic permutations, that is, cyclic permutations with no letters in common. We say that $[\alpha] \in S_n^c$ is a cyclic shuffle of two cyclic permutations $[\sigma]$ and $[\pi]$ if both $[\sigma]$ and $[\pi]$ are circular subsequences of $[\alpha]$. Recall that a cyclic permutation $[\pi]$ is called a circular subsequence of $[\alpha]$ if there exists a rotation of $[\alpha]$, which contains $\pi$ linearly. The set of cyclic shuffles of $[\sigma]$ and $[\pi]$ is denoted $S^c([\sigma], [\pi])$. For example,

$$S^c([63], [41]) = \{[63\ 1\ 4], [63\ 4\ 1], [61\ 4\ 3], [64\ 1\ 3], [61\ 3\ 4], [64\ 3\ 1]\}. \quad (1.3)$$
Evidently, 
\[
\#\mathcal{S}^c([\sigma], [\pi]) = \left(\frac{m + n - 2}{m - 1}\right).
\]
for two disjoint cyclic permutations \([\sigma] \in \mathcal{S}^c_m\) and \([\pi] \in \mathcal{S}^c_n\), see [5, Eq. (7)].

In order to study Solomon’s descent algebra, Cellini [3, 4] introduced the cyclic descent set. Let \(\pi = \pi_1 \pi_2 \ldots \pi_n\) be a linear permutation. The cyclic descent set of \(\pi\) is defined to be
\[
c\text{Des}(\pi) = \{1 \leq i \leq n : \pi_i > \pi_{i+1}\}
\]
with the convention \(\pi_{n+1} = \pi_1\). The number of its cyclic descents is called the cyclic descent number, denoted \(c\text{des}(\pi)\), viz.,
\[
c\text{des}(\pi) := \#c\text{Des}(\pi).
\]
Let \([\pi]\) be a cyclic permutation of length \(n\). Note that all linear permutations corresponding to \([\pi]\) have the same number of cyclic descents, so we may define the cyclic descent number of \([\pi]\) as
\[
c\text{des}([\pi]) = c\text{des}(\pi),
\]
where \(\pi\) is any linear permutation corresponding to \([\pi]\).

Based on their setting of cyclic quasi-symmetric functions, Adin, Gessel, Reiner and Roichman [1] established the following enumerative formula for the cyclic shuffles with a given cyclic descent number.

**Theorem 1.2 (Adin-Gessel-Reiner-Roichman).** Let \([\sigma] \in \mathcal{S}^c_m\) and \([\pi] \in \mathcal{S}^c_n\) be disjoint cyclic permutations, where \(c\text{des}([\sigma]) = r\) and \(c\text{des}([\pi]) = s\). Let \(\mathcal{S}^c([\sigma], [\pi], k)\) denote the set of cyclic shuffles of \([\sigma]\) and \([\pi]\) with cyclic descent number \(k\). Then
\[
\#\mathcal{S}^c([\sigma], [\pi], k) = \frac{k(m - r)(n - s) + (m + n - k)rs}{(m - r + s)(n - s + r)} \left(\frac{m - r + s}{k - r}\right) \left(\frac{n - s + r}{k - s}\right).
\]
Summing (1.6) over all \(k\) gives (1.4) upon using the Chu-Vandermonde identity [13, p. 135, Ex. 100]. At the end of their paper, Adin, Gessel, Reiner and Roichman [1] asked a question about looking for a notion of cyclic major index, which provides a bivariate analogue of Theorem 1.2. This question has been posed by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema in [5, Question 4.1] again.

In this paper, we introduce the cyclic major index of a cyclic permutation \([\pi]\). Let \([\pi]\) be a cyclic permutation of length \(n\). Suppose that the representative of \([\pi]\) is \(\hat{\pi} = \hat{\pi}_1 \hat{\pi}_2 \cdots \hat{\pi}_n\), where \(\hat{\pi}_1\) is the largest element in \([\pi]\). The cyclic major index of the cyclic permutation \([\pi]\) is defined to be

\[
\text{maj}([\pi]) = \text{maj}(\hat{\pi}).
\]

(1.7)

For example, the representative of the cyclic permutation \([4 1 3 2]\) is \(\hat{\pi} = 4 1 3 2\), and so its cyclic major index is defined to be the major index of \(\hat{\pi} = 4 1 3 2\). It gives that \(\text{maj}([4 1 3 2]) = 1 + 3 = 4\).

In order to state the cyclic analogue of Stanley’s shuffling theorem, we will need to introduce the cyclic descent-bottom set of a cyclic permutation and recall the splitting map \(S_i\) defined by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema in [5], which maps a cyclic permutation to a linear permutation. Let \([\pi]\) be a cyclic permutation of length \(n\). The cyclic descent-bottom set of \([\pi]\) is defined as:

\[
\text{cB}_d([\pi]) = \{\pi_{i+1} : \pi_i > \pi_{i+1}, \text{ for } 1 \leq i \leq n\}
\]

(1.8)

with the convention \(\pi_{n+1} = \pi_1\). It should be mentioned that the descent-bottom set of a linear permutation has been studied by Haglund and Visontai [7] and Hall and Remmel [8, 9].

It is manifest from (1.5) and (1.8) that

\[
\#\text{cB}_d([\pi]) = \text{cdes}([\pi]).
\]

For example,

\[
\text{cB}_d([6 4 1 3]) = \{1, 4\}.
\]

Let \([\pi]\) be a cyclic permutation of length \(n\). For \(i \in [\pi]\), Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema [5] defined the map \(S_i([\pi])\) to be the unique permutation corresponding to \([\pi]\) which starts with \(i\). For example,

\[
S_5([5 1 3 4]) = 5 1 3 4, S_1([5 1 3 4]) = 1 3 4 5, S_6([5 1 3 4]) = 3 4 5 1,
\]

and

\[
S_4([5 1 3 4]) = 4 5 1 3.
\]

We obtain the following generating function of the number of cyclic shuffles of two disjoint cyclic permutations with a given cyclic descent number and a given cyclic major index.

**Theorem 1.3** (Cyclic Stanley’s shuffling theorem). Let \([\sigma] \in \mathcal{S}_m^c\) and let \([\pi] \in \mathcal{S}_n^c\) be disjoint cyclic permutations, where \(\text{cdes}([\sigma]) = r\) and \(\text{cdes}([\pi]) = s\). Suppose that the largest element of \([\sigma]\) and \([\pi]\) is in \([\sigma]\). Then

\[
\sum_{\begin{array}{c}[\alpha] \in \mathcal{S}_m^c([\sigma],[\pi]) \\ \text{cdes}([\alpha]) = k \end{array}} q^{\text{maj}([\alpha])}
\]

4
\[ \begin{align*}
&= \left[ \frac{m - r + s}{k - r} \right] \left[ \frac{n - s + r - 1}{k - s - 1} \right] q^{\text{maj}([\sigma]) + (k-s)(k-r)} \sum_{i \in B_i([\pi])} q^{\text{maj}(S_i([\pi]))} \\
&\quad + \left[ \frac{m - r + s - 1}{k - r} \right] \left[ \frac{n - s + r}{k - s} \right] q^{\text{maj}([\sigma]) + (k-s+1)(k-r)} \sum_{i \in \text{chu}([\pi])} q^{\text{maj}(S_i([\pi]))}.
\end{align*} \tag{1.9} \]

Setting \( q \to 1 \) in Theorem 1.3, we obtain (1.6), that is,
\[ \#S^c([\sigma], [\pi], k) \]
\[= \sum_{i \in \text{chu}([\pi])} \left( \frac{m - r + s}{k - r} \right) \left( \frac{n - s + r - 1}{k - s - 1} \right) + \sum_{i \in B_i([\pi])} \left( \frac{m - r + s - 1}{k - r} \right) \left( \frac{n - s + r}{k - s} \right) \]
\[= (n - s) \left( \frac{m - r + s}{k - r} \right) \left( \frac{n - s + r - 1}{k - s - 1} \right) + s \left( \frac{m - r + s - 1}{k - r} \right) \left( \frac{n - s + r}{k - s} \right) \]
\[= \frac{k(m-r)(n-s)+(m+n-k)rs}{(m-r+s)(n-s+r)} \left( \frac{m - r + s}{k - r} \right) \left( \frac{n - s + r}{k - s} \right). \]

\section{Proof of Theorem 1.3}

This section is devoted to the proof of Theorem 1.3 with the aid of Stanley’s shuffling theorem.

\textbf{Proof of Theorem 1.3.} Let \([\sigma] \in S_m^c\) and let \([\pi] \in S_n^c\) be two disjoint cyclic permutations, where \(c\text{des}([\sigma]) = r\) and \(c\text{des}([\pi]) = s\). Suppose that the largest element of \([\sigma]\) and \([\pi]\) is in \([\sigma]\). Let \(\hat{\sigma} = \hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_m\) be the representative of the cyclic permutation \([\sigma]\), that is, \(\hat{\sigma}_1\) is the largest element of \([\sigma]\). Under the hypothesis of this theorem, we see that \(\hat{\sigma}_1\) is greater than all elements in \([\pi]\). Define
\[\hat{\sigma}' = \hat{\sigma}_2 \cdots \hat{\sigma}_m.\] \tag{2.1}

Obviously,
\[c\text{des}([\sigma]) = \text{des}(\hat{\sigma}') + 1\] \tag{2.2}
and
\[\text{maj}([\sigma]) = \text{maj}(\hat{\sigma}') + \text{des}(\hat{\sigma}') + 1.\] \tag{2.3}

Let \(S^c([\sigma], [\pi])\) denote the set of cyclic shuffles of \([\sigma]\) and \([\pi]\), and let \(S(\hat{\sigma}', S_i([\pi]))\) denote the set of linear shuffles of \(\hat{\sigma}'\) and \(S_i([\pi])\), where \(\hat{\sigma}'\) is defined in (2.1) and \(S_i([\pi])\) is the unique permutation corresponding to \([\pi]\) which starts with \(i \in [\pi]\). We claim that there is a bijection \(\psi\) between the set \(S^c([\sigma], [\pi])\) and the set \(\bigcup_{i \in [\pi]} S(\hat{\sigma}', S_i([\pi]))\). Moreover, for \([\alpha] \in S^c([\sigma], [\pi])\), we have \(\psi(\alpha) = \hat{\alpha}'\) such that
\[c\text{des}([\alpha]) = \text{des}(\hat{\alpha}') + 1\] \tag{2.4}
and
\[\text{maj}([\alpha]) = \text{maj}(\hat{\alpha}') + \text{des}(\hat{\alpha}') + 1.\] \tag{2.5}

Let \([\alpha] \in S^c([\sigma], [\pi])\) and let \(\hat{\alpha} = \hat{\alpha}_1 \hat{\alpha}_2 \cdots \hat{\alpha}_{n+m}\) be the representative of \([\alpha]\), which is a linear permutation corresponding to \([\alpha]\) such that \(\hat{\alpha}_1\) is the largest element in \([\alpha]\). Since \(\hat{\sigma}_1\) is the largest element in \([\sigma]\) and \([\pi]\), we deduce that \(\hat{\alpha}_1 = \hat{\sigma}_1\) and \(c\text{des}([\alpha]) = \text{des}(\hat{\alpha})\). Define
\[\hat{\alpha}' = \hat{\alpha}_2 \hat{\alpha}_3 \cdots \hat{\alpha}_{n+m}.\]
From the construction of $\hat{\alpha}'$, it is evident that $\hat{\alpha}' \in \bigcup_{i \in [\pi]} S(\hat{\alpha}', S_i([\pi]))$ and $[\alpha]$ and $\hat{\alpha}'$ satisfy (2.4) and (2.5). Moreover, this process is clearly reversible. This proved the claim. We therefore obtain

$$\sum_{[\alpha] \in S^C([\pi], [\pi])} q^{\text{maj}([\alpha])} = \sum_{i \in [\pi]} \sum_{\alpha' \in S(\hat{\alpha}', S_i([\pi]))} q^{\text{maj}(\hat{\alpha}') + k}$$

$$= \sum_{i \in [\pi]} \sum_{\alpha' \in S(\hat{\alpha}', S_i([\pi]))} q^{\text{maj}(\hat{\alpha}') + k} + \sum_{i \in cB_4([\pi])} \sum_{\alpha' \in S(\hat{\alpha}', S_i([\pi]))} q^{\text{maj}(\hat{\alpha}') + k}.$$  (2.6)

By (2.2) and (2.3), we see that

$$\text{des}(\hat{\sigma}') = c\text{des}([\sigma']) - 1 = r - 1 \quad \text{and} \quad \text{maj}(\hat{\sigma}') = \text{maj}([\sigma]) - r. \quad \text{(2.7)}$$

Observe that $\text{des}(S_i([\pi])) = \text{cdes}([\pi]) = s$ if $i \not\in cB_4([\pi])$. Hence, by Theorem 1.1, we obtain

$$\sum_{i \in cB_4([\pi])} \sum_{\alpha' \in S(\hat{\alpha}', S_i([\pi]))} q^{\text{maj}(\hat{\alpha}') + k}$$

$$= \sum_{i \in cB_4([\pi])} \left[ m - r + s \right] \left[ n - s + r - 1 \right] q^{(k)q(\text{maj}(\hat{\sigma}') + \text{maj}(S_i([\pi])) + (k-s-1)(k-r)+k}$$

$$\text{(2.7)}$$

$$= \left[ m - r + s \right] \left[ n - s + r - 1 \right] q^{(k-s)(k-r)+\text{maj}([\sigma]) + q^{\text{maj}(S_i([\pi]))}}. \quad \text{(2.8)}$$

Since $\text{des}(S_i([\pi])) = \text{cdes}([\pi]) - 1 = s - 1$ when $i \in cB_4([\pi])$, it follows from Theorem 1.1 that

$$\sum_{i \in cB_4([\pi])} \sum_{\alpha' \in S(\hat{\alpha}', S_i([\pi]))} q^{\text{maj}(\hat{\alpha}') + k}$$

$$= \sum_{i \in cB_4([\pi])} \left[ m - r + s - 1 \right] \left[ n - s + r \right] q^{(k)q(\text{maj}(\hat{\sigma}') + \text{maj}(S_i([\pi])) + (k-s)(k-r)+k}$$

$$\text{(2.7)}$$

$$= \left[ m - r + s - 1 \right] \left[ n - s + r \right] q^{(k-s+1)(k-r)+\text{maj}([\sigma]) + q^{\text{maj}(S_i([\pi]))}}. \quad \text{(2.9)}$$

Substituting (2.8) and (2.9) into (2.6), we obtain (1.9). This completes the proof.

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**References**


