



Fractional differential operators, fractional Sobolev spaces and fractional variation on homogeneous Carnot groups

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Abstract

We establish a novel definition of the fractional gradient and divergence of order $\alpha \in (0, 1)$ through the use of Riesz potential on homogeneous Carnot groups. We introduce and investigate the distributional fractional Sobolev space and the space of fractional BV functions in this context. Additionally, we provide a definition of fractional Caccioppoli sets on homogeneous Carnot groups and demonstrate their blow-up property, using similar methods as outlined in [13].

Keywords Homogeneous Carnot groups · Fractional gradient · Fractional divergence · Fractional Sobolev spaces · Fractional BV functions · Fractional Caccioppoli sets

Mathematics Subject Classification 43A80 · 26A33 · 26B30 · 46E36

1 Introduction

Fractional differential operators, a topic of significant interest in the field of PDEs, have been widely investigated by numerous authors. M. Šilhavý's recent work [46] examines the notion of the fractional gradient ∇^α and the fractional divergence $\operatorname{div}^\alpha$ on $\mathbb{R}^n$, where $\alpha \in \mathbb{R}$ is the order of the operators. If $\alpha \in (0, 1)$, ∇^α and $\operatorname{div}^\alpha$ are

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explicitly given by

$$\nabla^\alpha f(x) := c_{n,\alpha} \lim_{\varepsilon \rightarrow 0} \int_{\{|x-y| \geq \varepsilon\}} \frac{(x-y)f(y)}{|x-y|^{n+\alpha+1}} dy, \quad \forall f \in C_c^\infty(\mathbb{R}^n), \quad (1.1)$$

and

$$\operatorname{div}^\alpha \varphi(x) := c_{n,\alpha} \lim_{\varepsilon \rightarrow 0} \int_{\{|x-y| \geq \varepsilon\}} \frac{(x-y) \cdot \varphi(y)}{|x-y|^{n+\alpha+1}} dy, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \quad (1.2)$$

where $|\cdot|$ is the Euclidean norm and $c_{n,\alpha} > 0$ is a normalized constant. As shown in [46], the fractional operators defined above should be considered as intrinsic objects. Indeed, Theorems 2.2 and 2.4 of [46] demonstrate that, under some natural requirements such as translation and rotation invariance, α -homogeneity and certain weak continuity, the operators (1.1) and (1.2) are uniquely determined, up to a multiplicative constant. For more historical background on fractional differential operators, see [45, Section 1] and references therein. The fractional operators (1.1) and (1.2) have a wide range of applications in analysis. One notable application is the definition of fractional Sobolev spaces on $\mathbb{R}^n$. There are several different definitions of these spaces in the literature (see [18] for a comprehensive overview). In [44], the authors introduced a definition of fractional Sobolev spaces in the distributional sense using these operators, denoted by $S_0^{\alpha,p}(\mathbb{R}^n)$ for $0 < \alpha < 1$ and $1 \leq p < \infty$ (as described in Section 3.5 below). They also established the relation

$$S_0^{\alpha,p}(\mathbb{R}^n) = L_\alpha^p(\mathbb{R}^n), \quad \forall \alpha \in (0, 1), \quad p \in (1, \infty),$$

where $L_\alpha^p(\mathbb{R}^n)$ is the fractional Sobolev space defined through the use of Riesz potential (or equivalently, Bessel potential). For the case $p = 2$, [44, Theorem 2.2] states that

$$S_0^{\alpha,2}(\mathbb{R}^n) = W^{\alpha,2}(\mathbb{R}^n), \quad \forall \alpha \in (0, 1),$$

where $W^{\alpha,2}(\mathbb{R}^n)$ is the fractional Gagliardo-Sobolev space (see Section 2.3 below). Further properties of $S_0^{\alpha,p}(\mathbb{R}^n)$ could be found in [13, Section 3].

The fractional gradient and divergence have also been applied to define the fractional variation of functions, as shown in the Euclidean case in [13]. This development is based on the duality: for any $f \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi dx = - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f dx.$$

Then the authors of [13] introduced the space $BV^\alpha(\mathbb{R}^n)$, defined as follows:

$$BV^\alpha(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n); \|D^\alpha f\|(\mathbb{R}^n) < \infty\},$$

where

$$\|D^\alpha f\|(\mathbb{R}^n) := \sup \left\{ \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx; \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \sup_{x \in \mathbb{R}^n} |\varphi(x)| \leq 1 \right\}.$$

This space could be regarded as a fractional analogue of $BV(\mathbb{R}^n)$. Similarly to the relation $W^{1,1}(\mathbb{R}^n) \subset BV(\mathbb{R}^n)$, it can be shown that

$$W^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n).$$

In [13, Sections 4–5], the fractional variation is used as a major tool to study the geometric measure theoretic properties of sets with finite fractional perimeter, and the existence of blow-ups for fractional Caccioppoli sets is proved. For some subsequent developments in this area, we refer the reader to [8, 9, 11, 14–17, 36, 47].

Homogeneous Carnot groups, which generalize classical Euclidean spaces, have attracted considerable attention in recent decades due to their rich structures and the potential for extending objects and results of classical analysis. Researchers have focused on the study of Sobolev spaces, functions with bounded variation and sets of finite perimeter in the context of homogeneous Carnot groups, as seen in works including [19, 23, 25, 38, 39, 43, 49] (just to mention a few). The present paper is devoted to the construction of fractional differential operators and the development of the theory of fractional Sobolev spaces and fractional variation within the framework of homogeneous Carnot groups.

Inspired by the work [46], we propose the following formal definitions for the fractional (horizontal) gradient and divergence on a homogeneous Carnot group $\mathbb{G}$:

$$\nabla_{\mathbb{G}}^\alpha = \nabla_{\mathbb{G}} \circ I_{1-\alpha} \quad \text{and} \quad \operatorname{div}_{\mathbb{G}}^\alpha = I_{1-\alpha} \circ \operatorname{div}_{\mathbb{G}} \quad \alpha \in (0, 1),$$

where $\nabla_{\mathbb{G}}$ and $\operatorname{div}_{\mathbb{G}}$ denote the horizontal gradient and divergence on $\mathbb{G}$, respectively, and $I_{1-\alpha}$ is the Riesz potential (as explained below in Section 2.2). In contrast to the classical Euclidean case, the operators $\nabla_{\mathbb{G}}$ and $\operatorname{div}_{\mathbb{G}}$ do not generally commute with the sub-Laplacian $\mathcal{L}$ or the Riesz potential $I_{1-\alpha}$ on $\mathbb{G}$. To address this issue, we utilize the theory of the Riesz potential and the heat kernel on $\mathbb{G}$ and then derive the principal value integral expressions of $\nabla_{\mathbb{G}}^\alpha$ and $\operatorname{div}_{\mathbb{G}}^\alpha$ as follows

$$\nabla_{\mathbb{G}}^\alpha f(x) = \int_{\mathbb{G}} (f(y) - f(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y), \quad \forall f \in C_c^\infty(\mathbb{G}),$$

and

$$\operatorname{div}_{\mathbb{G}}^\alpha \varphi(x) = \int_{\mathbb{G}} (\phi(y) - \phi(x)) \cdot \widetilde{\nabla}_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y), \quad \forall \varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G}),$$

where $R_{1-\alpha}$ is the convolution kernel of $I_{1-\alpha}$. These expressions give rise to (1.1) and (1.2) in the case of $\mathbb{R}^n$. In the subsequent sections of this work, we extend the study of fractional Sobolev spaces and fractional variation on $\mathbb{G}$ following the approach

outlined in [13]. The main results of this paper could be divided into three aspects. In the first aspect, Sections 3.1–3.4 deal with definitions and properties of fractional differential operators on $\mathbb{G}$, and the results are as follows:

- For any $f \in C_c^\infty(\mathbb{G})$ and $\phi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$, the paths $(\nabla_{\mathbb{G}}^\alpha f, 0 \leq \alpha \leq 1)$ and $(\operatorname{div}_{\mathbb{G}}^\alpha \phi, 0 \leq \alpha \leq 1)$ are continuous in $L^p(\mathbb{G}; H\mathbb{G})$ and $L^p(\mathbb{G})$ respectively, where $\frac{Q}{Q-1} < p \leq \infty$;
- $\nabla_{\mathbb{G}}^\alpha$ and $\operatorname{div}_{\mathbb{G}}^\alpha$ are invariant under left-translations and α -homogeneous;
- Some equivalent characterizations of $\nabla_{\mathbb{G}}^\alpha$ and $\operatorname{div}_{\mathbb{G}}^\alpha$ as well as their extensions to wider class of functions are developed;
- $\nabla_{\mathbb{G}}^\alpha$ and $\operatorname{div}_{\mathbb{G}}^\alpha$ share duality relation and the Leibniz's rules.

It is worth noting that the uniqueness of the fractional gradient and divergence is not discussed in this paper. Afterwards, in the second aspect, Section 3.5 is devoted to distributional fractional Sobolev space $S^{\alpha,p}(\mathbb{G})$. The main results are as follows:

- $C_c^\infty(\mathbb{G})$ is dense in $S^{\alpha,p}(\mathbb{G})$ for $1 \leq p < \frac{Q}{1-\alpha}$;
- $S^{\alpha,p}(\mathbb{G})$ coincides with the fractional Sobolev space $L_\alpha^p(\mathbb{G})$ for $1 < p < \frac{Q}{1-\alpha}$.

The final aspect, which consists of Sections 4 and 5, focuses on the space $BV^\alpha(\mathbb{G})$ and sets with finite Caccioppoli α -perimeter. The results on $BV^\alpha(\mathbb{G})$, which are presented in Section 4, are as follows:

- Both $C_c^\infty(\mathbb{G})$ and $C^\infty(\mathbb{G}) \cap BV^\alpha(\mathbb{G})$ are dense in $BV^\alpha(\mathbb{G})$;
- The Gagliardo-Nirenberg-Sobolev inequality for functions in $BV^\alpha(\mathbb{G})$:¹

$$\|f\|_{L^{\frac{Q}{Q-\alpha}}(\mathbb{G})} \lesssim \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G});$$

- The compactness theorem: every bounded sequence in $BV^\alpha(\mathbb{G})$ has a subsequence that converges in $L_{\text{loc}}^1(\mathbb{G})$;
- For $0 < \alpha < \beta \leq 1$ $BV^\beta(\mathbb{G}) \subset W^{\alpha,1}(\mathbb{G}) \subset L_\alpha^1(\mathbb{G})$ and $W^{\alpha,1}(\mathbb{G}) \subset S^{\alpha,1}(\mathbb{G}) \subsetneq BV^\alpha(\mathbb{G})$ with continuous embeddings.

As for any set $E \subset \mathbb{G}$ with finite Caccioppoli α -perimeter, whose fractional reduced boundary $\mathcal{F}_{\mathbb{G}}^\alpha E$ is defined with respect to a smooth homogeneous distance d_s , the main results, which are presented in Section 5, are as follows:

- For any open set $U \subset \mathbb{G}$, $\|D_{\mathbb{G}}^\alpha \chi_E\|(U) \lesssim P_\alpha(E; U)$ under the condition that $P_\alpha(E; U) < \infty$;
- We establish an integration by parts formula on d_s -balls with centers on $\mathcal{F}_{\mathbb{G}}^\alpha E$;
- We prove the existence of the blow-ups for fractional Caccioppoli sets.

2 Preliminaries

2.1 Homogeneous Carnot groups

Homogeneous Carnot groups are a class of spaces with rich structures and have been involved in several mathematical areas. This section provides a brief exposition of

¹ Throughout this study, the notation $a \lesssim b$ means that there exists a universal constant $C > 0$ such that $a \leq Cb$. And we write $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$ hold.

homogeneous Carnot groups and related objects (see e.g. [10, 41] for more backgrounds).

Definition 1 A Lie group on $\mathbb{R}^n$, $\mathbb{G} = (\mathbb{R}^n, \circ)$, is called a homogeneous Carnot group (or a homogeneous stratified group) if it satisfies the following conditions:

- (1) $\mathbb{R}^n$ can be decomposed as $\mathbb{R}^n = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \dots \times \mathbb{R}^{m_k}$ ($m_i \in \mathbb{N}$, $i = 1, \dots, k$), and the dilation $\delta_r : \mathbb{R}^n \rightarrow \mathbb{R}^n$, which is defined by

$$\delta_r(x) = (rx^{(1)}, r^2x^{(2)}, \dots, r^kx^{(k)}), \quad \forall x^{(i)} \in \mathbb{R}^{m_i}, i = 1, \dots, k,$$

is an automorphism of the group $\mathbb{G}$ for all $r > 0$.

- (2) We write $m := m_1$ for sake of simplicity. Let $X_1, X_2, \dots, X_m$ be the left-invariant vector fields on $\mathbb{G}$ satisfying $X_j(0) = \frac{\partial}{\partial x_j}|_0$ for $j = 1, 2, \dots, m$. Then

$$\text{rank}(\text{Lie}\{X_1, X_2, \dots, X_m\}(x)) = n, \quad \forall x \in \mathbb{R}^n, \quad (2.1)$$

i.e. the iterated commutators of $X_1, X_2, \dots, X_m$ span the whole Lie algebra of $\mathbb{G}$.

Remark 1 The number $Q := \sum_{i=1}^k i m_i$ is called the homogeneous dimension of $\mathbb{G}$. We can always assume $Q \geq 3$. Since the case $Q \leq 2$ corresponds to the classical Euclidean spaces, which have already been discussed in [13].

For any $z \in \mathbb{G}$, we denote by τ_z the left translation by z on $\mathbb{G}$ (i.e. $\tau_z(x) = z \circ x$). It should be noted that for $j = 1, \dots, m$, the operator X_j satisfies

$$X_j(f \circ \tau_z) = (X_j f) \circ \tau_z \quad \text{and} \quad X_j(f \circ \delta_r) = r(X_j f) \circ \delta_r, \quad \forall f \in C^1(\mathbb{G}) \quad (2.2)$$

for any $z \in \mathbb{G}$ and $r > 0$, where the symbol $\circ$ denotes the composition of mappings. For later purposes, we introduce the corresponding right-invariant vector fields of $X_1, \dots, X_m$. Let $\tilde{X}_j$ be the right-invariant vector field on $\mathbb{G}$ satisfying $\tilde{X}_j(0) = \frac{\partial}{\partial x_j}|_0$. For any $f \in C^1(\mathbb{G})$, we set $\tilde{f}(x) := f(x^{-1})$. Then $\tilde{X}_j$ and X_j are related by (see e.g. [24, Section 1.3])

$$\tilde{X}_j f(x) = -X_j \tilde{f}(x^{-1}), \quad \forall x \in \mathbb{G}. \quad (2.3)$$

In what follows, we shall use the sub-Riemannian structure of $\mathbb{G}$. The horizontal bundle $H\mathbb{G}$ is the subbundle of the tangent bundle $T\mathbb{G}$ spanned by the vector fields $X_1, \dots, X_m$, i.e. the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span}\{X_1(x), X_2(x), \dots, X_m(x)\}, \quad \forall x \in \mathbb{G}.$$

Then for any $x \in \mathbb{G}$, we can endow $H\mathbb{G}_x$ with a norm $|\cdot|_x$ and a scalar product $\langle \cdot, \cdot \rangle_x$ making $\{X_1(x), X_2(x), \dots, X_m(x)\}$ an orthonormal basis.² Now we are in a

² In the sequel, if the identification of the base point is clear or not important, we often omit x in subscripts.

position to define the Carnot-Carathéodory distance on $\mathbb{G}$. Let $\mathcal{C}$ be the set of absolutely continuous horizontal curves $\gamma : [0, 1] \rightarrow \mathbb{G}$ (i.e. for almost all $t \in [0, 1]$, $\dot{\gamma}(t) = \sum_{j=1}^m a_j(t) X_j(\gamma(t)) \in H\mathbb{G}_{\gamma(t)}$, where $a_1, \dots, a_m$ are measurable functions). Then for any $x, y \in \mathbb{G}$, the Carnot-Carathéodory distance between them is defined by

$$d(x, y) := \inf \left\{ \int_0^1 |\dot{\gamma}(t)|_{\gamma(t)} dt ; \gamma \in \mathcal{C} \text{ with } \gamma(0) = x, \gamma(1) = y \right\}.$$

By Carathéodory-Chow-Rashevsky Theorem (see e.g. [10, Theorem 9.1.3]), d is a distance function and induces the topology of $\mathbb{G}$. Moreover, d is left-invariant and 1-homogeneous, i.e.

$$d(\tau_z(x), \tau_z(y)) = d(x, y) \quad \text{and} \quad d(\delta_r(x), \delta_r(y)) = rd(x, y)$$

hold for any $x, y, z \in \mathbb{G}$ and $r > 0$. Another useful property of the Carnot-Carathéodory distance is the following one (see e.g. [4, Section 4]): for any $x, y \in \mathbb{G}$, there exists a horizontal curve $\gamma : [0, d(x, y)] \rightarrow \mathbb{G}$ jointing them, which satisfies $\gamma(0) = x, \gamma(d(x, y)) = y$ and

$$d(\gamma(a), \gamma(b)) = |a - b|, \quad \forall a, b \in [0, d(x, y)]. \quad (2.4)$$

In the sequel, we set $d(x) = d(0, x)$, which satisfies $d(y^{-1} \circ x) = d(x, y)$ and defines a homogeneous norm on $\mathbb{G}$. We denote by $B(x, R)$ the open ball $\{y \in \mathbb{G}; d(x, y) < R\}$ and $\partial B(x, R)$ the sphere $\{y \in \mathbb{G}; d(x, y) = R\}$. Besides, the Lipschitz functions considered in this work are all defined with respect to d .

However, the sphere associated with the Carnot-Carathéodory distance is non-smooth in general, due to the existence of abnormal geodesics (see [1] for more details), which poses a challenge for developing calculus on metric balls. In order to avoid this issue, later on we will also use another homogeneous distance on $\mathbb{G}$, denoted by d_s , whose unit sphere is smooth. The existence of such a homogeneous distance has been proved in [34], and [34, Theorem 2] allows us to construct a homogeneous distance whose unit ball coincides with the Euclidean ball with a sufficiently small radius. We note that d and d_s are equivalent, i.e., there exist $C_1, C_2 > 0$, such that

$$C_1 d(x, y) \leq d_s(x, y) \leq C_2 d(x, y), \quad \forall x, y \in \mathbb{G}.$$

As above we set $d_s(x) = d_s(0, x)$, which is smooth on $\mathbb{G} \setminus \{0\}$. Let us denote by $B_s(x, R)$ the open ball $\{y \in \mathbb{G}; d_s(x, y) < R\}$ and $\partial B_s(x, R)$ the sphere $\{y \in \mathbb{G}; d_s(x, y) = R\}$.

The Haar measure for $\mathbb{G}$ is the Lebesgue measure on $\mathbb{R}^n$, denoted by μ . That is to say, for any μ -measurable set $A \subset \mathbb{G}$, we have

$$\mu(x \circ A) = \mu(A \circ x) = \mu(A), \quad \forall x \in \mathbb{G}.$$

By changing variable, we can also see that

$$\mu(\delta_r(A)) = r^Q \mu(A), \quad \forall r > 0.$$

Note that all the L^p spaces we consider are defined with respect to μ .

The $H\mathbb{G}$ -valued operator

$$\nabla_{\mathbb{G}} f := \sum_{j=1}^m (X_j f) X_j, \quad \forall f \in C^1(\mathbb{G})$$

is called the horizontal gradient. By definition of the norm $|\cdot|$ on $H\mathbb{G}$,

$$|\nabla_{\mathbb{G}} f(x)| = \left(\sum_{j=1}^m (X_j f)^2(x) \right)^{\frac{1}{2}}.$$

And the real-valued operator

$$\operatorname{div}_{\mathbb{G}} \phi := \sum_{j=1}^m X_j \phi_j, \quad \forall \phi = \sum_{j=1}^m \phi_j X_j \in C^1(\mathbb{G}; H\mathbb{G})$$

is called the horizontal divergence. Moreover, a Rademacher-type Theorem holds: for any $f \in \operatorname{Lip}(\mathbb{G})$ and $\varphi \in \operatorname{Lip}(\mathbb{G}; H\mathbb{G})$, $\nabla_{\mathbb{G}} f \in L^\infty(\mathbb{G}; H\mathbb{G})$ and $\operatorname{div}_{\mathbb{G}} \varphi \in L^\infty(\mathbb{G})$ respectively (see [3, 6] and the references therein). By the integration by parts formula, $\nabla_{\mathbb{G}}$ and $\operatorname{div}_{\mathbb{G}}$ share the following duality: for any $f \in \operatorname{Lip}(\mathbb{G})$ and $\phi \in \operatorname{Lip}(\mathbb{G}; H\mathbb{G})$, if either of them has compact support, then

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}} \phi \, d\mu = - \int_{\mathbb{G}} \langle \phi, \nabla_{\mathbb{G}} f \rangle \, d\mu. \quad (2.5)$$

Finally, for later convenience, we need the following $T\mathbb{G}$ -valued operator

$$\tilde{\nabla}_{\mathbb{G}} f := \sum_{j=1}^m (\tilde{X}_j f) \tilde{X}_j, \quad \forall f \in C^1(\mathbb{G}).$$

For every $f \in C_c^\infty(\mathbb{G})$ and $g \in L^1_{\operatorname{loc}}(\mathbb{G})$, the convolution $f * g$ is defined by

$$(f * g)(x) := \int_{\mathbb{G}} f(y) g(y^{-1} \circ x) \, d\mu(y), \quad \forall x \in \mathbb{G}.$$

By Young's convolution inequality, $f * g$ is well-defined for any $f \in L^p(\mathbb{G})$ and $g \in L^q(\mathbb{G})$, where $1 \leq p, q \leq \infty$ and $p^{-1} + q^{-1} \geq 1$ (see also [26, Proposition 1.10]).

As for the interaction of convolution and differential operators, for $j = 1, \dots, m$, we have

$$X_j(f * g) = f * (X_j g), \quad \widetilde{X}_j(f * g) = (\widetilde{X}_j f) * g, \quad (X_j f) * g = f * (\widetilde{X}_j g), \quad (2.6)$$

where f and g are smooth functions with suitable decay at infinity (see [27, Chapter 1]). And by a standard approximation argument, (2.6) remains valid if $f, g \in \text{Lip}(\mathbb{G})$ and either of them has compact support.

Generally, the convolution on $\mathbb{G}$ is not commutative. However, we have the following simple fact: given μ -measurable functions $f, g, h : \mathbb{G} \rightarrow \mathbb{R}$, suppose that two of them belong to $\text{Lip}_c(\mathbb{G})$ and the other one is locally integrable, if additionally the function h satisfies $h(x) = h(x^{-1})$ for any $x \in \mathbb{G}$, then using Fubini's Theorem we have

$$\int_{\mathbb{G}} f(h * g) d\mu = \int_{\mathbb{G}} g(h * f) d\mu \quad \text{and} \quad \int_{\mathbb{G}} f(g * h) d\mu = \int_{\mathbb{G}} g(f * h) d\mu. \quad (2.7)$$

2.2 Heat kernel, Riesz potential and Riesz transform

This subsection collects some basic properties of the Riesz potential on $\mathbb{G}$, which is necessary for our definitions of fractional differential operators. The second-order differential operator $\mathcal{L} := -\sum_{j=1}^m X_j^2$ is called the sub-Laplacian on $\mathbb{G}$. The semigroup of operators $P_t := e^{-t\mathcal{L}}$ ($t \geq 0$) is called the heat semigroup on $\mathbb{G}$, which is contractive on $L^p(\mathbb{G})$ for every $1 \leq p \leq \infty$. Then for all $t > 0$, the heat kernel p_t is defined by

$$P_t f(x) = (f * p_t)(x) = \int_{\mathbb{G}} f(y) p_t(y^{-1} \circ x) d\mu(y), \quad \forall f \in C_c^\infty(\mathbb{G}).$$

The following proposition lists some properties of p_t which will be used throughout this work. We refer the reader to [20, 27, 48] for more details.

Proposition 1 *For every $x \in \mathbb{G}$, $t > 0$ and $r > 0$,*

- (1) $p_t(x) = p_t(x^{-1})$, thus $\widetilde{X}_j p_t(x) = -X_j p_t(x^{-1})$ for $j = 1, \dots, m$;
- (2) $p_{r^2 t}(\delta_r(x)) = r^{-Q} p_t(x)$, $X_j p_{r^2 t}(\delta_r(x)) = r^{-Q-1} X_j p_t(x)$;
- (3) *There exists $c \in (0, 1)$ such that*

$$c t^{-\frac{Q}{2}} e^{-\frac{d^2(x)}{ct}} \leq p_t(x) \leq c^{-1} t^{-\frac{Q}{2}} e^{-\frac{cd^2(x)}{t}},$$

and for $1 \leq i, j \leq m$,

$$|X_j p_t(x)| + |\widetilde{X}_j p_t(x)| \lesssim t^{-\frac{Q+1}{2}} e^{-\frac{cd^2(x)}{t}},$$

$$|X_i X_j p_t(x)| + |X_i \tilde{X}_j p_t(x)| \lesssim t^{-\frac{Q+2}{2}} e^{-\frac{cd^2(x)}{t}}.$$

By means of p_t , the Riesz potential of order $\alpha \in (0, Q)$ is defined through

$$I_\alpha f(x) := (f * R_\alpha)(x) = \int_{\mathbb{G}} f(y) R_\alpha(y^{-1} \circ x) d\mu(y), \quad \forall f \in C_c^\infty(\mathbb{G}), \quad (2.8)$$

where

$$R_\alpha(x) := \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1} p_t(x) dt.$$

It follows from spectral analysis that $I_\alpha = \mathcal{L}^{-\frac{\alpha}{2}}$. By simple computation, the following properties of R_α are easily deduced from Proposition 1.

Proposition 2 For every $x \in \mathbb{G}$ and $r > 0$,

- (1) $R_\alpha(x) = R_\alpha(x^{-1})$, thus $\tilde{X}_j R_\alpha(x) = -X_j R_\alpha(x^{-1})$ for $j = 1, \dots, m$;
- (2) $R_\alpha(\delta_r(x)) = r^{\alpha-Q} R_\alpha(x)$, $X_j R_\alpha(\delta_r(x)) = r^{\alpha-Q-1} X_j R_\alpha(x)$;
- (3) We have

$$|R_\alpha(x)| \sim d^{\alpha-Q}(x), \quad (2.9)$$

and for $1 \leq i, j \leq m$,

$$\begin{aligned} |X_j R_\alpha(x)| + |\tilde{X}_j R_\alpha(x)| &\lesssim d^{\alpha-Q-1}(x), \\ |X_i X_j R_\alpha(x)| + |X_i \tilde{X}_j R_\alpha(x)| &\lesssim d^{\alpha-Q-2}(x). \end{aligned} \quad (2.10)$$

The relation (2.9) implies that $R_\alpha \in L^1_{\text{loc}}(\mathbb{G})$, which guarantees (2.8) makes sense. Moreover, it follows from [26, Proposition 1.10, 1.11] that I_α has the following boundedness properties:

$$\|I_\alpha f\|_{L^{\frac{Qp}{Q-\alpha p}}(\mathbb{G})} \lesssim \|f\|_{L^p(\mathbb{G})}, \quad \forall f \in L^p(\mathbb{G}), \quad (2.11)$$

where $1 < p < Q/\alpha$, and

$$\mu(\{x; |I_\alpha f(x)| > \lambda\})^{\frac{Q-\alpha}{Q}} \lesssim \lambda^{-1} \|f\|_{L^1(\mathbb{G})}, \quad \forall f \in L^1(\mathbb{G}), \quad \lambda > 0. \quad (2.12)$$

Combining this boundedness with Fubini's Theorem, we obtain a generalization of (2.7) for the case $h = R_\alpha$.

Lemma 1 Let $1 \leq p < Q/\alpha$. Then for every $f \in L^p(\mathbb{G})$ and $g \in \text{Lip}_c(\mathbb{G})$,

$$\int_{\mathbb{G}} f(I_\alpha g) d\mu = \int_{\mathbb{G}} g(I_\alpha f) d\mu.$$

And combining the Kolmogorov condition (see [33, Chapter 3]) with the weak boundedness (2.12), we obtain the following simple lemma.

Lemma 2 *Let $A \subset \mathbb{G}$ be a μ -measurable set with finite measure. Then there exists $C = C(A, \alpha) > 0$ such that*

$$\|I_\alpha f\|_{L^1(A)} \leq C \|f\|_{L^1(\mathbb{G})}, \quad \forall f \in L^1(\mathbb{G}).$$

Consequently, for any $f \in L^1(\mathbb{G})$, $I_\alpha f \in L^1_{\text{loc}}(\mathbb{G})$.

To develop the principal value integral form of fractional differential operators, we need the following lemma.

Lemma 3 *Suppose that $\Phi \in L^\infty([0, +\infty))$ is identically equal to 0 outside some interval $[a, b]$, where $0 < a < b < \infty$. Then for $j = 1, \dots, m$, we have*

$$\int_{\mathbb{G}} \Phi(d(x)) X_j R_\alpha(x) d\mu(x) = 0 \quad \text{and} \quad \int_{\mathbb{G}} \Phi(d(x)) X_j(d)(x) R_\alpha(x) d\mu(x) = 0.$$

Moreover, if $0 < \alpha < 1$,

$$\text{p.v.} \int_{\mathbb{G}} X_j R_\alpha(x) d\mu(x) := \lim_{\varepsilon \rightarrow 0} \int_{\{d(x) \geq \varepsilon\}} X_j R_\alpha(x) d\mu(x) = 0.$$

And the same conclusions also hold if X_j is replaced by $\tilde{X}_j$.

Proof By Proposition 2, all integrands considered belong to $L^1(\mathbb{G})$. To prove this lemma, we need the “semicheck” map on $\mathbb{G}$, which is defined by

$$w(x) := w(x^{(1)}, x^{(2)}, \dots, x^{(k)}) = (-x^{(1)}, x^{(2)}, \dots, (-1)^k x^{(k)}), \quad \forall x \in \mathbb{G},$$

where the decomposition $x = (x^{(1)}, x^{(2)}, \dots, x^{(k)})$ comes from Definition 1. This map has the following properties (see [22, Section 3]):

- (i) $X_j(f \circ w) = -(X_j f) \circ w$, where $f \in C^1(\mathbb{G})$;
- (ii) $d(x, y) = d(w(x), w(y))$;
- (iii) $R_\alpha(x) = R_\alpha(w(x))$ and $X_j R_\alpha(x) = -X_j R_\alpha(w(x))$.

Since the function d is 1-Lipschitz, by (i)-(ii) and approximation, it follows that $X_j(d)(x) = -X_j(d)(w(x))$ holds almost everywhere. Notice that $\Phi(d(x)) = \Phi(d(w(x)))$. This lemma is easily obtained after changing the variable $y = w(x)$. $\square$

The Riesz transform on $\mathbb{G}$ is defined by $\mathcal{R} := \nabla_{\mathbb{G}} \mathcal{L}^{-\frac{1}{2}}$. And we may write

$$\mathcal{R}f = \sum_{j=1}^m (\mathcal{R}_j f) X_j, \quad \forall f \in C_c^\infty(\mathbb{G}),$$

where $\mathcal{R}_j = X_j \mathcal{L}^{-\frac{1}{2}}$. We will make use of the L^p -boundness of the Riesz transform on $\mathbb{G}$: for every $1 < p < \infty$, $\mathcal{R}$ is bounded from $L^p(\mathbb{G})$ to $L^p(\mathbb{G}; H(\mathbb{G}))$. This L^p -boundness result was established in [26].

2.3 Function spaces on homogeneous Carnot groups

The purpose of this subsection is to briefly introduce some of the main function spaces considered in this research. We begin with the fractional Sobolev spaces on $\mathbb{G}$, which have been studied extensively in the literature (see e.g. [26, 42]).

Definition 2 Let $1 \leq p < \infty$ and $0 < \alpha < 1$. The fractional Sobolev space $L_\alpha^p(\mathbb{G})$ is defined by

$$L_\alpha^p(\mathbb{G}) := \{f \in L^p(\mathbb{G}); \mathcal{L}^{\frac{\alpha}{2}} f \in L^p(\mathbb{G})\},$$

endowed with the norm

$$\|f\|_{L_\alpha^p(\mathbb{G})} := (\|f\|_{L^p(\mathbb{G})}^p + \|\mathcal{L}^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{G})}^p)^{\frac{1}{p}},$$

where $\mathcal{L}^{\frac{\alpha}{2}}$ is the fractional sub-Laplacian, which has the following expression (see [22, Theorem 3.11])

$$\mathcal{L}^{\frac{\alpha}{2}} f(x) := \lim_{\varepsilon \rightarrow 0} \int_{\{d(x,y) \geq \varepsilon\}} (f(y) - f(x)) \tilde{R}_{-\alpha}(y^{-1} \circ x) d\mu(y), \quad (2.13)$$

for every $f \in C_c^\infty(\mathbb{G})$. The kernel $\tilde{R}_{-\alpha}$ is given by

$$\tilde{R}_{-\alpha}(x) = -\frac{\alpha}{2\Gamma(-\frac{\alpha}{2})} \int_0^\infty t^{-\frac{\alpha}{2}-1} p_t(x) dt \sim d^{-\alpha-Q}(x).$$

On the other hand, it is well-known that in the classical Euclidean case there are other different kinds of the fractional Sobolev spaces, for example, the fractional Gagliardo-Sobolev spaces (see e.g. [18]). Analogously, we can define the fractional Gagliardo-Sobolev spaces on $\mathbb{G}$ (see [23]). For $1 \leq p < \infty$ and $0 < \alpha < 1$, the Gagliardo semi-norm is defined by

$$[f]_{W^{\alpha,p}(\mathbb{G})}^p := \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(x) - f(y)|^p}{d(x,y)^{Q+\alpha p}} d\mu(x) d\mu(y).$$

Definition 3 Let $1 \leq p < \infty$ and $0 < \alpha < 1$. The fractional Gagliardo-Sobolev space $W^{\alpha,p}(\mathbb{G})$ is defined by

$$W^{\alpha,p}(\mathbb{G}) := \{f \in L^p(\mathbb{G}); [f]_{W^{\alpha,p}(\mathbb{G})} < \infty\},$$

equipped with the norm

$$\|f\|_{W^{\alpha,p}(\mathbb{G})} := (\|f\|_{L^p(\mathbb{G})}^p + [f]_{W^{\alpha,p}(\mathbb{G})}^p)^{\frac{1}{p}}.$$

Remark 2 By the equivalence of homogeneous norms (see [24, Section 3.1.6]), this definition is independent of the chosen homogeneous distance.

We also set

$$w^{\alpha,p}(\mathbb{G}) := \{f \in L^1_{\text{loc}}(\mathbb{G}); [f]_{W^{\alpha,p}(\mathbb{G})} < \infty\}.$$

Then it is obvious that $\text{Lip}_c(\mathbb{G}) \subset W^{\alpha,p}(\mathbb{G}) \subset w^{\alpha,p}(\mathbb{G})$. And for $0 < \alpha < 1$ and $1 \leq p < \infty$, we denote by

$$W^{\alpha,p}(\mathbb{G}; H\mathbb{G}) := \left\{ \varphi = \sum_{j=1}^m \varphi_j X_j \in L^p(\mathbb{G}; H\mathbb{G}); [\varphi]_{W^{\alpha,p}(\mathbb{G}; H\mathbb{G})} < \infty \right\},$$

the space of all the fractional Gagliardo-Sobolev $H\mathbb{G}$ -valued functions on $\mathbb{G}$, where

$$[\varphi]_{W^{\alpha,p}(\mathbb{G}; H\mathbb{G})}^p := \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{(\sum_{j=1}^m (\varphi_j(x) - \varphi_j(y))^2)^{\frac{p}{2}}}{d(x,y)^{Q+\alpha p}} d\mu(x) d\mu(y).$$

Similar to the preceding definition, we set

$$w^{\alpha,p}(\mathbb{G}; H\mathbb{G}) = \{ \varphi \in L^1_{\text{loc}}(\mathbb{G}; H\mathbb{G}); [\varphi]_{W^{\alpha,p}(\mathbb{G}; H\mathbb{G})} < \infty \}.$$

Then $\text{Lip}_c(\mathbb{G}; H\mathbb{G}) \subset W^{\alpha,p}(\mathbb{G}; H\mathbb{G}) \subset w^{\alpha,p}(\mathbb{G}; H\mathbb{G})$.

Another crucial function space involved in this study is $BV(\mathbb{G})$. We refer the reader to [3] for more details. The total (horizontal) variation of function f is defined by

$$\|D_{\mathbb{G}} f\|(\mathbb{G}) := \sup \left\{ \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}} \varphi d\mu; \varphi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G}), \sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1 \right\}.$$

And $BV(\mathbb{G})$ is the set of all the L^1 -functions with bounded variation. $BV(\mathbb{G})$ is endowed with the norm

$$\|f\|_{BV(\mathbb{G})} := \|f\|_{L^1(\mathbb{G})} + \|D_{\mathbb{G}} f\|(\mathbb{G}).$$

For each $f \in BV(\mathbb{G})$, by Riesz Representation Theorem, we can find a non-negative Radon measure, denoted by $\|D_{\mathbb{G}} f\|$, and a Borel section of $H\mathbb{G}$, denoted by ν_f , satisfying $|\nu_f| = 1$, such that for every $\varphi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$,

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}} \varphi d\mu = - \int_{\mathbb{G}} \langle \varphi, \nu_f \rangle d\|D_{\mathbb{G}} f\|.$$

Let $\mathcal{M}(\mathbb{G}; H\mathbb{G})$ be the set all the $H\mathbb{G}$ -valued Radon measures on $\mathbb{G}$. Then for $f \in BV(\mathbb{G})$, we define $D_{\mathbb{G}} f \in \mathcal{M}(\mathbb{G}; H\mathbb{G})$ satisfying $dD_{\mathbb{G}} f = \nu_f d\|D_{\mathbb{G}} f\|$, so that for every $\varphi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$,

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}} \varphi d\mu = - \int_{\mathbb{G}} \langle \varphi, dD_{\mathbb{G}} f \rangle.$$

And for later purposes, we set

$$bv(\mathbb{G}) := \{f \in L^1_{\text{loc}}(\mathbb{G}); \|D_{\mathbb{G}} f\|(\mathbb{G}) < \infty\}.$$

3 Fractional gradient and divergence on homogeneous Carnot groups

3.1 Definition of $\nabla_{\mathbb{G}}^{\alpha}$ and $\text{div}_{\mathbb{G}}^{\alpha}$

In the spirit of [46], for $0 < \alpha < 1$, we can give the definition of α -gradient $\nabla_{\mathbb{G}}^{\alpha}$ and the α -divergence $\text{div}_{\mathbb{G}}^{\alpha}$ by letting

$$\nabla_{\mathbb{G}}^{\alpha} f := \nabla_{\mathbb{G}}(I_{1-\alpha} f)$$

and

$$\text{div}_{\mathbb{G}}^{\alpha} \varphi := I_{1-\alpha}(\text{div}_{\mathbb{G}} \varphi)$$

for each $f \in C_c^{\infty}(\mathbb{G})$ and $\varphi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$. Then applying the properties of the Riesz potential, we can establish the principal value integral expressions of $\nabla_{\mathbb{G}}^{\alpha}$ and $\text{div}_{\mathbb{G}}^{\alpha}$.

Proposition 3 *Let $0 < \alpha < 1$. Then*

$$\begin{aligned} \nabla_{\mathbb{G}}^{\alpha} f(x) &= \sum_{j=1}^m \left(\text{p.v.} \int_{\mathbb{G}} f(y) X_j R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \right) X_j \\ &:= \sum_{j=1}^m \left(\lim_{\varepsilon \rightarrow 0} \int_{\{d(x,y) \geq \varepsilon\}} f(y) X_j R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \right) X_j, \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \text{div}_{\mathbb{G}}^{\alpha} \varphi(x) &= \sum_{j=1}^m \text{p.v.} \int_{\mathbb{G}} \varphi_j(y) \tilde{X}_j R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \\ &:= \sum_{j=1}^m \lim_{\varepsilon \rightarrow 0} \int_{\{d(x,y) \geq \varepsilon\}} \varphi_j(y) \tilde{X}_j R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \end{aligned} \quad (3.2)$$

for each $f \in C_c^{\infty}(\mathbb{G})$ and $\phi = \sum_{j=1}^m \varphi_j X_j \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$. With a slight abuse of notations, (3.1) and (3.2) could be expressed in the following more condensed form

$$\nabla_{\mathbb{G}}^{\alpha} f(x) = \text{p.v.} \int_{\mathbb{G}} f(y) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y)$$

and

$$\operatorname{div}_{\mathbb{G}}^{\alpha} \varphi(x) = \text{p.v.} \int_{\mathbb{G}} \varphi(y) \cdot \widetilde{\nabla}_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y).$$

Here and below, for any $u = \sum_{j=1}^m u_j X_j \in H\mathbb{G}$ and $v = \sum_{j=1}^m v_j \widetilde{X}_j \in T\mathbb{G}$, the notation $u \cdot v$ denotes $\sum_{j=1}^m u_j v_j$.

Proof We need to show that for each $j = 1, \dots, m$,

$$X_j(I_{1-\alpha} f)(x) = \lim_{\varepsilon \rightarrow 0} \int_{\{d(x,y) \geq \varepsilon\}} f(y) X_j R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y) \quad (3.3)$$

and

$$I_{1-\alpha}(X_j \phi_j)(x) = \lim_{\varepsilon \rightarrow 0} \int_{\{d(x,y) \geq \varepsilon\}} \phi_j(y) \widetilde{X}_j R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y) \quad (3.4)$$

For this purpose, we choose a function $\Psi \in C_c^1([0, +\infty))$ such that $0 \leq \Psi \leq 1$ and

$$\Psi(r) = \begin{cases} 1 & \text{if } r \in [0, \frac{1}{2}], \\ 0 & \text{if } r \in [1, +\infty). \end{cases}$$

For $\varepsilon > 0$, we set $\psi_{\varepsilon}(x) := \Psi(\varepsilon^{-1}d(x))$, which is a radial function on $\mathbb{G}$. Then we have

$$\begin{aligned} I_{1-\alpha} f(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{G}} f(y) (1 - \psi_{\varepsilon}(y^{-1} \circ x)) R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y) \\ &= \lim_{\varepsilon \rightarrow 0} (f * (1 - \psi_{\varepsilon}) R_{1-\alpha})(x). \end{aligned}$$

Since $R_{1-\alpha} \in L_{\text{loc}}^1(\mathbb{G})$, it follows from the dominated convergence Theorem that

$$X_j(I_{1-\alpha} f) = \lim_{\varepsilon \rightarrow 0} X_j(f * (1 - \psi_{\varepsilon}) R_{1-\alpha}).$$

Then by the chain rule and (2.6), we obtain

$$\begin{aligned} &X_j(f * (1 - \psi_{\varepsilon}) R_{1-\alpha})(x) \\ &= f * X_j((1 - \psi_{\varepsilon}) R_{1-\alpha})(x) \\ &= \int_{\mathbb{G}} f(y) (1 - \psi_{\varepsilon}(y^{-1} \circ x)) X_j R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y) \\ &\quad - \varepsilon^{-1} \int_{\mathbb{G}} f(y) \Psi'(\varepsilon^{-1}d(x, y)) X_j d(y^{-1} \circ x) R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y). \end{aligned}$$

Therefore for each $\varepsilon > 0$,

$$\left| X_j(f * (1 - \psi_\varepsilon)R_{1-\alpha})(x) - \int_{\{d(x,y) \geq \varepsilon\}} f(y) X_j R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \right| \leq \text{I} + \text{II},$$

where

$$\begin{aligned} \text{I} &= \left| \int_{\{\varepsilon/2 \leq d(z) < \varepsilon\}} f(x \circ z^{-1}) (1 - \psi_\varepsilon(z)) X_j R_{1-\alpha}(z) d\mu(z) \right|, \\ \text{II} &= \varepsilon^{-1} \left| \int_{\{\varepsilon/2 \leq d(z) < \varepsilon\}} f(x \circ z^{-1}) \Psi'(\varepsilon^{-1}d(z)) X_j d(z) R_{1-\alpha}(z) d\mu(z) \right|. \end{aligned}$$

Using Proposition 2 and Lemma 3,

$$\begin{aligned} \text{I} &= \left| \int_{\{\varepsilon/2 \leq d(z) < \varepsilon\}} (f(x \circ z^{-1}) - f(x)) (1 - \psi_\varepsilon(z)) X_j R_{1-\alpha}(z) d\mu(z) \right| \\ &\lesssim \|f\|_{\text{Lip}} \varepsilon \cdot \varepsilon^{-\alpha-Q} \cdot \mu(B(0, \varepsilon)) = O(\varepsilon^{1-\alpha}) \end{aligned}$$

and

$$\begin{aligned} \text{II} &= \varepsilon^{-1} \left| \int_{\{\varepsilon/2 \leq d(z) < \varepsilon\}} (f(x \circ z^{-1}) - f(x)) \Psi'(\varepsilon^{-1}d(z)) X_j d(z) R_{1-\alpha}(z) d\mu(z) \right| \\ &\lesssim \varepsilon^{-1} \cdot \|f\|_{\text{Lip}} \varepsilon \cdot \varepsilon^{1-\alpha-Q} \cdot \mu(B(0, \varepsilon)) = O(\varepsilon^{1-\alpha}), \end{aligned}$$

where

$$\|f\|_{\text{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty.$$

Combining the above estimates yields

$$\begin{aligned} X_j(f * (1 - \psi_\varepsilon)R_{1-\alpha})(x) &= \int_{\{d(x,y) \geq \varepsilon\}} f(y) X_j R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \\ &\quad + O(\varepsilon^{1-\alpha}). \end{aligned}$$

We end the proof of (3.3) by letting $\varepsilon \rightarrow 0$. (3.4) can be proved by the same argument after noticing that

$$I_{1-\alpha}(X_j \phi_j) = \lim_{\varepsilon \rightarrow 0} X_j \phi_j * (1 - \psi_\varepsilon)R_{1-\alpha} = \lim_{\varepsilon \rightarrow 0} \phi_j * \tilde{X}_j((1 - \psi_\varepsilon)R_{1-\alpha}).$$

□

Combining Lemma 3 and Proposition 3, the following corollary is immediate.

Corollary 1 *Let $0 < \alpha < 1$. Then for each $f \in C_c^\infty(\mathbb{G})$ and $\phi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,*

$$\nabla_{\mathbb{G}}^\alpha f(x) = \int_{\mathbb{G}} (f(y) - f(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \quad (3.5)$$

and

$$\operatorname{div}_{\mathbb{G}}^\alpha \varphi(x) = \int_{\mathbb{G}} (\phi(y) - \phi(x)) \cdot \widetilde{\nabla}_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y). \quad (3.6)$$

Remark 3 (1) Proposition 3 and Corollary 1 are still valid if we only assume $f \in \operatorname{Lip}_c(\mathbb{G})$ and $\varphi \in \operatorname{Lip}_c(\mathbb{G}; H\mathbb{G})$. In addition from the proof of Proposition 3, it is known that for each $f \in \operatorname{Lip}_c(\mathbb{G})$,

$$I_{1-\alpha}(X_j f)(x) = \int_{\mathbb{G}} (f(y) - f(x)) \widetilde{X}_j R_{1-\alpha}(y^{-1} \circ x) d\mu(y), \quad j = 1, \dots, m. \quad (3.7)$$

(2) If we define $\nabla_{\mathbb{G}}^0 = \nabla_{\mathbb{G}} \mathcal{L}^{-\frac{1}{2}} = \mathcal{R}$, Proposition 3 and Corollary 1 still hold in the case $\alpha = 0$.

3.2 Properties of $\nabla_{\mathbb{G}}^\alpha$ and $\operatorname{div}_{\mathbb{G}}^\alpha$

For each $f \in C_c^\infty(\mathbb{G})$, it is obvious that $\nabla_{\mathbb{G}}^\alpha f$ and $\operatorname{div}_{\mathbb{G}}^\alpha f$ are continuous on $(0, 1)$ as functions of α (for a rigorous proof, one may simply use Lemma 4 below). As for their asymptotic behavior as $\alpha \rightarrow 1^-$, not surprisingly we have the following continuity property. This result suggests that, as the order α increases, the operators $\nabla_{\mathbb{G}}^\alpha$ and $\operatorname{div}_{\mathbb{G}}^\alpha$ become increasingly “local”.

Theorem 1 (Strong continuity in α) *For each $f \in C_c^\infty(\mathbb{G})$ and $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,*

$$\lim_{\alpha \rightarrow 1^-} \nabla_{\mathbb{G}}^\alpha f(x) = \nabla_{\mathbb{G}} f(x), \quad \lim_{\alpha \rightarrow 1^-} \operatorname{div}_{\mathbb{G}}^\alpha \varphi(x) = \operatorname{div}_{\mathbb{G}} \varphi(x), \quad \forall x \in \mathbb{G},$$

and this convergence is uniform in $x \in \mathbb{G}$.

To prove this assertion, we begin with the following lemma on the heat flow.

Lemma 4 *For any $f \in C_c^\infty(\mathbb{G})$, $x \in \mathbb{G}$ and $t > 0$,*

- (i) $|P_t f(x)| \lesssim t^{-\frac{Q}{2}} \|f\|_{L^1(\mathbb{G})}$;
- (ii) $|\nabla_{\mathbb{G}} P_t f(x)| \lesssim t^{-\frac{Q+1}{2}} \|f\|_{L^1(\mathbb{G})}$;
- (iii) $|P_t f(x) - f(x)| \leq \|\mathcal{L}f\|_{L^\infty(\mathbb{G})} t$;
- (iv) $|\nabla_{\mathbb{G}} P_t f(x) - \nabla_{\mathbb{G}} f(x)| \lesssim \|\mathcal{L}f\|_{L^\infty(\mathbb{G})} \sqrt{t}$.

Proof (i)-(iii) can be verified directly from Proposition 1 and the fact

$$f(x) - P_t f(x) = \int_0^t P_s(\mathcal{L}f)(x) ds.$$

Then it suffices to prove (iv). Notice that

$$|\nabla_{\mathbb{G}} P_t f(x) - \nabla_{\mathbb{G}} f(x)| \leq \int_0^t |\nabla_{\mathbb{G}} P_s(\mathcal{L}f)(x)| ds. \quad (3.8)$$

Applying the reverse Poincaré inequality on $\mathbb{G}$ (see [7, Proposition 2.5.]), we have

$$|\nabla_{\mathbb{G}} P_s(\mathcal{L}f)(x)| \lesssim \frac{P_s(\mathcal{L}f)^2(x)^{1/2}}{\sqrt{s}} \lesssim \frac{\|\mathcal{L}f\|_{L^\infty(\mathbb{G})}}{\sqrt{s}}. \quad (3.9)$$

Combining (3.8) and (3.9) yields (iv), which ends the proof. $\square$

By the definition of $\nabla_{\mathbb{G}}^\alpha$ and $\operatorname{div}_{\mathbb{G}}^\alpha$, Theorem 1 could be deduced from the following lemma.

Lemma 5 For each $f \in C_c^\infty(\mathbb{G})$,

$$\lim_{\beta \rightarrow 0^+} I_\beta f(x) = f(x), \quad \lim_{\beta \rightarrow 0^+} \nabla_{\mathbb{G}} I_\beta f(x) = \nabla_{\mathbb{G}} f(x), \quad \forall x \in \mathbb{G},$$

and this convergence is uniform in $x \in \mathbb{G}$.

Proof We only prove the first one. Since the other could be treated in the same way. Recalling the definition of I_β ,

$$I_\beta f(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta}{2}-1} P_t f(x) dt.$$

Using Lemma 4, we have

$$\left| \frac{1}{\Gamma(\beta/2)} \int_0^1 t^{\frac{\beta}{2}-1} P_t f(x) dt - \frac{2}{\beta \Gamma(\beta/2)} f(x) \right| \leq \frac{2}{(\beta+2)\Gamma(\beta/2)} \|\mathcal{L}f\|_{L^\infty(\mathbb{G})}$$

and

$$\left| \frac{1}{\Gamma(\beta/2)} \int_1^\infty t^{\frac{\beta}{2}-1} P_t f(x) dt \right| \lesssim \frac{2}{(Q-\beta)\Gamma(\beta/2)} \|f\|_{L^1(\mathbb{G})}.$$

Since $\lim_{\beta \rightarrow 0^+} \frac{2}{\beta \Gamma(\beta/2)} = 1$ and $\lim_{\beta \rightarrow 0^+} \frac{1}{\Gamma(\beta/2)} = 0$, this lemma is established by letting $\beta \rightarrow 0^+$, which ends the proof of Theorem 1. $\square$

If we further define $\nabla_{\mathbb{G}}^0 = \mathcal{R}$ and $\nabla_{\mathbb{G}}^1 = \nabla_{\mathbb{G}}$, then Theorem 1 illustrates that for each $f \in C_c^\infty(\mathbb{G})$, $(\nabla_{\mathbb{G}}^\beta f, 0 \leq \beta \leq 1)$ forms a continuous path in $C_b(\mathbb{G}; H\mathbb{G})$ connecting $\mathcal{R}f$ and $\nabla_{\mathbb{G}} f$. In the following theorem, we discuss the L^p -continuity properties of I_β and $\nabla_{\mathbb{G}} I_\beta$, and then we address the continuity of $(\nabla_{\mathbb{G}}^\beta f, 0 \leq \beta \leq 1)$ in $L^p(\mathbb{G}; H\mathbb{G})$.

Theorem 2 For any $f \in C_c^\infty(\mathbb{G})$,

- (i) $\lim_{\beta \rightarrow 0^+} \|I_\beta f - f\|_{L^p(\mathbb{G})} = 0$ for $1 < p \leq \infty$;
- (ii) $\lim_{\beta \rightarrow \alpha} \|I_\beta f - I_\alpha f\|_{L^p(\mathbb{G})} = 0$ for $0 < \alpha < Q$ and $\frac{Q}{Q-\alpha} < p \leq \infty$;
- (iii) $\lim_{\beta \rightarrow 0^+} \|\nabla_{\mathbb{G}} I_\beta f - \nabla_{\mathbb{G}} f\|_{L^p(\mathbb{G}; H\mathbb{G})} = 0$ for $1 \leq p \leq \infty$;
- (iv) $\lim_{\beta \rightarrow \alpha} \|\nabla_{\mathbb{G}} I_\beta f - \nabla_{\mathbb{G}} I_\alpha f\|_{L^p(\mathbb{G}; H\mathbb{G})} = 0$ for $0 < \alpha \leq 1 (< Q)$ and $\frac{Q}{Q-\alpha} < p \leq \infty$.
In particular, if $\alpha = 1$, this limit should be understood as a left limit.

Proof (i) The case $p = \infty$ has already been proved in Lemma 5. Therefore we may assume $1 < p < \infty$. Notice that

$$\begin{aligned} \left\| I_\beta f - \frac{2}{\beta \Gamma(\beta/2)} f \right\|_{L^p(\mathbb{G})} &= \left\| I_\beta f - \frac{1}{\Gamma(\beta/2)} \int_0^1 t^{\frac{\beta}{2}-1} dt \cdot f \right\|_{L^p(\mathbb{G})} \\ &\leq \frac{1}{\Gamma(\beta/2)} \int_0^1 t^{\frac{\beta}{2}-1} \|P_t f - f\|_{L^p(\mathbb{G})} dt + \frac{1}{\Gamma(\beta/2)} \int_1^\infty t^{\frac{\beta}{2}-1} \|P_t f\|_{L^p(\mathbb{G})} dt \\ &\leq \frac{1}{\Gamma(\beta/2)} \int_0^1 t^{\frac{\beta}{2}} \|\mathcal{L} f\|_{L^p(\mathbb{G})} dt + \frac{1}{\Gamma(\beta/2)} \int_1^\infty t^{\frac{\beta}{2}-1} \|P_t f\|_{L^p(\mathbb{G})} dt \end{aligned}$$

Then using the hypercontractivity $\|P_t\|_{L^1 \rightarrow L^p} \lesssim t^{-\frac{Q(p-1)}{2p}}$ and the same argument used in Lemma 5, we obtain (i).

(ii) By the boundedness of I_α and the continuity of the Gamma function, it suffices to show that

$$\lim_{\beta \rightarrow \alpha} \|\Gamma(\beta/2)I_\beta f - \Gamma(\alpha/2)I_\alpha f\|_{L^p(\mathbb{G})} = \lim_{\beta \rightarrow \alpha} \|f * R_{\alpha, \beta}\|_{L^p(\mathbb{G})} = 0, \quad (3.10)$$

where

$$R_{\alpha, \beta}(x) := \Gamma(\beta/2)R_\beta(x) - \Gamma(\alpha/2)R_\alpha(x).$$

For the case $p = \infty$, since $f \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$, we have

$$\begin{aligned} \|\Gamma(\beta/2)I_\beta f - \Gamma(\alpha/2)I_\alpha f\|_{L^\infty(\mathbb{G})} &= \sup_{x \in \mathbb{G}} \left| \int_{\mathbb{G}} f(y) R_{\alpha, \beta}(y^{-1} \circ x) d\mu(y) \right| \\ &\leq \|f\|_{L^1(\mathbb{G})} \|R_{\alpha, \beta}\|_{L^\infty(B^c(0,1))} + \|f\|_{L^\infty(\mathbb{G})} \|R_{\alpha, \beta}\|_{L^1(B(0,1))}. \end{aligned}$$

[12, Proposition 2.2] tells us that there exists $C_\alpha > 0$, such that

$$|R_{\alpha, \beta}(x)| \leq C_\alpha |\beta - \alpha| (1 + |\log d(x)|) (d^{\alpha-Q}(x) + d^{\beta-Q}(x)), \quad (3.11)$$

which implies that

$$\lim_{\beta \rightarrow \alpha} (\|R_{\alpha, \beta}\|_{L^\infty(B^c(0,1))} + \|R_{\alpha, \beta}\|_{L^1(B(0,1))}) = 0,$$

Thus (3.10) is obtained.

Now we turn to the case $\frac{Q}{Q-\alpha} < p < \infty$. By (3.11), for $\varepsilon > 0$, there exists $C_{\alpha,\varepsilon} > 0$ such that for any $\beta \in (0, Q)$ with $|\beta - \alpha| < \varepsilon$,

$$|R_{\alpha,\beta}(x)| \leq C_{\alpha,\varepsilon} |\beta - \alpha| (d^{\alpha-2\varepsilon-Q}(x) + d^{\alpha+2\varepsilon-Q}(x)). \quad (3.12)$$

Now fix $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2}(Q - \alpha - \frac{Q}{p})$. Combining (3.12) and [26, Proposition 1.10], it follows that for any $\beta \in (0, Q)$ with $|\beta - \alpha| < \varepsilon$,

$$\|f * R_{\alpha,\beta}\|_{L^p(\mathbb{G})} \lesssim |\beta - \alpha| (\|f\|_{L^{p_1}(\mathbb{G})} + \|f\|_{L^{p_2}(\mathbb{G})}),$$

where $p_1 = \frac{Qp}{Q+(\alpha-2\varepsilon)p}$ and $p_2 = \frac{Qp}{Q+(\alpha+2\varepsilon)p}$. The proof of (3.10) is complete.

(iii) The case $p = \infty$ has been proved in Lemma 5. We assume $1 \leq p < \infty$. For $j = 1, \dots, m$,

$$\begin{aligned} \left\| X_j I_\beta f - \frac{2}{\beta \Gamma(\beta/2)} X_j f \right\|_{L^p(\mathbb{G})} &\leq \frac{1}{\Gamma(\beta/2)} \int_0^1 t^{\frac{\beta}{2}-1} \|X_j P_t f - X_j f\|_{L^p(\mathbb{G})} dt \\ &\quad + \frac{1}{\Gamma(\beta/2)} \int_1^\infty t^{\frac{\beta}{2}-1} \|f * (X_j p_t)\|_{L^p(\mathbb{G})} dt. \end{aligned}$$

By Young's convolution inequality and Proposition 2,

$$\begin{aligned} \|X_j P_t f - X_j f\|_{L^p(\mathbb{G})} &\leq \int_0^t \|X_j P_s(\mathcal{L}f)\|_{L^p(\mathbb{G})} ds \\ &\leq \int_0^t \|X_j p_s\|_{L^1(\mathbb{G})} \|\mathcal{L}f\|_{L^p(\mathbb{G})} ds \lesssim \|\mathcal{L}f\|_{L^p(\mathbb{G})} \sqrt{t} \end{aligned}$$

and

$$\|f * (X_j p_t)\|_{L^p(\mathbb{G})} \leq \|X_j p_t\|_{L^1(\mathbb{G})} \|f\|_{L^p(\mathbb{G})} \lesssim t^{-\frac{1}{2}} \|f\|_{L^p(\mathbb{G})}.$$

Then by the same argument used in Lemma 5, we obtain (iii).

(iv) By the similar argument used in (ii), it suffices to show that

$$\lim_{\beta \rightarrow \alpha} \|\nabla_{\mathbb{G}}(\Gamma(\beta/2)I_\beta - \Gamma(\alpha/2)I_\alpha)f\|_{L^p(\mathbb{G}; H\mathbb{G})} = 0. \quad (3.13)$$

By Corollary 1 and the remark therein, for $0 < \alpha, \beta \leq 1$ and $f \in C_c^\infty(\mathbb{G})$,

$$\nabla_{\mathbb{G}}(\Gamma(\beta/2)I_\beta - \Gamma(\alpha/2)I_\alpha)f(x) = \int_{\mathbb{G}} (f(y) - f(x)) \nabla_{\mathbb{G}} R_{\alpha,\beta}(y^{-1} \circ x) d\mu(y).$$

Now we assume $\text{supp } f \subset B(0, M)$ for some $M > 0$. Then³

³ Here and subsequently, χ_A denotes the characteristic function of the set A .

$$\begin{aligned}
 & \left| \nabla_{\mathbb{G}} \left(\Gamma(\beta/2) I_{\beta} - \Gamma(\alpha/2) I_{\alpha} \right) f(x) \right| \\
 & \leq \|f\|_{\text{Lip}} \int_{B(0,M)} d(y^{-1} \circ x) |\nabla_{\mathbb{G}} R_{\alpha,\beta}(y^{-1} \circ x)| \, d\mu(y) \\
 & = \|f\|_{\text{Lip}} \left(\chi_{B(0,M)} * (d |\nabla_{\mathbb{G}} R_{\alpha,\beta}|) \right)(x).
 \end{aligned} \tag{3.14}$$

By the definition of $R_{\alpha,\beta}$,

$$\begin{aligned}
 \nabla_{\mathbb{G}} R_{\alpha,\beta}(x) &= \int_0^{\infty} (t^{\frac{\beta}{2}} - t^{\frac{\alpha}{2}}) \nabla_{\mathbb{G}} p_t(x) \frac{dt}{t} \\
 &= \frac{1}{2} \int_{\alpha}^{\beta} \left(\int_0^{\infty} t^{\frac{s}{2}-1} (\ln t) \nabla_{\mathbb{G}} p_t(x) \, dt \right) ds.
 \end{aligned}$$

Using Proposition 1, there exists $C > 0$, which is independent of $s \in [0, 1]$, such that

$$\begin{aligned}
 & \left| \int_0^{\infty} t^{\frac{s}{2}-1} (\ln t) \nabla_{\mathbb{G}} p_t(x) \, dt \right| \leq \int_0^{\infty} t^{\frac{s}{2}-1} (\ln t) |\nabla_{\mathbb{G}} p_t(x)| \, dt \\
 & \leq C \int_0^{\infty} t^{\frac{s-Q-1}{2}} (\ln t) e^{-\frac{cd^2(x)}{t}} \frac{dt}{t} \leq C(1 + |\log d(x)|) d^{s-Q-1}(x).
 \end{aligned}$$

Then we obtain for any $0 < \alpha, \beta \leq 1$,

$$|\nabla_{\mathbb{G}} R_{\alpha,\beta}(x)| \lesssim |\beta - \alpha| (1 + |\log d(x)|) (d^{\alpha-Q-1}(x) + d^{\beta-Q-1}(x)). \tag{3.15}$$

Combining (3.14) and (3.15), (3.13) is established by the same argument we used in (ii). $\square$

Corollary 2 *If we define $\nabla_{\mathbb{G}}^0 = \mathcal{R}$ and $\nabla_{\mathbb{G}}^1 = \nabla_{\mathbb{G}}$, then for any $f \in C_c^{\infty}(\mathbb{G})$ and $0 \leq \alpha < 1$, $(\nabla_{\mathbb{G}}^{\beta} f, \alpha \leq \beta \leq 1)$ is a continuous path in $L^p(\mathbb{G}; H\mathbb{G})$, where $Q/(Q + \alpha - 1) < p \leq \infty$. In particular, $(\nabla_{\mathbb{G}}^{\beta} f, 0 \leq \beta \leq 1)$ is a continuous path in $L^p(\mathbb{G}; H\mathbb{G})$ for $Q/(Q - 1) < p \leq \infty$. If we define $\text{div}_{\mathbb{G}}^0 = I_1 \circ \text{div}$ and $\text{div}_{\mathbb{G}}^1 = \text{div}_{\mathbb{G}}$, an analogous conclusion holds for $\text{div}_{\mathbb{G}}^{\beta}$.*

In analogy with the classical Euclidean case (see [46]), the fractional operators we just defined satisfy some natural requirements, including the left-translation invariance and α -homogeneity, which reflect isotropic properties of $\mathbb{G}$.

Proposition 4 *Let $0 < \alpha < 1$. The operators $\nabla_{\mathbb{G}}^{\alpha}$ and $\text{div}_{\mathbb{G}}^{\alpha}$ are invariant under left translations and α -homogeneous, i.e. for each $f \in C_c^{\infty}(\mathbb{G})$ and $\phi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$,*

$$\nabla_{\mathbb{G}}^{\alpha}(f \circ \tau_z) = (\nabla_{\mathbb{G}}^{\alpha} f) \circ \tau_z, \quad \nabla_{\mathbb{G}}^{\alpha}(f \circ \delta_r) = r^{\alpha} (\nabla_{\mathbb{G}}^{\alpha} f) \circ \delta_r$$

and

$$\text{div}_{\mathbb{G}}^{\alpha}(\phi \circ \tau_z) = (\text{div}_{\mathbb{G}}^{\alpha} \phi) \circ \tau_z, \quad \text{div}_{\mathbb{G}}^{\alpha}(\phi \circ \delta_r) = r^{\alpha} (\text{div}_{\mathbb{G}}^{\alpha} \phi) \circ \delta_r$$

hold for any $z \in \mathbb{G}$ and $r > 0$.

Proof We give the proof only for $\nabla_{\mathbb{G}}^{\alpha}$; the case for $\operatorname{div}_{\mathbb{G}}^{\alpha}$ is almost the same.

Using Proposition 3, we have

$$\begin{aligned}\nabla_{\mathbb{G}}^{\alpha}(f \circ \tau_z)(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\{d(x,y) \geq \varepsilon\}} f(z \circ y) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\{d(\tau_z(x), w) \geq \varepsilon\}} f(w) \nabla_{\mathbb{G}} R_{1-\alpha}(w^{-1} \circ \tau_z(x)) \, d\mu(w) = (\nabla_{\mathbb{G}}^{\alpha} f)(\tau_z(x)),\end{aligned}$$

where in the last line we have used the left-invariance of d .

Combined with Proposition 2 (2), it follows that

$$\begin{aligned}\nabla_{\mathbb{G}}^{\alpha}(f \circ \delta_r)(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\{d(x,y) \geq \varepsilon\}} f(\delta_r(y)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y) \\ &= r^{Q+\alpha} \lim_{\varepsilon \rightarrow 0} \int_{\{d(x,y) \geq \varepsilon\}} f(\delta_r(y)) \nabla_{\mathbb{G}} R_{1-\alpha}(\delta_r(y)^{-1} \circ \delta_r(x)) \, d\mu(y) \\ &= r^{\alpha} \lim_{\varepsilon \rightarrow 0} \int_{\{d(\delta_r(x), w) \geq r\varepsilon\}} f(w) \nabla_{\mathbb{G}} R_{1-\alpha}(w^{-1} \circ \delta_r(x)) \, d\mu(w) \\ &= r^{\alpha} (\nabla_{\mathbb{G}}^{\alpha} f)(\delta_r(x)).\end{aligned}$$

□

3.3 Extensions

The fractional operators $\nabla_{\mathbb{G}}^{\alpha}$ and $\operatorname{div}_{\mathbb{G}}^{\alpha}$ are initially defined on $C_c^{\infty}(\mathbb{G})$ and $C_c^{\infty}(\mathbb{G}; H\mathbb{G})$ respectively. In this subsection, we discuss their extensions to wider classes of functions. Notice that (3.5) makes sense for bounded Lipschitz functions. Indeed, by Proposition 2 (3), for any $f \in \operatorname{Lip}_b(\mathbb{G})$ and $x \in \mathbb{G}$,

$$\begin{aligned}& \left| \int_{\mathbb{G}} (f(y) - f(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y) \right| \\ & \leq \sum_{k=0}^{\infty} \int_{\{2^{-(k+1)} \leq d(x,y) < 2^{-k}\}} |f(y) - f(x)| |\nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x)| \, d\mu(y) \\ & \quad + \int_{\{d(x,y) \geq 1\}} |f(y) - f(x)| |\nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x)| \, d\mu(y) \\ & \lesssim \sum_{k=0}^{\infty} \|f\|_{\operatorname{Lip}} 2^{-k} \cdot 2^{(Q+\alpha)k} \mu(B(0, 2^{-k})) + \|f\|_{L^{\infty}(\mathbb{G})} \int_{\{d(z) \geq 1\}} d^{-Q-\alpha}(z) \, d\mu(z) \\ & \lesssim \|f\|_{\operatorname{Lip}} + \|f\|_{L^{\infty}(\mathbb{G})}.\end{aligned}\tag{3.16}$$

Therefore we can extend $\nabla_{\mathbb{G}}^{\alpha}$ to $\operatorname{Lip}_b(\mathbb{G})$ by using (3.5) as its definition, and (3.16) implies that $\nabla_{\mathbb{G}}^{\alpha} f \in L^{\infty}(\mathbb{G}; H\mathbb{G})$ for $f \in \operatorname{Lip}_b(\mathbb{G})$. A similar extension can be done for $\operatorname{div}_{\mathbb{G}}^{\alpha}$.

Moreover, combined with Proposition 2 (3), it follows that for each $f \in w^{\alpha,1}(\mathbb{G})$,

$$\begin{aligned} & \int_{\mathbb{G}} \left| \int_{\mathbb{G}} (f(y) - f(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \right| d\mu(x) \\ & \leq \int_{\mathbb{G}} \int_{\mathbb{G}} \left| (f(y) - f(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \right| d\mu(y) d\mu(x) \\ & \lesssim \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(x) - f(y)|}{d(x, y)^{Q+\alpha}} d\mu(x) d\mu(y) = [f]_{W^{\alpha,1}(\mathbb{G})}. \end{aligned}$$

Based on this prior estimate, we can extend the definition of $\nabla_{\mathbb{G}}^{\alpha}$ to the space $w^{\alpha,1}(\mathbb{G})$, i.e. given $f \in w^{\alpha,1}(\mathbb{G})$, it holds that for μ -a.e. $x \in \mathbb{G}$,

$$\nabla_{\mathbb{G}}^{\alpha} f(x) := \int_{\mathbb{G}} (f(y) - f(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \in L^1(\mathbb{G}; H\mathbb{G}), \quad (3.17)$$

and

$$\|\nabla_{\mathbb{G}}^{\alpha} f\|_{L^1(\mathbb{G}; H\mathbb{G})} \lesssim [f]_{W^{\alpha,1}(\mathbb{G})}. \quad (3.18)$$

Similarly, the definition of $\operatorname{div}_{\mathbb{G}}^{\alpha}$ can be extended to $w^{\alpha,1}(\mathbb{G}; H\mathbb{G})$, i.e. given $\varphi \in w^{\alpha,1}(\mathbb{G}; H\mathbb{G})$, it holds that for μ -a.e. $x \in \mathbb{G}$,

$$\operatorname{div}_{\mathbb{G}}^{\alpha} \varphi(x) := \int_{\mathbb{G}} (\varphi(y) - \varphi(x)) \cdot \tilde{\nabla}_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \in L^1(\mathbb{G}),$$

and

$$\|\operatorname{div}_{\mathbb{G}}^{\alpha} \varphi\|_{L^1(\mathbb{G})} \lesssim [\varphi]_{W^{\alpha,1}(\mathbb{G}; H\mathbb{G})}. \quad (3.19)$$

On the other hand, recalling the definition of the Riesz transform, $\nabla_{\mathbb{G}}^{\alpha}$ has the following expression: for any $f \in C_c^{\infty}(\mathbb{G})$,

$$\nabla_{\mathbb{G}}^{\alpha} f = \nabla_{\mathbb{G}} I_{1-\alpha} f = \mathcal{R}(\mathcal{L}^{\frac{\alpha}{2}} f).$$

Therefore $\nabla_{\mathbb{G}}^{\alpha}$ can be extended to the fractional Sobolev space $L_{\alpha}^p(\mathbb{G})$, where $1 < p < \infty$. For every $f \in L_{\alpha}^p(\mathbb{G})$, it is reasonable to define

$$\nabla_{\mathbb{G}}^{\alpha} f := \mathcal{R}(\mathcal{L}^{\frac{\alpha}{2}} f) \in L^p(\mathbb{G}, H\mathbb{G}), \quad (3.20)$$

where we have used the L^p -boundness of the Riesz transform and $\mathcal{L}^{\frac{\alpha}{2}} f \in L^p(\mathbb{G})$.

Notice that $\operatorname{Lip}_c(\mathbb{G}) \subset w^{\alpha,1}(\mathbb{G}) \cap \operatorname{Lip}_b(\mathbb{G})$, combining (3.16) and (3.18) (the same argument works for $\operatorname{div}_{\mathbb{G}}^{\alpha}$), we have the following proposition.

Proposition 5 *Let $0 < \alpha < 1$. Then*

$$\nabla_{\mathbb{G}}^{\alpha} f \in L^1(\mathbb{G}; H\mathbb{G}) \cap L^{\infty}(\mathbb{G}; H\mathbb{G})$$

and

$$\operatorname{div}_{\mathbb{G}}^{\alpha} \varphi \in L^1(\mathbb{G}) \cap L^{\infty}(\mathbb{G})$$

hold for every $f \in \operatorname{Lip}_c(\mathbb{G})$ and $\varphi \in \operatorname{Lip}_c(\mathbb{G}; H\mathbb{G})$.

3.4 Duality and Leibniz's rules

As in the Euclidean case (see [13, Section 2.4]), in the context of homogeneous Carnot groups $\nabla_{\mathbb{G}}^{\alpha}$ and $\operatorname{div}_{\mathbb{G}}^{\alpha}$ also share duality relation and the Leibniz's rules, which will be used later.

Proposition 6 (Duality) *Let $0 < \alpha < 1$. For each $f \in \operatorname{Lip}_c(\mathbb{G})$ and $\varphi \in \operatorname{Lip}_c(\mathbb{G}; H\mathbb{G})$,*

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi \, d\mu = - \int_{\mathbb{G}} \langle \phi, \nabla_{\mathbb{G}}^{\alpha} f \rangle \, d\mu.$$

Proof For $\varphi \in \operatorname{Lip}_c(\mathbb{G}; H\mathbb{G})$, it is clear that $\operatorname{div}_{\mathbb{G}} \varphi \in L^{\infty}(\mathbb{G})$ and has compact support. Combined with Lemma 1, it follows that

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi \, d\mu = \int_{\mathbb{G}} f I_{1-\alpha}(\operatorname{div}_{\mathbb{G}} \varphi) \, d\mu = \int_{\mathbb{G}} \operatorname{div}_{\mathbb{G}} \varphi (I_{1-\alpha} f) \, d\mu.$$

As a consequence of Proposition 5, we know that $I_{1-\alpha} f$ is Lipschitz. Then using (2.5), we complete the proof. $\square$

Proposition 7 (Leibniz's rules) *For each $f, g \in \operatorname{Lip}_c(\mathbb{G})$ and $x \in \mathbb{G}$,*

$$\nabla_{\mathbb{G}}^{\alpha}(fg)(x) = f(x) \nabla_{\mathbb{G}}^{\alpha} g(x) + g(x) \nabla_{\mathbb{G}}^{\alpha} f(x) + \nabla_{\mathbb{G}, \text{NL}}^{\alpha}(f, g)(x),$$

where

$$\nabla_{\mathbb{G}, \text{NL}}^{\alpha}(f, g)(x) := \int_{\mathbb{G}} (f(y) - f(x))(g(y) - g(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y).$$

Moreover,

$$\begin{aligned} \|\nabla_{\mathbb{G}, \text{NL}}^{\alpha}(f, g)\|_{L^1(\mathbb{G}; H\mathbb{G})} &\lesssim \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(y) - f(x)| |g(y) - g(x)|}{d(x, y)^{Q+\alpha}} \, d\mu(y) d\mu(x) \\ &\lesssim [f]_{W^{\alpha, 1}(\mathbb{G})} \|g\|_{L^{\infty}(\mathbb{G})}. \end{aligned}$$

Similarly, for each $f \in \text{Lip}_c(\mathbb{G})$ and $\varphi \in \text{Lip}_c(\mathbb{G}; H\mathbb{G})$,

$$\text{div}_{\mathbb{G}}^{\alpha}(f\varphi)(x) = f(x) \text{div}_{\mathbb{G}}^{\alpha} \varphi(x) + \langle \varphi(x), I_{1-\alpha}(\nabla_{\mathbb{G}} f)(x) \rangle + \text{div}_{\mathbb{G},\text{NL}}^{\alpha}(f, \varphi)(x),$$

where

$$\text{div}_{\mathbb{G},\text{NL}}^{\alpha}(f, \varphi)(x) := \int_{\mathbb{G}} (f(y) - f(x)) (\varphi(y) - \varphi(x)) \cdot \widetilde{\nabla}_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y).$$

Moreover,

$$\begin{aligned} \|\text{div}_{\mathbb{G},\text{NL}}^{\alpha}(f, \varphi)\|_{L^1(\mathbb{G})} &\lesssim \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(y) - f(x)| |\varphi(y) - \varphi(x)|}{d(x, y)^{Q+\alpha}} d\mu(y) d\mu(x) \\ &\lesssim [f]_{W^{\alpha,1}(\mathbb{G})} \|\varphi\|_{L^{\infty}(\mathbb{G}; H\mathbb{G})}. \end{aligned}$$

Proof We only proof the conclusions about $\nabla_{\mathbb{G}}^{\alpha}$. For each $f, g \in \text{Lip}_c(\mathbb{G})$, it is clear that $fg \in \text{Lip}_c(\mathbb{G})$. Then the Leibniz's rule is a direct consequence of Corollary 1. And simply using Proposition 2 (3) leads to

$$\|\nabla_{\mathbb{G},\text{NL}}^{\alpha}(f, g)\|_{L^1(\mathbb{G}; H\mathbb{G})} \lesssim [f]_{W^{\alpha,1}(\mathbb{G})} \|g\|_{L^{\infty}(\mathbb{G})}.$$

□

Remark 4 Applying Theorem 1, it follows that for each $f, g \in C_c^{\infty}(\mathbb{G})$ and $\varphi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$,

$$\lim_{\alpha \rightarrow 1^-} \nabla_{\mathbb{G},\text{NL}}^{\alpha}(f, g)(x) = 0, \quad \lim_{\alpha \rightarrow 1^-} \text{div}_{\mathbb{G},\text{NL}}^{\alpha}(f, \varphi)(x) = 0, \quad \forall x \in \mathbb{G}.$$

3.5 Weak α -gradient and the distributional fractional Sobolev spaces

In this subsection, we define the weak α -gradient and the distributional fractional Sobolev space (denoted by $S^{\alpha,p}(\mathbb{G})$) on the homogeneous Carnot groups. Then we discuss some approximation properties of $S^{\alpha,p}(\mathbb{G})$, and indicate the relationship between $L_{\alpha}^p(\mathbb{G})$ and $S^{\alpha,p}(\mathbb{G})$.

Definition 4 Let $0 < \alpha < 1$ and $1 \leq p \leq \infty$. For each $f \in L^p(\mathbb{G})$, $g \in L_{\text{loc}}^1(\mathbb{G}; H\mathbb{G})$ is said to be a weak α -gradient of f , denoted by $\nabla_{\mathbb{G},w}^{\alpha} f$, if

$$\int_{\mathbb{G}} f \text{div}_{\mathbb{G}}^{\alpha} \phi d\mu = - \int_{\mathbb{G}} \langle \phi, g \rangle d\mu \quad (3.21)$$

holds for any $\phi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$.

Then the distributional fractional Sobolev space $S^{\alpha,p}(\mathbb{G})$ is defined by

$$S^{\alpha,p}(\mathbb{G}) := \{f \in L^p(\mathbb{G}) \text{ with a weak } \alpha\text{-gradient } \nabla_{\mathbb{G},w}^{\alpha} f \in L^p(\mathbb{G}; H\mathbb{G})\},$$

endowed with the norm

$$\|f\|_{S^{\alpha,p}(\mathbb{G})} := \|f\|_{L^p(\mathbb{G})} + \|\nabla_{\mathbb{G},w}^\alpha f\|_{L^p(\mathbb{G};H\mathbb{G})}.$$

To investigate approximation properties, we need to introduce the concept of mollifiers. Let $\rho \in C_c^\infty(\mathbb{G})$ be a non-negative function satisfying the following conditions

$$\text{supp } \rho \subset B(0, 1), \quad \rho(x) = \rho(x^{-1}) \quad \text{and} \quad \int_{\mathbb{G}} \rho(x) \, d\mu(x) = 1.$$

For $\varepsilon > 0$, we set $\rho_\varepsilon(x) := \varepsilon^{-Q} \rho(\delta_{\varepsilon^{-1}}(x))$. Then $\{\rho_\varepsilon\}_{\varepsilon>0}$ is called a family of standard mollifiers on $\mathbb{G}$, which forms an “approximation to the identity” (see e.g. [41, Proposition 1.2.15]). We will use mollifiers to study approximation properties of $S^{\alpha,p}(\mathbb{G})$. For this purpose, we introduce

$$S_0^{\alpha,p}(\mathbb{G}) := \overline{C_c^\infty(\mathbb{G})}^{\|\cdot\|_{S^{\alpha,p}(\mathbb{G})}}.$$

We begin with the case $p = 1$.

Proposition 8 *Let $0 < \alpha < 1$. Then*

- (1) $S^{\alpha,1}(\mathbb{G}) \cap C^\infty(\mathbb{G})$ is dense in $S^{\alpha,1}(\mathbb{G})$;
- (2) $S^{\alpha,1}(\mathbb{G}) = S_0^{\alpha,1}(\mathbb{G})$.

Proof For each $f \in S^{\alpha,1}(\mathbb{G})$, we consider a family of smooth functions $\{f_\varepsilon\}_{\varepsilon>0}$, where $f_\varepsilon = \rho_\varepsilon * f$. Then by the familiar property of the mollifier, it is known that $f_\varepsilon \rightarrow f$ in $L^1(\mathbb{G})$ as $\varepsilon \rightarrow 0$.

For any $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$, combining (2.6), (2.7), Lemma 1 and the definition of $\nabla_{\mathbb{G},w}^\alpha f$ yields

$$\begin{aligned} \int_{\mathbb{G}} f_\varepsilon \, \text{div}_{\mathbb{G}}^\alpha \varphi \, d\mu &= \int_{\mathbb{G}} f_\varepsilon \, I_{1-\alpha}(\text{div}_{\mathbb{G}} \varphi) \, d\mu = \int_{\mathbb{G}} \text{div}_{\mathbb{G}} \varphi \, I_{1-\alpha} f_\varepsilon \, d\mu \\ &= \int_{\mathbb{G}} \text{div}_{\mathbb{G}} \varphi \, (\rho_\varepsilon * I_{1-\alpha} f) \, d\mu = \int_{\mathbb{G}} I_{1-\alpha} f \, (\rho_\varepsilon * \text{div}_{\mathbb{G}} \varphi) \, d\mu \\ &= \int_{\mathbb{G}} I_{1-\alpha} f \, \text{div}_{\mathbb{G}}(\rho_\varepsilon * \varphi) \, d\mu = \int_{\mathbb{G}} f \, \text{div}_{\mathbb{G}}^\alpha(\rho_\varepsilon * \varphi) \, d\mu \\ &= - \int_{\mathbb{G}} \langle \rho_\varepsilon * \varphi, \nabla_{\mathbb{G},w}^\alpha f \rangle \, d\mu = - \int_{\mathbb{G}} \langle \varphi, \rho_\varepsilon * \nabla_{\mathbb{G},w}^\alpha f \rangle \, d\mu. \end{aligned}$$

In other words,

$$\nabla_{\mathbb{G},w}^\alpha f_\varepsilon = \rho_\varepsilon * \nabla_{\mathbb{G},w}^\alpha f.$$

Consequently, we conclude that $f_\varepsilon \rightarrow f$ in $S^{\alpha,1}(\mathbb{G})$ as $\varepsilon \rightarrow 0$. The first statement is proved.

For the second one, it suffices to prove that for each $f \in S^{\alpha,1}(\mathbb{G}) \cap C^\infty(\mathbb{G})$, there exists $\{f_k\}_{k=1}^\infty \subset C_c^\infty(\mathbb{G})$ such that $f_k \rightarrow f$ in $S^{\alpha,1}(\mathbb{G})$ as $k \rightarrow \infty$, which can be proved by using smooth cut-off functions (see the proof of Proposition 10 below) together with Proposition 7. This argument is similar to the case of Euclidean space (see [13, Theorem 3.23]), therefore we omit it. $\square$

Now we turn to the case $p > 1$. We have the following assertion, which could be viewed as an extension of Meyers-Serrin type theorem presented in [28].

Theorem 3 *Let $0 < \alpha < 1$ and $1 < p < \frac{Q}{1-\alpha}$. Then $S^{\alpha,p}(\mathbb{G}) = S_0^{\alpha,p}(\mathbb{G}) = L_\alpha^p(\mathbb{G})$.*

Proof To begin with, it is not difficult to verify that $S^{\alpha,p}(\mathbb{G})$ is a Banach space. Consequently, the whole proof can be divided into three steps below.

Step 1. $L_\alpha^p(\mathbb{G}) \subset S_0^{\alpha,p}(\mathbb{G})$.

By [26, Theorem (4.5)], $C_c^\infty(\mathbb{G})$ is dense in $L_\alpha^p(\mathbb{G})$. Therefore for each $f \in L_\alpha^p(\mathbb{G})$, we can find $\{f_k\}_{k=1}^\infty \subset C_c^\infty(\mathbb{G})$ such that

$$f_k \rightarrow f \text{ in } L^p(\mathbb{G}) \quad \text{and} \quad \mathcal{L}^{\frac{\alpha}{2}} f_k \rightarrow \mathcal{L}^{\frac{\alpha}{2}} f \text{ in } L^p(\mathbb{G}) \quad \text{as } k \rightarrow \infty.$$

Recalling the definition of $\nabla_{\mathbb{G}}^\alpha$ on $L_\alpha^p(\mathbb{G})$, and the fact that $\nabla_{\mathbb{G}}^\alpha$ and $\nabla_{\mathbb{G},w}^\alpha$ coincide on $C_c^\infty(\mathbb{G})$,

$$\nabla_{\mathbb{G},w}^\alpha f_k = \nabla_{\mathbb{G}}^\alpha f_k = \mathcal{R}(\mathcal{L}^{\frac{\alpha}{2}} f_k).$$

Then by the L^p -boundness of the Riesz transform, it follows that

$$\nabla_{\mathbb{G},w}^\alpha f_k \rightarrow \mathcal{R}(\mathcal{L}^{\frac{\alpha}{2}} f) = \nabla_{\mathbb{G}}^\alpha f \text{ in } L^p(\mathbb{G}; H\mathbb{G}) \quad \text{as } k \rightarrow \infty.$$

In order to show that $f_k \rightarrow f$ in $S^{\alpha,p}(\mathbb{G})$ as $k \rightarrow \infty$, it suffices to indicate that $\nabla_{\mathbb{G},w}^\alpha f = \nabla_{\mathbb{G}}^\alpha f$. Indeed, for any $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,

$$\begin{aligned} \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu &= \lim_{k \rightarrow \infty} \int_{\mathbb{G}} f_k \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu = - \lim_{k \rightarrow \infty} \int_{\mathbb{G}} \langle \varphi, \nabla_{\mathbb{G}}^\alpha f_k \rangle \, d\mu \\ &= - \int_{\mathbb{G}} \langle \varphi, \nabla_{\mathbb{G}}^\alpha f \rangle \, d\mu, \end{aligned}$$

where we have used Proposition 6.

Step 2. The closure of $L_\alpha^p(\mathbb{G})$ in $S^{\alpha,p}(\mathbb{G})$ is the whole space $L_\alpha^p(\mathbb{G})$.

For every $f \in \overline{L_\alpha^p(\mathbb{G})}^{\|\cdot\|_{S^{\alpha,p}(\mathbb{G})}}$, by definition there exists $\{f_k\}_{k=1}^\infty \subset L_\alpha^p(\mathbb{G})$ such that

$$f_k \rightarrow f \text{ in } L^p(\mathbb{G}) \quad \text{and} \quad \nabla_{\mathbb{G},w}^\alpha f_k \rightarrow \nabla_{\mathbb{G},w}^\alpha f \text{ in } L^p(\mathbb{G}; H\mathbb{G}) \quad \text{as } k \rightarrow \infty.$$

Since $\{f_k\}_{k=1}^\infty \subset L_\alpha^p(\mathbb{G})$, as already explained in Step 1,

$$\nabla_{\mathbb{G},w}^\alpha f_k = \nabla_{\mathbb{G}}^\alpha f_k = \mathcal{R}(\mathcal{L}^{\frac{\alpha}{2}} f_k),$$

so that $\{\mathcal{R}_j \mathcal{L}^{\frac{\alpha}{2}} f_k\}_{k=1}^{\infty}$ forms a Cauchy sequence in $L^p(\mathbb{G})$ for each $j = 1, \dots, m$.

From the proof of [26, Theorem (4.10)], $\mathcal{L}^{\frac{1}{2}}$ has the following decomposition

$$\mathcal{L}^{\frac{1}{2}} = \sum_{j=1}^m T_j X_j, \quad (3.22)$$

where $\{T_j\}_{j=1}^m$ are bounded operators on $L^p(\mathbb{G})$. This decomposition allows us to write the identity operator on $L^p(\mathbb{G})$ as $\sum_{j=1}^m T_j \mathcal{R}_j$, which is due to the L^p -boundness of the Riesz transform and a standard approximation argument. As a result, we conclude that $\{\mathcal{L}^{\frac{\alpha}{2}} f_k\}_{k=1}^{\infty}$ is also a Cauchy sequence in $L^p(\mathbb{G})$. Indeed, for every $k, l \in \mathbb{N}$,

$$\mathcal{L}^{\frac{\alpha}{2}} f_k - \mathcal{L}^{\frac{\alpha}{2}} f_l = \sum_{j=1}^m T_j (\mathcal{R}_j \mathcal{L}^{\frac{\alpha}{2}} f_k - \mathcal{R}_j \mathcal{L}^{\frac{\alpha}{2}} f_l),$$

so that

$$\|\mathcal{L}^{\frac{\alpha}{2}} f_k - \mathcal{L}^{\frac{\alpha}{2}} f_l\|_{L^p(\mathbb{G})} \lesssim \sum_{j=1}^m \|\mathcal{R}_j \mathcal{L}^{\frac{\alpha}{2}} f_k - \mathcal{R}_j \mathcal{L}^{\frac{\alpha}{2}} f_l\|_{L^p(\mathbb{G})}.$$

Therefore there exists a function $g \in L^p(\mathbb{G})$ such that

$$\mathcal{L}_p^{\frac{\alpha}{2}} f_k = \mathcal{L}^{\frac{\alpha}{2}} f_k \rightarrow g \text{ in } L^p(\mathbb{G}) \text{ as } k \rightarrow \infty,$$

where $\mathcal{L}_p$ is the maximal restriction of $\mathcal{L}$ to $L^p(\mathbb{G})$. By [26, Theorem (3.15)] (or [24, Theorem 4.36]), $\mathcal{L}_p^{\frac{\alpha}{2}}$ is a closed operator on $L^p(\mathbb{G})$. Then we have

$$g = \mathcal{L}_p^{\frac{\alpha}{2}} f = \mathcal{L}^{\frac{\alpha}{2}} f.$$

In conclusion, $f \in L_{\alpha}^p(\mathbb{G})$, which completes the proof.

Step 3. $S^{\alpha,p}(\mathbb{G}) \subset L_{\alpha}^p(\mathbb{G})$.

Let $\{\rho_{\varepsilon}\}_{\varepsilon>0}$ be a family of standard mollifiers. As already explained in the proof of Proposition 8, for each $f \in S^{\alpha,p}(\mathbb{G})$,

$$\nabla_{\mathbb{G},w}^{\alpha}(\rho_{\varepsilon} * f) = \rho_{\varepsilon} * \nabla_{\mathbb{G},w}^{\alpha} f.$$

Then combined with the approximation property of the mollifier, it follows that the set $H := \{\rho_{\varepsilon} * f; f \in S^{\alpha,p}(\mathbb{G}), \varepsilon > 0\}$ is dense in $S^{\alpha,p}(\mathbb{G})$.

As a result, it suffices to prove that $H \subset L_{\alpha}^p(\mathbb{G})$. Since after that, combining with the preceding two steps, we can establish the following relationship

$$S^{\alpha,p}(\mathbb{G}) = \overline{H}^{\|\cdot\|_{S^{\alpha,p}(\mathbb{G})}} \subset \overline{L_{\alpha}^p(\mathbb{G})}^{\|\cdot\|_{S^{\alpha,p}(\mathbb{G})}} = L_{\alpha}^p(\mathbb{G}) \subset S_0^{\alpha,p}(\mathbb{G}) \subset S^{\alpha,p}(\mathbb{G}),$$

which finishes the whole proof.

For each $f_\varepsilon := \rho_\varepsilon * f \in H \subset C^\infty(\mathbb{G}) \cap L^p(\mathbb{G})$ and $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$, by the integration by parts formula and Lemma 1, we have

$$\begin{aligned} \int_{\mathbb{G}} f_\varepsilon \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu &= \int_{\mathbb{G}} f_\varepsilon I_{1-\alpha}(\operatorname{div}_{\mathbb{G}} \varphi) \, d\mu = \int_{\mathbb{G}} \operatorname{div}_{\mathbb{G}} \varphi I_{1-\alpha} f_\varepsilon \, d\mu \\ &= - \int_{\mathbb{G}} \langle \varphi, \nabla_{\mathbb{G}} I_{1-\alpha} f_\varepsilon \rangle \, d\mu, \end{aligned}$$

where we have used the fact that $I_{1-\alpha} f_\varepsilon = \rho_\varepsilon * I_{1-\alpha} f \in C^\infty(\mathbb{G})$.

By the definition of $\nabla_{\mathbb{G},w}^\alpha$,

$$\nabla_{\mathbb{G}} I_{1-\alpha} f_\varepsilon = \nabla_{\mathbb{G},w}^\alpha f_\varepsilon \in L^p(\mathbb{G}; H\mathbb{G}),$$

so that for $j = 1, \dots, m$, $X_j(I_{1-\alpha} f_\varepsilon) \in L^p(\mathbb{G})$.

Therefore using (3.22),

$$\mathcal{L}^{\frac{1}{2}}(I_{1-\alpha} f_\varepsilon) = \sum_{j=1}^m T_j X_j(I_{1-\alpha} f_\varepsilon) \in L^p(\mathbb{G}).$$

Notice that $\mathcal{L}^{\frac{\alpha}{2}} f_\varepsilon = \mathcal{L}^{\frac{1}{2}}(I_{1-\alpha} f_\varepsilon)$, we have $f_\varepsilon \in L_\alpha^p(\mathbb{G})$, which is the desired conclusion. $\square$

Remark 5 In the Euclidean case, [11, Corollary 1] states that

$$S^{\alpha,p}(\mathbb{R}^n) = L_\alpha^p(\mathbb{R}^n), \quad \forall \alpha \in (0, 1), \quad p \in (1, \infty).$$

For more on relationship between fractional Sobolev spaces in the setting of $\mathbb{R}^n$, see [8, 9, 11, 13, 36, 44]. The extension of Theorem 3 to the case where $1 < p < \infty$ might be an interesting direction for future research, but is beyond the scope of the present work, which does not address the mapping properties of I_α on $L^p(\mathbb{G})$ for $p \geq Q/\alpha$.

4 Fractional BV functions

4.1 Definition of the fractional BV functions

Definition 5 Let $0 < \alpha < 1$. The total (horizontal) α -variation of function f is defined by

$$\|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}) := \sup \left\{ \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu; \varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G}), \sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1 \right\}.$$

$BV^\alpha(\mathbb{G})$ denotes the set of all the L^1 -functions with bounded α -variation.

As in the classical Euclidean case (see [13, Theorem 3.2]), we have the following structure theorem for functions in $BV^\alpha(\mathbb{G})$.

Theorem 4 *Let $0 < \alpha < 1$ and $f \in L^1(\mathbb{G})$. Then $f \in BV^\alpha(\mathbb{G})$ if and only if there is an $H\mathbb{G}$ -valued Radon measure on $\mathbb{G}$, denoted by $D_{\mathbb{G}}^\alpha f$, such that for any $\phi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,*

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \phi \, d\mu = - \int_{\mathbb{G}} \langle \phi, dD_{\mathbb{G}}^\alpha f \rangle.$$

Moreover, it holds that for any open set $U \subset \mathbb{G}$,

$$\|D_{\mathbb{G}}^\alpha f\|(U) = \sup \left\{ \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \phi \, d\mu; \phi \in C_c^\infty(U; H\mathbb{G}), \sup_{x \in U} |\phi(x)| \leq 1 \right\} < \infty.$$

Proof One direction follows immediately from Definition 5. The converse implication holds thanks to Riesz Representation Theorem. The proof is the same as that for [21, Theorem 5.1]. $\square$

Remark 6 (1) $D_{\mathbb{G}}^\alpha f$ is called the α -variation measure of f . Similar to $D_{\mathbb{G}} f$ defined in Section 2.3, we can find a non-negative Radon measure $\|D_{\mathbb{G}}^\alpha f\|$ and a Borel section of $H\mathbb{G}$, denoted by v_f^α , satisfying $|v_f^\alpha| = 1$, such that $dD_{\mathbb{G}}^\alpha f = v_f^\alpha d\|D_{\mathbb{G}}^\alpha f\|$.

(2) Combined with Definition 4, it follows that $S^{\alpha,1}(\mathbb{G}) \subset BV^\alpha(\mathbb{G})$, and for any $f \in S^{\alpha,1}(\mathbb{G})$,

$$dD_{\mathbb{G}}^\alpha f = \nabla_{\mathbb{G},w}^\alpha f \, d\mu \quad \text{and} \quad \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}) = \|\nabla_{\mathbb{G},w}^\alpha f\|_{L^1(\mathbb{G}; H\mathbb{G})}. \quad (4.1)$$

Using Theorem 4, we get the following theorem on the lower semicontinuity of the fractional variation measures.

Theorem 5 *Let $0 < \alpha < 1$ and $f \in L^1(\mathbb{G})$. If $\{f_k\}_{k=1}^\infty \subset BV^\alpha(\mathbb{G})$ satisfies $f_k \rightarrow f$ in $L^1(\mathbb{G})$ as $k \rightarrow \infty$, then $f \in BV^\alpha(\mathbb{G})$, and for any open set $U \subset \mathbb{G}$,*

$$\|D_{\mathbb{G}}^\alpha f\|(U) \leq \liminf_{k \rightarrow \infty} \|D_{\mathbb{G}}^\alpha f_k\|(U).$$

Proof For any $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$, by Proposition 5, $\operatorname{div}_{\mathbb{G}}^\alpha \varphi \in L^\infty(\mathbb{G})$. Then for any open set $U \subset \mathbb{G}$,

$$\begin{aligned} \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu &= \lim_{k \rightarrow \infty} \int_{\mathbb{G}} f_k \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu = - \lim_{k \rightarrow \infty} \int_{\mathbb{G}} \langle \varphi, dD_{\mathbb{G}}^\alpha f_k \rangle \\ &\leq \liminf_{k \rightarrow \infty} \|D_{\mathbb{G}}^\alpha f_k\|(U), \end{aligned}$$

where $\varphi \in C_c^\infty(U; H\mathbb{G})$ satisfies $\sup_{x \in U} |\varphi(x)| \leq 1$. Thus

$$\|D_{\mathbb{G}}^\alpha f\|(U) \leq \liminf_{k \rightarrow \infty} \|D_{\mathbb{G}}^\alpha f_k\|(U).$$

By Definition 5, we conclude that $f \in BV^\alpha(\mathbb{G})$. $\square$

As a simple application of Theorem 5, we know that for $0 < \alpha < 1$, $BV^\alpha(\mathbb{G})$ endowed with the norm

$$\|f\|_{BV^\alpha(\mathbb{G})} := \|f\|_{L^1(\mathbb{G})} + \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G})$$

is a Banach space.

4.2 Properties of the fractional BV functions

4.2.1 Smooth approximation and Gagliardo-Nirenberg-Sobolev inequality

Lemma 6 *Let $0 < \alpha < 1$ and let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers. Then for each $\varphi \in \text{Lip}_c(\mathbb{G}; H\mathbb{G})$,*

$$\text{div}_{\mathbb{G}}^\alpha(\rho_\varepsilon * \varphi) = \rho_\varepsilon * \text{div}_{\mathbb{G}}^\alpha \varphi$$

holds for any $\varepsilon > 0$.

Proof For each $\varphi = \sum_{j=1}^m \varphi_j X_j \in \text{Lip}_c(\mathbb{G}; H\mathbb{G})$, by (2.6),

$$\text{div}_{\mathbb{G}}(\rho_\varepsilon * \varphi) = \sum_{j=1}^m X_j(\rho_\varepsilon * \varphi_j) = \sum_{j=1}^m \rho_\varepsilon * X_j \varphi_j = \rho_\varepsilon * \text{div}_{\mathbb{G}} \varphi.$$

Then by the definition of $\text{div}_{\mathbb{G}}^\alpha$,

$$\text{div}_{\mathbb{G}}^\alpha(\rho_\varepsilon * \varphi) = \text{div}_{\mathbb{G}}(\rho_\varepsilon * \varphi) * R_{1-\alpha} = \rho_\varepsilon * (\text{div}_{\mathbb{G}} \varphi * R_{1-\alpha}) = \rho_\varepsilon * \text{div}_{\mathbb{G}}^\alpha \varphi.$$

□

Combining with Fubini's Theorem and Proposition 6 yields the following two corollaries.

Corollary 3 *Let $0 < \alpha < 1$. For each $f \in BV^\alpha(\mathbb{G})$, $\rho_\varepsilon * f \in BV^\alpha(\mathbb{G})$. Moreover,*

$$D_{\mathbb{G}}^\alpha(\rho_\varepsilon * f) = \rho_\varepsilon * D_{\mathbb{G}}^\alpha f \text{ in } \mathcal{M}(\mathbb{G}; H\mathbb{G}).$$

Corollary 4 *Let $0 < \alpha < 1$. For each $f \in BV^\alpha(\mathbb{G})$ and $\varphi \in \text{Lip}_c(\mathbb{G}; H\mathbb{G})$,*

$$\int_{\mathbb{G}} f \text{div}_{\mathbb{G}}^\alpha \varphi \, d\mu = - \int_{\mathbb{G}} \langle \varphi, dD_{\mathbb{G}}^\alpha f \rangle.$$

In analogy with [28, Theorem 2.2.2], the following proposition states the smooth approximation of fractional BV functions using mollifiers.

Proposition 9 *Let $0 < \alpha < 1$ and let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers. Then for each $f \in BV^\alpha(\mathbb{G})$,*

$$\rho_\varepsilon * f \rightarrow f \text{ in } L^1(\mathbb{G}) \text{ and } \|D_{\mathbb{G}}^\alpha(\rho_\varepsilon * f)\|(\mathbb{G}) \rightarrow \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}) \text{ as } \varepsilon \rightarrow 0.$$

Proof It is clear that $\rho_\varepsilon * f \rightarrow f$ in $L^1(\mathbb{G})$ as $\varepsilon \rightarrow 0$. Then by Theorem 5,

$$\|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}) \leq \liminf_{\varepsilon \rightarrow 0} \|D_{\mathbb{G}}^\alpha(\rho_\varepsilon * f)\|(\mathbb{G}). \quad (4.2)$$

On the other hand, by definition,

$$\|D_{\mathbb{G}}^\alpha(\rho_\varepsilon * f)\|(\mathbb{G}) = \sup \left\{ \int_{\mathbb{G}} (\rho_\varepsilon * f) \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu; \varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G}), \sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1 \right\}.$$

By Proposition 5, $\operatorname{div}_{\mathbb{G}}^\alpha \varphi \in L^\infty(\mathbb{G})$. Then combining Fubini's Theorem and Lemma 6 yields

$$\begin{aligned} \int_{\mathbb{G}} (\rho_\varepsilon * f) \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu &= \int_{\mathbb{G}} (\rho_\varepsilon * \operatorname{div}_{\mathbb{G}}^\alpha \varphi) f \, d\mu = \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha(\rho_\varepsilon * \varphi) \, d\mu \\ &\leq \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}). \end{aligned}$$

As a result

$$\limsup_{\varepsilon \rightarrow 0} \|D_{\mathbb{G}}^\alpha(\rho_\varepsilon * f)\|(\mathbb{G}) \leq \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}). \quad (4.3)$$

Combining (4.2) and (4.3), the proof is complete. $\square$

Based on Proposition 9, we further obtain the following proposition on approximating the fractional BV functions by C_c^∞ functions.

Proposition 10 *Let $0 < \alpha < 1$. For each $f \in BV^\alpha(\mathbb{G})$, there exists $\{f_k\}_{k=1}^\infty \subset C_c^\infty(\mathbb{G})$ such that*

$$f_k \rightarrow f \text{ in } L^1(\mathbb{G}) \text{ and } \|D_{\mathbb{G}}^\alpha f_k\|(\mathbb{G}) \rightarrow \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}) \text{ as } k \rightarrow \infty.$$

Proof By Proposition 9, we need only consider the case $f \in C^\infty(\mathbb{G}) \cap BV^\alpha(\mathbb{G})$. Consider a family of smooth cut-off functions $\{g_R\}_{R>0}$ satisfying

$$0 \leq g_R \leq 1, \quad g_R = 1 \text{ on } B(0, R), \quad \operatorname{supp} g_R \subset B(0, R+1) \text{ and } \|g_R\|_{\operatorname{Lip}} \leq 2, \quad (4.4)$$

and set $f_k = fg_k$ for $k \in \mathbb{N}$. Then it is obvious that $f_k \rightarrow f$ in $L^1(\mathbb{G})$ as $k \rightarrow \infty$. By Theorem 5, in order to prove that $\|D_{\mathbb{G}}^\alpha f_k\|(\mathbb{G}) \rightarrow \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G})$, it suffices to verify

$$\limsup_{k \rightarrow \infty} \|D_{\mathbb{G}}^\alpha f_k\|(\mathbb{G}) \leq \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}).$$

By Proposition 7, for any $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,

$$g_k \operatorname{div}_{\mathbb{G}}^\alpha \varphi = \operatorname{div}_{\mathbb{G}}^\alpha(g_k \varphi) - \langle \varphi, I_{1-\alpha}(\nabla_{\mathbb{G}} g_k) \rangle - \operatorname{div}_{\mathbb{G}, \operatorname{NL}}^\alpha(g_k, \varphi).$$

Combined with (3.7) and Proposition 2 (3), if the test function φ additionally satisfies $\sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1$, then

$$|g_k(x) \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi(x) - \operatorname{div}_{\mathbb{G}}^{\alpha} (g_k \varphi)(x)| \lesssim \int_{\mathbb{G}} \frac{|g_k(x) - g_k(y)|}{d(x, y)^{Q+\alpha}} d\mu(y).$$

Consequently, there exists $c > 0$ such that for any $\varphi \in C_c^{\infty}(\mathbb{G}; H(\mathbb{G}))$ with $\sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1$,

$$\begin{aligned} \int_{\mathbb{G}} f_k \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi d\mu &\leq \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^{\alpha} (g_k \varphi) d\mu + c \int_{\mathbb{G}} |f(x)| \int_{\mathbb{G}} \frac{|g_k(x) - g_k(y)|}{d(x, y)^{Q+\alpha}} d\mu(y) d\mu(x) \\ &\leq \|D_{\mathbb{G}}^{\alpha} f\|(\mathbb{G}) + c \int_{\mathbb{G}} |f(x)| \int_{\mathbb{G}} \frac{|g_k(x) - g_k(y)|}{d(x, y)^{Q+\alpha}} d\mu(y) d\mu(x), \quad \forall k \in \mathbb{N}. \end{aligned}$$

Using (4.4), the dominated convergence Theorem implies that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{G}} |f(x)| \int_{\mathbb{G}} \frac{|g_k(x) - g_k(y)|}{d(x, y)^{Q+\alpha}} d\mu(y) d\mu(x) = 0.$$

By Definition 5, we complete the proof. $\square$

To establish the Gagliardo-Nirenberg-Sobolev inequality, we need to introduce the so-called Lorentz spaces $L^{p,q}(\mathbb{G})$ (see e.g. [32, Section 1.4] for more details).

Definition 6 For $1 < p, q < \infty$ and μ -measurable function f , we define

$$\|f\|_{L^{p,q}(\mathbb{G})} = \left(\int_0^{\infty} (s^{\frac{1}{p}} f^*(s))^q \frac{ds}{s} \right)^{\frac{1}{q}},$$

where $f^* : [0, \infty) \rightarrow [0, \infty)$ is the decreasing rearrangement of f , which is given by

$$f^*(s) := \inf \{t > 0; \mu(\{x; |f(x)| > t\}) \leq s\}.$$

Then the Lorentz space $L^{p,q}(\mathbb{G})$ is the set of all the μ -measurable functions f on $\mathbb{G}$ satisfying $\|f\|_{L^{p,q}(\mathbb{G})} < \infty$. And we have the following relationship between Lorentz spaces.

Lemma 7 For any $1 < p < \infty$ and $1 \leq q < r < \infty$,

$$L^{p,p}(\mathbb{G}) = L^p(\mathbb{G}) \quad \text{and} \quad \|f\|_{L^{p,r}(\mathbb{G})} \lesssim \|f\|_{L^{p,q}(\mathbb{G})}.$$

Now we can establish the following Gagliardo-Nirenberg-Sobolev inequality for the fractional BV functions.

Theorem 6 (Gagliardo-Nirenberg-Sobolev inequality) Let $0 < \alpha < 1$. Then for each $f \in BV^{\alpha}(\mathbb{G})$,

$$\|f\|_{L^{\frac{Q}{Q-\alpha}}(\mathbb{G})} \lesssim \|D_{\mathbb{G}}^{\alpha} f\|(\mathbb{G}).$$

Proof By the proof of a recent result [35, p. 8-9], it is known that for any smooth g with $\|\nabla_{\mathbb{G}} g\|_{L^1(\mathbb{G}; H\mathbb{G})} < \infty$,

$$\|I_{\alpha} \mathcal{L}^{\frac{1}{2}} g\|_{L^{\frac{Q}{Q-\alpha}, 1}(\mathbb{G})} \lesssim \|\nabla_{\mathbb{G}} g\|_{L^1(\mathbb{G}; H\mathbb{G})}. \quad (4.5)$$

For each $f \in C_c^{\infty}(\mathbb{G})$, it is clear that $I_{1-\alpha} f \in C^{\infty}(\mathbb{G})$. And by Proposition 5, $\nabla_{\mathbb{G}} I_{1-\alpha} f = \nabla_{\mathbb{G}}^{\alpha} f \in L^1(\mathbb{G}; H\mathbb{G})$. Then applying (4.5) to $I_{1-\alpha} f$ yields

$$\begin{aligned} \|f\|_{L^{\frac{Q}{Q-\alpha}}(\mathbb{G})} &\lesssim \|f\|_{L^{\frac{Q}{Q-\alpha}, 1}(\mathbb{G})} = \|I_{\alpha} \mathcal{L}^{\frac{1}{2}} I_{1-\alpha} f\|_{L^{\frac{Q}{Q-\alpha}, 1}(\mathbb{G})} \\ &\lesssim \|\nabla_{\mathbb{G}} I_{1-\alpha} f\|_{L^1(\mathbb{G})} = \|D_{\mathbb{G}}^{\alpha} f\|(\mathbb{G}), \end{aligned} \quad (4.6)$$

where we have also used (4.1) and Lemma 7.

As for a general $f \in BV^{\alpha}(\mathbb{G})$, by Proposition 10, there exists $\{f_k\}_{k=1}^{\infty} \subset C_c^{\infty}(\mathbb{G})$ satisfying

$$f_k \rightarrow f \text{ in } L^1(\mathbb{G}), \quad f_k \rightarrow f \text{ } \mu - \text{a.e.} \quad \text{and} \quad \|D_{\mathbb{G}}^{\alpha} f_k\|(\mathbb{G}) \rightarrow \|D_{\mathbb{G}}^{\alpha} f\|(\mathbb{G}) \text{ as } k \rightarrow \infty.$$

Therefore using (4.6) and Fatou's Lemma,

$$\begin{aligned} \|f\|_{L^{\frac{Q}{Q-\alpha}}(\mathbb{G})} &\leq \liminf_{k \rightarrow \infty} \|f_k\|_{L^{\frac{Q}{Q-\alpha}}(\mathbb{G})} \lesssim \liminf_{k \rightarrow \infty} \|\nabla_{\mathbb{G}}^{\alpha} f_k\|_{L^1(\mathbb{G})} \\ &= \lim_{k \rightarrow \infty} \|D_{\mathbb{G}}^{\alpha} f_k\|(\mathbb{G}) = \|D_{\mathbb{G}}^{\alpha} f\|(\mathbb{G}), \end{aligned}$$

which is the desired conclusion. $\square$

4.2.2 Compactness

Lemma 8 *Let $0 < \alpha < 1$. Then for each $f \in C_c^{\infty}(\mathbb{G})$ and $j = 1, \dots, m$, $X_j^2(I_{1-\alpha} f) \in L^1(\mathbb{G})$.*

Proof For $f \in C_c^{\infty}(\mathbb{G})$, we assume that $\text{supp } f \subset B(0, M)$ for some $M > 0$. Since $I_{1-\alpha} f \in C^{\infty}(\mathbb{G})$, we have $X_j^2(I_{1-\alpha} f) \in L^1(B(0, M+1))$.

On the other hand, for every $x \in B(0, M+1)^c$, the function $R_{1-\alpha}(x \circ \cdot)$ is smooth on a neighborhood of $\text{supp } f$. Then applying (2.6), we obtain

$$X_j^2(I_{1-\alpha} f)(x) = \int_{\mathbb{G}} f(y) X_j^2 R_{1-\alpha}(x \circ y) d\mu(y).$$

Combined with Proposition 2, it follows that

$$\begin{aligned} \|X_j^2(I_{1-\alpha} f)\|_{L^1(B(0, M+1)^c)} &\lesssim \int_{B(0, M+1)^c} \int_{B(0, M)} \frac{|f(y)|}{d(x, y)^{Q+\alpha+1}} d\mu(y) d\mu(x) \\ &\lesssim \|f\|_{L^1(\mathbb{G})} < \infty. \end{aligned}$$

$\square$

As a consequence, we have the following fractional version of the Fundamental Theorem of Calculus.

Lemma 9 (Fractional Fundamental Theorem of Calculus) *Let $0 < \alpha < 1$. Then for each $f \in C_c^\infty(\mathbb{G})$,*

$$f(y) - f(x) = \int_{\mathbb{G}} \nabla_{\mathbb{G}}^\alpha f(z) \cdot (\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(z^{-1} \circ x) - \tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(z^{-1} \circ y)) \, d\mu(z).$$

Proof Clearly, it suffices to prove that for $f \in C_c^\infty(\mathbb{G})$,

$$f(x) = - \int_{\mathbb{G}} \nabla_{\mathbb{G}}^\alpha f(z) \cdot \tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(z^{-1} \circ x) \, d\mu(z). \quad (4.7)$$

For $0 < \alpha < 1$, we define the divergence operator of order $-\alpha$ by

$$\operatorname{div}_{\mathbb{G}}^{-\alpha} \varphi := I_{1+\alpha}(\operatorname{div}_{\mathbb{G}} \varphi)$$

for suitable $\varphi : \mathbb{G} \rightarrow H\mathbb{G}$. Then we observe that

$$-\operatorname{div}_{\mathbb{G}}^{-\alpha} \circ \nabla_{\mathbb{G}}^\alpha = \operatorname{Id}_{C_c^\infty(\mathbb{G})}.$$

Indeed, for $f \in C_c^\infty(\mathbb{G})$,

$$-\operatorname{div}_{\mathbb{G}}^{-\alpha}(\nabla_{\mathbb{G}}^\alpha f) = -I_{1+\alpha}(\operatorname{div}_{\mathbb{G}}(\nabla_{\mathbb{G}}^\alpha f)) = I_{1+\alpha} \mathcal{L} I_{1-\alpha} f = f,$$

where we have used $\operatorname{div}_{\mathbb{G}} \circ \nabla_{\mathbb{G}} = -\mathcal{L}$. Consequently, we have

$$f(x) = - \int_{\mathbb{G}} \operatorname{div}_{\mathbb{G}}(\nabla_{\mathbb{G}}^\alpha f)(z) R_{1+\alpha}(z^{-1} \circ x) \, d\mu(z). \quad (4.8)$$

By the same argument as in Proposition 3, we can prove the following statement. Assume that $g : \mathbb{G} \rightarrow \mathbb{R}$ is smooth and belongs to $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. In addition, $X_j g \in L^1(\mathbb{G})$ for some $j = 1, \dots, m$. Then

$$\int_{\mathbb{G}} X_j g(z) R_{1+\alpha}(z^{-1} \circ x) \, d\mu(z) = \int_{\mathbb{G}} g(z) \tilde{X}_j R_{1+\alpha}(z^{-1} \circ x) \, d\mu(z).$$

Applying this identity to $X_j(I_{1-\alpha} f)$, combining with (4.8), Proposition 5 and Lemma 8, (4.7) is established. $\square$

Proposition 11 *Let $0 < \alpha < 1$. Then for each $f \in C_c^\infty(\mathbb{G})$ and $y \in \mathbb{G}$,*

$$\int_{\mathbb{G}} |f(x \circ y) - f(x)| \, d\mu(x) \lesssim d^\alpha(y) \|\nabla_{\mathbb{G}}^\alpha f\|_{L^1(\mathbb{G}; H\mathbb{G})}. \quad (4.9)$$

Proof Applying Lemma 9, for each $f \in C_c^\infty(\mathbb{G})$ and $y \in \mathbb{G}$,

$$\begin{aligned}
& \int_{\mathbb{G}} |f(x \circ y) - f(x)| \, d\mu(x) \\
& \leq \int_{\mathbb{G}} \int_{\mathbb{G}} |\nabla_{\mathbb{G}}^{\alpha} f(x \circ w) \cdot (\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1} \circ y) - \tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1}))| \, d\mu(w) d\mu(x) \\
& \leq \|\nabla_{\mathbb{G}}^{\alpha} f\|_{L^1(\mathbb{G}; H\mathbb{G})} \cdot \int_{\mathbb{G}} |\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1} \circ y) - \tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1})| \, d\mu(w).
\end{aligned}$$

Then it suffices to show that

$$\int_{\mathbb{G}} |\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1} \circ y) - \tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1})| \, d\mu(w) \lesssim d^{\alpha}(y).$$

To this end, we split this integral into two pieces. By using Proposition 2, we have

$$\begin{aligned}
& \int_{\{d(w,y) \leq 3d(y)\}} |\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1} \circ y) - \tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1})| \, d\mu(w) \\
& \leq \int_{\{d(w,y) \leq 3d(y)\}} |\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1} \circ y)| + |\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1})| \, d\mu(w) \\
& \leq 2 \int_{\{d(w) \leq 4d(y)\}} |\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w)| \, d\mu(w) \lesssim \int_{\{d(w) \leq 4d(y)\}} \frac{1}{d^{Q-\alpha}(w)} \, d\mu(w) \\
& \lesssim d^{\alpha}(y). \tag{4.10}
\end{aligned}$$

On the other hand, for every w with $d(w, y) > 3d(y)$, by (2.4) we can find a horizontal curve $\gamma : [0, d(y)] \rightarrow \mathbb{G}$, satisfying $\gamma(0) = w^{-1}$, $\gamma(d(y)) = w^{-1} \circ y$, and

$$d(\gamma(t)) \geq d(\gamma(0)) - t \geq d(w) - d(y) > \frac{1}{2}d(w), \quad \forall t \in [0, d(y)],$$

where we have used $d(w) \geq d(w, y) - d(y) > 2d(y)$. Then for $j = 1, \dots, m$, combined with Proposition 2, it follows that

$$\begin{aligned}
|\tilde{X}_j R_{1+\alpha}(w^{-1} \circ y) - \tilde{X}_j R_{1+\alpha}(w^{-1})| & \leq \int_0^{d(y)} |\nabla_{\mathbb{G}} \tilde{X}_j R_{1+\alpha}(\gamma(t))| \, dt \\
& \lesssim \frac{d(y)}{d^{Q+1-\alpha}(w)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_{\{d(w,y) > 3d(y)\}} |\tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1} \circ y) - \tilde{\nabla}_{\mathbb{G}} R_{1+\alpha}(w^{-1})| \, d\mu(w) \\
& \leq \sum_{j=1}^m \int_{\{d(w) > 2d(y)\}} |\tilde{X}_j R_{1+\alpha}(w^{-1} \circ y) - \tilde{X}_j R_{1+\alpha}(w^{-1})| \, d\mu(w) \\
& \lesssim d(y) \cdot \int_{\{d(w) > 2d(y)\}} \frac{1}{d^{Q+1-\alpha}(w)} \, d\mu(w) \lesssim d^{\alpha}(y). \tag{4.11}
\end{aligned}$$

Combining (4.10) and (4.11), we complete the proof. $\square$

Remark 7 The analogue of Proposition 11 for the case $\alpha = 1$ is also valid. Indeed, for every $y \in \mathbb{G}$, let $\gamma_y : [0, d(y)] \rightarrow \mathbb{G}$ be the horizontal curve jointing 0 and y , which satisfies (2.4). Then for each $f \in C_c^\infty(\mathbb{G})$, by Fubini's Theorem,

$$\begin{aligned} \int_{\mathbb{G}} |f(x \circ y) - f(x)| d\mu(x) &\leq \int_{\mathbb{G}} \left(\int_0^{d(y)} |\nabla_{\mathbb{G}} f(x \circ \gamma_y(t))| dt \right) d\mu(x) \\ &\leq d(y) \|\nabla_{\mathbb{G}} f\|_{L^1(\mathbb{G}; H\mathbb{G})}. \end{aligned}$$

Corollary 5 Let $0 < \alpha < 1$ and let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers. Then for each $f \in BV^\alpha(\mathbb{G})$ and $\varepsilon > 0$,

$$\|f * \rho_\varepsilon - f\|_{L^1(\mathbb{G})} \lesssim \varepsilon^\alpha \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G})$$

Proof For each $f \in BV^\alpha(\mathbb{G})$, by Proposition 10, there exists $\{f_k\}_{k=1}^\infty \subset C_c^\infty(\mathbb{G})$ satisfying

$$f_k \rightarrow f \text{ in } L^1(\mathbb{G}) \quad \text{and} \quad \|D_{\mathbb{G}}^\alpha f_k\|(\mathbb{G}) \rightarrow \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}) \quad \text{as } k \rightarrow \infty.$$

By a change of variables,

$$(f * \rho_\varepsilon)(x) = \int_{\mathbb{G}} f(z) \rho_\varepsilon(z^{-1} \circ x) d\mu(z) = \int_{\mathbb{G}} f(x \circ \delta_\varepsilon(y)) \rho(y) d\mu(y).$$

Then applying Proposition 11, we have

$$\begin{aligned} \|f_k * \rho_\varepsilon - f_k\|_{L^1(\mathbb{G})} &\leq \int_{\mathbb{G}} \int_{\mathbb{G}} |f_k(x \circ \delta_\varepsilon(y)) - f_k(x)| \rho(y) d\mu(y) d\mu(x) \\ &= \int_{\mathbb{G}} \rho(y) \left(\int_{\mathbb{G}} |f_k(x \circ \delta_\varepsilon(y)) - f_k(x)| d\mu(x) \right) d\mu(y) \\ &\lesssim \varepsilon^\alpha \|\nabla_{\mathbb{G}}^\alpha f_k\|_{L^1(\mathbb{G}; H\mathbb{G})} \cdot \int_{\mathbb{G}} \rho(y) d^\alpha(y) d\mu(y) \\ &\lesssim \varepsilon^\alpha \|D_{\mathbb{G}}^\alpha f_k\|(\mathbb{G}). \end{aligned}$$

Thus

$$\begin{aligned} \|f * \rho_\varepsilon - f\|_{L^1(\mathbb{G})} &= \lim_{k \rightarrow \infty} \|f_k * \rho_\varepsilon - f\|_{L^1(\mathbb{G})} \\ &\lesssim \varepsilon^\alpha \lim_{k \rightarrow \infty} \|D_{\mathbb{G}}^\alpha f_k\|(\mathbb{G}) = \varepsilon^\alpha \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}). \end{aligned}$$

$\square$

Now we can prove the compactness theorem for $BV^\alpha(\mathbb{G})$ as follows.

Theorem 7 (Compactness) *Let $0 < \alpha < 1$. If $\{f_k\}_{k=1}^\infty \subset BV^\alpha(\mathbb{G})$ satisfies*

$$\sup_{k \in \mathbb{N}} \|f_k\|_{BV^\alpha(\mathbb{G})} < \infty,$$

then we can find a subsequence $\{f_{k_j}\}_{j=1}^\infty \subset BV^\alpha(\mathbb{G})$ and a function $f \in L^1(\mathbb{G})$ such that

$$f_{k_j} \rightarrow f \text{ in } L^1_{\text{loc}}(\mathbb{G}) \text{ as } j \rightarrow \infty.$$

Proof This result can be deduced by Corollary 5 together with the Arzelà-Ascoli Theorem. The proof for the classical Euclidean case (see [13, Theorem 3.16]) also works in our setting without major modification. We omit the details. $\square$

4.3 Relationship between $BV^\alpha(\mathbb{G})$ and other function spaces

4.3.1 $W^{\alpha,1}(\mathbb{G}) \subset S^{\alpha,1}(\mathbb{G}) \subset BV^\alpha(\mathbb{G})$

Recalling the definition of $\nabla_{\mathbb{G}}^\alpha$ on $W^{\alpha,1}(\mathbb{G})$, we have the following theorem.

Theorem 8 *Let $0 < \alpha < 1$. Then for each $f \in W^{\alpha,1}(\mathbb{G})$ and $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,*

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu = - \int_{\mathbb{G}} \langle \varphi, \nabla_{\mathbb{G}}^\alpha f \rangle \, d\mu. \quad (4.12)$$

Consequently, $f \in S^{\alpha,1}(\mathbb{G})$ and

$$\|\nabla_{\mathbb{G}}^\alpha f\|_{L^1(\mathbb{G}; H\mathbb{G})} \lesssim [f]_{W^{\alpha,1}(\mathbb{G})}. \quad (4.13)$$

Proof [13, Appendix A] with slight modification indicates that $C_c^\infty(\mathbb{G})$ is dense in $W^{\alpha,1}(\mathbb{G})$. Then for each $f \in W^{\alpha,1}(\mathbb{G})$, we can find $\{f_k\}_{k=1}^\infty \subset C_c^\infty(\mathbb{G})$ satisfying

$$f_k \rightarrow f \text{ in } L^1(\mathbb{G}) \text{ and } [f - f_k]_{W^{\alpha,1}(\mathbb{G})} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

And combining with (3.18), we have

$$\nabla_{\mathbb{G}}^\alpha f_k \rightarrow \nabla_{\mathbb{G}}^\alpha f \text{ in } L^1(\mathbb{G}; H\mathbb{G}) \text{ as } k \rightarrow \infty. \quad (4.14)$$

Since $f \in L^1(\mathbb{G})$, by Lemma 1, for every $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu = \int_{\mathbb{G}} f I_{1-\alpha}(\operatorname{div}_{\mathbb{G}} \varphi) \, d\mu = \int_{\mathbb{G}} \operatorname{div}_{\mathbb{G}} \varphi (I_{1-\alpha} f) \, d\mu.$$

By Lemma 2, it is known that $I_{1-\alpha} f_k \rightarrow I_{1-\alpha} f$ in $L^1_{\text{loc}}(\mathbb{G})$ as $k \rightarrow \infty$. Combined with (4.14), it follows that

$$\begin{aligned} \int_{\mathbb{G}} \operatorname{div}_{\mathbb{G}} \varphi(I_{1-\alpha} f) \, d\mu &= \lim_{k \rightarrow \infty} \int_{\mathbb{G}} \operatorname{div}_{\mathbb{G}} \varphi(I_{1-\alpha} f_k) \, d\mu = - \lim_{k \rightarrow \infty} \int_{\mathbb{G}} \langle \varphi, \nabla_{\mathbb{G}}^{\alpha} f_k \rangle \, d\mu \\ &= - \int_{\mathbb{G}} \langle \varphi, \nabla_{\mathbb{G}}^{\alpha} f \rangle \, d\mu. \end{aligned}$$

Then (4.12) is established. (4.13) immediately follows from (3.18) together with (4.14). $\square$

Remark 8 By approximation, (4.12) still holds for $f \in w^{\alpha,1}(\mathbb{G}) \cap L^{\infty}(\mathbb{G})$ and $\varphi \in \operatorname{Lip}_c(\mathbb{G}; H\mathbb{G})$. As a by-product, the Leibniz's rule discussed in Section 3.4 could be generalized as follows. For each $f \in w^{\alpha,1}(\mathbb{G}) \cap L^{\infty}(\mathbb{G})$ and $\varphi \in \operatorname{Lip}_c(\mathbb{G}; H\mathbb{G})$,

$$\operatorname{div}_{\mathbb{G}}^{\alpha}(f\varphi)(x) = f(x) \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi(x) + \langle \varphi(x), I_{1-\alpha}(\nabla_{\mathbb{G}} f)(x) \rangle + \operatorname{div}_{\mathbb{G},\text{NL}}^{\alpha}(f, \varphi)(x),$$

holds for μ -a.e. $x \in \mathbb{G}$, where $I_{1-\alpha}(\nabla_{\mathbb{G}} f) \in L^1(\mathbb{G}; H\mathbb{G})$ is understood as

$$I_{1-\alpha}(\nabla_{\mathbb{G}} f)(x) = \sum_{j=1}^m \left(\int_{\mathbb{G}} (f(y) - f(x)) \tilde{X}_j R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y) \right) X_j$$

and

$$\operatorname{div}_{\mathbb{G},\text{NL}}^{\alpha}(f, \varphi)(x) := \int_{\mathbb{G}} (f(y) - f(x)) (\varphi(y) - \varphi(x)) \cdot \tilde{\nabla}_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) \, d\mu(y). \quad (4.15)$$

Moreover,

$$\| \operatorname{div}_{\mathbb{G},\text{NL}}^{\alpha}(f, \varphi) \|_{L^1(\mathbb{G})} \lesssim [f]_{W^{\alpha,1}(\mathbb{G})} \| \varphi \|_{L^{\infty}(\mathbb{G}; H\mathbb{G})}. \quad (4.16)$$

This generalized Leibniz's rule will be used in Section 5.

Combining (4.1) and Theorem 8, the following corollary is immediate.

Corollary 6 $W^{\alpha,1}(\mathbb{G}) \subset S^{\alpha,1}(\mathbb{G}) \subset BV^{\alpha}(\mathbb{G})$ with continuous embeddings.

4.3.2 $BV^{\beta}(\mathbb{G}) \subset W^{\alpha,1}(\mathbb{G}) \subset L^1_{\alpha}(\mathbb{G})$ for $\alpha < \beta$

Theorem 9 Let $0 < \alpha < \beta \leq 1$. Then $BV^{\beta}(\mathbb{G}) \subset W^{\alpha,1}(\mathbb{G})$ with continuous embedding. By convention $BV^1(\mathbb{G}) = BV(\mathbb{G})$.

Proof For each $f \in C_c^{\infty}(\mathbb{G})$, applying Proposition 11 and Remark 7, we have

$$\int_{\mathbb{G}} |f(x \circ z) - f(x)| \, d\mu(x) \lesssim \min \{ d^{\beta}(z) \| \nabla_{\mathbb{G}}^{\beta} f \|_{L^1(\mathbb{G}; H\mathbb{G})}, \| f \|_{L^1(\mathbb{G})} \}.$$

Therefore

$$\begin{aligned} [f]_{W^{1,\alpha}(\mathbb{G})} &= \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|f(x \circ z) - f(x)|}{d^{Q+\alpha}(z)} d\mu(x) d\mu(z) \\ &\lesssim \int_{\mathbb{G}} \min \{ d^{-Q+\beta-\alpha}(z) \|\nabla_{\mathbb{G}}^{\beta} f\|_{L^1(\mathbb{G}; H\mathbb{G})}, d^{-Q-\alpha}(z) \|f\|_{L^1(\mathbb{G})} \} d\mu(z) \\ &\lesssim \|f\|_{L^1(\mathbb{G})}^{1-\frac{\alpha}{\beta}} \cdot \|\nabla_{\mathbb{G}}^{\beta} f\|_{L^1(\mathbb{G}; H\mathbb{G})}^{\frac{\alpha}{\beta}} \lesssim \|f\|_{BV^{\beta}(\mathbb{G})}. \end{aligned} \quad (4.17)$$

For a general $f \in BV^{\beta}(\mathbb{G})$, by Proposition 10 (and the analogue for $BV(\mathbb{G})$, see [28]), we can find $\{f_k\}_{k=1}^{\infty} \subset C_c^{\infty}(\mathbb{G})$ satisfying

$$f_k \rightarrow f \text{ in } L^1(\mathbb{G}), \quad f_k \rightarrow f \text{ } \mu - \text{a.e. and } \|D_{\mathbb{G}}^{\beta} f_k\|(\mathbb{G}) \rightarrow \|D_{\mathbb{G}}^{\beta} f\|(\mathbb{G}) \text{ as } k \rightarrow \infty.$$

Then by the Fatou's Lemma and (4.17), we obtain

$$[f]_{W^{1,\alpha}(\mathbb{G})} \leq \liminf_{k \rightarrow \infty} [f_k]_{W^{1,\alpha}(\mathbb{G})} \lesssim \liminf_{k \rightarrow \infty} \|f_k\|_{BV^{\beta}(\mathbb{G})} = \|f\|_{BV^{\beta}(\mathbb{G})}.$$

□

As for the relationship between $W^{\alpha,1}(\mathbb{G})$ and $L_{\alpha}^1(\mathbb{G})$, we have the following result.

Theorem 10 *Let $0 < \alpha < 1$. Then $W^{\alpha,1}(\mathbb{G}) \subset L_{\alpha}^1(\mathbb{G})$ with continuous embedding.*

Proof As already explained in the proof of Theorem 8, for each $f \in W^{\alpha,1}(\mathbb{G})$, we can find $\{f_k\}_{k=1}^{\infty} \subset C_c^{\infty}(\mathbb{G})$ satisfying

$$f_k \rightarrow f \text{ in } L^1(\mathbb{G}), \quad f_k \rightarrow f \text{ } \mu - \text{a.e. and } [f - f_k]_{W^{\alpha,1}(\mathbb{G})} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (4.18)$$

Then using (2.13), for every $k, l \in \mathbb{N}$, we have

$$\begin{aligned} &\|\mathcal{L}^{\frac{\alpha}{2}} f_k - \mathcal{L}^{\frac{\alpha}{2}} f_l\|_{L^1(\mathbb{G})} \\ &\leq \int_{\mathbb{G}} \left| \int_{\mathbb{G}} ((f_k(y) - f_k(x)) - (f_l(y) - f_l(x))) \tilde{R}_{-\alpha}(y^{-1} \circ x) d\mu(y) \right| d\mu(x) \\ &\lesssim \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|(f_k - f_l)(x) - (f_k - f_l)(y)|}{d(x, y)^{Q+\alpha}} d\mu(x) d\mu(y) = [f_k - f_l]_{W^{\alpha,1}(\mathbb{G})}. \end{aligned} \quad (4.19)$$

Therefore $\{\mathcal{L}^{\frac{\alpha}{2}} f_k\}_{k=1}^{\infty}$ forms a Cauchy sequence in $L^1(\mathbb{G})$. Since $\mathcal{L}_1^{\frac{\alpha}{2}}$ is a closed operator on $L^1(\mathbb{G})$ (see [24, Theorem 4.36]). We know that $f \in \text{Dom}(\mathcal{L}_1^{\frac{\alpha}{2}})$ and $\mathcal{L}^{\frac{\alpha}{2}} f_k \rightarrow \mathcal{L}^{\frac{\alpha}{2}} f$ in $L^1(\mathbb{G})$ as $k \rightarrow \infty$, which implies that $f \in L_{\alpha}^1(\mathbb{G})$. And combining (4.18) and (4.19), we have

$$\|\mathcal{L}^{\frac{\alpha}{2}} f\|_{L^1(\mathbb{G})} \lesssim [f]_{W^{\alpha,1}(\mathbb{G})}. \quad (4.20)$$

So that

$$\|f\|_{L^1_\alpha(\mathbb{G})} = \|f\|_{L^1(\mathbb{G})} + \|\mathcal{L}^{\frac{\alpha}{2}} f\|_{L^1(\mathbb{G})} \lesssim \|f\|_{L^1(\mathbb{G})} + [f]_{W^{\alpha,1}(\mathbb{G})} = \|f\|_{W^{\alpha,1}(\mathbb{G})}.$$

□

4.3.3 Relationship between $BV^\alpha(\mathbb{G})$ and $BV(\mathbb{G})$

Applying Theorems 9 and 10, we can prove the following theorem.

Theorem 11 *Let $0 < \alpha < 1$. Then the following assertions hold.*

(1) *For each $f \in BV^\alpha(\mathbb{G})$, $I_{1-\alpha}f \in bv(\mathbb{G})$ with*

$$D_{\mathbb{G}}(I_{1-\alpha}f) = D_{\mathbb{G}}^\alpha f \text{ in } \mathcal{M}(\mathbb{G}; H\mathbb{G}).$$

(2) *For each $g \in BV(\mathbb{G})$, $\mathcal{L}^{\frac{1-\alpha}{2}}g \in BV^\alpha(\mathbb{G})$ with*

$$\|\mathcal{L}^{\frac{1-\alpha}{2}}g\|_{L^1(\mathbb{G})} \lesssim \|g\|_{BV(\mathbb{G})} \text{ and } D_{\mathbb{G}}^\alpha(\mathcal{L}^{\frac{1-\alpha}{2}}g) = D_{\mathbb{G}}g \text{ in } \mathcal{M}(\mathbb{G}; H\mathbb{G}).$$

Proof (1) For each $f \in BV^\alpha(\mathbb{G})$, combining (2.11) and Theorem 6, it follows that $I_{1-\alpha}f \in L^{\frac{Q}{Q-1}}(\mathbb{G}) (\subset L^1_{\text{loc}}(\mathbb{G}))$. Then by Lemma 1, for every $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu = \int_{\mathbb{G}} f I_{1-\alpha}(\operatorname{div}_{\mathbb{G}} \varphi) \, d\mu = \int_{\mathbb{G}} (I_{1-\alpha}f) \operatorname{div}_{\mathbb{G}} \varphi \, d\mu.$$

Thus by definition, $I_{1-\alpha}f \in bv(\mathbb{G})$ and $D_{\mathbb{G}}(I_{1-\alpha}f) = D_{\mathbb{G}}^\alpha f$ in $\mathcal{M}(\mathbb{G}; H\mathbb{G})$.

(2) For each $g \in BV(\mathbb{G})$, by Theorems 9 and 10, $g \in L^\beta_{\mathbb{G}}(\mathbb{G})$ for any $\beta \in (0, 1)$. As a result, $g \in \operatorname{Dom}(\mathcal{L}^\beta_1)$. Then combining (2.12) and (4.20),

$$I_{1-\alpha}\mathcal{L}^{\frac{1-\alpha}{2}}g = g \text{ in } L^1(\mathbb{G})$$

holds via approximation. And by Theorem 10,

$$\|\mathcal{L}^{\frac{1-\alpha}{2}}g\|_{L^1(\mathbb{G})} \lesssim \|g\|_{BV(\mathbb{G})}.$$

Then by Lemma 1, for every $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,

$$\int_{\mathbb{G}} \operatorname{div}_{\mathbb{G}}^\alpha \varphi (\mathcal{L}^{\frac{1-\alpha}{2}}g) \, d\mu = \int_{\mathbb{G}} \operatorname{div}_{\mathbb{G}} \varphi I_{1-\alpha}(\mathcal{L}^{\frac{1-\alpha}{2}}g) \, d\mu = \int_{\mathbb{G}} g \operatorname{div}_{\mathbb{G}} \varphi \, d\mu.$$

Thus $\mathcal{L}^{\frac{1-\alpha}{2}}g \in BV^\alpha(\mathbb{G})$ and $D_{\mathbb{G}}^\alpha(\mathcal{L}^{\frac{1-\alpha}{2}}g) = D_{\mathbb{G}}g$ in $\mathcal{M}(\mathbb{G}; H\mathbb{G})$. □

By the same argument used in [13, Theorem 3.30], Theorem 11 implies the following result.

Corollary 7 *Let $0 < \alpha < 1$. Then the inclusion $S^{\alpha,1}(\mathbb{G}) \subset BV^\alpha(\mathbb{G})$ is strict.*

Thanks to the inclusions discussed in the preceding sections, we can establish the following relation: for every $0 < \alpha < 1$,

$$\bigcup_{\alpha < \beta \leq 1} BV^\beta(\mathbb{G}) \subset W^{\alpha,1}(\mathbb{G}) \subsetneq BV^\alpha(\mathbb{G}) \subset \bigcap_{0 < \gamma < \alpha} W^{\gamma,1}(\mathbb{G}) = \bigcap_{0 < \gamma < \alpha} BV^\gamma(\mathbb{G}).$$

Now we define

$$BV^{1-}(\mathbb{G}) := \bigcap_{0 < \gamma < 1} W^{\gamma,1}(\mathbb{G}) = \bigcap_{0 < \gamma < 1} BV^\gamma(\mathbb{G}).$$

Then it is obvious that $BV(\mathbb{G}) \subset BV^{1-}(\mathbb{G})$. Conversely, we have the following criterion for functions in $BV^{1-}(\mathbb{G})$ to lie in $BV(\mathbb{G})$.

Theorem 12 *If $f \in BV^{1-}(\mathbb{G})$ satisfies*

$$\liminf_{\alpha \rightarrow 1^-} \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}) < \infty,$$

then $f \in BV(\mathbb{G})$ with

$$\|D_{\mathbb{G}} f\|(\mathbb{G}) \leq \liminf_{\alpha \rightarrow 1^-} \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G})$$

and

$$D_{\mathbb{G}} I_{1-\alpha} f \rightarrow D_{\mathbb{G}} f \text{ in } \mathcal{M}(\mathbb{G}; H\mathbb{G}) \text{ as } \alpha \rightarrow 1^-.$$

Proof By Theorem 1, for every $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$ with $\sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1$,

$$\int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}} \varphi \, d\mu = \lim_{\alpha \rightarrow 1^-} \int_{\mathbb{G}} f \operatorname{div}_{\mathbb{G}}^\alpha \varphi \, d\mu \leq \liminf_{\alpha \rightarrow 1^-} \|D_{\mathbb{G}}^\alpha f\|(\mathbb{G}) < \infty.$$

Therefore $f \in BV(\mathbb{G})$. Combined with Theorem 11, it is obvious that $D_{\mathbb{G}} I_{1-\alpha} f \rightarrow D_{\mathbb{G}} f$ in $\mathcal{M}(\mathbb{G}; H\mathbb{G})$. $\square$

5 Fractional Caccioppoli sets

In this section, we shall study the so-called fractional Caccioppoli sets, whose characteristic functions belong to $BV^\alpha(\mathbb{G})$. The notations and definitions are the same as those used for Euclidean cases in [13, Section 4]. Since proofs in this section are almost the same as the ones in [13, Section 4], we only sketch them.

5.1 Definition of fractional Caccioppoli sets and their properties

Definition 7 Let $0 < \alpha < 1$ and let $U \subset \mathbb{G}$ be an open set. For every μ -measurable set E , the fractional Caccioppoli (horizontal) α -perimeter in U is defined by

$$\|D_{\mathbb{G}}^{\alpha}\chi_E\|(U) := \sup \left\{ \int_E \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi \, d\mu; \varphi \in C_c^{\infty}(U; H\mathbb{G}), \sup_{x \in U} |\varphi(x)| \leq 1 \right\}. \quad (5.1)$$

E is said to have finite fractional Caccioppoli (horizontal) α -perimeter in U if $\|D_{\mathbb{G}}^{\alpha}\chi_E\|(U) < \infty$. And E is said to have locally finite fractional Caccioppoli (horizontal) α -perimeter in U if $\|D_{\mathbb{G}}^{\alpha}\chi_E\|(V) < \infty$ for any open set $V \Subset U$.

By the same argument as in [13, Theorem 4.2], we have the following theorem.

Theorem 13 Let $0 < \alpha < 1$ and let $U \subset \mathbb{G}$ be an open set. $E \subset \mathbb{G}$ satisfies $\|D_{\mathbb{G}}^{\alpha}\chi_E\|(U) < \infty$ if and only if we can find an $H\mathbb{G}$ -valued measure on U , denoted by $D_{\mathbb{G}}^{\alpha}\chi_E$, such that for any $\varphi \in C_c^{\infty}(U; H\mathbb{G})$,

$$\int_E \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi \, d\mu = - \int_U \langle \varphi, dD_{\mathbb{G}}^{\alpha}\chi_E \rangle.$$

Moreover, it holds that for any open set $V \subset U$,

$$\|D_{\mathbb{G}}^{\alpha}\chi_E\|(V) = \sup \left\{ \int_E \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi \, d\mu; \varphi \in C_c^{\infty}(V; H\mathbb{G}), \sup_{x \in V} |\varphi(x)| \leq 1 \right\}.$$

By applying this theorem, several theorems developed in Section 4.2 could be adapted to the setting of fractional Caccioppoli sets. We state the following conclusions without proofs.

Proposition 12 Let $0 < \alpha < 1$ and let $U \subset \mathbb{G}$ be an open set. If $\{E_k\}_{k=1}^{\infty}$ is a sequence of μ -measurable sets satisfying

$$\forall k \in \mathbb{N}, \quad \|D_{\mathbb{G}}^{\alpha}\chi_{E_k}\|(U) < \infty \quad \text{and} \quad \chi_{E_k} \rightarrow \chi_E \text{ in } L_{\text{loc}}^1(\mathbb{G}) \text{ as } k \rightarrow \infty,$$

then

$$\|D_{\mathbb{G}}^{\alpha}\chi_E\|(U) \leq \liminf_{k \rightarrow \infty} \|D_{\mathbb{G}}^{\alpha}\chi_{E_k}\|(U)$$

and

$$D_{\mathbb{G}}^{\alpha}\chi_{E_k} \rightharpoonup D_{\mathbb{G}}^{\alpha}\chi_E \text{ in } \mathcal{M}(U; H\mathbb{G}) \text{ as } k \rightarrow \infty.$$

Theorem 14 Let $0 < \alpha < 1$. Then for each μ -measurable set E with $\mu(E) < \infty$ and $\|D_{\mathbb{G}}^{\alpha}\chi_E\|(\mathbb{G}) < \infty$,

$$\mu(E)^{\frac{Q-\alpha}{Q}} \lesssim \|D_{\mathbb{G}}^{\alpha}\chi_E\|(\mathbb{G}).$$

Theorem 15 Let $0 < \alpha < 1$. If $\{E_k\}_{k=1}^\infty$ is a sequence of μ -measurable sets satisfying

$$\forall k \in \mathbb{N}, E_k \subset B(0, M) \text{ for some } M > 0 \text{ and } \sup_{k \in \mathbb{N}} \|D_{\mathbb{G}}^\alpha \chi_{E_k}\|(\mathbb{G}) < \infty,$$

then we can find a subsequence $\{E_{k_j}\}_{j=1}^\infty$ and a set $E \subset B(0, M)$ which satisfies $\|D_{\mathbb{G}}^\alpha \chi_E\|(\mathbb{G}) < \infty$, such that

$$\chi_{E_{k_j}} \rightarrow \chi_E \text{ in } L^1(\mathbb{G}) \text{ as } j \rightarrow \infty.$$

5.2 Fractional reduced boundary and the fractional perimeter

In this subsection, we define the fractional reduced boundary and the fractional perimeter, and then establish some related results. We mainly work with the smooth homogeneous distance d_s rather than Carnot-Carathéodory distance d in this and the next subsection.

Definition 8 Let $0 < \alpha < 1$ and let $U \subset \mathbb{G}$ be an open set. If $E \subset \mathbb{G}$ satisfies $\|D_{\mathbb{G}}^\alpha \chi_E\|(U) < \infty$, then we say a point $x \in U$ belongs to the fractional reduced boundary of E (inside U) if $x \in \text{supp } D_{\mathbb{G}}^\alpha \chi_E$ and

$$v_E^\alpha(x) := \lim_{r \rightarrow 0^+} \frac{D_{\mathbb{G}}^\alpha \chi_E(B_s(x, r))}{\|D_{\mathbb{G}}^\alpha \chi_E\|(B_s(x, r))} \in H_{\mathbb{G}_x} \text{ exists with } |v_E^\alpha(x)| = 1.$$

We denote by $\mathcal{F}_{\mathbb{G}}^\alpha E$ the fractional α -reduced boundary of E (inside U).

- Remark 9** (1) It is worth mentioning that the definition of fractional α -reduced boundary may vary depending on the choice of distance. In this section we adopt the smooth distance d_s rather than the Carnot-Carathéodory distance d due to the fact that calculus is easier to handle on d_s -balls.
- (2) As in the Euclidean case (see [13, (4.9)]), by applying Theorem 13, the following formula

$$\int_E \text{div}_{\mathbb{G}}^\alpha \varphi \, d\mu = - \int_{U \cap \mathcal{F}_{\mathbb{G}}^\alpha E} \langle \varphi, v_E^\alpha \rangle \, d\|D_{\mathbb{G}}^\alpha \chi_E\| \quad (5.2)$$

holds for any $\varphi \in \text{Lip}_c(U; H_{\mathbb{G}})$.

The notation of fractional α -perimeter was introduced in [23], in which some relations between fractional perimeter and the fractional heat semigroup are further explored.

Definition 9 Let $0 < \alpha < 1$ and let $U \subset \mathbb{G}$ be an open set. The fractional (horizontal) α -perimeter of $E \subset \mathbb{G}$ in U is defined by

$$P_\alpha(E; U)$$

$$:= \left(\int_{E^c \cap U} \int_{E \cap U} + \int_{E^c \cap U} \int_{E \cap U^c} + \int_{E \cap U} \int_{E^c \cap U^c} \right) \frac{1}{d(x, y)^{Q+\alpha}} d\mu(x) d\mu(y).$$

In particular, if $U = \mathbb{G}$,

$$P_\alpha(E) := P_\alpha(E; \mathbb{G}) = \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|\chi_E(x) - \chi_E(y)|}{d(x, y)^{Q+\alpha}} d\mu(x) d\mu(y).$$

Remark 10 (1) By an elementary calculation, we have

$$\begin{aligned} P_\alpha(E; U) &= \iint_{(\mathbb{G} \times \mathbb{G}) \setminus (U^c \times U^c)} \frac{|\chi_E(x) - \chi_E(y)|}{d(x, y)^{Q+\alpha}} d\mu(x) d\mu(y) \\ &= \left(\int_U \int_U + 2 \int_U \int_{\mathbb{G} \setminus U} \right) \frac{|\chi_E(x) - \chi_E(y)|}{d(x, y)^{Q+\alpha}} d\mu(x) d\mu(y). \end{aligned}$$

(2) Taking into account the equivalence of d and d_s , the fractional α -perimeter defined via d_s is equivalent to P_α defined above.

In analogy with $W^{\alpha,1}(\mathbb{G}) \subset BV^\alpha(\mathbb{G})$, the following proposition describes the relationship between the fractional Caccioppoli α -perimeter and the fractional α -perimeter.

Proposition 13 *Let $0 < \alpha < 1$ and let $U \subset \mathbb{G}$ be an open set. If $E \subset \mathbb{G}$ satisfies $P_\alpha(E; U) < \infty$, then E has finite fractional Caccioppoli α -perimeter in U with*

$$\|D_{\mathbb{G}}^\alpha \chi_E\|(U) \lesssim P_\alpha(E; U). \quad (5.3)$$

And for any $\varphi \in C_c^\infty(U; H\mathbb{G})$,

$$\int_E \operatorname{div}_{\mathbb{G}}^\alpha \varphi d\mu = - \int_U \langle \varphi, \nabla_{\mathbb{G}}^\alpha \chi_E \rangle d\mu. \quad (5.4)$$

Combined with Theorem 13 and (5.2), it follows that

$$dD_{\mathbb{G}}^\alpha \chi_E = v_E^\alpha d\|D_{\mathbb{G}}^\alpha \chi_E\| = \nabla_{\mathbb{G}}^\alpha \chi_E d\mu.$$

Proof By Proposition 2 (3), we have the following prior estimate.

$$\begin{aligned} & \int_U \left| \int_{\mathbb{G}} (\chi_E(y) - \chi_E(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \right| d\mu(x) \\ & \lesssim \int_U \int_{\mathbb{G}} \frac{|\chi_E(x) - \chi_E(y)|}{d(x, y)^{Q+\alpha}} d\mu(x) d\mu(y) \lesssim P_\alpha(E; U) < \infty. \end{aligned}$$

By the definition of $\nabla_{\mathbb{G}}^\alpha$, it holds that for μ -a.e. $x \in U$,

$$\nabla_{\mathbb{G}}^\alpha \chi_E(x) := \int_{\mathbb{G}} (\chi_E(y) - \chi_E(x)) \nabla_{\mathbb{G}} R_{1-\alpha}(y^{-1} \circ x) d\mu(y) \in L^1(U; H\mathbb{G}),$$

and

$$\|\nabla_{\mathbb{G}}^{\alpha} \chi_E\|_{L^1(U; H\mathbb{G})} \lesssim P_{\alpha}(E; U). \quad (5.5)$$

Then (5.4) can be established by a similar proof as Theorem 8. And (5.3) immediately follows from Theorem 13, (5.4) and (5.5). $\square$

5.3 Blow-ups for fractional Caccioppoli sets

The so-called blow-up theorem plays an important role in the proof of the rectifiability theorem for finite perimeter sets in the classical Euclidean setting, as originally established by De Giorgi. Unfortunately, its direct analogies do not hold true in the context of general Carnot groups. This is due to the fact that sets with constant horizontal normal are not necessarily half-spaces. However, there exists an extensive literature on the properties of tangent sets, as defined in Definition 11, for finite perimeter sets on Carnot groups. Additional information on this topic can be found in references [2, 29, 30, 37, 40]. In this section, we aim to prove the existence of blow-ups for fractional Caccioppoli sets. For such purpose, it is useful to introduce the divergence formula and polar coordinates on $\mathbb{G}$. To begin with, we need the notation of $\mathbb{G}$ -perimeter, which was first discussed in [31].

Definition 10 Let E be a bounded open set of $\mathbb{G}$ with C^1 boundary, then the $\mathbb{G}$ -perimeter measure of E , denoted by $\|\partial E\|$, is a Radon measure on $\mathbb{G}$ and satisfies, for any open set $U \subset \mathbb{G}$,

$$\|\partial E\|(U) = \int_{\partial E \cap U} \left(\sum_{j=1}^m \langle X_j, \mathbf{n} \rangle_{\mathbb{R}^n}^2 \right)^{\frac{1}{2}} d\mathcal{H}^{n-1},$$

where $\mathbf{n}$ is the Euclidean unit outward normal to ∂E , $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is the scalar product on $T\mathbb{R}^n$ and $\mathcal{H}^{n-1}$ is the Euclidean $(n-1)$ -dimensional Hausdorff measure.

The application of the divergence formula on $\mathbb{R}^n$ yields the following result.

Lemma 10 (Divergence formula on $\mathbb{G}$) *Let E be a regular bounded open set of $\mathbb{G}$ with C^1 boundary and let $\phi \in C^1(U; H\mathbb{G}) \cap C(\overline{U}; H\mathbb{G})$. Then*

$$\int_E \operatorname{div}_{\mathbb{G}} \phi \, d\mu = \int_{\partial E} \langle \phi, \mathbf{n}_{\mathbb{G}} \rangle \, d\|\partial E\|,$$

where $\mathbf{n}_{\mathbb{G}}(x)$ is the (normalized) projection of $\mathbf{n}(x)$ on $H\mathbb{G}_x$ for $x \in \partial E$, and is given by

$$\mathbf{n}_{\mathbb{G}}(x) = \begin{cases} \sum_{j=1}^m \frac{\langle X_j, \mathbf{n}(x) \rangle_{\mathbb{R}^n}}{\left(\sum_{j=1}^m \langle X_j, \mathbf{n}(x) \rangle_{\mathbb{R}^n}^2 \right)^{\frac{1}{2}}} X_j, & \text{if } \sum_{j=1}^m \langle X_j, \mathbf{n}(x) \rangle_{\mathbb{R}^n}^2 > 0; \\ 0, & \text{if } \sum_{j=1}^m \langle X_j, \mathbf{n}(x) \rangle_{\mathbb{R}^n}^2 = 0. \end{cases}$$

For the case $E = B_s(x, r)$, combining the above lemma and Riesz Representation Theorem, for $j = 1, \dots, m$, there exists a (signed) Borel measure, denoted by $D_j \chi_{B_s(x, r)}$, which is supported on $\partial B_s(x, r)$ and satisfies

$$\int_{B_s(x, r)} X_j h \, d\mu = - \int_{\mathbb{G}} h \, dD_j \chi_{B_s(x, r)}, \quad \forall h \in C^1(U) \cap C(\overline{U}). \quad (5.6)$$

Using (2.2) and the properties of μ , for any $x \in \mathbb{G}$, $r > 0$ and $y \in \partial B_s(0, 1)$,

$$dD_j \chi_{B_s(x, r)}(x \circ \delta_r(y)) = r^{Q-1} dD_j \chi_{B_s(0, 1)}(y). \quad (5.7)$$

The polar coordinates on $\mathbb{G}$ is stated in the following lemma (see e.g. [24, Section 3.1.7]).

Lemma 11 (Polar coordinates on $\mathbb{G}$) *There exists a family of Radon measures $(\sigma_{x, r})_{x \in \mathbb{G}, r \geq 0}$, such that*

- (1) *Each $\sigma_{x, r}$ is supported on $\partial B_s(x, r)$;*
- (2) *For any $x \in \mathbb{G}$, $r > 0$ and $y \in \partial B_s(0, 1)$,*

$$d\sigma_{x, r}(x \circ \delta_r(y)) = r^{Q-1} d\sigma_{0, 1}(y);$$

- (3) *For any $f \in L^1(\mathbb{G})$ and $x \in \mathbb{G}$,*

$$\int_{\mathbb{G}} f \, d\mu = \int_0^\infty \left(\int_{\partial B_s(x, s)} f \, d\sigma_{x, s} \right) ds.$$

Now, for any $-(Q-1)$ -homogeneous function $\omega \in C^\infty(\mathbb{G} \setminus \{0\})$ and $r, \epsilon > 0$, by (5.6) and (5.7),

$$\int_{B_s(0, r+\epsilon) \setminus B_s(0, r)} X_j \omega \, d\mu = \int_{\mathbb{G}} \omega \, dD_j \chi_{B_s(0, r)} - \int_{\mathbb{G}} \omega \, dD_j \chi_{B_s(0, r+\epsilon)} = 0.$$

Combined with Lemma 11, it follows that

$$\int_{\partial B_s(0, r)} X_j \omega \, d\sigma_{0, r} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{B_s(0, r+\epsilon) \setminus B_s(0, r)} X_j \omega \, d\mu = 0. \quad (5.8)$$

Given $x \in \mathbb{G}$, here and subsequently, we set $d_{s, x}(y) = d_s(y, x)$ for brevity. We take $h(y) = \omega(x^{-1} \circ y) d_{s, x}(y)$ and then apply the polar coordinates, by the homogeneity of ω and d_s ,

$$\int_{B_s(x, r)} X_j h \, d\mu = \int_0^r \left(\int_{\partial B_s(x, s)} X_j h \, d\sigma_{x, s} \right) ds = r \int_{\partial B_s(x, r)} X_j h \, d\sigma_{x, r},$$

Combining this with the chain rule and (5.8), we obtain

$$\int_{B_s(x,r)} X_j h \, d\mu = r \int_{\partial B_s(x,r)} \omega(x^{-1} \circ y) (X_j d_{s,x})(y) \, d\sigma_{x,r}(y).$$

Comparing with (5.6),⁴ since ω is arbitrary, we conclude that

$$-dD_j \chi_{B_s(x,r)}(y) = (X_j d_{s,x})(y) \, d\sigma_{x,r}(y). \quad (5.9)$$

Since $d_{s,x} \in C^\infty(\mathbb{G} \setminus \{x\})$, $\partial B_s(x, r)$ is an $(n-1)$ -dimensional smooth sub-manifold of $\mathbb{R}^n$ for any $r > 0$. As a result the above identity holds not only in the distributional sense but also pointwise. In fact, for the metric ball $B_s(x, r)$, $\partial B_s(x, r) = \{y \in \mathbb{G}; d_{s,x}(y) = r\}$. Then in Lemma 10, for each $y \in \partial B_s(x, r)$,

$$n(y) = \frac{\nabla_{\mathbb{R}^n} d_{s,x}(y)}{|\nabla_{\mathbb{R}^n} d_{s,x}(y)|_{\mathbb{R}^n}}, \quad \left(\sum_{j=1}^m \langle X_j, n(y) \rangle_{\mathbb{R}^n}^2 \right)^{\frac{1}{2}} = \frac{|\nabla_{\mathbb{G}} d_{s,x}(y)|}{|\nabla_{\mathbb{R}^n} d_{s,x}(y)|_{\mathbb{R}^n}}, \quad (5.10)$$

and

$$n_{\mathbb{G}}(y) = \begin{cases} \frac{\nabla_{\mathbb{G}} d_{s,x}(y)}{|\nabla_{\mathbb{G}} d_{s,x}(y)|}, & \text{if } |\nabla_{\mathbb{G}} d_{s,x}(y)| > 0; \\ 0, & \text{if } |\nabla_{\mathbb{G}} d_{s,x}(y)| = 0, \end{cases}$$

which implies that

$$n_{\mathbb{G}} d \|\partial B_s(x, r)\| = \frac{\nabla_{\mathbb{G}} d_{s,x}}{|\nabla_{\mathbb{R}^n} d_{s,x}|_{\mathbb{R}^n}} d\mathcal{H}^{n-1} \llcorner \partial B_s(x, r). \quad (5.11)$$

On the other hand, by the coarea formula on $\mathbb{R}^n$ (see e.g. [21, Section 3.4.4]),

$$d\sigma_{x,r} = \frac{1}{|\nabla_{\mathbb{R}^n} d_{s,x}|_{\mathbb{R}^n}} d\mathcal{H}^{n-1} \llcorner \partial B_s(x, r). \quad (5.12)$$

The comparison of (5.11) and (5.12) provides a more direct proof of (5.9). It can be seen from the preceding discussion that the requirement on the smoothness of d_s could be relaxed to a weaker condition $d_s \in C^1(\mathbb{G} \setminus \{0\})$.

Remark 11 Before proceeding, to ensure the validity of (5.10) rigorously, it remains to show that $\nabla_{\mathbb{R}^n} d_{s,x}(y) \neq 0$ for any $y \in \mathbb{G} \setminus \{x\}$. We first prove the special case $x = 0$, namely

$$\nabla_{\mathbb{R}^n} d_s(y) \neq 0, \quad \forall y \in \mathbb{G} \setminus \{0\}. \quad (5.13)$$

⁴ Since $X_j h$ is smooth except at x and locally integrable, we can employ (5.6) through the use of a standard approximation argument.

Indeed, by the homogeneity of d_s and elementary calculus, we have the following Euler's theorem. Under the decomposition given in Definition 1, for any $y_0 = (y_0^{(1)}, y_0^{(2)}, \dots, y_0^{(k)}) \in \mathbb{G} \setminus \{0\}$,

$$\sum_{i=1}^k i (\partial_{y^{(i)}} d_s)(y_0) \cdot y_0^{(i)} = d_s(y_0) > 0,$$

which demonstrates (5.13). Then we turn to the general case. Since $d_{s,x}(y) = d_s(\tau_{x^{-1}}(y))$, by the chain rule, for any $y \in \mathbb{G} \setminus \{x\}$,

$$\nabla_{\mathbb{R}^n} d_{s,x}(y) = \mathcal{J}_{\tau_{x^{-1}}}^T(y) \nabla_{\mathbb{R}^n} d_s(\tau_{x^{-1}}(y)), \quad (5.14)$$

where $\mathcal{J}_{\tau_{x^{-1}}}^T$ is the transpose of the Jacobian of the map $\tau_{x^{-1}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Applying the structure theorem for the composition law of $\mathbb{G}$ (see e.g. [10, Section 1.3.2]), $\mathcal{J}_{\tau_{x^{-1}}}^T(y)$ is an upper triangular matrix whose main diagonal entities are all 1, so that $\mathcal{J}_{\tau_{x^{-1}}}^T(y)$ is a non-singular matrix. As we have seen in (5.13), $\nabla_{\mathbb{R}^n} d_s(\tau_{x^{-1}}(y)) \neq 0$. Combining these facts with (5.14) yields the desired conclusion.

We can now begin the proof of the integration by parts formula on d_s -balls.

Proposition 14 *For every $E \subset \mathbb{G}$ with $\|D_{\mathbb{G}}^{\alpha} \chi_E\|(\mathbb{G}) < \infty$ and $x \in \mathcal{F}_{\mathbb{G}}^{\alpha} E$,*

$$\begin{aligned} - \int_{B_s(x,r)} \langle \varphi, dD_{\mathbb{G}}^{\alpha} \chi_E \rangle &= \int_{E \cap B_s(x,r)} \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi \, d\mu + \int_E \operatorname{div}_{\mathbb{G}, \text{NL}}^{\alpha}(\varphi, \chi_{B_s(x,r)}) \, d\mu \\ &\quad + \int_E \langle \varphi, I_{1-\alpha}(\nabla_{\mathbb{G}} \chi_{B_s(x,r)}) \rangle \, d\mu \end{aligned}$$

holds for any $\varphi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$ and $\mathcal{L}^1$ -a.e. $r > 0$.

Proof To begin with, since $\chi_{B_s(x,r)} \in W^{\alpha,1}(\mathbb{G}) \cap L^{\infty}(\mathbb{G})$, by Remark 8, it is known that

$$\begin{aligned} &I_{1-\alpha}(X_j \chi_{B_s(x,r)})(y) \\ &= \left(\int_{\mathbb{G}} (\chi_{B_s(x,r)}(z) - \chi_{B_s(x,r)}(y)) \tilde{X}_j R_{1-\alpha}(z^{-1} \circ y) \, d\mu(z) \right) X_j \end{aligned}$$

and

$$\|I_{1-\alpha}(X_j \chi_{B_s(x,r)})\|_{L^1(\mathbb{G})} \lesssim [\chi_{B_s(x,r)}]_{W^{\alpha,1}(\mathbb{G})} < \infty.$$

Combining Theorem 8, (5.6) and Fubini's Theorem, it follows that for any $h \in C_c^{\infty}(\mathbb{G})$,

$$\begin{aligned}
\int_{\mathbb{G}} h I_{1-\alpha}(X_j \chi_{B_s(x,r)}) d\mu &= - \int_{\mathbb{G}} (X_j I_{1-\alpha} h) \chi_{B_s(x,r)} d\mu \\
&= - \int_{B_s(x,r)} X_j I_{1-\alpha} h d\mu = \int_{\mathbb{G}} I_{1-\alpha} h dD_j \chi_{B_s(x,r)} \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} h(z) R_{1-\alpha}(z^{-1} \circ y) d\mu(z) \right) dD_j \chi_{B_s(x,r)}(y) \\
&= \int_{\mathbb{G}} h(z) \left(\int_{\mathbb{G}} R_{1-\alpha}(z^{-1} \circ y) dD_j \chi_{B_s(x,r)}(y) \right) d\mu(z).
\end{aligned}$$

Consequently, for μ -a.e. $z \in \mathbb{G}$,

$$I_{1-\alpha}(X_j \chi_{B_s(x,r)})(z) = \int_{\mathbb{G}} R_{1-\alpha}(z^{-1} \circ y) dD_j \chi_{B_s(x,r)}(y). \quad (5.15)$$

For each $\epsilon > 0$, we define

$$\ell_\epsilon(y) := \left(\frac{\epsilon - (d_s(y, x) - r)^+}{\epsilon} \right)^+ = \begin{cases} 1, & \text{on } \overline{B_s(x, r)}; \\ \frac{\epsilon + r - d_{s,x}(y)}{\epsilon}, & \text{on } B_s(x, r + \epsilon) \setminus B_s(x, r); \\ 0, & \text{on } B_s^c(x, r + \epsilon). \end{cases}$$

Then $\lim_{\epsilon \rightarrow 0} \ell_\epsilon = \chi_{\overline{B_s(x, r)}}$ pointwisely. We next show that for any $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$ and $j = 1, \dots, m$,

$$\lim_{\epsilon \rightarrow 0} \int_E \varphi I_{1-\alpha}(X_j \ell_\epsilon) d\mu = \int_E \varphi I_{1-\alpha}(X_j \chi_{B_s(x,r)}) d\mu. \quad (5.16)$$

Indeed, by the definition of ℓ_ϵ and the polar coordinates,

$$\begin{aligned}
&\int_E \varphi I_{1-\alpha}(X_j \ell_\epsilon) d\mu \\
&= -\frac{1}{\epsilon} \int_E \varphi(z) \int_{B_s(x, r+\epsilon) \setminus B_s(x, r)} X_j d_{s,x}(y) R_{1-\alpha}(z^{-1} \circ y) d\mu(y) d\mu(z) \\
&= -\frac{1}{\epsilon} \int_r^{r+\epsilon} \left(\int_{\partial B_s(x, s)} X_j d_{s,x}(y) \int_E \varphi(z) R_{1-\alpha}(z^{-1} \circ y) d\mu(z) d\sigma_{x,s}(y) \right) ds.
\end{aligned}$$

By the dominated convergence Theorem,

$$\int_{\partial B_s(x, s)} X_j d_{s,x}(y) \int_E \varphi(z) R_{1-\alpha}(z^{-1} \circ y) d\mu(z) d\sigma_{x,s}(y)$$

is a continuous function of s , which implies that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_E \varphi I_{1-\alpha}(X_j \ell_\epsilon) d\mu \\ &= - \int_{\partial B_s(x,r)} X_j d_{s,x}(y) \int_E \varphi(z) R_{1-\alpha}(z^{-1} \circ y) d\mu(z) d\sigma_{x,r}(y). \end{aligned}$$

Combining with (5.9) and (5.15), by Fubini's Theorem,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_E \varphi I_{1-\alpha}(X_j \ell_\epsilon) d\mu &= \int_{\partial B_s(x,r)} \int_E \varphi(z) R_{1-\alpha}(z^{-1} \circ y) d\mu(z) dD_j \chi_{B_s(x,r)}(y) \\ &= \int_E \varphi I_{1-\alpha}(X_j \chi_{B_s(x,r)}) d\mu, \end{aligned}$$

which is the desired result.

For any $x \in \mathcal{F}_{\mathbb{G}}^\alpha E$ and $r > 0$, $B_s(x, r) \cap (\text{supp } D_{\mathbb{G}}^\alpha \chi_E) \neq \emptyset$. And since $r \mapsto \|D_{\mathbb{G}}^\alpha \chi_E\|(B_s(x, r))$ is increasing and finite, it follows that for $\mathcal{L}^1$ -a.e. $r > 0$,

$$\|D_{\mathbb{G}}^\alpha \chi_E\|(\partial B_s(x, r)) = \lim_{\epsilon \rightarrow 0} \|D_{\mathbb{G}}^\alpha \chi_E\|(B_s(x, r) \setminus B_s(x, r - \epsilon)) = 0 \quad (5.17)$$

By Remark 8 and Corollary 4, we know that for any $\varphi \in C_c^\infty(\mathbb{G}; H\mathbb{G})$,

$$\begin{aligned} - \int_{B_s(x,r)} \langle \varphi, dD_{\mathbb{G}}^\alpha \chi_E \rangle &= - \lim_{\epsilon \rightarrow 0} \int_{\mathbb{G}} \langle \varphi \ell_\epsilon, dD_{\mathbb{G}}^\alpha \chi_E \rangle = - \lim_{\epsilon \rightarrow 0} \int_E \text{div}_{\mathbb{G}}^\alpha(\varphi \ell_\epsilon) d\mu \\ &= \lim_{\epsilon \rightarrow 0} \int_E \ell_\epsilon \text{div}_{\mathbb{G}}^\alpha \varphi d\mu + \lim_{\epsilon \rightarrow 0} \int_E \text{div}_{\mathbb{G},\text{NL}}^\alpha(\varphi, \ell_\epsilon) d\mu + \lim_{\epsilon \rightarrow 0} \int_E \langle \varphi, I_{1-\alpha}(\nabla_{\mathbb{G}} \ell_\epsilon) \rangle d\mu \end{aligned}$$

By the dominated convergence Theorem, it is clear that for $r > 0$ which satisfies (5.17),

$$\lim_{\epsilon \rightarrow 0} \int_E \ell_\epsilon \text{div}_{\mathbb{G}}^\alpha \varphi d\mu = \int_{E \cap B_s(x,r)} \text{div}_{\mathbb{G}}^\alpha \varphi d\mu.$$

For the second term, notice that $\phi \in W^{\alpha,1}(\mathbb{G})$ and $\ell_\epsilon \in L^\infty(\mathbb{G})$, by (4.15) and the dominated convergence Theorem,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_E \text{div}_{\mathbb{G},\text{NL}}^\alpha(\varphi, \ell_\epsilon) d\mu &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{G}} \chi_E \text{div}_{\mathbb{G},\text{NL}}^\alpha(\varphi, \ell_\epsilon) d\mu \\ &= \int_E \text{div}_{\mathbb{G},\text{NL}}^\alpha(\varphi, \chi_{B_s(x,r)}) d\mu. \end{aligned}$$

Combined with (5.16), the proof is complete. $\square$

Theorem 16 Let $0 < \alpha < 1$. There exists $A > 0$, which depends on $\mathbb{G}$ and α , with the following property. For each $E \subset \mathbb{G}$ with $\|D_{\mathbb{G}}^{\alpha} \chi_E\|(\mathbb{G}) < \infty$ and $x \in \mathcal{F}_{\mathbb{G}}^{\alpha} E$, we can find $r_x > 0$ such that for any $0 < r < r_x$,

$$\|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(x, r)) \leq Ar^{\mathcal{Q}-\alpha} \quad \text{and} \quad \|D_{\mathbb{G}}^{\alpha} \chi_{E \cap B_s(x, r)}\|(\mathbb{G}) \leq Ar^{\mathcal{Q}-\alpha}.$$

Proof For fixed $x \in \mathcal{F}_{\mathbb{G}}^{\alpha} E$, $v_E^{\alpha}(x) = \sum_{j=1}^m v_j X_j(x) \in H\mathbb{G}_x$ exists. With a slight abuse of notations, we also denote the constant horizontal vector field $\sum_{j=1}^m v_j X_j$ by $v_E^{\alpha}(x)$. Then using Definition 8,

$$\lim_{r \rightarrow 0^+} \frac{\int_{B_s(x, r)} \langle v_E^{\alpha}(x), dD_{\mathbb{G}}^{\alpha} \chi_E \rangle}{\|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(x, r))} = \lim_{r \rightarrow 0^+} \frac{\langle v_E^{\alpha}(x), D_{\mathbb{G}}^{\alpha} \chi_E(B_s(x, r)) \rangle}{\|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(x, r))} = 1. \quad (5.18)$$

Now we choose a $\phi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G})$ such that $\phi \equiv v_E^{\alpha}(x)$ on $B_s(x, 1)$ and $\sup_{x \in \mathbb{G}} |\phi(x)| \leq 1$. By (5.18), there exists $r_x \in (0, 1)$ such that for any $0 < r < r_x$,

$$\int_{B_s(x, r)} \langle \phi, dD_{\mathbb{G}}^{\alpha} \chi_E \rangle \geq \frac{1}{2} \|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(x, r)).$$

On the other hand, by Proposition 14, for $\mathcal{L}^1$ -a.e. $r \in (0, r_x)$,

$$\begin{aligned} \int_{B_s(x, r)} \langle \phi, dD_{\mathbb{G}}^{\alpha} \chi_E \rangle &\leq \left| \int_{E \cap B_s(x, r)} \operatorname{div}_{\mathbb{G}}^{\alpha} \phi \, d\mu \right| + \left| \int_E \operatorname{div}_{\mathbb{G}, \text{NL}}^{\alpha} (\phi, \chi_{B_s(x, r)}) \, d\mu \right| \\ &\quad + \left| \int_E \langle \phi, I_{1-\alpha}(\nabla_{\mathbb{G}} \chi_{B_s(x, r)}) \rangle \, d\mu \right|. \end{aligned}$$

For the first term of the right-hand side, since $\phi \equiv v_E^{\alpha}(x)$ on $B_s(x, r)$, by the definition of $\operatorname{div}_{\mathbb{G}}^{\alpha}$,

$$\begin{aligned} \left| \int_{E \cap B_s(x, r)} \operatorname{div}_{\mathbb{G}}^{\alpha} \phi \, d\mu \right| &\lesssim \int_{B_s(x, r)} \int_{B_s^c(x, r)} \frac{1}{d(y, z)^{\mathcal{Q}+\alpha}} \, d\mu(y) d\mu(z) \\ &\lesssim r^{\mathcal{Q}-\alpha} \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|\chi_{B_s(0, 1)}(y) - \chi_{B_s(0, 1)}(z)|}{d(y, z)^{\mathcal{Q}+\alpha}} \, d\mu(y) d\mu(z) \lesssim r^{\mathcal{Q}-\alpha}. \end{aligned}$$

For the other terms, by (4.16) and Definition 9, it is known that

$$\begin{aligned} &\left| \int_E \operatorname{div}_{\mathbb{G}, \text{NL}}^{\alpha} (\phi, \chi_{B_s(x, r)}) \, d\mu \right| + \left| \int_E \langle \phi, I_{1-\alpha}(\nabla_{\mathbb{G}} \chi_{B_s(x, r)}) \rangle \, d\mu \right| \\ &\lesssim \|\phi\|_{L^{\infty}(\mathbb{G}; H\mathbb{G})} P_{\alpha}(B_s(x, r)) \lesssim r^{\mathcal{Q}-\alpha}, \end{aligned} \quad (5.19)$$

where we have used the scaling property $P_{\alpha}(B_s(x, r)) = r^{\mathcal{Q}-\alpha} P_{\alpha}(B_s(0, 1))$. Combining the above estimates, we obtain for $0 < r < r_x$,

$$\|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(x, r)) \lesssim r^{\mathcal{Q}-\alpha}. \quad (5.20)$$

As for the second part, combining Proposition 14, (5.19) and (5.20), it follows that

$$\begin{aligned}
 & \|D_{\mathbb{G}}^{\alpha} \chi_{E \cap B_s(x,r)}\|(\mathbb{G}) \\
 &= \sup \left\{ \int_{E \cap B_s(x,r)} \operatorname{div}_{\mathbb{G}}^{\alpha} \varphi \, d\mu; \varphi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G}), \sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1 \right\} \\
 &\lesssim \|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(x,r)) \\
 &\quad + \sup_{\substack{\varphi \in C_c^{\infty}(\mathbb{G}; H\mathbb{G}) \\ \sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1}} \left| \int_E \operatorname{div}_{\mathbb{G}, \text{NL}}^{\alpha} (\varphi, \chi_{B_s(x,r)}) \, d\mu + \int_E \langle \varphi, I_{1-\alpha}(\nabla_{\mathbb{G}} \chi_{B_s(x,r)}) \rangle \, d\mu \right| \\
 &\lesssim \|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(x,r)) + P_{\alpha}(B_s(x,r)) \lesssim r^{Q-\alpha}.
 \end{aligned} \tag{5.21}$$

Finally, since the maps $r \mapsto \|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(x,r))$ and $r \mapsto \|D_{\mathbb{G}}^{\alpha} \chi_{E \cap B_s(x,r)}\|(\mathbb{G})$ are monotone, (5.20) and (5.21) hold for every $r \in (0, r_x)$, which ends the proof. $\square$

As a direct consequence of Theorem 16, we know that

$$\|D_{\mathbb{G}}^{\alpha} \chi_E\| \ll \mathcal{H}_s^{Q-\alpha} \llcorner \mathcal{F}_{\mathbb{G}}^{\alpha} E,$$

where $\mathcal{H}_s^{Q-\alpha}$ denotes the $(Q-\alpha)$ -dimensional Hausdorff measure of the metric space $(\mathbb{G}, d_s)$.

Definition 11 (Tangent set) Let $0 < \alpha < 1$ and let $E \subset \mathbb{G}$ be a μ -measurable set with $\|D_{\mathbb{G}}^{\alpha} \chi_E\|(\mathbb{G}) < \infty$. For each $x \in \mathcal{F}_{\mathbb{G}}^{\alpha} E$, we define $\operatorname{Tan}(E, x)$ to be the set of all tangent sets of E at x , i.e. the set of all limit points in the topology of $L_{\text{loc}}^1(\mathbb{G})$, of the family of sets $\{\delta_{\frac{1}{r}}(\tau_{x^{-1}}(E))\}_{r>0}$ as $r \rightarrow 0$.

Now we are ready to establish the existence of the blow-ups for fractional Caccioppoli sets, which asserts the non-emptiness of $\operatorname{Tan}(E, x)$. The proof follows the argument used in Theorem 5.8 and Proposition 5.9 of [13].

Theorem 17 Let $0 < \alpha < 1$ and let $E \subset \mathbb{G}$ be a μ -measurable set with $\|D_{\mathbb{G}}^{\alpha} \chi_E\|(\mathbb{G}) < \infty$. Then for each $x \in \mathcal{F}_{\mathbb{G}}^{\alpha} E$, we can find a sequence of positive numbers $\{r_j\}_{j=1}^{\infty}$ satisfying $\lim_{j \rightarrow \infty} r_j = 0$, and a non-empty set, denoted by E_x^* , such that

$$\chi_{E_{x,r_j}} \rightarrow \chi_{E_x^*} \text{ in } L_{\text{loc}}^1(\mathbb{G}) \text{ as } j \rightarrow \infty,$$

where for $r > 0$, $E_{x,r}$ denotes the set $\delta_{\frac{1}{r}}(\tau_{x^{-1}}(E))$. Besides, the set E_x^* has locally finite α -perimeter in $\mathbb{G}$, and

$$\nu_{E_x^*}^{\alpha}(y) = \nu_E^{\alpha}(x) \tag{5.22}$$

holds for $\|D_{\mathbb{G}}^{\alpha} \chi_{E_x^*}\|$ -a.e. $y \in \mathcal{F}_{\mathbb{G}}^{\alpha} E_x^*$.

Proof After making a left translation, we can assume $x = 0 \in \mathcal{F}_{\mathbb{G}}^{\alpha} E$ without loss of generality, and from now on we abbreviate $E_{0,r}$ by E_r . By Theorem 16, there exists $r_0 > 0$ such that for any $r \in (0, r_0)$,

$$\|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(0, r)) \leq Ar^{Q-\alpha} \quad \text{and} \quad \|D_{\mathbb{G}}^{\alpha} \chi_{E \cap B_s(0, r)}\|(\mathbb{G}) \leq Ar^{Q-\alpha}. \quad (5.23)$$

For each $N \in \mathbb{N}^+$ and $r > 0$, by the α -homogeneity of $\text{div}_{\mathbb{G}}^{\alpha}$, for any $\varphi \in C_c^{\infty}(\mathbb{G}; H(\mathbb{G}))$ with $\sup_{x \in \mathbb{G}} |\varphi(x)| \leq 1$,

$$\begin{aligned} \int_{E_r \cap B_s(0, N)} \text{div}_{\mathbb{G}}^{\alpha} \varphi(x) \, d\mu(x) &= r^{-Q} \int_{E \cap B_s(0, rN)} \text{div}_{\mathbb{G}}^{\alpha} \varphi(\delta_{\frac{1}{r}}(y)) \, d\mu(y) \\ &= r^{\alpha-Q} \int_{E \cap B_s(0, rN)} \text{div}_{\mathbb{G}}^{\alpha} (\varphi \circ \delta_{\frac{1}{r}})(y) \, d\mu(y), \end{aligned} \quad (5.24)$$

which implies that

$$\|D_{\mathbb{G}}^{\alpha} \chi_{E_r \cap B_s(0, N)}\|(\mathbb{G}) = r^{\alpha-Q} \|D_{\mathbb{G}}^{\alpha} \chi_{E \cap B_s(0, rN)}\|(\mathbb{G}). \quad (5.25)$$

Combining (5.23) and (5.25), it follows that for any $r \in (0, \frac{r_0}{N})$,

$$\|D_{\mathbb{G}}^{\alpha} \chi_{E_r \cap B_s(0, N)}\|(\mathbb{G}) \leq AN^{Q-\alpha} < \infty.$$

Therefore using Theorem 15, we can find a subsequence $\{r_{N,j}\}_{j=1}^{\infty}$ satisfying $\lim_{j \rightarrow \infty} r_{N,j} = 0$ and a set $E_N^* \subset B_s(0, N)$ with $\|D_{\mathbb{G}}^{\alpha} \chi_{E_N^*}\|(\mathbb{G}) < \infty$ such that

$$\chi_{E_{r_{N,j}} \cap B_s(0, N)} \rightarrow \chi_{E_N^*} \text{ in } L^1(\mathbb{G}) \quad \text{as } j \rightarrow \infty.$$

By a standard diagonal argument, we can extract a subsequence, denoted by $\{r_j\}_{j=1}^{\infty}$, satisfying $\lim_{j \rightarrow \infty} r_j = 0$, and a sequence of sets $\{E_N^*\}_{N \in \mathbb{N}^+}$ such that for any $N \in \mathbb{N}^+$,

$$\chi_{E_{r_j}} \rightarrow \chi_{E_N^*} \text{ in } L^1(B_s(0, N)) \quad \text{as } j \rightarrow \infty.$$

By construction, $\{E_N^*\}_{N \in \mathbb{N}^+}$ is an increasing sequence of sets (up to a set of zero μ -measure). So if we set $E^* := \bigcup_{N \in \mathbb{N}^+} E_N^*$, then

$$\chi_{E_{r_j}} \rightarrow \chi_{E^*} \text{ in } L_{\text{loc}}^1(\mathbb{G}) \quad \text{as } j \rightarrow \infty,$$

and E^* has locally finite α -perimeter in $\mathbb{G}$.

Now we turn to the proof of (5.22). By Proposition 12,

$$D_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}} \rightarrow D_{\mathbb{G}}^{\alpha} \chi_{E^*} \text{ in } \mathcal{M}_{\text{loc}}(\mathbb{G}; H(\mathbb{G})) \quad \text{as } j \rightarrow \infty,$$

Therefore for $\mathcal{L}^1$ -a.e. $R > 0$,

$$D_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}}(B_s(0, R)) \rightarrow D_{\mathbb{G}}^{\alpha} \chi_{E^*}(B_s(0, R)) \text{ in } H\mathbb{G} \text{ as } j \rightarrow \infty. \quad (5.26)$$

And using Proposition 12 again, we obtain that for any $R > 0$,

$$\|D_{\mathbb{G}}^{\alpha} \chi_{E^*}\|(B_s(0, R)) \leq \liminf_{j \rightarrow \infty} \|D_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}}\|(B_s(0, R)). \quad (5.27)$$

On the other hand, by the same scaling argument used in (5.24),

$$\|D_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}}\|(B_s(0, R)) = r_j^{\alpha-Q} \|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(0, r_j R))$$

and

$$D_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}}(B_s(0, R)) = r_j^{\alpha-Q} D_{\mathbb{G}}^{\alpha} \chi_E(B_s(0, r_j R)).$$

By the definition of $\mathcal{F}_{\mathbb{G}}^{\alpha} E$, it is known that

$$\lim_{j \rightarrow \infty} \frac{D_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}}(B_s(0, R))}{\|D_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}}\|(B_s(0, R))} = \lim_{j \rightarrow \infty} \frac{D_{\mathbb{G}}^{\alpha} \chi_E(B_s(0, r_j R))}{\|D_{\mathbb{G}}^{\alpha} \chi_E\|(B_s(0, r_j R))} = v_E^{\alpha}(0). \quad (5.28)$$

Combining (5.26), (5.27), (5.28) and Theorem 5.2, we have

$$\begin{aligned} \|D_{\mathbb{G}}^{\alpha} \chi_{E^*}\|(B_s(0, R)) &\leq \liminf_{j \rightarrow \infty} \|D_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}}\|(B_s(0, R)) \\ &= \lim_{j \rightarrow \infty} \int_{B_s(0, R)} \langle v_E^{\alpha}(0), dD_{\mathbb{G}}^{\alpha} \chi_{E_{r_j}} \rangle = \int_{B_s(0, R)} \langle v_E^{\alpha}(0), dD_{\mathbb{G}}^{\alpha} \chi_{E^*} \rangle \\ &= \int_{B_s(0, R)} \langle v_E^{\alpha}(0), v_{E^*}^{\alpha} \rangle d\|D_{\mathbb{G}}^{\alpha} \chi_{E^*}\| \leq \|D_{\mathbb{G}}^{\alpha} \chi_{E^*}\|(B_s(0, R)) \end{aligned}$$

for $\mathcal{L}^1$ -a.e. $R > 0$, which means that $v_{E^*}^{\alpha}(y) = v_E^{\alpha}(0)$ holds for $\|D_{\mathbb{G}}^{\alpha} \chi_{E^*}\|$ -a.e. $y \in \mathcal{F}_{\mathbb{G}}^{\alpha} E^* \cap B_s(0, R)$ and $\mathcal{L}^1$ -a.e. $R > 0$. Then (5.22) is obtained. $\square$

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