

Rigidity of homogeneous holomorphic two-spheres in a complex Grassmann manifold $G(2, N)$

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Abstract In this paper, we give a local rigid characterization of all homogeneous holomorphic two-spheres in a complex Grassmann manifold $G(2, N)$. Let κ denote the new global invariant defined in terms of the square norm of $(1, 0)$ part of the second order covariant differential of the first ∂ -transform for a holomorphic curve in $G(2, N)$. It is proved that a linearly full non-degenerate holomorphic curve of constant curvature and constant square norm of the second fundamental form in $G(2, N)$ with $\kappa = 0$ must be homogeneous.

Keywords and Phrases. Rigidity, holomorphic curves, Gaussian curvature, the second fundamental form, complex Grassmann manifolds.

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1 Introduction

Let $G(k, N)$ be the complex Grassmann manifold of all k -dimensional subspaces in a complex number space $\mathbb{C}^N$ of dimension N . In particular, $G(1, N)$ is the complex projective space $\mathbb{C}P^{N-1}$. The construction theory of harmonic maps or minimal immersions from the two-sphere S^2 into $G(k, N)$ were obtained in a sequence of papers [9, 12, 3, 6, 28, 30], etc. A holomorphic map $S^2 \rightarrow G(k, N)$ is harmonic. Chern and Wolfson called it a classical harmonic map of the first kind [6]. This kind of harmonic maps plays an important role in the construction of all harmonic maps $S^2 \rightarrow G(k, N)$. We study the geometry of holomorphic immersions from the two-sphere (holomorphic two-spheres) into $G(2, N)$, such as the first fundamental form (induced metric), Gaussian curvature, the second fundamental form, and so on.

For the particular case of the complex projective space, beautiful rigidity and classification results were established. Calabi [4] proved that a linearly full holomorphic curve in $\mathbb{C}P^n$ is uniquely determined by its first fundamental form, up to a rigid motion of $\mathbb{C}P^n$, which is well known as a special case of Calabi's rigidity theorem for complex submanifolds of a Kaehler manifold. Based on this rigidity result, Bolton et al. classified all linearly full minimal immersions of the two-sphere into $\mathbb{C}P^n$ with constant curvature, which belong to the *Veronese sequence* up to a rigid motion [2]. Bando and Ohnita [1] showed that the

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elements of the Veronese sequence are *homogeneous* (its image is an orbit of an isometry subgroup of the target space).

If the target space is a general $G(k, N)$, the situations become much more complicated. Chi and Zheng [8] classified all holomorphic immersions from the two-sphere into $G(2, 4)$ with constant curvature 2 into two classes, up to unitary equivalence. This result shows that the rigidity of holomorphic curves in a general $G(k, N)$ ($k \geq 2$) fails if we only assume they have the same first fundamental form. Therefore, one natural question is to consider the classification of minimal two-spheres in $G(k, N)$ with constant curvature. To the authors' knowledge, there are only the classification result for the case $G(2, 4)$ [25, 26] and partial result for the case $G(2, 5)$ [23, 24], and the classification for the homogeneous holomorphic two-spheres in $G(2, N)$ and $G(3, N)$ [27, 17]. Theoretical physicists are also interested in this problem. Delisle, Hussin and Zakrzewski recovered the classification results in $G(2, 4)$ and $G(2, 5)$ by their approach from the viewpoint of Grassmannian sigma models. One can refer to [10, 11] and references therein for details.

It is worthwhile to mention one important fact that holomorphic two-spheres in a general $G(k, N)$ is not always homogeneous. The examples of non-homogeneous holomorphic two-spheres with constant curvature in $G(2, 4), G(2, 5), G(2, 6)$ were obtained in [26, 14, 21, 22]. So it is interesting to consider the rigidity of homogeneous holomorphic two-spheres in $G(k, N)$, that is, under what conditions, a holomorphic two-sphere in $G(k, N)$ must be homogeneous. The theory of Lie groups and the method of moving frames play a powerful role in studying such rigidity problem. Griffiths [18] proved the special case of Calabi's rigidity theorem mentioned above by these methods, and also proved that two non-degenerate holomorphic curves in $G(2, 4)$ are congruent if they have the same first and second fundamental forms. We generalized the Griffiths' rigidity result to the case of non-degenerate holomorphic curves in $G(2, 6)$ and $G(3, 6)$ [16, 29]. The main idea is that for a holomorphic immersion from a Riemann surface M in $G(k, N)$, there naturally exists a good frame over M by the theory of singular values decomposition of complex matrices. Restricting the Maurer-Cartan forms to this "natural frame" gives a complete set of invariants of M . Exploring this idea, we gave a characterisation of homogeneous holomorphic two-spheres in $G(3, 6)$ [29]. In the present paper, we will give a local characterisation of homogeneous holomorphic two-spheres in a general $G(2, N)$. To describe our main result, we begin with examples of homogeneous holomorphic two-spheres in $G(2, N)$ classified by the first named author in [17]. Let $\mathcal{V}_0^{(n)}, \dots, \mathcal{V}_n^{(n)}$ denote the Veronese sequence in $\mathbb{C}P^n$, which will be described explicitly in Section 3.

Example 1.1 For each $0 < t < \pi/2$ and $N = 2n$, the homogeneous holomorphic immersion $f_t : S^2 \rightarrow G(2, N)$ [17] given by

$$z \mapsto \begin{bmatrix} \mathcal{V}_0^{(n)} & 0 \\ \cos t \mathcal{V}_1^{(n)} & \sin t \mathcal{V}_0^{(n-2)} \end{bmatrix}$$

has constant Gaussian curvature $K = 4/(N - 2) = 2/(n - 1)$ and constant square norm of the second fundamental form

$$S = \frac{4n^2 \sin^4 t - (8n^2 - 8n) \sin^2 t + 8n^2 - 20n + 12}{(n - 1)^2}.$$

Example 1.2 For each pair of positive integers $n_1 > n_2$ and $N = n_1 + n_2 + 2$, the homogeneous holomorphic immersion $f_{n_1, n_2} : S^2 \rightarrow G(2, N)$ [17] given by

$$z \mapsto \begin{bmatrix} \mathcal{V}_0^{(n_1)} & 0 \\ 0 & \mathcal{V}_0^{(n_2)} \end{bmatrix}$$

has constant Gaussian curvature $K = 4/(N - 2) = 4/(n_1 + n_2)$ and constant square norm of the second fundamental form

$$S = \frac{8(n_1^2 + n_2^2 - n_1 - n_2)}{(n_1 + n_2)^2}.$$

Remark 1.3 If t takes the value $\pi/2$ in Example 1.1, the immersion $f_{\pi/2}$ is just $f_{n, n-2}$ in Example 1.2. When $n_1 = n_2 = n - 1$ in Example 1.2, we will show that the map $f_{n-1, n-1}$ is unitary equivalent to the map $f_{\arctan \sqrt{n-1}}$ in Example 1.1 (See Lemma 4.4).

To determine these homogeneous holomorphic two-spheres, a new global invariant κ for a holomorphic immersion from a Riemann surface M into $G(2, N)$ is defined in terms of the square norm of the $(1, 0)$ part of the second order covariant differential of the first ∂ -transform. The main result is stated as follows.

Theorem 1.4 The immersions f_t in Example 1.1 and f_{n_1, n_2} in Example 1.2 are the only linearly full non-degenerate holomorphic two-spheres in $G(2, N)$ satisfying $K = \text{const.}$, $S = \text{const.}$ and $\kappa = 0$.

The proof is by local argument and the corresponding local result also holds.

Theorem 1.5 A linearly full non-degenerate holomorphic curve of constant curvature and constant square norm of the second fundamental form in $G(2, N)$ with κ vanishing identically is locally unitary equivalent to one of the immersion f_t in Example 1.1 or f_{n_1, n_2} in Example 1.2. So it must be homogeneous.

The main strategy for our proof has the same philosophy as the one used by Chern, Do Carmo and Kobayashi [5] to determine all minimal submanifolds of a sphere S^{n+p} satisfying $S = n/(2 - 1/p)$. In terms of Veronese sequence, we shall choose good unitary frames of the immersions defined in Example 1.1 and Example 1.2 by induction and determine the pull back of Maurer-Cartan forms by a series of recursion formulae. Surprisingly, for any linearly full non-degenerate holomorphic curve of constant K and constant S in $G(2, N)$, $\kappa = 0$ makes it possible to choose a good unitary frame by induction so that the pull back of Maurer-Cartan forms satisfying the same recursion formulae, which follows that the pull back of Maurer-Cartan forms coincide with those forms of Examples 1.1 and 1.2. So the rigidity result yields by virtue of the lemma (1.3) in [18].

Our paper is organized as follows. In Section 2, we recall some basic facts about the geometry of holomorphic curves in $G(2, N)$ and define a complete set of invariants. In Section 3, we devote to describe the Veronese sequence in $\mathbb{CP}^n$ explicitly. In Section 4, we choose natural unitary frames to compute the geometric invariants of these homogeneous holomorphic two-spheres in Example 1.1 and Example 1.2. In Section 5, we give a local characterization of these homogeneous holomorphic two-spheres in $G(2, N)$ by using the method of moving frames (See Theorem 5.2 and Theorem 5.3). In Section 6, we give some remarks and propose some questions.

2 Geometry of holomorphic curves in $G(2, N)$

Let f be a linearly full holomorphic immersion from a Riemann surface M into $G(2, N)$, where $G(2, N)$ is endowed with the standard Kaehlerian metric ds^2 . Then M acquires an induced Riemannian metric

$$ds_M^2 = f^* ds^2 = \phi \bar{\phi}, \quad (2.1)$$

where ϕ is a local unitary coframe on M , defined up to a factor of norm one. The structure equations of M with respect to the induced metric can be written as

$$d\phi = -\sqrt{-1}\rho \wedge \phi, \quad (2.2)$$

$$d(\sqrt{-1}\rho) = \frac{K}{2}\phi \wedge \bar{\phi}, \quad (2.3)$$

where ρ is the real-valued connection form and K is the Gaussian curvature of the induced metric.

According to the theory of harmonic sequence [6], f can generate an orthogonal harmonic sequence through ∂ -transform

$$f = f_1 \xrightarrow{\partial} f_2 \xrightarrow{\partial} \cdots \xrightarrow{\partial} f_n \xrightarrow{\partial} f_{n+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} f_{n+l} \xrightarrow{\partial} 0 \quad (2.4)$$

with $N = 2n + l$ since f is linearly full, where $f_k : M \rightarrow G(2, N)$, $1 \leq k \leq n$ and $f_{n+j} : M \rightarrow \mathbb{C}P^{N-1}$, $1 \leq j \leq l$ are harmonic maps. We say f is *totally non-degenerate* if $l = 0$, f *degenerates at position n* if $l \geq 1$, and f is *non-degenerate* if f degenerates at position n with $n \geq 2$.

We can choose a local unitary frame $e = \{Z_1 \cdots, Z_N\}$ along f so that

$$f_k = [Z_{2k-1} \wedge Z_{2k}], 1 \leq k \leq n, \quad f_{n+j} = [Z_{2n+j}], 1 \leq j \leq l.$$

Such unitary frames are called *natural unitary frames* and are determined up to a transformation of the group $\underbrace{U(2) \times \cdots \times U(2)}_k \times \underbrace{U(1) \times \cdots \times U(1)}_l$. The pull back of Maurer-

Cartan forms ω_{AB} (or denoted by $\omega_{A,B}$) of $U(N)$ under such frames are then determined by $dZ_A = \sum_{B=1}^N \omega_{AB} Z_B$ and take the block matrix form

$$\begin{aligned} \Omega = (\omega_{AB}) &= \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ -\Omega_{12}^* & \Omega_{22} \end{pmatrix} \text{ with} \\ \Omega_{11} &= \begin{pmatrix} \Omega_1 & A_1\phi & 0 & 0 & 0 \\ -A_1^*\bar{\phi} & \Omega_2 & A_2\phi & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -A_{n-2}^*\bar{\phi} & \Omega_{n-1} & A_{n-1}\phi \\ 0 & 0 & 0 & -A_{n-1}^*\bar{\phi} & \Omega_n \end{pmatrix}, \\ \Omega_{12} &= \begin{pmatrix} 0 & 0 \\ A_n\phi & 0 \end{pmatrix}_{2n \times l}, \\ \Omega_{22} &= \begin{pmatrix} \omega_{2n+1,2n+1} & a_{n+1}\phi & 0 & 0 & 0 \\ -\bar{a}_{n+1}\bar{\phi} & \omega_{2n+2,2n+2} & a_{n+2}\phi & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & -\bar{a}_{n+l-2}\bar{\phi} & \omega_{2n+l-1,2n+l-1} & a_{n+l-1}\phi \\ 0 & 0 & 0 & -\bar{a}_{n+l-1}\bar{\phi} & \omega_{2n+l,2n+l} \end{pmatrix}, \end{aligned} \quad (2.5)$$

where $\Omega_k, 1 \leq k \leq n$ are $\mathfrak{u}(2)$ -valued 1-forms,

$$A_k = \begin{pmatrix} p_k & q_k \\ r_k & s_k \end{pmatrix}, \quad 1 \leq k \leq n-1, \quad A_n = \begin{pmatrix} p_n & 0 \\ r_n & 0 \end{pmatrix},$$

are locally complex 2×2 matrices-valued smooth functions, and $a_{n+j}, 1 \leq j \leq l-1$ are locally complex-valued smooth functions. For each $1 \leq k \leq n-1$, the matrix A_k is the matrix representation of the k -th ∂ -transform of the harmonic sequence (2.4) with respect to the chosen unitary frame.

By using the Maurer-Cartan structure equations $d\Omega = \Omega \wedge \Omega$ and (2.2), we get

$$dA_k \equiv (\sqrt{-1}\rho A_k + \Omega_k A_k - A_k \Omega_{k+1}) \mod \phi, \quad 1 \leq k \leq n-1, \quad (2.6)$$

and

$$d\Omega_k = A_{k-1}^* A_{k-1} \phi \wedge \bar{\phi} - A_k A_k^* \phi \wedge \bar{\phi} + \Omega_k \wedge \Omega_k, \quad 1 \leq k \leq n \quad (2.7)$$

with $A_0 = 0$. From (2.6), we can set

$$DA_k := dA_k - \Omega_k A_k + A_k \Omega_{k+1} - \sqrt{-1}\rho A_k = P_k \phi, \quad 1 \leq k \leq n-1. \quad (2.8)$$

where DA_k is the covariant differential of A_k and P_k are locally complex 2×2 matrices-valued smooth functions. The quadratic differential form $\text{II} = (P_1 \ A_1 A_2 \ 0)\phi^2$ is the “second fundamental form” of the map f [6, 16].

Now taking the exterior derivatives of (2.8) and using (2.2), (2.3), (2.7), (2.8), we obtain

$$dP_k \equiv (2\sqrt{-1}\rho P_k + \Omega_k P_k - P_k \Omega_{k+1}) + B_k \bar{\phi} \mod \phi, \quad 1 \leq k \leq n-1, \quad (2.9)$$

where

$$B_k = \frac{K}{2} A_k + A_{k-1}^* A_{k-1} A_k + A_k A_{k+1} A_{k+1}^* - 2A_k A_k^* A_k.$$

Then we can set

$$DP_k := dP_k - \Omega_k P_k + P_k \Omega_{k+1} - 2\sqrt{-1}\rho P_k = P_{k,1} \phi + B_k \bar{\phi}, \quad 1 \leq k \leq n-1. \quad (2.10)$$

where DP_k is the covariant differential of P_k and $P_{k,1}$ are locally complex 2×2 matrices-valued smooth functions.

Let $\tilde{e}$ be another natural unitary frame along f such that

$$f_k = [\tilde{Z}_{2k-1} \wedge \tilde{Z}_{2k}], \quad 1 \leq k \leq n, \quad f_{n+j} = [\tilde{Z}_{2n+j}], \quad 1 \leq j \leq l.$$

Then $\tilde{e} = Ue$, where $U = \text{diag}\{U_1, \dots, U_n, e^{\sqrt{-1}\tau_1}, \dots, e^{\sqrt{-1}\tau_l}\}$ and $U_1, \dots, U_n$ are local $U(2)$ -valued smooth functions, $\tau_1, \dots, \tau_l$ are local smooth real functions. We can check that the transformation rules on $A_k, P_k, P_{k,1}, a_{n+j}$ and Ω_k are given by

$$\begin{aligned} \tilde{A}_k &= U_k A_k U_{k+1}^{-1}, \quad \tilde{P}_k = U_k P_k U_{k+1}^{-1}, \quad \tilde{P}_{k,1} = U_k P_{k,1} U_{k+1}^{-1}, \quad 1 \leq k \leq n-1, \\ \tilde{a}_{n+j} &= a_{n+j} e^{\sqrt{-1}(\tau_j - \tau_{j+1})}, \quad 1 \leq j \leq l-1 \end{aligned} \quad (2.11)$$

and

$$\tilde{\Omega}_k = dU_k \cdot U_k^{-1} + U_k \Omega_k U_k^{-1}, \quad 1 \leq k \leq n, \quad (2.12)$$

which implies that

$$\begin{aligned} & |\det A_k|, \quad L_k = \operatorname{tr} A_k A_k^* = \operatorname{tr} A_k^* A_k, \quad 1 \leq k \leq n-1, \\ & |\det P_k|, \quad |\det P_{k,1}|, \quad \operatorname{tr} P_k P_k^*, \quad \operatorname{tr} P_{k,1} P_{k,1}^*, \quad 1 \leq k \leq n-1, \\ & |a_{n+j}|, \quad 1 \leq j \leq l-1 \end{aligned}$$

and the singular values of the matrices A_k , $1 \leq k \leq n-1$ are all globally defined invariants on M . Particularly, the square norm $\operatorname{tr} P_{1,1} P_{1,1}^*$ of $(1,0)$ part of the second order covariant differential of the first ∂ -transform plays an important role in our later proof, so we denote it by

$$\kappa = \operatorname{tr} P_{1,1} P_{1,1}^*$$

in this paper. Another important invariant is the square norm of the second fundamental form of f given by

$$S = 4(\operatorname{tr} P_1 P_1^* + \operatorname{tr}(A_1^* A_1 A_2 A_2^*)).$$

The Gaussian curvature K , the square norm of the second fundamental form S and $|\det A_1|$ are related by the following Gauss equation.

Theorem 2.1 ([17]) *Let $f : M \rightarrow G(2, N)$ be a holomorphic immersion and S be the square norm of the second fundamental form. Then the Gauss equation of f is*

$$K = 4 - 8|\det A_1|^2 - \frac{1}{2}S,$$

where K is the Gaussian curvature with respect to the induced metric.

3 The Veronese Sequence in $\mathbb{C}P^n$

In this section, we describe the Veronese sequence in $\mathbb{C}P^n$ explicitly. Let $\psi^{(n)} : S^2 \rightarrow \mathbb{C}P^n$ be the *Veronese map* defined by

$$\psi^{(n)}(z) = \left[1, \sqrt{\binom{n}{1}}z, \dots, \sqrt{\binom{n}{l}}z^l, \dots, z^n \right]$$

in terms of the holomorphic coordinate z in S^2 . It is well known [2] that this is a linearly full holomorphic one which has constant curvature $4/n$, and is uniquely determined up to a rigid motion of $\mathbb{C}P^n$.

Let $\psi_0^{(n)}, \psi_1^{(n)}, \dots, \psi_n^{(n)}$ be the harmonic sequence generated by $\psi^{(n)} = \psi_0^{(n)}$ and $\psi_l^{(n)} = [f_l^n] = [f_{l,0}^{(n)}, \dots, f_{l,p}^{(n)}, \dots, f_{l,n}^{(n)}]$, where $f_{l,p}^{(n)}$ is explicitly given by

$$f_{l,p}^{(n)}(z) = \frac{l!}{(1+z\bar{z})^l} \sqrt{\binom{n}{p}} z^{p-l} \sum_k (-1)^k \binom{p}{l-k} \binom{n-p}{k} (z\bar{z})^k.$$

Set $e_l^{(n)} = \frac{f_l^{(n)}}{\|f_l^{(n)}\|}$, $0 \leq l \leq n$. Then $\{e_0^{(n)}, \dots, e_n^{(n)}\}$ is a Frenet frame of $\psi_0^{(n)}$ and we have [2, 17]

$$de_l^{(n)} = -\sqrt{l(n-l+1)}\bar{\varphi}e_{l-1}^{(n)} + (n-2l)\sqrt{-1}\omega e_l^{(n)} + \sqrt{(l+1)(n-l)}\varphi e_{l+1}^{(n)}, \quad (3.1)$$

where $0 \leq l \leq n$, $\varphi\bar{\varphi}$ is the standard metric on S^2 with constant curvature 4 and -2ω is the real connection form of the metric.

Such a map $\psi_l^{(n)} : S^2 \rightarrow \mathbb{C}P^n$ is a conformal minimal immersion with constant curvature and constant Kaehler angle, which are given by

$$K_l^{(n)} = \frac{4}{n+2l(n-l)}, \quad \cos \alpha_l^{(n)} = \frac{n-2l}{n+2l(n-l)}.$$

This harmonic sequence is well known as the *Veronese sequence* in $\mathbb{C}P^n$ [2], which will be denoted by $\mathcal{V}_0^{(n)}, \dots, \mathcal{V}_n^{(n)} : S^2 \rightarrow \mathbb{C}P^n$ correspondingly.

4 Homogeneous holomorphic two-spheres in $G(2, N)$

In the paper [17], the first named author gave a complete classification of homogeneous holomorphic immersions from two-sphere into $G(2, N)$ by using unitary representations of $SU(2)$. These immersions can be expressed explicitly in terms of the Veronese sequence in $\mathbb{C}P^n$. In this section, we will compute the geometric invariants of these immersions in more details under suitable natural unitary frames.

Firstly, let's consider the immersion f_t given in Example 1.1:

$$f_t : S^2 \rightarrow G(2, 2n), \quad z \mapsto \begin{bmatrix} \mathcal{V}_0^{(n)} & 0 \\ \cos t \mathcal{V}_1^{(n)} & \sin t \mathcal{V}_0^{(n-2)} \end{bmatrix}, \quad t \in (0, \frac{\pi}{2}).$$

We can choose a natural unitary frame $e = \{Z_A\}$ along f_t by induction as follows:

Step 1 We set

$$\begin{aligned} Z_1 &= (e_0^{(n)}, 0) & Z_2 &= (\cos t e_1^{(n)}, \sin t e_0^{(n-2)}), \\ Z_3 &= (\sin t e_1^{(n)}, -\cos t e_0^{(n-2)}). \end{aligned}$$

Using (3.1) and $a_k^{(n)} := \sqrt{(k+1)(n-k)}$, we compute that

$$\begin{aligned} dZ_1 &= n\sqrt{-1}\omega Z_1 + a_0^{(n)} \cos t \varphi Z_2 + a_0^{(n)} \sin t \varphi Z_3, \\ dZ_2 &= -a_0^{(n)} \cos t \bar{\varphi} Z_1 + (n-2)\sqrt{-1}\omega Z_2 + (a_1^{(n)} \cos t e_2^{(n)}, a_0^{(n-2)} \sin t e_1^{(n-2)})\varphi. \end{aligned}$$

Step 2 Set $t_1 = t$, $\gamma_1 = \sqrt{(a_1^{(n)} \cos t_1)^2 + (a_0^{(n-2)} \sin t_1)^2}$ and the angle $t_2 \in (0, \pi/2)$ satisfies

$$\cos t_2 = \frac{a_1^{(n)} \cos t_1}{\gamma_1}, \quad \sin t_2 = \frac{a_0^{(n-2)} \sin t_1}{\gamma_1},$$

then we can set

$$\begin{aligned} Z_4 &= (\cos t_2 e_2^{(n)}, \sin t_2 e_1^{(n-2)}), \\ Z_5 &= (\sin t_2 e_2^{(n)}, -\cos t_2 e_1^{(n-2)}). \end{aligned}$$

Direct computation gives

$$\begin{aligned} dZ_3 &= (n-2)\sqrt{-1}\omega Z_3 + \alpha_2\varphi Z_4 + \beta_2\varphi Z_5, \quad \text{mod } Z_1, Z_2, \\ dZ_4 &= -\gamma_2\bar{\varphi}Z_3 + (n-4)\sqrt{-1}\omega Z_4 + (a_2^{(n)}\cos t_2 e_3^{(n)}, a_1^{(n-2)}\sin t_2 e_2^{(n-2)})\varphi, \quad \text{mod } Z_1, Z_2, \end{aligned}$$

where

$$\begin{aligned} \alpha_2 &= a_1^{(n)}\sin t_1\cos t_2 - a_0^{(n-2)}\cos t_1\sin t_2, \\ \beta_2 &= a_1^{(n)}\sin t_1\sin t_2 + a_0^{(n-2)}\cos t_1\cos t_2. \end{aligned}$$

Step 3 Set $\gamma_2 = \sqrt{(a_2^{(n)}\cos t_2)^2 + (a_1^{(n-2)}\sin t_2)^2}$ and the angle $t_3 \in (0, \pi/2)$ satisfies

$$\cos t_3 = \frac{a_2^{(n)}\cos t_2}{\gamma_2}, \quad \sin t_3 = \frac{a_1^{(n-2)}\sin t_2}{\gamma_2},$$

then we can set

$$\begin{aligned} Z_6 &= (\cos t_3 e_3^{(n)}, \sin t_3 e_2^{(n-2)}), \\ Z_7 &= (\sin t_3 e_3^{(n)}, -\cos t_3 e_2^{(n-2)}). \end{aligned}$$

Step k Induction step Set $\gamma_{k-1} = \sqrt{(a_{k-1}^{(n)}\cos t_{k-1})^2 + (a_{k-2}^{(n-2)}\sin t_{k-1})^2}$ and the angle $t_k \in (0, \pi/2)$ satisfies

$$\cos t_k = \frac{a_{k-1}^{(n)}\cos t_{k-1}}{\gamma_{k-1}}, \quad \sin t_k = \frac{a_{k-2}^{(n-2)}\sin t_{k-1}}{\gamma_{k-1}},$$

then we can set

$$\begin{aligned} Z_{2k} &= (\cos t_k e_k^{(n)}, \sin t_k e_{k-1}^{(n-2)}), \\ Z_{2k+1} &= (\sin t_k e_k^{(n)}, -\cos t_k e_{k-1}^{(n-2)}). \end{aligned}$$

Through direct computation by (3.1), we get

$$\begin{aligned} dZ_{2k-1} &= (n-2k+2)\sqrt{-1}\omega Z_{2k-1} + \alpha_k\varphi Z_{2k} + \beta_k\varphi Z_{2k+1}, \quad \text{mod } Z_1, \dots, Z_{2k-2}, \\ dZ_{2k} &= -\gamma_k\bar{\varphi}Z_{2k-1} + (n-2k)\sqrt{-1}\omega Z_{2k} \\ &\quad + (a_k^{(n)}\cos t_k e_k^{(n)}, a_{k-1}^{(n-2)}\sin t_k e_k^{(n-2)})\varphi, \quad \text{mod } Z_1, \dots, Z_{2k-2}, \end{aligned}$$

where

$$\alpha_k = a_{k-1}^{(n)}\sin t_{k-1}\cos t_k - a_{k-2}^{(n-2)}\cos t_{k-1}\sin t_k, \quad (4.1)$$

$$\beta_k = a_{k-1}^{(n)}\sin t_{k-1}\sin t_k + a_{k-2}^{(n-2)}\cos t_{k-1}\cos t_k. \quad (4.2)$$

Now we set

$$\gamma_k = \sqrt{(a_k^{(n)}\cos t_k)^2 + (a_{k-1}^{(n-2)}\sin t_k)^2}. \quad (4.3)$$

Step n Set $\gamma_{n-1} = \sqrt{(a_{n-1}^{(n)} \cos t_{n-1})^2 + (a_{n-2}^{(n-2)} \sin t_{n-1})^2} = a_{n-1}^{(n)} \cos t_{n-1}$ and $Z_{2n} = (e_n^{(n)}, 0)$. Then we have

$$\begin{aligned} dZ_{2n-1} &= (2-n)\sqrt{-1}\omega Z_{2n-1} + \alpha_n \varphi Z_{2n}, \quad \text{mod } Z_1, \dots, Z_{2n-2}, \\ dZ_{2n} &= -\alpha_n \bar{\varphi} Z_{2n-1} - n\sqrt{-1}\omega Z_{2n}, \quad \text{mod } Z_1, \dots, Z_{2n-2}. \end{aligned}$$

where $\alpha_n = a_{n-1}^{(n)} \sin t_{n-1}$. In summary, we have

Proposition 4.1 *The natural unitary frame $e = \{Z_1, \dots, Z_{2n}\}$ along the immersion f_t can be chosen as*

$$\begin{aligned} Z_1 &= (e_0^{(n)}, 0), \quad Z_{2n} = (e_n^{(n)}, 0), \\ Z_{2k} &= (\cos t_k e_k^{(n)}, \sin t_k e_{k-1}^{(n-2)}), \\ Z_{2k+1} &= (\sin t_k e_k^{(n)}, -\cos t_k e_{k-1}^{(n-2)}), \quad 1 \leq k \leq n-1, \end{aligned}$$

where

$$t_1 = t, \quad \cos t_k = \frac{a_{k-1}^{(n)} \cos t_{k-1}}{\gamma_{k-1}}, \quad \sin t_k = \frac{a_{k-2}^{(n-2)} \sin t_{k-1}}{\gamma_{k-1}}, \quad 2 \leq k \leq n-1.$$

The pull back of Maurer-Cartan forms under such frame are given by

$$\begin{pmatrix} \Omega_1 & A_1 \varphi & 0 & 0 & 0 \\ -A_1^* \bar{\varphi} & \Omega_2 & A_2 \varphi & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -A_{n-2}^* \bar{\varphi} & \Omega_{n-1} & A_{n-1} \varphi \\ 0 & 0 & 0 & -A_{n-1}^* \bar{\varphi} & \Omega_n \end{pmatrix}, \quad (4.4)$$

$$\Omega_k = \begin{pmatrix} (n-2k+2)\sqrt{-1}\omega & \alpha_k \varphi \\ -\alpha_k \bar{\varphi} & (n-2k)\sqrt{-1}\omega \end{pmatrix}, \quad 1 \leq k \leq n, \quad (4.5)$$

$$A_k = \begin{pmatrix} \beta_k & 0 \\ 0 & \gamma_k \end{pmatrix}, \quad 1 \leq k \leq n-1, \quad (4.6)$$

where α_k, β_k and γ_k are given by the formulae (4.1)-(4.3) with $t_0 = \pi/2$.

The induced metric is

$$f_t^* ds^2 = (\beta_1^2 + \gamma_1^2) \varphi \bar{\varphi} = (a_1^{(n)})^2 \varphi \bar{\varphi} = \phi \bar{\phi},$$

so the Gaussian curvature $K = 4/(a_1^{(n)})^2 = 2/(n-1)$ and $\varphi = \phi/a_1^{(n)}$. We set

$$\hat{\alpha}_k = \frac{\alpha_k}{a_1^{(n)}}, \quad \hat{\beta}_k = \frac{\beta_k}{a_1^{(n)}}, \quad \hat{\gamma}_k = \frac{\gamma_k}{a_1^{(n)}}.$$

Using the formulae (4.1)-(4.3) and the relation

$$(a_k^{(n)})^2 + (a_{k-2}^{(n-2)})^2 = (a_{k-1}^{(n)})^2 + (a_{k-1}^{(n-2)})^2,$$

one can check directly that $\hat{\beta}_1 = \frac{a_0^{(n)}}{a_1^{(n)}} \sin t = \frac{\sqrt{K+2}}{2} \sin t$, and $\hat{\alpha}_k, \hat{\beta}_k, \hat{\gamma}_k$ satisfy the following recursion formulae:

$$\hat{\beta}_1^2 + \hat{\gamma}_1^2 = 1, \quad \hat{\alpha}_1^2 + \hat{\beta}_1^2 = \left(\frac{a_0^{(n)}}{a_1^{(n)}} \right)^2 = \frac{K+2}{4}, \quad (4.7)$$

$$\hat{\beta}_k^2 + \hat{\gamma}_k^2 = \frac{(a_k^{(n)})^2 + (a_{k-2}^{(n-2)})^2}{(a_1^{(n)})^2} = k - \frac{(k-1)k}{2} K := L_k, \quad (4.8)$$

$$\hat{\beta}_{k+1}^2 \hat{\gamma}_k^2 = \left(\frac{a_k^{(n)} a_{k-1}^{(n-2)}}{a_1^{(n)} a_1^{(n)}} \right)^2 = \frac{L_k L_{k+1}}{4}, \quad \hat{\alpha}_{k+1} = \frac{\hat{\beta}_k}{\hat{\gamma}_k} \hat{\alpha}_k. \quad (4.9)$$

In light with Proposition 4.1 and the recursion formulae (4.7)-(4.9), we obtain a one-parameter family of sequences of harmonic two-spheres in $G(2, 2n)$ with the same constant Gaussian curvatures and the same constant Kaehler angles.

Proposition 4.2 *For each $t \in (0, \pi/2)$, the maps $\psi_{t,k} = [Z_{2k-1} \wedge Z_{2k}] : S^2 \rightarrow G(2, 2n)$, $1 \leq k \leq n$ are a family of conformal minimal immersions with constant curvature $K(\psi_{t,k})$ and constant Kaehler angle $\theta(\psi_{t,k})$ given by*

$$K(\psi_{t,k}) = \frac{K}{L_{k-1} + L_k} = \frac{2}{(n+1)(2k-1) - 2k^2},$$

$$\cos \theta(\psi_{t,k}) = \frac{L_k - L_{k-1}}{L_k + L_{k-1}} = \frac{n - 2k + 1}{(n+1)(2k-1) - 2k^2},$$

for $1 \leq k \leq n$.

Now we consider the immersion f_{n_1, n_2} in Example 1.2:

$$f_{n_1, n_2} : S^2 \rightarrow G(2, N), \quad z \mapsto \begin{bmatrix} \mathcal{V}_0^{(n_1)} & 0 \\ 0 & \mathcal{V}_0^{(n_2)} \end{bmatrix}.$$

From (3.1), we have

Proposition 4.3 *The natural unitary frame $e = \{Z_1, \dots, Z_N\}$ along the immersion f_{n_1, n_2} can be chosen as*

$$Z_{2k+1} = (e_k^{(n_1)}, 0), \quad Z_{2k+2} = (0, e_k^{(n_2)}), \quad 0 \leq k \leq n_2,$$

$$Z_{2n+j} = (e_{n_2+j}^{(n_1)}, 0), \quad 1 \leq j \leq l,$$

where $n = n_2 + 1$, $n_1 = n_2 + l$, $N = n_1 + n_2 - 2 = 2n + l$. The pull back of Maurer-Cartan forms under such frame are given by

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ -\Omega_{12}^* & \Omega_{22} \end{pmatrix},$$

(1) Ω_{11} has the same form as (4.4) with

$$A_k = \begin{pmatrix} \beta_k & 0 \\ 0 & \gamma_k \end{pmatrix} = \begin{pmatrix} a_{k-1}^{(n_1)} & 0 \\ 0 & a_{k-1}^{(n_2)} \end{pmatrix}, \quad 1 \leq k \leq n_2 = n-1,$$

$$\Omega_k = \begin{pmatrix} (n_1 - 2k + 2)\sqrt{-1}\omega & 0 \\ 0 & (n_2 - 2k + 2)\sqrt{-1}\omega \end{pmatrix}, \quad 1 \leq k \leq n_2 + 1 = n.$$

(2)

$$\Omega_{12} = \begin{pmatrix} 0 & 0 \\ A_n \varphi & 0 \end{pmatrix}_{2n \times l}, \quad A_n = \begin{pmatrix} \beta_n & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{n_2}^{(n_1)} & 0 \\ 0 & 0 \end{pmatrix}.$$

(3)

$$\Omega_{22} = \begin{pmatrix} \omega_{2n+1,2n+1} & \beta_{n+1}\varphi & 0 & 0 & 0 \\ -\beta_{n+1}\bar{\varphi} & \omega_{2n+2,2n+2} & \beta_{n+2}\varphi & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -\beta_{n+l-2}\bar{\varphi} & \omega_{2n+l-1,2n+l-1} & \beta_{n+l-1}\varphi \\ 0 & 0 & 0 & -\beta_{n+l-1}\bar{\varphi} & \omega_{2n+l,2n+l} \end{pmatrix},$$

where

$$\omega_{2n+j,2n+j} = (n_1 - 2n_2 - 2j)\sqrt{-1}\omega, \quad \beta_{n+j} = a_{n_2+j}^{(n_1)}, \quad 1 \leq j \leq l.$$

The induced metric is

$$f_{n_1, n_2}^* ds^2 = (\beta_1^2 + \gamma_1^2)\varphi\bar{\varphi} = ((a_0^{(n_1)})^2 + (a_0^{(n_2)})^2)\varphi\bar{\varphi} = (n_1 + n_2)\varphi\bar{\varphi} = \phi\bar{\phi},$$

so the Gaussian curvature $K = 4/(n_1 + n_2) = 4/(N - 2)$ and $\varphi = \phi/\sqrt{n_1 + n_2}$. We set

$$\hat{\beta}_k = \frac{\beta_k}{\sqrt{n_1 + n_2}}, \quad \hat{\gamma}_k = \frac{\gamma_k}{\sqrt{n_1 + n_2}}.$$

Then

$$\hat{\beta}_k = \frac{k(n_1 - k + 1)}{n_1 + n_2}, \quad 1 \leq k \leq n_1, \quad \hat{\gamma}_k = \frac{k(n_2 - k + 1)}{n_1 + n_2}, \quad 1 \leq k \leq n_2. \quad (4.10)$$

Lemma 4.4 When $n_1 = n_2 = n-1$ in Example 1.2, the map $f_{n-1, n-1}$ is unitary equivalent to the map $f_{\arctan \sqrt{n-1}}$ in Example 1.1.

Proof: For the map $f_{n-1, n-1}$ and the unitary frame chosen as in Proposition 4.3, we choose a new unitary frame $\tilde{e} = \text{diag}(U, \dots, U)e$, where

$$U = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix},$$

and z is a local complex coordinate on S^2 so that $\varphi = \frac{1}{1+|z|^2}dz$. Then one can check by straightforward computations that the pull back of Maurer-Cartan forms via $\tilde{e}$ are the same as the forms in Proposition 4.1 by setting $t = \arctan \sqrt{n-1}$, which implies that $f_{n-1, n-1}$ and $f_{\arctan \sqrt{n-1}}$ are unitary equivalent. $\square$

5 Characterization of homogeneous holomorphic two-spheres in $G(2, N)$

In this section, we will give a local characterization of these homogeneous holomorphic two-spheres in $G(2, N)$ given in Examples 1.1 and 1.2. The following algebraic lemma will be used in later proof.

Lemma 5.1 *Let $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be a 2 by 2 complex matrix with $\text{rank} A = 2$ and $A^* A = \begin{pmatrix} \mu^2 & 0 \\ 0 & \nu^2 \end{pmatrix}$, where μ, ν are two positive real numbers. If $p\bar{r} + q\bar{s} = 0$, then one of the following three cases must hold*

- (1) $|q| = |r| \neq 0$, $|p| = |s| \neq 0$, $\mu^2 = |p|^2 + |q|^2 = |r|^2 + |s|^2 = \nu^2$,
- (2) $q = r = 0$, $|p|^2 = \mu^2$ and $|s|^2 = \nu^2$,
- (3) $p = s = 0$, $|r|^2 = \mu^2$ and $|q|^2 = \nu^2$.

Proof: Using $A^* A = \text{diag}\{\mu^2, \nu^2\}$ gives $|p|^2 + |r|^2 = \mu^2$, $|q|^2 + |s|^2 = \nu^2$ and $p\bar{q} + r\bar{s} = 0$. Thus

$$|p||q| = |r||s|. \quad (5.1)$$

On the other hand, since $p\bar{r} + q\bar{s} = 0$, it follows that

$$|p||r| = |q||s|. \quad (5.2)$$

If $p \neq 0$, then from (5.2) and $\det A = ps - qr \neq 0$, we get $s \neq 0$. Then both q and r are zero or nonzero together. If $q \neq 0$ and $r \neq 0$, then from (5.1) and (5.2), we obtain $|q| = |r|$ and $|p| = |s|$, so the case (1) holds. If $q = r = 0$, then we have the case (2).

If $p = 0$, from (5.2) and $\det A = ps - qr \neq 0$, we have $s = 0$ and case (3) holds. So the lemma is proved. $\square$

5.1 f is totally non-degenerate

Theorem 5.2 *Let $f : S^2 \rightarrow G(2, 2n)$, ($n \geq 3$) be a linearly full totally non-degenerate holomorphic immersion. If both K and S are constant and $\kappa = \text{tr} P_{1,1} P_{1,1}^*$ vanishes identically, then f is congruent to one of the immersions f_t in Example 1.1.*

Proof: By the first equation of (2.11) and the theory of singular values decomposition of complex matrices, we can further choose a local unitary frame e such that

$$A_1 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \nu_1 \end{pmatrix}, A_k^* A_k = \begin{pmatrix} \mu_k^2 & 0 \\ 0 & \nu_k^2 \end{pmatrix}, 2 \leq k \leq n-1, \quad (5.3)$$

where μ_k, ν_k are singular values of A_k and $\mu_k \geq \nu_k \geq 0$. Since S^2 acquires the induced metric, then

$$L_1 = \text{tr} A_1 A_1^* = \mu_1^2 + \nu_1^2 = 1, \quad (5.4)$$

which implies $1 > \mu_1 \geq \nu_1 > 0$. From our assumptions that K and S are constant, then it follows from Gauss equation that $|\det A_1| = \mu_1 \nu_1$ is constant. From (5.4) we conclude that both μ_1 and ν_1 are positive constant. We set

$$P_k = \begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix}, \quad P_{k,1} = \begin{pmatrix} p_{k1} & p_{k2} \\ p_{k3} & p_{k4} \end{pmatrix},$$

for some local smooth functions $x_k, y_k, z_k, w_k, p_{k1}, p_{k2}, p_{k3}, p_{k4}$, $1 \leq k \leq n-1$.

Case I $\mu_1 > \nu_1$.

Step 1 Now we rewrite the equation (2.8) with $k = 1$ in terms of its components, we have

$$d\mu_1 - \mu_1(\sqrt{-1}\rho + \omega_{11} - \omega_{33}) = x_1\phi, \quad (5.5)$$

$$d\nu_1 - \nu_1(\sqrt{-1}\rho + \omega_{22} - \omega_{44}) = w_1\phi, \quad (5.6)$$

$$\mu_1\omega_{34} - \nu_1\omega_{12} = y_1\phi, \quad (5.7)$$

$$\nu_1\omega_{43} - \mu_1\omega_{21} = z_1\phi. \quad (5.8)$$

Then by using (5.5), (5.6) and observing that $\sqrt{-1}\rho, \omega_{11}, \omega_{22}, \omega_{33}$ and ω_{44} are purely imaginary forms, we have $x_1 = w_1 = 0$ and

$$-\sqrt{-1}\rho = \omega_{11} - \omega_{33} = \omega_{22} - \omega_{44}. \quad (5.9)$$

By solving the equations (5.7) and (5.8), we obtain

$$\omega_{12} = \frac{\nu_1 y_1 \phi + \mu_1 \bar{z}_1 \bar{\phi}}{\mu_1^2 - \nu_1^2}, \quad \omega_{34} = \frac{\mu_1 y_1 \phi + \nu_1 \bar{z}_1 \bar{\phi}}{\mu_1^2 - \nu_1^2}. \quad (5.10)$$

Since $x_1 = w_1 = 0$, using (5.9), we can also rewrite the equation (2.10) with $k = 1$ in terms of components as follows

$$0 = dx_1 = z_1\omega_{12} - y_1\omega_{43} + \left(\frac{K}{2}\mu_1 + \mu_1(|p_2|^2 + |q_2|^2) - 2\mu_1^3\right)\bar{\phi} + p_{11}\phi, \quad (5.11)$$

$$0 = dw_1 = y_1\omega_{21} - z_1\omega_{34} + \left(\frac{K}{2}\nu_1 + \nu_1(|r_2|^2 + |s_2|^2) - 2\nu_1^3\right)\bar{\phi} + p_{14}\phi, \quad (5.12)$$

$$dy_1 = y_1(\sqrt{-1}\rho + \omega_{11} - \omega_{22}) + p_{12}\phi + \mu_1(p_2\bar{r}_2 + q_2\bar{s}_2)\bar{\phi}, \quad (5.13)$$

$$dz_1 = z_1(\sqrt{-1}\rho - \omega_{11} + \omega_{22}) + p_{13}\phi + \nu_1(\bar{p}_2 r_2 + \bar{q}_2 s_2)\bar{\phi}, \quad (5.14)$$

The coefficients of the (1,0) type and (0,1) type of the right sides of the equations (5.11) and (5.12) should be zero. Then using $\kappa = \text{tr} P_{1,1} P_{1,1}^* = 0$, that is $p_{11} = p_{12} = p_{13} = p_{14} = 0$ and (5.10), we have $y_1 z_1 = 0$ and

$$\frac{|y_1|^2 + |z_1|^2}{\mu_1^2 - \nu_1^2} + \frac{K}{2} - 2\mu_1^2 + |p_2|^2 + |q_2|^2 = 0, \quad (5.15)$$

$$-\frac{|y_1|^2 + |z_1|^2}{\mu_1^2 - \nu_1^2} + \frac{K}{2} - 2\nu_1^2 + |r_2|^2 + |s_2|^2 = 0. \quad (5.16)$$

Thus $y_1 = 0$ or $z_1 = 0$.

Subcase II $y_1 \neq 0$ and $z_1 = 0$.

From (5.14), we have $\bar{p}_2 r_2 + \bar{q}_2 s_2 = 0$ or equivalently $p_2 \bar{r}_2 + q_2 \bar{s}_2 = 0$. Then using (5.13) and $p_{12} = 0$ give

$$dy_1 = y_1(\sqrt{-1}\rho + \omega_{11} - \omega_{22}). \quad (5.17)$$

By rotating Z_1 and Z_3 through an angle θ_1 , we can take y_1 to be nonnegative real. Since $\sqrt{-1}\rho$, ω_{11} and ω_{22} are purely imaginary forms, then taking the complex conjugate of (5.17) gives

$$dy_1 = d\bar{y}_1 = -y_1(\sqrt{-1}\rho + \omega_{11} - \omega_{22}), \quad (5.18)$$

which implies that $dy_1 = 0$, so y_1 is a positive constant. Therefore, from (5.17),

$$\sqrt{-1}\rho + \omega_{11} - \omega_{22} = 0. \quad (5.19)$$

Taking exterior derivative of (5.19) and using (2.8) with $k = 1$, we get

$$\frac{K}{2} - \frac{2\nu_1^2 y_1^2}{(\mu_1^2 - \nu_1^2)^2} - (\mu_1^2 - \nu_1^2) = 0, \quad (5.20)$$

or

$$y_1^2 = \frac{K(\mu_1^2 - \nu_1^2)^2 - 2(\mu_1^2 - \nu_1^2)^3}{4\nu_1^2} > 0, \quad (5.21)$$

which implies that $\mu_1^2 < \frac{K+2}{4}$. Using (5.21), we solve from (5.15), (5.16) to get

$$\begin{aligned} |p_2|^2 + |q_2|^2 &= 2\mu_1^2 - \frac{K}{2} - \frac{y_1^2}{\mu_1^2 - \nu_1^2} = \frac{2-K}{4\nu_1^2}, \\ |r_2|^2 + |s_2|^2 &= 2\nu_1^2 - \frac{K}{2} + \frac{y_1^2}{\mu_1^2 - \nu_1^2} = \frac{(2-K)(4\nu_1^2 - 1)}{4\nu_1^2}. \end{aligned}$$

Note that $\nu_1^2 < 1/2$, then $|p_2|^2 + |q_2|^2 > |r_2|^2 + |s_2|^2$. From $p_2 \bar{r}_2 + q_2 \bar{s}_2 = 0$ and Lemma 5.1, the case (1) could not happen. As $\mu_2 \geq \nu_2$, we must have $q_2 = r_2 = 0$ and

$$\mu_2^2 = |p_2|^2 = 2\mu_1^2 - \frac{K}{2} - \frac{y_1^2}{\mu_1^2 - \nu_1^2}, \quad \nu_2^2 = |s_2|^2 = 2\nu_1^2 - \frac{K}{2} + \frac{y_1^2}{\mu_1^2 - \nu_1^2}. \quad (5.22)$$

So

$$L_2 = \mu_2^2 + \nu_2^2 = 2 - K, \quad \mu_2^2 - \nu_2^2 = \frac{(\mu_1^2 - \nu_1^2)(2 - K)}{2\nu_1^2}. \quad (5.23)$$

Now, by suitable choices of Z_5 and Z_6 , we can set $p_2 = \mu_2$ and $s_2 = \nu_2$, that is

$$A_2 = \begin{pmatrix} \mu_2 & 0 \\ 0 & \nu_2 \end{pmatrix},$$

where μ_2, ν_2 are two positive constants given by (5.22). Since $z_1 = 0$ in this subcase, from (5.10) we set

$$\omega_{12} = \lambda_1 \phi, \quad \omega_{34} = \lambda_2 \phi, \quad \text{where } \lambda_1 = \frac{\nu_1 y_1}{\mu_1^2 - \nu_1^2}, \quad \lambda_2 = \frac{\mu_1 y_1}{\mu_1^2 - \nu_1^2} = \frac{\mu_1}{\nu_1} \lambda_1. \quad (5.24)$$

From the first equation of (5.24) and (5.21), we have

$$\lambda_1^2 + \mu_1^2 = \frac{K+2}{4}. \quad (5.25)$$

Step 2 Using the equation (2.8) with $k = 2$ in terms of its components, we have

$$d\mu_2 - \mu_2(\sqrt{-1}\rho + \omega_{33} - \omega_{55}) = x_2 \phi, \quad (5.26)$$

$$d\nu_2 - \nu_2(\sqrt{-1}\rho + \omega_{44} - \omega_{66}) = w_2 \phi, \quad (5.27)$$

$$\mu_2 \omega_{56} - \nu_2 \omega_{34} = y_2 \phi, \quad (5.28)$$

$$\nu_2 \omega_{65} - \mu_2 \omega_{43} = z_2 \phi. \quad (5.29)$$

Then by using (5.26), (5.27) and observing that $\sqrt{-1}\rho, \omega_{33}, \omega_{44}, \omega_{55}$ and ω_{66} are purely imaginary forms, we have $x_2 = w_2 = 0$ and

$$-\sqrt{-1}\rho = \omega_{33} - \omega_{55} = \omega_{44} - \omega_{66}. \quad (5.30)$$

From (5.28), we know

$$\omega_{56} = \lambda_3 \phi, \quad \text{where } \lambda_3 = \frac{y_2 + \nu_2 \lambda_2}{\mu_2}. \quad (5.31)$$

Using (5.29), we have $z_2 = 0$ and

$$\bar{\lambda}_3 = \frac{\mu_2}{\nu_2} \lambda_2 = \lambda_3. \quad (5.32)$$

Therefore, $y_2 = \frac{(\mu_2^2 - \nu_2^2)\lambda_2}{\nu_2}$ is also a positive constant. From (5.9) and (5.19), we get

$$\sqrt{-1}\rho + \omega_{33} - \omega_{44} = 0. \quad (5.33)$$

Taking exterior derivative of (5.33) and using (2.8) with $k = 2$, we obtain

$$y_2^2 = \frac{K(\mu_2^2 - \nu_2^2)^2 + 2(\mu_1^2 - \nu_1^2)(\mu_2^2 - \nu_2^2)^2 - 2(\mu_2^2 - \nu_2^2)^3}{4\nu_2^2}. \quad (5.34)$$

Using the equation (2.10) with $k = 2$ together with (5.30), we have $P_{2,1} = 0, p_3 \bar{r}_3 + q_3 \bar{s}_3 = 0$ and

$$y_2 \lambda_3 + \frac{K}{2} \mu_2 + \mu_1^2 \mu_2 - 2\mu_2^3 + \mu_2(|p_3|^2 + |q_3|^2) = 0, \quad (5.35)$$

$$-y_2 \lambda_2 + \frac{K}{2} \nu_2 + \nu_1^2 \nu_2 - 2\nu_2^3 + \nu_2(|r_3|^2 + |s_3|^2) = 0. \quad (5.36)$$

Substituting $\lambda_3 = \mu_2\lambda_2/\nu_2$ and $\lambda_2 = y_2\nu_2/(\mu_2^2 - \nu_2^2)$ into (5.35) and (5.36) gives

$$\frac{y_2^2}{\mu_2^2 - \nu_2^2} + \frac{K}{2} + \mu_1^2 - 2\mu_2^2 + |p_3|^2 + |q_3|^2 = 0, \quad (5.37)$$

$$-\frac{y_2^2}{\mu_2^2 - \nu_2^2} + \frac{K}{2} + \nu_1^2 - 2\nu_2^2 + |r_3|^2 + |s_3|^2 = 0. \quad (5.38)$$

Note that $p_3\bar{r}_3 + q_3\bar{s}_3 = 0$, then using Lemma 5.1 and the same arguments as in Step 1, we can choose unitary frames Z_7 and Z_8 so that

$$A_3 = \begin{pmatrix} \mu_3 & 0 \\ 0 & \nu_3 \end{pmatrix},$$

where μ_3, ν_3 are two positive constants given by

$$\mu_3^2 = 2\mu_2^2 - \mu_1^2 - \frac{K}{2} - \frac{y_2^2}{\mu_2^2 - \nu_2^2}, \quad \nu_3^2 = 2\nu_2^2 - \nu_1^2 - \frac{K}{2} + \frac{y_2^2}{\mu_2^2 - \nu_2^2}. \quad (5.39)$$

Using (5.34), (5.22) and through straightforward computations, the equations (5.39) can be simplified to

$$\mu_3^2 = \frac{(2-K)(3-3K)}{4\nu_2^2}, \quad \nu_3^2 = \frac{(3-3K)(4\nu_2^2 + K - 2)}{4\nu_2^2}, \quad (5.40)$$

which yields

$$L_3 = \mu_3^2 + \nu_3^2 = 3 - 3K, \quad \mu_3^2 - \nu_3^2 = \frac{(\mu_2^2 - \nu_2^2)(3 - 3K)}{2\nu_2^2}. \quad (5.41)$$

Step $k-1$ Assumption Step Now we assume that we can choose unitary frames so that

$$A_j = \begin{pmatrix} \mu_j & 0 \\ 0 & \nu_j \end{pmatrix}, \quad \omega_{2j-1,2j} = \lambda_j\phi, \quad 1 \leq j \leq k. \quad (5.42)$$

$$-\sqrt{-1}\rho = \omega_{11} - \omega_{22} = \omega_{2j-1,2j-1} - \omega_{2j+1,2j+1} \quad (5.43)$$

$$= \omega_{2j,2j} - \omega_{2j+2,2j+2}, \quad 1 \leq j \leq k-1. \quad (5.44)$$

where $\mu_j > \nu_j$, λ_j are positive constants, and

$$L_j = \mu_j^2 + \nu_j^2 = j - \frac{j(j-1)}{2}K, \quad \mu_j^2 - \nu_j^2 = \frac{L_j(\mu_{j-1}^2 - \nu_{j-1}^2)}{2\nu_{j-1}^2}, \quad (5.45)$$

$$\mu_j^2 = \frac{L_{j-1}L_j}{4\nu_{j-1}^2}, \quad \nu_j^2 = \frac{L_j(4\nu_{j-1}^2 - L_{j-1})}{4\nu_{j-1}^2}, \quad \lambda_j = \frac{\mu_{j-1}}{\nu_{j-1}}\lambda_{j-1} \quad 1 \leq j \leq k. \quad (5.46)$$

$$y_j = \frac{(\mu_j^2 - \nu_j^2)\lambda_j}{\nu_j}, \quad y_j \text{ are positive constants, } 1 \leq j \leq k-1. \quad (5.47)$$

Step k Induction Step Using the equation (2.8) with general k in terms of its components, we have

$$d\mu_k - \mu_k(\sqrt{-1}\rho + \omega_{2k-1,2k-1} - \omega_{2k+1,2k+1}) = x_k\phi, \quad (5.48)$$

$$d\nu_k - \nu_k(\sqrt{-1}\rho + \omega_{2k,2k} - \omega_{2k+2,2k+2}) = w_k\phi, \quad (5.49)$$

$$\mu_k\omega_{2k+1,2k+2} - \nu_k\omega_{2k-1,2k} = y_k\phi, \quad (5.50)$$

$$\nu_k\omega_{2k+2,2k+1} - \mu_k\omega_{2k,2k-1} = z_k\phi. \quad (5.51)$$

Then by using (5.48), (5.49) and observing that $\sqrt{-1}\rho$, $\omega_{2k-1,2k-1}$, $\omega_{2k,2k}$, $\omega_{2k+1,2k+1}$ and $\omega_{2k+2,2k+2}$ are purely imaginary forms, we have $x_k = w_k = 0$ and

$$-\sqrt{-1}\rho = \omega_{2k-1,2k-1} - \omega_{2k+1,2k+1} = \omega_{2k,2k} - \omega_{2k+2,2k+2}. \quad (5.52)$$

From (5.50), we have

$$\omega_{2k+1,2k+2} = \lambda_{k+1}\phi, \text{ where } \lambda_{k+1} = \frac{y_k + \nu_k\lambda_k}{\mu_k}. \quad (5.53)$$

Using (5.51), we have $z_k = 0$ and

$$\bar{\lambda}_{k+1} = \frac{\mu_k}{\nu_k}\lambda_k = \lambda_{k+1}. \quad (5.54)$$

Therefore, $y_k = \frac{(\mu_k^2 - \nu_k^2)\lambda_k}{\nu_k}$ is also a positive constant. From assumption step, we know

$$\sqrt{-1}\rho + \omega_{2k-1,2k-1} - \omega_{2k,2k} = 0. \quad (5.55)$$

Taking exterior derivative of (5.55) and using (2.8) with general k , we obtain

$$y_k^2 = \frac{K(\mu_k^2 - \nu_k^2)^2 + 2(\mu_{k-1}^2 - \nu_{k-1}^2)(\mu_k^2 - \nu_k^2)^2 - 2(\mu_k^2 - \nu_k^2)^3}{4\nu_k^2}. \quad (5.56)$$

Using the equation (2.10) with general k , we have $P_{k,1} = 0$, $p_{k+1}\bar{r}_{k+1} + q_{k+1}\bar{s}_{k+1} = 0$ and

$$y_k\lambda_{k+1} + \frac{K}{2}\mu_k + \mu_1^2\mu_k - 2\mu_k^3 + \mu_k(|p_{k+1}|^2 + |q_{k+1}|^2) = 0, \quad (5.57)$$

$$-y_k\lambda_k + \frac{K}{2}\nu_k + \nu_1^2\nu_k - 2\nu_k^3 + \nu_k(|r_{k+1}|^2 + |s_{k+1}|^2) = 0. \quad (5.58)$$

Substituting $\lambda_{k+1} = \mu_k\lambda_k/\nu_k$ and $\lambda_k = y_k\nu_k/(\mu_k^2 - \nu_k^2)$ into (5.57) and (5.58) gives

$$\frac{y_k^2}{\mu_k^2 - \nu_k^2} + \frac{K}{2} + \mu_{k-1}^2 - 2\mu_k^2 + |p_{k+1}|^2 + |q_{k+1}|^2 = 0, \quad (5.59)$$

$$-\frac{y_k^2}{\mu_k^2 - \nu_k^2} + \frac{K}{2} + \nu_{k-1}^2 - 2\nu_k^2 + |r_{k+1}|^2 + |s_{k+1}|^2 = 0. \quad (5.60)$$

Note that $p_{k+1}\bar{r}_{k+1} + q_{k+1}\bar{s}_{k+1} = 0$, then using Lemma 5.1 and the same arguments as in Step 1, we can choose unitary frames Z_{2k+1} and Z_{2k+2} so that

$$A_{k+1} = \begin{pmatrix} \mu_{k+1} & 0 \\ 0 & \nu_{k+1} \end{pmatrix},$$

where μ_{k+1}, ν_{k+1} are two positive constants given by

$$\mu_{k+1}^2 = 2\mu_k^2 - \mu_{k-1}^2 - \frac{K}{2} - \frac{y_k^2}{\mu_k^2 - \nu_k^2}, \quad \nu_{k+1}^2 = 2\nu_k^2 - \nu_{k-1}^2 - \frac{K}{2} + \frac{y_k^2}{\mu_k^2 - \nu_k^2}. \quad (5.61)$$

Then from (5.45), we check that

$$\begin{aligned} L_{k+1} &= \mu_{k+1}^2 + \nu_{k+1}^2 = 2(\mu_k^2 + \nu_k^2) - (\mu_{k-1}^2 + \nu_{k-1}^2) - K \\ &= 2L_k - L_{k-1} - K = (k+1) - \frac{k(k+1)}{2}K. \end{aligned} \quad (5.62)$$

Using (5.56) together with (5.45), (5.46) and through straightforward computations, the equations (5.61) can be simplified to

$$\mu_{k+1}^2 = \frac{L_k L_{k+1}}{4\nu_k^2}, \quad \nu_{k+1}^2 = \frac{L_{k+1}(4\nu_k^2 - L_k)}{4\nu_k^2}, \quad (5.63)$$

which yields

$$\mu_{k+1}^2 - \nu_{k+1}^2 = \frac{(\mu_k^2 - \nu_k^2)L_{k+1}}{2\nu_k^2} > 0. \quad (5.64)$$

Thus, we have showed that the equations (5.42)-(5.47) also hold with k replaced by $k+1$.

Step $n-1$ Final Step The equations (5.48)-(5.56) also hold when $k = n-1$ and (5.59), (5.60) are reduced to

$$\frac{y_{n-1}^2}{\mu_{n-1}^2 - \nu_{n-1}^2} + \frac{K}{2} + \mu_{n-2}^2 - 2\mu_{n-1}^2 = 0, \quad (5.65)$$

$$-\frac{y_{n-1}^2}{\mu_{n-1}^2 - \nu_{n-1}^2} + \frac{K}{2} + \nu_{n-2}^2 - 2\nu_{n-1}^2 = 0. \quad (5.66)$$

The two equations can be viewed as

$$\begin{aligned} 0 = \mu_n^2 &= 2\mu_{n-1}^2 - \mu_{n-2}^2 - \frac{K}{2} - \frac{y_{n-1}^2}{\mu_{n-1}^2 - \nu_{n-1}^2}, \\ 0 = \nu_n^2 &= 2\nu_{n-1}^2 - \nu_{n-2}^2 - \frac{K}{2} + \frac{y_{n-1}^2}{\mu_{n-1}^2 - \nu_{n-1}^2}. \end{aligned}$$

Then by similar computations, we can show that (5.62) also holds for $k = n-1$, that is

$$\begin{aligned} 0 = L_n &= \mu_n^2 + \nu_n^2 = 2(\mu_{n-1}^2 + \nu_{n-1}^2) - (\mu_{n-2}^2 + \nu_{n-2}^2) - K \\ &= 2L_{n-1} - L_{n-2} - K = n - \frac{(n-1)n}{2}K, \end{aligned}$$

which gives

$$K = \frac{2}{n-1}.$$

In summary, for this subcase I(i), we have obtained that we can choose unitary frames $e = \{Z_1, \dots, Z_{2n}\}$ so that the pull back of Maurer-Cartan forms are given by

$$\begin{pmatrix} \Omega_1 & A_1\phi & 0 & 0 & 0 \\ -A_1^*\bar{\phi} & \Omega_2 & A_2\phi & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -A_{n-2}^*\bar{\phi} & \Omega_{n-1} & A_{n-1}\phi \\ 0 & 0 & 0 & -A_{n-1}^*\bar{\phi} & \Omega_n \end{pmatrix}, \quad (5.67)$$

$$\Omega_k = \begin{pmatrix} \omega_{2k-1,2k-1} & \lambda_k\phi \\ -\lambda_k\bar{\phi} & \omega_{2k,2k} \end{pmatrix}, \quad 1 \leq k \leq n, \quad (5.68)$$

$$A_k = \begin{pmatrix} \mu_k & 0 \\ 0 & \nu_k \end{pmatrix}, \quad 1 \leq k \leq n-1, \quad (5.69)$$

$$-\sqrt{-1}\rho = \omega_{11} - \omega_{22} = \omega_{2k-1,2k-1} - \omega_{2k+1,2k+1} \quad (5.70)$$

$$= \omega_{2k,2k} - \omega_{2k+2,2k+2}, \quad 1 \leq k \leq n-1, \quad (5.71)$$

where $\lambda_k, \mu_k > \nu_k$ are all positive constants and satisfy the recursion formulae

$$\mu_1^2 + \nu_1^2 = 1, \quad \lambda_1^2 + \mu_1^2 = \frac{K+2}{4}, \quad (5.72)$$

$$\mu_k^2 + \nu_k^2 = L_k = k - \frac{(k-1)k}{2}K, \quad \mu_{k+1}^2 = \frac{L_k L_{k+1}}{4\nu_k^2}, \quad \lambda_{k+1} = \frac{\mu_k}{\nu_k} \lambda_k. \quad (5.73)$$

Such unitary frames are determined up to the same rotations $\tilde{Z}_k = e^{\sqrt{-1}\tau} Z_k$, $1 \leq k \leq 2n$, where τ is a local smooth real function on M , which induces the change

$$\sum_{k=1}^{2n} \tilde{\omega}_{kk} = \sum_{k=1}^{2n} \omega_{kk} + 2n\sqrt{-1}d\tau. \quad (5.74)$$

On the other hand, through direction computations, we have

$$d \sum_{k=1}^{2n} \omega_{kk} = 0, \quad (5.75)$$

which means $\sum_{k=1}^{2n} \omega_{kk}$ is a closed form. Hence, we can take a suitable local function τ so that

$$\sum_{k=1}^{2n} \omega_{kk} = 0. \quad (5.76)$$

From (5.70), (5.71) and (5.76), we solve that

$$\omega_{11} = -\frac{\sqrt{-1}\rho n}{2}, \quad \omega_{2k,2k} = \omega_{2k+1,2k+1} = \omega_{11} + k\sqrt{-1}\rho. \quad (5.77)$$

Now from (5.21), we can set

$$\mu_1 = \frac{\sqrt{K+2}}{2} \sin t, \quad t \in (0, \pi/2).$$

From $\mu_1 > \nu_1$ and $\mu_1^2 + \nu_1^2 = 1$, we derive that $\tan t > \sqrt{2/K}$, that is $t \in (\arctan \sqrt{2/K}, \pi/2)$. Note that $\rho = -2\omega$, $K = 2/(n-1)$ and comparing the Maurer-Cartan forms with (4.4)-(4.9), we see that $\lambda_k = \hat{\alpha}_k$, $1 \leq k \leq n$ and $\mu_k = \hat{\beta}_k$, $\nu_k = \hat{\gamma}_k$, $1 \leq k \leq n-1$, so all Maurer-Cartan forms agree, which follows that f is unitary equivalent to the immersion f_t defined in Example 1.1 for some $t \in (\arctan \sqrt{n-1}, \pi/2)$.

Subcase Iii $z_1 \neq 0$ and $y_1 = 0$.

Similar computations are applied to the case, we can also determine the pull back of Maurer-Cartan forms with respect to a suitable unitary frame e as in (5.67), (5.69)-(5.73), only Ω_k should be changed to

$$\Omega_k = \begin{pmatrix} \omega_{2k-1,2k-1} & -\lambda_k \bar{\phi} \\ \lambda_k \phi & \omega_{2k,2k} \end{pmatrix}.$$

If we put

$$\tilde{Z}_{2k-1} = Z_{2k}, \quad \tilde{Z}_{2k} = Z_{2k-1}, \quad 1 \leq k \leq n,$$

then the pull back of Maurer-Cartan forms via the new unitary frame $\tilde{e}$ has the form in (5.67) and (5.68) with

$$A_k = \begin{pmatrix} \nu_k & 0 \\ 0 & \mu_k \end{pmatrix}, \quad 1 \leq k \leq n-1.$$

Now we set

$$\nu_1 = \frac{\sqrt{K+2}}{2} \sin t, \quad t \in (0, \pi/2),$$

then $t \in (0, \arctan \sqrt{n-1})$ due to $\mu_1 > \nu_1$. So for this case, f is unitary equivalent to the immersion f_t defined in Example 1.1 for some $t \in (0, \arctan \sqrt{n-1})$.

Subcase Iiii $y_1 = z_1 = 0$.

For this case, by similar computations as in subcase Ii, we can show that $y_k = z_k = 0$, $1 \leq k \leq n$ and the equations (5.61) also hold for $1 \leq k \leq n$ with setting $\mu_0 = \nu_0 = \mu_n = \nu_n = 0$. Then we derive that

$$2(\mu_k^2 - \nu_k^2) = (\mu_{k+1}^2 - \nu_{k+1}^2) - (\mu_{k-1}^2 - \nu_{k-1}^2), \quad 1 \leq k \leq n-1, \quad (5.78)$$

which implies

$$\mu_1^2 - \nu_1^2 = -(\mu_{n-1}^2 - \nu_{n-1}^2).$$

Since $\mu_1 > \nu_1$, $\mu_{n-1} > \nu_{n-1}$ in this case, the above equality leads to a contradiction, which means when $\mu_1 > \nu_1$, the subcase Iiii does not happen.

Case II $\mu_1 = \nu_1 = \frac{\sqrt{2}}{2}$.

Step 1 For this case, we can choose unitary frames such that

$$A_2 = \begin{pmatrix} \mu_2 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad \mu_2 > 0, \quad \nu_2 > 0.$$

From (5.5)-(5.8), we have $P_1 = 0$, $\omega_{12} = \omega_{34}$ and (5.9). Then using (5.11)-(5.16), we get $P_{1,1} = 0$ and

$$\begin{aligned}\frac{K}{2} - 2\mu_1^2 + \mu_2^2 &= 0, \\ \frac{K}{2} - 2\nu_1^2 + \nu_2^2 &= 0,\end{aligned}$$

which yields $\mu_2^2 = \nu_2^2 = 2\mu_1^2 - \frac{K}{2} = \frac{2-K}{2} = \frac{L_2}{2}$.

Step 2 Now we can choose unitary frames such that

$$A_3 = \begin{pmatrix} \mu_3 & 0 \\ 0 & \nu_3 \end{pmatrix}, \quad \mu_3 > 0, \quad \nu_3 > 0.$$

From (5.26)-(5.29), we have $P_2 = 0$, $\omega_{34} = \omega_{56}$ and (5.30). Similarly, using the equation (2.10) with $k = 2$, we get $P_{2,1} = 0$ and

$$\begin{aligned}\frac{K}{2} + \mu_1^2 - 2\mu_2^2 + \mu_3^2 &= 0, \\ \frac{K}{2} + \nu_1^2 - 2\nu_2^2 + \nu_3^2 &= 0,\end{aligned}$$

then $\mu_3^2 = \nu_3^2 = 2\mu_2^2 - \mu_1^2 - \frac{K}{2} = \frac{3-3K}{2} = \frac{L_3}{2}$.

Step k Induction Step By induction, we obtain that the unitary frame e can be chosen so that

$$\begin{aligned}\omega_{12} &= \omega_{34} = \cdots = \omega_{2n-1,2n}, \\ -\sqrt{-1}\rho &= \omega_{2k-1,2k-1} - \omega_{2k+1,2k+1} = \omega_{2k,2k} - \omega_{2k+2,2k+2}, \\ A_k &= \begin{pmatrix} \mu_k & 0 \\ 0 & \nu_k \end{pmatrix}, \quad 1 \leq k \leq n-1,\end{aligned}\tag{5.79}$$

where $\mu_k^2 = \nu_k^2 = \frac{L_k}{2}$ and $K = 2/(n-1)$. Such unitary frames are then determined up to n same $U(2)$ transformation. Using the same proof of (6.23) in [18] (page 810), the unitary frame e can be chosen so that

$$\omega_{12} = \omega_{34} = \cdots = \omega_{2n-1,2n} = 0.$$

Such unitary frames are then determined up to rotations

$$\tilde{Z}_{2k-1} = e^{\sqrt{-1}\tau_1} Z_{2k-1}, \quad \tilde{Z}_{2k} = e^{\sqrt{-1}\tau_2} Z_{2k},$$

where τ_1, τ_2 are local smooth functions. By similar arguments as (5.74)-(5.76), we can choose suitable local functions τ_1, τ_2 so that

$$\sum_{k=1}^n \omega_{2k-1,2k-1} = \sum_{k=1}^n \omega_{2k,2k} = 0.\tag{5.80}$$

We solve from (5.79) and (5.80) to get

$$\omega_{11} = \omega_{22} = -\frac{\sqrt{-1}\rho n}{2}, \quad \omega_{2k+1,2k+1} = \omega_{2k+2,2k+2} = \omega_{11} + k\sqrt{-1}\rho, \quad 1 \leq k \leq n-1. \quad (5.81)$$

Then the pull back of Maurer-Cartan forms via $\tilde{e}$ are given by (5.67)-(5.69) and (5.77), with $\mu_k^2 = \nu_k^2 = \frac{L_k}{2}$ and $\lambda_k = \frac{1}{2(n-1)} = \frac{K}{4}$. Then all Maurer-Cartan forms coincides with the forms in Proposition 4.3 when $n_1 = n_2 = n-1, l=0$, which follows that f is unitary equivalent to the curve $f_{n-1,n-1}$, and then unitary equivalent to $f_{\arctan \sqrt{n-1}}$ defined in Example 1.1 by Lemma 4.4 in Section 4. So we complete the proof of the theorem. $\square$

5.2 f degenerates at position n

Theorem 5.3 *Let $f : S^2 \rightarrow G(2, N)$ be a linearly full holomorphic immersion which generates at position $n \geq 2$. If both K and S are constant and $\kappa = \text{tr} P_{1,1} P_{1,1}^*$ vanishes identically, then f is congruent to one of the immersions f_{n_1, n_2} in Example 1.2.*

Proof: It is easy to see that the first $n-1$ steps are the same as the subcase Ii.

Step $n-1$ Using (2.10) with $k = n-1$, we have $p_n \bar{r}_n = 0$ and

$$\begin{aligned} \mu_n^2 &:= |p_n|^2 = 2\mu_{n-1}^2 - \mu_{n-2}^2 - \frac{K}{2} - \frac{y_{n-1}^2}{\mu_{n-1}^2 - \nu_{n-1}^2}, \\ \nu_n^2 &:= |r_n|^2 = 2\nu_{n-1}^2 - \nu_{n-2}^2 - \frac{K}{2} + \frac{y_{n-1}^2}{\mu_{n-1}^2 - \nu_{n-1}^2}. \end{aligned}$$

Then we must have $r_n = 0$, $\nu_n = 0$ and

$$\mu_n^2 = |p_n|^2 = L_n = n - \frac{(n-1)n}{2}K = \frac{L_{n-1}(\mu_{n-1}^2 - \nu_{n-1}^2)}{2\mu_{n-1}^2} > 0,$$

which means only the case $\mu_1 > \nu_1$ happens. We may choose frames so that $p_n = \mu_n$ is positive constant.

Step n Using the equation (2.8) with $k = n$ and

$$\Omega_{n+1} = \begin{pmatrix} \omega_{2n+1,2n+1} & a_{n+1}\phi \\ -\bar{a}_{n+1}\bar{\phi} & \omega_{2n+2,2n+2} \end{pmatrix},$$

we obtain that $\lambda_n \mu_n = 0$ and

$$-\sqrt{-1}\rho = \omega_{2n-1,2n-1} - \omega_{2n+1,2n+1}. \quad (5.82)$$

So $\lambda_n = 0$. From the recursion formula $\lambda_{k+1} = \frac{\mu_k}{\nu_k} \lambda_k$, $1 \leq k \leq n-1$, we must have $\lambda_k = 0$, $1 \leq k \leq n-1$. Exterior differentiation of (5.82), we get

$$\mu_{n+1}^2 = |a_{n+1}|^2 = 2\mu_n^2 - \mu_{n-1}^2 - \frac{K}{2}. \quad (5.83)$$

Therefore, μ_{n+1}^2 is also constant. We may choose frames so that $a_{n+1} = \mu_{n+1}$ is positive constant. So far the unitary frames have been chosen so that

$$\begin{aligned}\omega_{2k-1,2k} &= 0, \quad 1 \leq k \leq n, \\ -\sqrt{-1}\rho &= \omega_{2k-1,2k-1} - \omega_{2k+1,2k+1} = \omega_{2k,2k} - \omega_{2k+2,2k+2}, \quad 1 \leq k \leq n-1, \\ &= \omega_{2n-1,2n-1} - \omega_{2n+1,2n+1} \\ A_k &= \begin{pmatrix} \mu_k & 0 \\ 0 & \nu_k \end{pmatrix}, \quad 1 \leq k \leq n-1, \quad A_n = \begin{pmatrix} \mu_n & 0 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

where μ_k, ν_k satisfy

$$\mu_{k+1}^2 = 2\mu_k^2 - \mu_{k-1}^2 - \frac{K}{2}, \quad \nu_{k+1}^2 = 2\nu_k^2 - \nu_{k-1}^2 - \frac{K}{2}, \quad (5.84)$$

with $\mu_0 = \nu_0 = \nu_n = 0$.

Step $n+j$ Induction Step Now we can obtain by induction that

$$-\sqrt{-1}\rho = \omega_{2n+j,2k+j} - \omega_{2n+j+1,2n+j+1}, \quad 1 \leq j \leq l-1,$$

and $a_{n+j+1} = \mu_{n+j+1}$, $1 \leq j \leq l-2$ are positive constants and satisfy

$$\mu_{n+j+1}^2 = 2\mu_{n+j}^2 - \mu_{n+j-1}^2 - \frac{K}{2}, \quad 1 \leq j \leq l-1 \quad (5.85)$$

with $\mu_{n+l} = 0$. Set $n_1 = n + l - 1$, $n_2 = n - 1$. From (5.84) and (5.85), we solve that

$$\mu_k = \frac{k(n_1 - k + 1)}{4}K, \quad 1 \leq k \leq n_1 - 1, \quad \nu_k = \frac{k(n_2 - k + 1)}{4}K, \quad 1 \leq k \leq n_2 - 1.$$

Using $\mu_1^2 + \nu_1^2 = 1$, we have $K = \frac{4}{n_1 + n_2}$. Then μ_k, ν_k coincides with $\hat{\beta}_k, \hat{\gamma}_k$ in (4.10). By similar arguments as (5.74)-(5.76), we can further choose unitary frame so that the diagonal entries in Ω are the same as those in Proposition 4.3. Therefore, we conclude that f is unitary equivalent to f_{n_1, n_2} in Example 1.2. The proof is completed. $\square$

Combining Theorem 5.2 and Theorem 5.3, we obtain our main result.

6 Some remarks and questions

Remark 6.1 The first named author, Jiao and Xu in [13] and the second named author in her Ph.D. Thesis [19] showed that for a linearly full holomorphic immersion $f : S^2 \rightarrow G(2, N)$ with constant curvature K , if it degenerates at position 1 (i.e., f degenerates), then f is unitary equivalent to $\mathcal{V}_0^{(N-1)} \oplus \mathcal{V}_1^{(N-1)}$ with $K = 2/(N-2)$ or $\mathcal{V}_0^{(N-2)} \oplus \mathbb{C}\mathbf{v}$ with $K = 4/(N-2)$, where $\mathbf{v}$ is a non-zero constant vector. Now if we further assume that the square norm of the second fundamental form S is constant, then the corresponding local result also holds by Calabi's Rigidity Theorem and Plücker formulae. The proof is similar as the arguments in subsection 4.3 in [29], which is omitted here.

Remark 6.2 For the case $G(2, 4)$ and f is non-degenerate, the two conditions that K is constant and f is unramified ($|\det A_1| = \mu_1 \nu_1$ has no zeros) are equivalent to $K = 2$, which implies that $|\det A_1| = \lambda_1 \nu_1$ is a constant and $|\det P_1| = |y_1 z_1| = 0$ [8]. One can check that κ vanishes automatically in this case. Therefore, Theorem 5.2 holds when we need only to assume that K is constant and f is unramified when $N = 4$. In this sense, we generalize the classification result of Chi and Zheng [8].

Remark 6.3 From the Gauss equation, if K is constant, then $S = \text{constant}$ is equivalent to $|\det A_1| = \text{constant}$. When $N \geq 5$, the condition of constancy of S is necessary. The second named author et al. [21] determined two classes of linearly full non-homogeneous holomorphic two-spheres in $G(2, 5)$ with the same constant curvature $4/3$. Jiao gave an example of a totally non-degenerate holomorphic two-sphere in $G(2, 6)$ with constant curvature 1 [22], which is explicitly given as follows

$$f : S^2 \rightarrow G(2, 6), \quad z \mapsto \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}}z & \frac{\sqrt{31}}{2\sqrt{7}}z^2 & \frac{9}{2\sqrt{7}}z^2 & 0 \\ 0 & 1 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{2}}z & \frac{1}{2}z^2 \end{bmatrix},$$

and he computed that

$$|\det A_1|^2 = \frac{112 + 1024z\bar{z} + 1176z^2\bar{z}^2 + 376z^3\bar{z}^3 + 31z^4\bar{z}^4}{1024(1 + z\bar{z})^4},$$

which is not a constant.

In the end, we would like to propose the following two interesting problems which are deserved to be considered further.

The two theorems 5.2 and 5.3 are completely local in nature, where no topological assumptions are needed. We also need not to assume the harmonic sequence (2.4) is totally unramified.

Question 1 With the global topological assumption $M = S^2$ and the harmonic sequence is totally unramified, does the condition $\kappa = 0$ is essential or not? Do there exist examples of non-homogeneous holomorphic two-spheres in $G(2, 6)$ or general $G(2, N)$ with K and S being constant and $\kappa \neq 0$?

Remark 6.4 The first and second named authors [15] showed that for a linearly full unramified holomorphic curve $f : S^2 \rightarrow G(2, N)$ ($N \geq 5$) of constant curvature and constant length of second fundamental form, if the harmonic sequence generated by f degenerates at position 2, then f is unitary equivalent to $\mathcal{V}_0^{(n-3)} \oplus \mathcal{V}_0^{(1)}$ with $K = 4/(N-2)$.

To state the second problem, we summarize some known results. For the case of $G(2, 4)$, the Gaussian curvature K for linearly full holomorphic two-spheres can take the values $4/r$ with positive integer $r = 2, 3, 4$, and linearly full holomorphic two-spheres in $G(2, 4)$ with constant curvature were explicitly classified [25]. For linearly full non-singular holomorphic two-spheres in $G(2, 5)$, Jiao and Peng determine the value distribution of $K = 4/r$

with positive integer $r = 3, 4, 5$ and give an explicit classification by using Plücker embedding [23, 24]. In a recent paper [7], Chi, Xie and Xu classified linearly full holomorphic two-spheres in $G(2, 5)$ with constant curvature $4/6$ by exploring the projectively equivalent Fano 3-folds of index 2 and degree 5. Recently, the second named author recovered the results for the value distribution of Gaussian curvature $K = 4/r$ of linearly full holomorphic two-spheres in $G(2, 4)$ ($r = 2, 3, 4$) and $G(2, 5)$ ($r = 3, 4, 5, 6$), and got a lower bound $r \geq N - 2$ for the general case $G(2, N)$ [20].

Question 2 Classify all linearly full holomorphic two-spheres in $G(2, 6)$ with constant curvature 1 and determine the value distribution of Gaussian curvature for all linearly full holomorphic two-spheres in $G(2, 6)$.

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