

A FIRST-ORDER FOURIER INTEGRATOR FOR THE NONLINEAR SCHRÖDINGER EQUATION ON \mathbb{T} WITHOUT LOSS OF REGULARITY

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ABSTRACT. In this paper, we propose a first-order Fourier integrator for solving the cubic nonlinear Schrödinger equation in one dimension. The scheme is explicit and can be implemented using the fast Fourier transform. By a rigorous analysis, we prove that the new scheme provides the first order accuracy in H^γ for any initial data belonging to H^γ , for any $\gamma > \frac{3}{2}$. That is, up to some fixed time T , there exists some constant $C = C(\|u\|_{L^\infty([0,T];H^\gamma)}) > 0$, such that

$$\|u^n - u(t_n)\|_{H^\gamma(\mathbb{T})} \leq C\tau,$$

where u^n denotes the numerical solution at $t_n = n\tau$. Moreover, the mass of the numerical solution $M(u^n)$ verifies

$$|M(u^n) - M(u_0)| \leq C\tau^5.$$

In particular, our scheme does not cost any additional derivative for the first-order convergence and the numerical solution obeys the almost mass conservation law. Furthermore, if $u_0 \in H^1(\mathbb{T})$, we rigorously prove that

$$\|u^n - u(t_n)\|_{H^1(\mathbb{T})} \leq C\tau^{\frac{1}{2}-},$$

where $C = C(\|u_0\|_{H^1(\mathbb{T})}) > 0$.

1. INTRODUCTION

In this paper, we are concerned with the numerical integration of the cubic nonlinear Schrödinger equation (NLS) on a torus:

$$\begin{cases} i\partial_t u(t, x) + \partial_{xx} u(t, x) + \lambda |u(t, x)|^2 u(t, x) = 0, & t > 0, x \in \mathbb{T}, \\ u(0, x) = u_0(x), & x \in \mathbb{T}, \end{cases} \quad (1.1)$$

where $\mathbb{T} = (0, 2\pi)$, $\lambda = \pm 1$, $u = u(t, x) : \mathbb{R}^+ \times \mathbb{T} \rightarrow \mathbb{C}$ is the unknown and $u_0 \in H^\gamma(\mathbb{T})$ with some $\gamma \geq 0$ is a given initial data. Here we only consider the case $\lambda = -1$, and the case $\lambda = 1$ is similar.

It is known that the local well-posedness of (1.1) has been established in H^γ for $\gamma \geq 0$ in one dimension space, referring to [2]. Moreover, for L^2 solution of (1.1), we have the following mass conservation law:

$$M(u(t)) = \frac{1}{2\pi} \int_{\mathbb{T}} |u(t, x)|^2 dx = M(u_0). \quad (1.2)$$

Then the global well-posedness in L^2 is followed directly by the mass conservation law and the local theory.

There has been substantial research undertaken in numerical analysis of (1.1). In order to do numerical discretizations in space and time, many methods have been proposed and extensively studied by assuming that the exact solution is smooth enough, for example in finite difference methods, operator splitting, spectral methods, discontinuous Galerkin methods and exponential integrators. Much of the literature focusses on these classical numerical schemes to establish the convergence results of the solution, referring to [12, 17, 19, 28]. For the nonlinear Schrödinger equation, we can further refer to [1, 4, 5, 7, 9, 10, 22, 23, 27, 34]. For the Korteweg-de Vries equation, refer to [8, 11, 18, 20, 21, 32] for recent works.

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For splitting methods, one of the most popular splitting is Strang splitting, which can speed up calculation for problems involving operators on very different time scales. In particular, splitting methods are especially effective if the equation splits into two equations which can be directly integrated such as the Korteweg-de Vries equation and the nonlinear Schrödinger equation. For example, for the nonlinear Schrödinger equation, the first-order and the second-order convergence in H^γ was achieved for the initial data in $H^{\gamma+2}$ and $H^{\gamma+4}$ respectively, see [27].

For exponential integrators, to the best of our knowledge, which were earlier considered by Hersch in [13], Certainé in [6] and Pope in [33]. Then Hochbruck, Lubich, and Selhofer [14] put forward the term “exponential integrator”, which created a powerful push of the exponential integrator. Furthermore, this work was regarded as the first actual implementation of the exponential integrator. Recently, exponential integrators have become an active area of research and have a good development, more on the rich history and research results of exponential integrators can be found in the literature [17] by Hochbruck and Ostermann. Originally developed for solving stiff differential equations by Hochbruck and Ostermann in [15,16], the methods have been used to solve partial differential equations including hyperbolic, as well as parabolic problems such as the heat equation.

As mentioned above, the schemes above were constructed under the assumption that the exact solution is smooth enough. However, in practice the initial data may not be ideally smooth due to multiple reasons such as measurements or noise. Rough data may appear naturally in some applications: initial data corrupted with noise (as in nonlinear optics applications). When the solution of the equation is not sufficiently smooth in space, the convergence of a certain order only holds under sufficient additional regularity assumptions on the solution. It can be regarded as the following error structure of the scheme

$$\tau^\nu (-\Delta)^\lambda u(t), \quad \nu, \lambda \geq 0,$$

where τ denotes the time step size. The structure explains that there are 2λ derivatives loss to reach the ν -order convergence. For example, for the nonlinear Schrödinger equation, the error structure of the scheme in [27] is

$$\tau^\nu (-\Delta)^\nu u(t), \quad \nu \geq 0.$$

It implies that in order to obtain ν -order convergence, 2ν derivatives loss is needed. Then the essential work is to design a numerical scheme such that ν -order convergence is achieved with λ as small as possible. To bring down the regularity requirements, more recent attention has focused on so-called low-regularity integrators (LRIs) that based on the exponential integrators. Unlike the classical numerical schemes, this method can break the natural order barrier to reach the optimal convergence rate. Of course, it will encounter many difficulties, the main difficulty is the design of LRIs, which needs the scheme being defined point-wise in the physical space while requiring as few additional derivatives as possible. Moreover, this point-wise evaluation requires $O(N \log N)$ operations, in general. The low-regularity integrators have already been considered for some important models such as the nonlinear Schrödinger equation (NLS), the Korteweg-de Vries equation.

For the Korteweg-de Vries equation, Hofmanová and Schratz [18] introduced an exponential integrator to obtain first-order convergence in H^1 with initial data in H^3 . Later, Wu and Zhao [35] established the second-order convergence result in H^γ for initial data in $H^{\gamma+4}$, which proved rigorously in theory the validity of the scheme that was proposed in [18]. Very recently, Wu and Zhao [36] proposed the Embedded exponential-type low-regularity integrators and proved the first-order and second-order convergence in H^γ under $H^{\gamma+1}$ -data and $H^{\gamma+3}$ -data respectively.

For the nonlinear Schrödinger equations, Ostermann and Schratz [31] introduced a new exponential-type numerical scheme, and the first order convergence was achieved under the requirement of only one additional derivative. That is

$$\|u^n - u(t_n)\|_{H^\gamma(\mathbb{T}^d)} \lesssim \tau,$$

up to some fixed time for the initial data $u_0 \in H^{\gamma+1}(\mathbb{T}^d)$, $\gamma > \frac{d}{2}$, where u^n denotes the numerical solution at $t_n = n\tau$. More precisely, the error behavior of the numerical scheme is dominated by

$$\tau(-\Delta)^{\frac{1}{2}} u(t),$$

which breaks the “natural order barrier” of $\tau^{\frac{1}{2}}$ for $(-\Delta)^{\frac{1}{2}}$ -loss. This presents a lower regularity assumptions on the data compared to the splitting or exponential integrator schemes. Later, for the

second-order convergence, Knöller, Ostermann and Schratz [25] gave a new type of integrator and the scheme requires two additional derivatives of the solution in one space dimension and three derivatives in higher space dimensions. Whereafter, Ostermann, Rousset and Schratz [29, 30] considered H^s , $0 < s \leq 1$ solutions in L^2 with order $\nu < 1$ in dimensions $d \leq 3$. For the quadratic nonlinear Schrödinger equation and the nonlinear Dirac equation, the first-order convergence in H^γ without any loss of derivatives in one space dimension, see [31].

In this paper, we are aiming to get the first-order convergence of (1.1) without any loss of derivatives by introducing a new type low-regularity exponential integrator. That is, we obtain the following result

$$\|u^n - u(t_n)\|_{H^\gamma(\mathbb{T})} \lesssim \tau,$$

up to some fixed time for the initial data $u_0 \in H^\gamma(\mathbb{T})$, $\gamma > \frac{3}{2}$, where u^n denotes the numerical solution at $t_n = n\tau$.

Now we explain our argument briefly. Our designation is based on the Phase-Space analysis of the nonlinear dynamics. In particular, we focus our attention on the following time integral,

$$\int_0^\tau e^{is(k^2+k_1^2-k_2^2-k_3^2)} ds, \quad \text{with } k = k_1 + k_2 + k_3. \quad (1.3)$$

Using the following formula of the phase function,

$$k^2 + k_1^2 - k_2^2 - k_3^2 = 2(k_1 + k_2)(k_1 + k_3),$$

we write that for any $k \neq 0$,

$$e^{is(k^2+k_1^2-k_2^2-k_3^2)} = \sum_{j=2,3} \frac{k_1 + k_j}{k} e^{is(k^2+k_1^2-k_2^2-k_3^2)} - \frac{k_1}{k} e^{is(k^2+k_1^2-k_2^2-k_3^2)}.$$

Then the first term is integrable. Indeed, by direct calculation we have that

$$\sum_{j=2,3} \int_0^\tau \frac{k_1 + k_j}{k} e^{is(k^2+k_1^2-k_2^2-k_3^2)} ds = \sum_{j=2,3} \frac{1}{2ik(k_1 + k_j)} \left(e^{i\tau(k^2+k_1^2-k_2^2-k_3^2)} - 1 \right).$$

Unfortunately, the second term can not be integrated in the physical space exactly. To overcome this difficulty, we use another formula of the phase function,

$$k^2 + k_1^2 - k_2^2 - k_3^2 = 2kk_1 + 2k_2k_3.$$

Therefore we have the formula

$$e^{is(k^2+k_1^2-k_2^2-k_3^2)} = e^{2isk_1} + O(s|k_2||k_3|).$$

This yields that for any $k \neq 0$,

$$\int_0^\tau \frac{k_1}{k} e^{is(k^2+k_1^2-k_2^2-k_3^2)} ds = \frac{1}{2ik^2} (e^{2i\tau k_1} - 1) + \tau^2 O(|k|^{-1}|k_1||k_2||k_3|).$$

Based on the rigorous analysis, we construct the following numerical solution of (1.1) as

$$u^{n+1} = \Psi(u^n), \quad n = 0, 1, \dots, \frac{T}{\tau} - 1, \quad (1.4)$$

with $u^0 = u_0$, where

$$\begin{aligned} \Psi(u^n) = & e^{i\tau(-2M_0-2P_0\partial_x^{-1}+\partial_x^2)} u^n - i\tau\Pi_0(|u^n|^2 u^n) + 2i\tau M_0\Pi_0(u^n) \\ & - \frac{1}{2}\partial_x^{-2} \left[(e^{-i\tau\partial_x^2} \bar{u}^n) \cdot e^{i\tau\partial_x^2} (u^n)^2 \right] + \frac{1}{2}e^{i\tau\partial_x^2} \partial_x^{-2} [|u^n|^2 u^n] \\ & + \partial_x^{-1} \left[(e^{i\tau\partial_x^2} u^n) \cdot \partial_x^{-1} \left(|e^{i\tau\partial_x^2} u^n|^2 \right) \right] - e^{i\tau\partial_x^2} \partial_x^{-1} \left[u^n \cdot \partial_x^{-1} (|u^n|^2) \right]. \end{aligned} \quad (1.5)$$

Here we denote $\Pi_0(f)$ to be the zero mode of the function f , that is,

$$\Pi_0(f) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) dx. \quad (1.6)$$

Moreover, M_0, P_0 are defined by

$$M_0 = M(u_0) = \Pi_0(|u_0|^2); \quad P_0 = \frac{1}{2\pi} \int_{\mathbb{T}} u_0 \partial_x \bar{u}_0 dx = \Pi_0(u_0 \partial_x \bar{u}_0).$$

Now, we state the convergence theorem of the presented (semi-discretized) LRI method given in (1.4) in one space dimension.

Theorem 1.1. *Let u^n be the numerical solution (1.4) of the equation (1.1) up to some fixed time $T > 0$. Under assumption that $u_0 \in H^\gamma(\mathbb{T})$ for some $\gamma > \frac{3}{2}$, there exist constants $\tau_0, C > 0$ such that for any $0 < \tau \leq \tau_0$,*

$$\|u(t_n, \cdot) - u^n\|_{H^\gamma} \leq C\tau, \quad n = 0, 1, \dots, \frac{T}{\tau}, \quad (1.7)$$

where the constants τ_0 and C depend only on T and $\|u\|_{L^\infty((0,T);H^\gamma)}$.

Our theorem above improves the result in [31] in one space dimension. In particular, our scheme does not loss any regularity in this case, which is the best one can expect in this sense. We believe that the idea used in this paper could be applied to the other models which will be studied in the forthcoming papers.

Based on the above theorem, we also obtain $\frac{1}{2}$ - order convergence in $H^1(\mathbb{T})$ with initial data in $H^1(\mathbb{T})$, where $\frac{1}{2}$ - denotes $\frac{1}{2} - \epsilon$ for any arbitrary small $\epsilon > 0$. It is practically reasonable to obtain lower convergence under lower regularity assumptions, because the accuracy order of the scheme in time and in space need to be rather equal.

Corollary 1.2. *Let u^n be the numerical solution (1.4) of the equation (1.1) up to some fixed time $T > 0$. Under assumption that $u_0 \in H^1(\mathbb{T})$, there exist constants $\tau_0, C > 0$ such that for any $0 < \tau \leq \tau_0$,*

$$\|u(t_n, \cdot) - u^n\|_{H^1} \leq C\tau^{\frac{1}{2}-}, \quad n = 0, 1, \dots, \frac{T}{\tau}, \quad (1.8)$$

where the constants τ_0 and C depend only on T and $\|u\|_{L^\infty((0,T);H^1)}$.

Furthermore, we continue to pursue a scheme such that it could be almost conserved in mass which meanwhile requires as less regularity as possible. To this purpose, we define a modified numerical scheme of (1.4) as follows. Let Ψ be defined in (1.5) and

$$F(U^n) = \Psi(U^n) - e^{i\tau\partial_x^2} U^n.$$

Then we denote the functionals G_1, G_2 to be

$$G_1(U) = H(U) e^{i\tau\partial_x^2} U; \quad (1.9)$$

$$G_2(U) = -\frac{1}{2} (H(U))^2 e^{i\tau\partial_x^2} U - M_0^{-1} H(U) \operatorname{Re} \Pi_0 \left(F(U) e^{-i\tau\partial_x^2} \bar{U} \right) e^{i\tau\partial_x^2} U; \quad (1.10)$$

and

$$H(U) = -M_0^{-1} \left[\operatorname{Re} \Pi_0 \left(F(U) e^{-i\tau\partial_x^2} \bar{U} \right) + \frac{1}{2} \Pi_0 \left(|F(U)|^2 \right) \right]. \quad (1.11)$$

Now the modified numerical scheme (NLRI) of (1.4) is defined by

$$U^{n+1} = \Psi(U^n) + G_1(U^n) + G_2(U^n), \quad (1.12)$$

for $n = 0, 1, \dots, \frac{T}{\tau} - 1$, and $U^0 = u_0$,

Then we obtain that

Theorem 1.3. *Let U^n be the numerical solution (1.12) of the equation (1.1) up to some fixed time $T > 0$. Under assumption that $u_0 \in H^\gamma(\mathbb{T})$ for some $\gamma > \frac{3}{2}$, there exist constants $\tau_0, C > 0$ such that for any $0 < \tau \leq \tau_0$ we have*

$$\|u(t_n, \cdot) - U^n\|_{H^\gamma} \leq C\tau, \quad n = 0, 1, \dots, \frac{T}{\tau}. \quad (1.13)$$

Moreover,

$$|M(U^n) - M(u_0)| \leq C\tau^5, \quad (1.14)$$

where the constants τ_0 and C depend only on T and $\|u\|_{L^\infty((0,T);H^\gamma)}$.

Furthermore, if $u_0 \in H^1(\mathbb{T})$, there exist constants $\tau_0, C > 0$ such that for any $0 < \tau \leq \tau_0$,

$$\|u(t_n, \cdot) - U^n\|_{H^1} \leq C\tau^{\frac{1}{2}-}, \quad n = 0, 1, \dots, \frac{T}{\tau}, \quad (1.15)$$

where the constants τ_0 and C depend only on T and $\|u\|_{L^\infty((0,T);H^1)}$.

To the best of our knowledge, this is the first attempt to consider the conservation laws of the numerical solution for the exponential-type integrators.

Remark 1.4. In this paper, we present fifth-order mass convergence. However, our method is also applicable to solve the equation (1.1) with arbitrary order mass convergence by suitably adding correction terms.

Now we slightly explain the key ingredient of the construction. Denote F to be

$$u^{n+1} = e^{i\tau\partial_x^2}u^n + F(u^n),$$

where u^n is the numerical solution (1.4). Then we can find a functional G such that

$$\langle G(u^n), e^{i\tau\partial_x^2}u^n \rangle = -\langle F(u^n), e^{i\tau\partial_x^2}u^n \rangle,$$

and

$$\|G(u^n)\|_{H^\gamma} \leq C\tau^2.$$

The key point is that we only have the first-order estimate of $F^n(u^n)$ which reads

$$\|F(u^n)\|_{H^\gamma} \leq C\tau.$$

Hence we can not choose $G(u^n) = -F(u^n)$ directly, and the cancellation in the L^2 -inner product plays a great role in ensuring the second-order estimate of $G(u^n)$. Based on the nice feature of G , we can modify the numerical solution u^n and define the new scheme by

$$\tilde{u}^{n+1} = e^{i\tau\partial_x^2}\tilde{u}^n + F(\tilde{u}^n) + G(\tilde{u}^n).$$

Then we can prove that

$$\|u(t_n, \cdot) - \tilde{u}^n\|_{H^\gamma} \leq C\tau, \quad |M(\tilde{u}^n) - M(u_0)| \leq C\tau^3.$$

Repeating the same process, we can design a new scheme U^n verifying the required accuracy as in Theorem 1.3. More details will be given in Section 5.

The paper is organized as follows. In Section 2, we give some notations and some useful lemmas. In Section 3, we give the main process of the construction of the first-order scheme. In Section 4, we devote to prove Theorem 1.1. Further discussion on the almost mass-conserved scheme is presented in Section 5. Numerical confirmations are reported in Section 6 and conclusions are drawn in Section 7.

2. PRELIMINARY

2.1. Some notations. We use $A \lesssim B$ or $B \gtrsim A$ to denote the statement that $A \leq CB$ for some absolute constant $C > 0$ which may vary from line to line but is independent of τ or n , and we denote $A \sim B$ for $A \lesssim B \lesssim A$. We use $O(Y)$ to denote any quantity X such that $X \lesssim Y$. Moreover, we denote $\langle \cdot, \cdot \rangle$ to be the L^2 -inner product, that is

$$\langle f, g \rangle = \operatorname{Re} \int_{\mathbb{T}} f(x) \overline{g(x)} dx.$$

The Fourier transform of a function f on \mathbb{T} is defined by

$$\hat{f}_k = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx} f(x) dx,$$

and thus the Fourier inversion formula

$$f(x) = \sum_{k \in \mathbb{Z}} e^{ikx} \hat{f}_k.$$

Then the following usual properties of the Fourier transform hold:

$$\|f\|_{L^2(\mathbb{T})}^2 = 2\pi \sum_{k \in \mathbb{Z}} |\hat{f}_k|^2 \quad (\text{Plancherel});$$

$$\widehat{(fg)}(k) = \sum_{k_1 \in \mathbb{Z}} \hat{f}_{k-k_1} \hat{g}_{k_1} \quad (\text{Convolution}).$$

The Sobolev space $H^s(\mathbb{T})$ for $s \geq 0$ has the equivalent norm,

$$\|f\|_{H^s(\mathbb{T})}^2 = \|J^s f\|_{L^2(\mathbb{T})}^2 = 2\pi \sum_{k \in \mathbb{Z}} (1+k^2)^s |\hat{f}_k|^2,$$

where we denote the operator

$$J^s = (1 - \partial_{xx})^{\frac{s}{2}}.$$

Moreover, we denote ∂_x^{-1} to be the operator defined by

$$\widehat{\partial_x^{-1} f}(k) = \begin{cases} (ik)^{-1} \hat{f}_k, & \text{when } k \neq 0, \\ 0, & \text{when } k = 0. \end{cases} \quad (2.1)$$

We denote $T_m(M; v)$ to be a class of quantities which is defined in the Fourier space by

$$\mathcal{F}T_m(M; v)(k) = O\left(\sup_{t \in [0, T]} \sum_{k=k_1+\dots+k_m} |M(k, k_1, \dots, k_m)| |\hat{v}_{k_1}(t)| \cdots |\hat{v}_{k_m}(t)| \right). \quad (2.2)$$

Here we regards \bar{v} and v as the same since there is no influence in the whole of analysis.

2.2. Some preliminary estimates. First, we will frequently apply the following Kato-Ponce inequality (simple version), which was originally proved in [24] and an important progress in the endpoint case was made in [3, 26] very recently.

Lemma 2.1. (*Kato-Ponce inequality*) *The following inequalities hold:*

(i) *For any $\gamma > \frac{1}{2}$, $f, g \in H^\gamma$, then*

$$\|J^\gamma(fg)\|_{L^2} \lesssim \|f\|_{H^\gamma} \|g\|_{H^\gamma}.$$

(ii) *For any $\gamma \geq 0, \gamma_1 > \frac{1}{2}$, $f \in H^{\gamma+\gamma_1}, g \in H^\gamma$, then*

$$\|J^\gamma(fg)\|_{L^2} \lesssim \|f\|_{H^{\gamma+\gamma_1}} \|g\|_{H^\gamma}.$$

To prove our main result below, we need the following two specific estimates.

Lemma 2.2. *The following inequalities hold:*

(i) *Let $\gamma > \frac{3}{2}$, and $v \in L^\infty((0, T); H^\gamma)$, then*

$$\|T_3(k^{-1}k_1k_2k_3; v)\|_{H^\gamma} \lesssim \|v\|_{L^\infty((0, T); H^\gamma)}^3.$$

(ii) *Let $\gamma \geq 1$, and $v \in L^\infty((0, T); H^\gamma)$, then*

$$\|T_3(k^{-1}k_1k_2^{\frac{1}{2}-}k_3^{\frac{1}{2}-}; v)\|_{H^\gamma} \lesssim \|v\|_{L^\infty((0, T); H^\gamma)}^3.$$

(iii) *Let $\gamma \geq 1$, and $v \in L^\infty((0, T); H^\gamma)$, then*

$$|\mathcal{F}T_3(k_2k_3; v)(0)| \lesssim \|v\|_{L^\infty((0, T); H^\gamma)}^3.$$

(iv) *Let $\gamma > \frac{1}{2}$, $m \geq 1$ and $v \in L^\infty((0, T); H^\gamma)$, then*

$$\|T_m(1; v)\|_{H^\gamma} \lesssim \|v\|_{L^\infty((0, T); H^\gamma)}^m.$$

Proof. We assume that $\hat{v}_{k_j}(t), j = 1, \dots, m$ are positive for any $t \in [0, T]$, otherwise one may replace them by $|\hat{v}_{k_j}(t)|$.

(i) Using the definition of $T_m(M)$ in (2.2), we have

$$|\mathcal{F}T_3(k^{-1}k_1k_2k_3; v)(k)| \lesssim \sup_{t \in [0, T]} \sum_{k=k_1+k_2+k_3 \neq 0} |k|^{-1} |k_1| |k_2| |k_3| \hat{v}_{k_1}(t) \hat{v}_{k_2}(t) \hat{v}_{k_3}(t).$$

By Plancherel's identity, we get

$$\begin{aligned} \|T_3(k^{-1}k_1k_2k_3; v)\|_{H^\gamma} &\lesssim \left\| \sum_{k=k_1+k_2+k_3 \neq 0} |k|^{\gamma-1} |k_1| |k_2| |k_3| \hat{v}_{k_1}(t) \hat{v}_{k_2}(t) \hat{v}_{k_3}(t) \right\|_{L^\infty((0,T);L^2)} \\ &\lesssim \|(|\nabla|v)^3\|_{L^\infty((0,T);H^{\gamma-1})}. \end{aligned}$$

Therefore, by Lemma 2.1 (i), we obtain that for any $\gamma > \frac{3}{2}$,

$$\|T_3(k^{-1}k_1k_2k_3; v)\|_{H^\gamma} \lesssim \|(|\nabla|v)\|_{L^\infty((0,T);H^{\gamma-1})}^3 \lesssim \|v\|_{L^\infty((0,T);H^\gamma)}^3.$$

(ii) By the same argument to the proof of (i), we have

$$\begin{aligned} \|T_3(k^{-1}k_1k_2^{\frac{1}{2}-}k_3^{\frac{1}{2}-}; v)\|_{H^\gamma} &\lesssim \left\| \sum_{k=k_1+k_2+k_3 \neq 0} |k|^{\gamma-1} |k_1| |k_2|^{\frac{1}{2}-} |k_3|^{\frac{1}{2}-} \hat{v}_{k_1}(t) \hat{v}_{k_2}(t) \hat{v}_{k_3}(t) \right\|_{L^\infty((0,T);L^2)} \\ &\lesssim \|(|\nabla|v)(|\nabla|^{\frac{1}{2}-}v)(|\nabla|^{\frac{1}{2}-}v)\|_{L^\infty((0,T);H^{\gamma-1})}. \end{aligned}$$

Therefore, by Lemma 2.1 (ii), we obtain that for any $\gamma \geq 1$,

$$\|T_3(k^{-1}k_1k_2^{\frac{1}{2}-}k_3^{\frac{1}{2}-}; v)\|_{H^\gamma} \lesssim \|(|\nabla|v)(|\nabla|^{\frac{1}{2}-}v)(|\nabla|^{\frac{1}{2}-}v)\|_{L^\infty((0,T);H^{\gamma-1})} \lesssim \|v\|_{L^\infty((0,T);H^\gamma)}^3.$$

(iii) From the definition of $T_m(M)$, we have that for any $\gamma \geq 1$,

$$\begin{aligned} |\mathcal{F}T_3(k_2k_3; v)(0)| &\lesssim \sup_{t \in [0, T]} \sum_{0=k_1+\dots+k_m} \hat{v}_{k_1}(t) |k_2| \hat{v}_{k_2}(t) |k_3| \hat{v}_{k_3}(t) \\ &\lesssim \sup_{t \in [0, T]} \int_{\mathbb{T}} v (|\nabla|v)^2 dx \lesssim \|v\|_{L^\infty((0,T);H^\gamma)}^3. \end{aligned}$$

(iv) Similarly, we have

$$|\mathcal{F}T_m(1; v)| \lesssim \sup_{t \in [0, T]} \sum_{k=k_1+\dots+k_m} \hat{v}_{k_1}(t) \cdots \hat{v}_{k_m}(t).$$

By Plancherel's identity and Lemma 2.1 (i), we obtain that for any $\gamma > \frac{1}{2}$,

$$\|T_m(1; v)\|_{H^\gamma} \lesssim \|v^m\|_{L^\infty((0,T);H^\gamma)} \lesssim \|v\|_{L^\infty((0,T);H^\gamma)}^m.$$

Hence we get the desired result. \square

3. THE FIRST ORDER SCHEME

By Duhamel formula, we write

$$u(t_{n+1}) = e^{i\tau\partial_x^2} u(t_n) - i \int_0^\tau e^{i(t_{n+1}-(t_n+s))\partial_x^2} [|u(t_n+s)|^2 u(t_n+s)] ds.$$

Let $v(t) := e^{-it\partial_x^2} u(t)$, then

$$v(t_{n+1}) = v(t_n) - i \int_0^\tau e^{-i(t_n+s)\partial_x^2} [|e^{i(t_n+s)\partial_x^2} v(t_n+s)|^2 e^{i(t_n+s)\partial_x^2} v(t_n+s)] ds. \quad (3.1)$$

Taking Fourier transform, we get

$$\hat{v}_k(t_{n+1}) = \hat{v}_k(t_n) - i \int_0^\tau \sum_{k=k_1+k_2+k_3} e^{i(t_n+s)\phi} \hat{v}_{k_1}(t_n+s) \hat{v}_{k_2}(t_n+s) \hat{v}_{k_3}(t_n+s) ds.$$

Here we denote $\hat{v}_k(t)$ to be the k -th Fourier coefficients of $v(t)$, and the phase function

$$\phi = \phi(k, k_1, k_2, k_3) = k^2 + k_1^2 - k_2^2 - k_3^2.$$

By (3.1), we find that for any $s \in [0, \tau]$,

$$v(t_n+s) = v(t_n) + \tau T_3(1; v). \quad (3.2)$$

Hence, we have that

$$\hat{v}_k(t_{n+1}) = \hat{v}_k(t_n) - i \sum_{k=k_1+k_2+k_3} \int_0^\tau e^{i(t_n+s)\phi} ds \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} + \tau^2 \mathcal{F}T_5(1; v)(k). \quad (3.3)$$

Here and below, we denote \hat{v}_k to be $\hat{v}_k(t_n)$ for short.

Now we split into the following two cases.

Case 1, $k = 0$. Then by (3.3), we get

$$\hat{v}_0(t_{n+1}) = \hat{v}_0(t_n) - i \sum_{k_1+k_2+k_3=0} \int_0^\tau e^{i(t_n+s)(k_1^2-k_2^2-k_3^2)} ds \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} + \tau^2 \mathcal{F}T_5(1; v)(0). \quad (3.4)$$

Note that under the condition of $k_1 + k_2 + k_3 = 0$, we can transform the phase function

$$k_1^2 - k_2^2 - k_3^2 = 2k_2k_3.$$

Therefore, we have

$$\int_0^\tau \left(e^{is(k_1^2-k_2^2-k_3^2)} - 1 \right) ds = \tau^2 O(k_2k_3). \quad (3.5)$$

According to (3.5), we can freeze the phase function in the integrand in (3.4) and obtain that

$$\begin{aligned} \hat{v}_0(t_{n+1}) &= \hat{v}_0(t_n) - i\tau \sum_{k_1+k_2+k_3=0} e^{it_n(k_1^2-k_2^2-k_3^2)} \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} + \tau^2 (\mathcal{F}T_3(k_2k_3; v)(0) + \mathcal{F}T_5(1; v)(0)) \\ &= \hat{v}_0(t_n) - i\tau \Pi_0 \left(\left| e^{it_n \partial_x^2} v(t_n) \right|^2 e^{it_n \partial_x^2} v(t_n) \right) + \tau^2 (\mathcal{F}T_3(k_2k_3; v)(0) + \mathcal{F}T_5(1; v)(0)). \end{aligned} \quad (3.6)$$

Case 2, $k \neq 0$. For (3.3), we only consider the term

$$-i \sum_{k=k_1+k_2+k_3} \int_0^\tau e^{i(t_n+s)\phi} ds \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3}. \quad (3.7)$$

Note that

$$1 = \frac{(k_1 + k_2) + (k_1 + k_3) - k_1}{k}.$$

Then by symmetry, it allows us to split (3.7) into two parts:

$$-2i \sum_{k=k_1+k_2+k_3} \int_0^\tau \frac{k_1 + k_2}{k} e^{i(t_n+s)\phi} ds \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} \quad (3.8a)$$

$$+i \sum_{k=k_1+k_2+k_3} \int_0^\tau \frac{k_1}{k} e^{i(t_n+s)\phi} ds \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3}. \quad (3.8b)$$

For (3.8a), we need the following equality: If $k = k_1 + k_2 + k_3$, then

$$\phi = 2(k_1 + k_2)(k_1 + k_3).$$

We note that if $k_1 + k_3 \neq 0$, then

$$\int_0^\tau \frac{k_1 + k_2}{k} e^{i(t_n+s)\phi} ds = \frac{1}{2ik(k_1 + k_3)} (e^{it_{n+1}\phi} - e^{it_n\phi}); \quad (3.9)$$

if $k_1 + k_3 = 0$, then $\phi = 0$, $k = k_2$ and thus

$$\int_0^\tau \frac{k_1 + k_2}{k} e^{i(t_n+s)\phi} ds = \tau \left(\frac{k_1}{k} + 1 \right). \quad (3.10)$$

Therefore, we get

$$\begin{aligned} (3.8a) &= - \sum_{\substack{k=k_1+k_2+k_3 \\ k_1+k_3 \neq 0}} \frac{1}{k(k_1 + k_3)} (e^{it_{n+1}\phi} - e^{it_n\phi}) \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} \\ &\quad - 2i\tau \sum_{k_1+k_3=0} \left(\frac{k_1}{k} + 1 \right) \widehat{v}_{k_1} \widehat{v}_k \widehat{v}_{k_3}. \end{aligned}$$

Now we need the following momentum conservation law:

$$P(u(t)) = \frac{1}{2\pi} \int_{\mathbb{T}} u(t) \bar{u}_x(t) dx = P_0. \quad (3.11)$$

Note that by (1.2) and (3.11), we have that

$$\begin{aligned} -2i\tau \sum_{\substack{k_1+k_3=0 \\ k_1+k_3 \neq 0}} \left(\frac{k_1}{k} + 1\right) \widehat{v}_{k_1} \widehat{v}_k \widehat{v}_{k_3} &= -\frac{i\tau}{\pi} \int_{\mathbb{T}} \partial_x \bar{u}(t_n) u(t_n) dx (ik)^{-1} \widehat{v}_k - \frac{i\tau}{\pi} \int_{\mathbb{T}} |u(t_n)|^2 dx \widehat{v}_k \\ &= -2i\tau P_0 (ik)^{-1} \widehat{v}_k - 2i\tau M_0 \widehat{v}_k. \end{aligned}$$

Therefore, we further obtain

$$(3.8a) = - \sum_{\substack{k=k_1+k_2+k_3 \\ k_1+k_3 \neq 0}} \frac{1}{k(k_1+k_3)} (e^{it_{n+1}\phi} - e^{it_n\phi}) \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} - 2i\tau P_0 (ik)^{-1} \widehat{v}_k - 2i\tau M_0 \widehat{v}_k. \quad (3.12)$$

For (3.8b), we note that it can not be integrated in the physical space exactly. Now we need the following two equalities: If $k = k_1 + k_2 + k_3$, then

$$\phi(k, k_1, k_2, k_3) = 2kk_1 + 2k_2k_3; \quad (3.13)$$

$$2kk_1 = k^2 + k_1^2 - (k_2 + k_3)^2. \quad (3.14)$$

Putting (3.13) into (3.8b), we decompose (3.8b) into two subparts again:

$$i \sum_{k=k_1+k_2+k_3} \int_0^\tau \frac{k_1}{k} e^{it_n\phi} (e^{2isk_2k_3} - 1) e^{2isk_1k_1} ds \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} \quad (3.15-1)$$

$$+ i \sum_{k=k_1+k_2+k_3} \int_0^\tau \frac{k_1}{k} e^{it_n\phi} e^{2isk_1k_1} ds \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3}. \quad (3.15-2)$$

For (3.15-1), applying the inequality

$$|e^{2isk_2k_3} - 1| \leq 2\tau |k_2| |k_3|, \quad \text{for any } s \in [0, \tau],$$

we have that

$$(3.15-1) = \tau^2 \mathcal{F}T_3(k^{-1}k_1k_2k_3; v)(k). \quad (3.16)$$

For (3.15-2), from (3.14) we have

$$\frac{k_1}{k} \int_0^\tau e^{2isk_1k_1} ds = \frac{1}{2ik^2} \left(e^{i\tau(k^2+k_1^2-(k_2+k_3)^2)} - 1 \right),$$

we get

$$(3.15-2) = \sum_{k=k_1+k_2+k_3} \frac{1}{2k^2} e^{it_n\phi} \left(e^{i\tau(k^2+k_1^2-(k_2+k_3)^2)} - 1 \right) \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3}.$$

Therefore, collecting the two finding in (3.15), we obtain

$$(3.8b) = \sum_{k=k_1+k_2+k_3} \frac{1}{2k^2} e^{it_n\phi} \left(e^{i\tau(k^2+k_1^2-(k_2+k_3)^2)} - 1 \right) \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} + \tau^2 \mathcal{F}T_3(k^{-1}k_1k_2k_3; v)(k). \quad (3.17)$$

Combining with (3.12) and (3.17), we derive that when $k \neq 0$,

$$\begin{aligned} \widehat{v}_k(t_{n+1}) &= \widehat{v}_k(t_n) - \sum_{\substack{k=k_1+k_2+k_3 \\ k_1+k_3 \neq 0}} \frac{1}{k(k_1+k_3)} (e^{it_{n+1}\phi} - e^{it_n\phi}) \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} \\ &\quad - 2i\tau P_0 (ik)^{-1} \widehat{v}_k - 2i\tau M_0 \widehat{v}_k \\ &\quad + \sum_{k=k_1+k_2+k_3} \frac{1}{2k^2} e^{it_n\phi} \left(e^{i\tau(k^2+k_1^2-(k_2+k_3)^2)} - 1 \right) \widehat{v}_{k_1} \widehat{v}_{k_2} \widehat{v}_{k_3} \\ &\quad + \tau^2 \left(\mathcal{F}T_3(k^{-1}k_1k_2k_3; v) + \mathcal{F}T_5(1; v) \right)(k). \end{aligned} \quad (3.18)$$

Now together with (3.6) and (3.18), and using the formula

$$f - 2i\tau M_0 f - 2i\tau P_0 \partial_x^{-1} f = e^{-2i\tau M_0 - 2i\tau P_0 \partial_x^{-1}} f + \tau^2 T_1(1; f),$$

and the inverse Fourier transform, we get

$$v(t_{n+1}) = \Phi^n(v(t_n)) + \tau^2 \left(T_1(1; v) + \mathcal{F}T_3(k_2 k_3; v)(0) + T_3(k^{-1} k_1 k_2 k_3; v) + T_5(1; v) \right), \quad (3.19)$$

where Φ^n is defined by

$$\begin{aligned} \Phi^n(f) = & e^{-2i\tau M_0 - 2i\tau P_0 \partial_x^{-1}} f + 2i\tau M_0 \Pi_0(f) - i\tau \Pi_0 \left(\left| e^{it_n \partial_x^2} f \right|^2 e^{it_n \partial_x^2} f \right) \\ & + e^{-it_{n+1} \partial_x^2} \partial_x^{-1} \left[(e^{it_{n+1} \partial_x^2} f) \cdot \partial_x^{-1} \left(\left| e^{it_{n+1} \partial_x^2} f \right|^2 \right) \right] - e^{-it_n \partial_x^2} \partial_x^{-1} \left[(e^{it_n \partial_x^2} f) \cdot \partial_x^{-1} \left(\left| e^{it_n \partial_x^2} f \right|^2 \right) \right] \\ & - \frac{1}{2} e^{-it_{n+1} \partial_x^2} \partial_x^{-2} \left[(e^{-it_{n+1} \partial_x^2} \bar{f}) \cdot e^{i\tau \partial_x^2} (e^{it_n \partial_x^2} f)^2 \right] + \frac{1}{2} e^{-it_n \partial_x^2} \partial_x^{-2} \left[\left| e^{it_n \partial_x^2} f \right|^2 e^{it_n \partial_x^2} f \right]. \end{aligned} \quad (3.20)$$

Accordingly, we define the numerical solution of (1.1) by

$$v^{n+1} = \Phi^n(v^n), \quad n = 0, 1, \dots, \frac{T}{\tau} - 1; \quad v^0 = u_0. \quad (3.21)$$

Let $u^n := e^{it_n \partial_x^2} v^n$, this gives the scheme (1.4)–(1.5) and thus finishes the construction of the numerical scheme.

4. THE PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

4.1. The proof of the Theorem 1.1. From (3.21), we have

$$\begin{aligned} v(t_{n+1}) - v^{n+1} = & v(t_{n+1}) - \Phi^n(v(t_n)) + \Phi^n(v(t_n)) - \Phi^n(v^n) \\ \triangleq & \mathcal{L}^n + \Phi^n(v(t_n)) - \Phi^n(v^n), \end{aligned}$$

where $\mathcal{L}^n = v(t_{n+1}) - \Phi^n(v(t_n))$.

Furthermore, from (3.19) we get

$$\mathcal{L}^n = \tau^2 \left(T_1(1; v) + \mathcal{F}T_3(k_2 k_3; v)(0) + T_3(k^{-1} k_1 k_2 k_3; v) + T_5(1; v) \right).$$

Then from Lemma 2.2, we have

$$\|\mathcal{L}^n\|_{H^\gamma} \leq C\tau^2, \quad (4.1)$$

where the constant C depends only on $\|u\|_{L^\infty((0, T); H^\gamma)}$.

Note that $\Phi^n(f)$ defined in (3.20) can be read as the following integral form:

$$\begin{aligned} \Phi^n(f) = & f - i\tau \Pi_0 \left(\left| e^{it_n \partial_x^2} f \right|^2 e^{it_n \partial_x^2} f \right) \\ & - 2i \int_0^\tau e^{-i(t_n+s) \partial_x^2} \partial_x^{-1} \left(e^{-i(t_n+s) \partial_x^2} \partial_x \bar{f} \cdot (e^{i(t_n+s) \partial_x^2} f)^2 \right) ds \\ & - 2i \int_0^\tau e^{-i(t_n+s) \partial_x^2} \partial_x^{-1} \left(\left| e^{i(t_n+s) \partial_x^2} f \right|^2 e^{i(t_n+s) \partial_x^2} \partial_x f \right) ds \\ & + i \int_0^\tau e^{-i(t_n+s) \partial_x^2} \partial_x^{-1} \left(e^{-i(t_n+s) \partial_x^2} \partial_x \bar{f} \cdot e^{is \partial_x^2} (e^{it_n \partial_x^2} f)^2 \right) ds \\ & + 2i\tau (M(f) - M(v(t_n)))f + 2i\tau (P(f) - P(v(t_n))) \partial_x^{-1} f \\ & + \left[e^{-2i\tau (M_0 + P_0 \partial_x^{-1})} - 1 + 2i\tau (M_0 + P_0 \partial_x^{-1}) \right] f. \end{aligned}$$

Therefore, we obtain

$$\Phi^n(v(t_n)) - \Phi^n(v^n) = v(t_n) - v^n + \Phi_1^n + \Phi_2^n + \Phi_3^n + \Phi_4^n + \Phi_5^n,$$

where

$$\begin{aligned}
\Phi_1^n &= -i\tau\Pi_0 \left(\left| e^{it_n\partial_x^2} v(t_n) \right|^2 e^{it_n\partial_x^2} v(t_n) \right) + i\tau\Pi_0 \left(\left| e^{it_n\partial_x^2} v^n \right|^2 e^{it_n\partial_x^2} v^n \right); \\
\Phi_2^n &= -2i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \partial_x^{-1} \left(e^{-i(t_n+s)\partial_x^2} \partial_x \bar{v}(t_n) \cdot \left(e^{i(t_n+s)\partial_x^2} v(t_n) \right)^2 \right) ds \\
&\quad + 2i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \partial_x^{-1} \left(e^{-i(t_n+s)\partial_x^2} \partial_x \bar{v}^n \cdot \left(e^{i(t_n+s)\partial_x^2} v^n \right)^2 \right) ds; \\
\Phi_3^n &= -2i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \partial_x^{-1} \left(\left| e^{i(t_n+s)\partial_x^2} v(t_n) \right|^2 e^{i(t_n+s)\partial_x^2} \partial_x v(t_n) \right) ds \\
&\quad + 2i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \partial_x^{-1} \left(\left| e^{i(t_n+s)\partial_x^2} v^n \right|^2 e^{i(t_n+s)\partial_x^2} \partial_x v^n \right) ds; \\
\Phi_4^n &= i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \partial_x^{-1} \left(e^{-i(t_n+s)\partial_x^2} \partial_x \bar{v}(t_n) \cdot e^{is\partial_x^2} \left(e^{it_n\partial_x^2} v(t_n) \right)^2 \right) ds \\
&\quad - i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \partial_x^{-1} \left(e^{-i(t_n+s)\partial_x^2} \partial_x \bar{v}^n \cdot e^{is\partial_x^2} \left(e^{it_n\partial_x^2} v^n \right)^2 \right) ds; \\
\Phi_5^n &= -2i\tau \left(M(v^n) - M(v(t_n)) v^n - 2i\tau \left(P(v^n) - P(v(t_n)) \right) \partial_x^{-1} v^n \right. \\
&\quad \left. + \left[e^{-2i\tau(M_0+P_0\partial_x^{-1})} - 1 + 2i\tau \left(M_0 + P_0\partial_x^{-1} \right) \right] (v(t_n) - v^n) \right).
\end{aligned}$$

Next we estimate the above terms. Φ_1^n can be divided into three parts

$$\begin{aligned}
\Phi_1^n &= -i\tau\Pi_0 \left[\left| e^{it_n\partial_x^2} v(t_n) \right|^2 e^{it_n\partial_x^2} (v(t_n) - v^n) \right] \\
&\quad - i\tau\Pi_0 \left[\left(\left| e^{it_n\partial_x^2} v(t_n) \right|^2 - \left| e^{it_n\partial_x^2} v^n \right|^2 \right) e^{it_n\partial_x^2} v(t_n) \right] \\
&\quad - i\tau\Pi_0 \left[\left(\left| e^{it_n\partial_x^2} v(t_n) \right|^2 - \left| e^{it_n\partial_x^2} v^n \right|^2 \right) e^{it_n\partial_x^2} (v^n - v(t_n)) \right].
\end{aligned}$$

Then by the Hölder and Sobolev inequalities, we obtain that

$$\|\Phi_1^n\|_{H^\gamma} \leq C|\Phi_1^n| \leq C\tau (\|v^n - v(t_n)\|_{H^\gamma} + \|v^n - v(t_n)\|_{H^\gamma}^3). \quad (4.2)$$

Similarly, by Lemma 2.1, we have that for any $\gamma \geq 1$,

$$\|\Phi_j^n\|_{H^\gamma} \leq C\tau (\|v^n - v(t_n)\|_{H^\gamma} + \|v^n - v(t_n)\|_{H^\gamma}^3), \quad j = 2, \dots, 5. \quad (4.3)$$

Therefore for any $\gamma \geq 1$,

$$\|\Phi^n(v(t_n)) - \Phi^n(v^n)\|_{H^\gamma} \leq (1 + C\tau) \|v^n - v(t_n)\|_{H^\gamma} + C\tau \|v^n - v(t_n)\|_{H^\gamma}^3. \quad (4.4)$$

Combining the above estimates, we conclude that

$$\|v(t_{n+1}) - v^{n+1}\|_{H^\gamma} \leq C\tau^2 + (1 + C\tau) \|v^n - v(t_n)\|_{H^\gamma} + C\tau \|v^n - v(t_n)\|_{H^\gamma}^3.$$

By iteration and Gronwall's inequalities, we get

$$\|v(t_{n+1}) - v^{n+1}\|_{H^\gamma} \leq C\tau^2 \sum_{j=0}^n (1 + C\tau)^j \leq C\tau, \quad n = 0, 1, \dots, \frac{T}{\tau} - 1.$$

This finishes the proof of the convergence result.

4.2. The proof of the Corollary 1.2. For (3.15-1), applying the inequality

$$|e^{2isk_2k_3} - 1| \leq 2\tau^{\frac{1}{2}-} |k_2|^{\frac{1}{2}-} |k_3|^{\frac{1}{2}-}, \quad \text{for any } s \in [0, \tau],$$

then we can replace (3.16) with

$$(3.15-1) = \tau^{\frac{3}{2}-} \mathcal{F}T_3(k^{-1}k_1k_2^{\frac{1}{2}-}k_3^{\frac{1}{2}-}; v)(k). \quad (4.5)$$

Therefore, we have

$$v(t_{n+1}) = \Phi^n(v(t_n)) + \tau^{\frac{3}{2}-} T_3(k^{-1}k_1k_2^{\frac{1}{2}-}k_3^{\frac{1}{2}-}; v) + \tau^2 \left(T_1(1; v) + \mathcal{F}T_3(k_2k_3; v)(0) + T_5(1; v) \right). \quad (4.6)$$

Similarly as before,

$$\begin{aligned} v(t_{n+1}) - v^{n+1} &= v(t_{n+1}) - \Phi^n(v(t_n)) + \Phi^n(v(t_n)) - \Phi^n(v^n) \\ &\triangleq \tilde{\mathcal{L}}^n + \Phi^n(v(t_n)) - \Phi^n(v^n), \end{aligned}$$

where

$$\tilde{\mathcal{L}}^n = \tau^{\frac{3}{2}-} T_3(k^{-1}k_1k_2^{\frac{1}{2}-}k_3^{\frac{1}{2}-}; v) + \tau^2 \left(T_1(1; v) + \mathcal{F}T_3(k_2k_3; v)(0) + T_5(1; v) \right).$$

Then from Lemma 2.2, we obtain that

$$\|\tilde{\mathcal{L}}^n\|_{H^1} \leq C\tau^{\frac{3}{2}-}, \quad (4.7)$$

where the constant $C > 0$ depending only on $\|u\|_{L^\infty((0,T);H^1)}$. This together with (4.4) yields

$$\|v(t_{n+1}) - v^{n+1}\|_{H^1} \leq C\tau^{\frac{3}{2}-} + (1 + C\tau)\|v^n - v(t_n)\|_{H^1} + C\tau\|v^n - v(t_n)\|_{H^1}^3.$$

By iteration and Gronwall's inequalities, we get

$$\|v(t_{n+1}) - v^{n+1}\|_{H^1} \leq C\tau^{\frac{3}{2}-} \sum_{j=0}^n (1 + C\tau)^j \leq C\tau^{\frac{1}{2}-}, \quad n = 0, 1, \dots, \frac{T}{\tau} - 1.$$

Hence, we get the desired convergence result.

5. FURTHER DISCUSSION ON THE ALMOST MASS-CONSERVED SCHEME

Let $V^n := e^{-it_n \partial_x^2} U^n$. Accordingly, from (1.9)–(1.12), we have that

$$V^{n+1} = V^n + \tilde{F}^n(V^n) + \tilde{G}_1^n(V^n) + \tilde{G}_2^n(V^n), \quad (5.1)$$

where Φ^n is defined in (3.20),

$$\tilde{F}^n(V^n) = \Phi^n(V^n) - V^n,$$

and the functionals $\tilde{G}_1^n, \tilde{G}_2^n$ are given by

$$\tilde{G}_1^n(V) = \tilde{H}^n(V)V; \quad \tilde{G}_2^n(V) = -\frac{1}{2}(\tilde{H}^n(V))^2V - M(u_0)^{-1} \tilde{H}^n(V) \langle \tilde{F}^n(V), V \rangle V, \quad (5.2)$$

and

$$\tilde{H}^n(V) = -M(u_0)^{-1} \left(\langle \tilde{F}^n(V), V \rangle + \frac{1}{2} \|\tilde{F}^n(V)\|_{L^2}^2 \right). \quad (5.3)$$

The proof of Theorem 1.3 depends on the following key lemmas.

Lemma 5.1. *Let \tilde{G}_1^n be defined in (5.2), then the following inequalities holds:*

(i) *If $\gamma > \frac{3}{2}$, then there exists some constant $C = C(\|V\|_{H^\gamma}, \|u_0\|_{L^2}) > 0$ such that*

$$2\langle \tilde{G}_1^n(V), V \rangle + 2\langle \tilde{F}^n(V), V \rangle + \|\tilde{F}^n(V)\|_{L^2}^2 \leq C\tau^2 |M(V) - M(u_0)|,$$

moreover,

$$\|\tilde{G}_1^n(V)\|_{H^\gamma} \leq C\tau^2.$$

(ii) *If $\gamma \geq 1$, then there exists some constant $C = C(\|V\|_{H^\gamma}, \|u_0\|_{L^2}) > 0$ such that*

$$2\langle \tilde{G}_1^n(V), V \rangle + 2\langle \tilde{F}^n(V), V \rangle + \|\tilde{F}^n(V)\|_{L^2}^2 \leq C\tau^{\frac{3}{2}-} |M(V) - M(u_0)|,$$

moreover,

$$\|\tilde{G}_1^n(V)\|_{H^\gamma} \leq C\tau^{\frac{3}{2}-}.$$

Proof. (i) According to (3.3) and (3.19), we find that

$$\begin{aligned} \tilde{F}^n(V) &= -i \int_0^\tau e^{-i(t_n+s)\partial_x^2} \left[|e^{i(t_n+s)\partial_x^2} V|^2 e^{i(t_n+s)\partial_x^2} V \right] ds \\ &\quad + 2i\tau(M(V) - M(v(t_n)))V + 2i\tau(P(V) - P(v(t_n)))\partial_x^{-1}V \\ &\quad - \tau^2 \left(T_1(1; V) + \mathcal{F}T_3(k_2k_3; V)(0) + T_3(k^{-1}k_1k_2k_3; V) \right). \end{aligned} \quad (5.4)$$

Then we have

$$\begin{aligned} \langle \tilde{F}^n(V), V \rangle &= \left\langle -i \int_0^\tau e^{-i(t_n+s)\partial_x^2} [|e^{i(t_n+s)\partial_x^2} V|^2 e^{i(t_n+s)\partial_x^2} V] ds, V \right\rangle \\ &\quad + 2\tau \left\langle i(M(V) - M(v(t_n)))V + i(P(V) - P(v(t_n)))\partial_x^{-1}V, V \right\rangle \\ &\quad - \tau^2 \left\langle T_1(1; V) + \mathcal{F}T_3(k_2k_3; V)(0) + T_3(k^{-1}k_1k_2k_3; V), V \right\rangle. \end{aligned}$$

The first term is obviously equal to 0. We claim that the second term is also equal to 0. Indeed, since

$$iP(f) = -\frac{1}{2\pi} \text{Im} \int_{\mathbb{T}} f \partial_x \bar{f} dx \in \mathbb{R},$$

we find that

$$\begin{aligned} &\left\langle i(M(V) - M(v(t_n)))V + i(P(V) - P(v(t_n)))\partial_x^{-1}V, V \right\rangle \\ &= (M(V) - M(v(t_n)))\langle iV, V \rangle + i(P(V) - P(v(t_n)))\langle \partial_x^{-1}V, V \rangle = 0. \end{aligned}$$

Then we get

$$\langle \tilde{F}^n(V), V \rangle = -\tau^2 \langle T_1(1; V) + \mathcal{F}T_3(k_2k_3; V)(0) + T_3(k^{-1}k_1k_2k_3; V), V \rangle.$$

From Lemma 2.2 (i) (iii), we obtain

$$\|T_1(1; V)\|_{L^2} + |\mathcal{F}T_3(k_2k_3; V)(0)| + \|T_3(k^{-1}k_1k_2k_3; V)\|_{L^2} \lesssim \|V\|_{H^\gamma} + \|V\|_{H^\gamma}^3. \quad (5.5)$$

This implies that

$$|\langle \tilde{F}^n(V), V \rangle| \lesssim \tau^2 (\|V\|_{H^\gamma}^2 + \|V\|_{H^\gamma}^4). \quad (5.6)$$

By (5.4) and (5.5), we have

$$\|\tilde{F}^n(V)\|_{L^2}^2 \lesssim \tau^2 (\|V\|_{H^\gamma}^2 + \|V\|_{H^\gamma}^6). \quad (5.7)$$

Hence, there exists $C > 0$ depending only on $\|V\|_{H^\gamma}$ and $\|u_0\|_{L^2}$, such that

$$|\tilde{H}^n(V)| \leq C\tau^2. \quad (5.8)$$

This yields that

$$\|\tilde{G}_1^n(V)\|_{H^\gamma} \leq C\tau^2.$$

Furthermore, we have

$$2\langle \tilde{G}_1^n(V), V \rangle + 2\langle \tilde{F}^n(V), V \rangle + \|\tilde{F}^n(V)\|_{L^2}^2 = 2\tilde{H}^n(V)(M(V) - M(u_0)). \quad (5.9)$$

From (5.8), the above equality is controlled by $C\tau^2 |M(V) - M(u_0)|$.

(ii) According to (3.3) and (4.6), we can replace $\tau^2 T_3(k^{-1}k_1k_2k_3; V)$ in (5.4) by

$$\tau^{\frac{3}{2}-} T_3(k^{-1}k_1k_2^{\frac{1}{2}-}k_3^{\frac{1}{2}-}; V).$$

Then arguing similarly as in the proof of (i) and applying Lemma 2.2 (ii) (iii) instead, it infers that

$$|\langle \tilde{F}^n(V), V \rangle| \leq C\tau^{\frac{3}{2}-}; \quad |\tilde{H}^n(V)| \leq C\tau^{\frac{3}{2}-}. \quad (5.10)$$

Hence we have for any $\gamma \geq 1$

$$\|\tilde{G}_1^n(V)\|_{H^\gamma} \leq C\tau^{\frac{3}{2}-}.$$

Moreover, by (5.9) and (5.10), we obtain

$$2\langle \tilde{G}_1^n(V), V \rangle + 2\langle \tilde{F}^n(V), V \rangle + \|\tilde{F}^n(V)\|_{L^2}^2 \leq C\tau^{\frac{3}{2}-} |M(V) - M(u_0)|.$$

This finishes the proof of the lemma. \square

Lemma 5.2. *Let the functionals $\tilde{G}_1^n, \tilde{G}_2^n$ be defined in (5.2).*

(i) *If $\gamma > \frac{3}{2}$, then there exists some constant $C = C(\|V\|_{H^\gamma}, \|u_0\|_{L^2}) > 0$ such that*

$$2\langle \tilde{G}_2^n(V), V \rangle + 2\langle \tilde{F}^n(V), \tilde{G}_1^n(V) \rangle + \|\tilde{G}_1^n(V)\|_{L^2}^2 \leq C\tau^4 |M(V) - M(u_0)|,$$

moreover,

$$\|\tilde{G}_2^n(V)\|_{H^\gamma} \leq C\tau^4.$$

- (ii) If $\gamma \geq 1$, then there exists some constant $C = C(\|V\|_{H^\gamma}, \|u_0\|_{L^2}) > 0$ such that
- $$2\langle \tilde{G}_2^n(V), V \rangle + 2\langle \tilde{F}^n(V), \tilde{G}_1^n(V) \rangle + \|\tilde{G}_1^n(V)\|_{L^2}^2 \leq C\tau^{3-} |M(V) - M(u_0)|,$$
- moreover,

$$\|\tilde{G}_2^n(V)\|_{H^\gamma} \leq C\tau^{3-}.$$

Proof. (i) From the definition of $\tilde{G}_1^n(V)$ in (5.2), we find

$$2\langle \tilde{F}^n(V), \tilde{G}_1^n(V) \rangle = 2\tilde{H}^n(V)\langle \tilde{F}^n(V), V \rangle; \quad \|\tilde{G}_1^n(V)\|_{L^2}^2 = (\tilde{H}^n(V))^2 M(V).$$

Note that

$$\begin{aligned} & 2\langle \tilde{G}_2^n(V), V \rangle + 2\langle \tilde{F}^n(V), \tilde{G}_1^n(V) \rangle + \|\tilde{G}_1^n(V)\|_{L^2}^2 \\ &= 2\tilde{H}^n(V)\langle \tilde{F}^n(V), V \rangle \frac{M(u_0) - M(V)}{M(u_0)}. \end{aligned} \quad (5.11)$$

By (5.6) and (5.8), the above equality is controlled by

$$C\tau^4 |M(V) - M(u_0)|.$$

Moreover, by (5.6) and (5.8) again, we obtain

$$\|\tilde{G}_2^n(V)\|_{H^\gamma} \leq C\tau^4.$$

- (ii) From (5.10) and (5.11), we get directly

$$2\langle \tilde{G}_2^n(V), V \rangle + 2\langle \tilde{F}^n(V), \tilde{G}_1^n(V) \rangle + \|\tilde{G}_1^n(V)\|_{L^2}^2 \leq C\tau^{3-} |M(V) - M(u_0)|,$$

and

$$\|\tilde{G}_2^n(V)\|_{H^\gamma} \leq C\tau^{3-}.$$

We obtain the conclusion of the lemma. \square

Proof of Theorem 1.3. Since $V^n = e^{-it_n \partial_x^2} U^n$, $v(t_n) = e^{-it_n \partial_x^2} u(t_n)$, we only need to prove the conclusion of Theorem 1.3 holds for V^n and $v(t_n)$.

From (5.1), we have

$$V^{n+1} = \Phi^n(V^n) + \tilde{G}_1^n(V^n) + \tilde{G}_2^n(V^n), \quad (5.12)$$

where $\Phi^n(V^n)$ is defined in (3.20). Then

$$\begin{aligned} V^{n+1} - v(t_{n+1}) &= \Phi^n(V^n) - \Phi^n(v(t_n)) + \tilde{G}_1^n(V^n) - \tilde{G}_1^n(v(t_n)) + \tilde{G}_2^n(V^n) - \tilde{G}_2^n(v(t_n)) \\ &\quad + \Phi^n(v(t_n)) - v(t_{n+1}) + \tilde{G}_1^n(v(t_n)) + \tilde{G}_2^n(v(t_n)). \end{aligned}$$

From the estimate on the functional Φ^n in (4.4), we obtain for $\gamma \geq 1$

$$\|\Phi^n(v(t_n)) - \Phi^n(V^n)\|_{H^\gamma} \leq (1 + C\tau) \|V^n - v(t_n)\|_{H^\gamma} + C\tau \|V^n - v(t_n)\|_{H^\gamma}^3.$$

For the term $\tilde{G}_j^n(V^n) - \tilde{G}_j^n(v(t_n))$, $j = 1, 2$, the similar treatment as in Section 4, we get

$$\|\tilde{G}_1^n(V^n) - \tilde{G}_1^n(v(t_n))\|_{H^\gamma} \leq C\tau \left(\|V^n - v(t_n)\|_{H^\gamma} + \|V^n - v(t_n)\|_{H^\gamma}^7 \right),$$

and

$$\|\tilde{G}_2^n(V^n) - \tilde{G}_2^n(v(t_n))\|_{H^\gamma} \leq C\tau^2 \left(\|V^n - v(t_n)\|_{H^\gamma} + \|V^n - v(t_n)\|_{H^\gamma}^{13} \right).$$

From (4.1), we have for $\gamma > \frac{3}{2}$

$$\|\Phi^n(v(t_n)) - v(t_{n+1})\|_{H^\gamma} \leq C\tau^2. \quad (5.13)$$

Furthermore, from Lemma 5.1 and Lemma 5.2, we find

$$\|\tilde{G}_1^n(v(t_n))\|_{H^\gamma} \leq C\tau^2, \quad \|\tilde{G}_2^n(v(t_n))\|_{H^\gamma} \leq C\tau^4.$$

Putting together with the above estimates, we conclude that for any $\tau \leq 1$,

$$\|V^{n+1} - v(t_{n+1})\|_{H^\gamma} \leq C\tau^2 + (1 + C\tau) \|V^n - v(t_n)\|_{H^\gamma} + C\tau \|V^n - v(t_n)\|_{H^\gamma}^{13},$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^\gamma)}$.

By the iteration and Gronwall inequalities, we get

$$\|v(t_{n+1}) - V^{n+1}\|_{H^\gamma} \leq C\tau^2 \sum_{j=0}^n (1 + C\tau)^j \leq C\tau, \quad n = 0, 1, \dots, \frac{T}{\tau} - 1.$$

This implies the first-order convergence and the following a priori estimate:

$$\|V^n\|_{H^\gamma} \leq C, \quad n = 0, 1, \dots, \frac{T}{\tau}. \quad (5.14)$$

Here the positive constant C depends only on T and $\|u\|_{L^\infty((0,T);H^\gamma)}$.

In addition, if $u_0 \in H^1$, from (4.7) we have

$$\|\Phi^n(v(t_n)) - v(t_{n+1})\|_{H^1} \leq C\tau^{\frac{3}{2}-}. \quad (5.15)$$

Furthermore, from Lemma 5.1 and Lemma 5.2, we find

$$\|\tilde{G}_1^n(v(t_n))\|_{H^1} \leq C\tau^{\frac{3}{2}-}, \quad \|\tilde{G}_2^n(v(t_n))\|_{H^1} \leq C\tau^{3-}.$$

Putting together with the above estimates, we conclude that for any $\tau \leq 1$,

$$\|V^{n+1} - v(t_{n+1})\|_{H^1} \leq C\tau^{\frac{3}{2}-} + (1 + C\tau)\|V^n - v(t_n)\|_{H^1} + C\tau\|V^n - v(t_n)\|_{H^1}^{13},$$

where the constant C depends only on $\|u\|_{L^\infty((0,T);H^1)}$.

By the iteration and Gronwall inequalities, we get

$$\|v(t_{n+1}) - V^{n+1}\|_{H^1} \leq C\tau^{\frac{3}{2}-} \sum_{j=0}^n (1 + C\tau)^j \leq C\tau^{\frac{1}{2}-}, \quad n = 0, 1, \dots, \frac{T}{\tau} - 1.$$

Next we prove the almost mass conservation law. From (5.1), we have

$$M(V^{n+1}) = M(V^n) + 2\langle \tilde{F}^n(V^n), V^n \rangle + 2\langle \tilde{G}_1^n(V^n), V^n \rangle + \|\tilde{F}^n(V^n)\|_{L^2}^2 \quad (5.16a)$$

$$+ 2\langle \tilde{G}_2^n(V^n), V^n \rangle + 2\langle \tilde{F}^n(V^n), \tilde{G}_1^n(V^n) \rangle + \|\tilde{G}_1^n(V^n)\|_{L^2}^2 \quad (5.16b)$$

$$+ 2\langle \tilde{F}^n(V^n), \tilde{G}_2^n(V^n) \rangle + 2\langle \tilde{G}_1^n(V^n), \tilde{G}_2^n(V^n) \rangle + \|\tilde{G}_2^n(V^n)\|_{L^2}^2. \quad (5.16c)$$

By Lemma 5.1 and Lemma 5.2, we get that

$$(5.16a) \leq C\tau^2 |M(V^n) - M(u_0)|; \quad (5.16b) \leq C\tau^4 |M(V^n) - M(u_0)|,$$

and

$$2 \left| \langle \tilde{G}_1^n(V^n), \tilde{G}_2^n(V^n) \rangle \right| + \|\tilde{G}_2^n(V^n)\|_{L^2}^2 \leq C\tau^6.$$

Therefore we deduce that

$$M(V^{n+1}) - M(V^n) \leq C\tau^6 + C\tau^2 |M(V^n) - M(u_0)| + 2\langle \tilde{F}^n(V^n), \tilde{G}_2^n(V^n) \rangle. \quad (5.17)$$

By the definitions of $G_2^n(V^n)$ in (5.2), we get

$$2 \left| \langle \tilde{F}^n(V^n), \tilde{G}_2^n(V^n) \rangle \right| \leq \tilde{H}^n(V^n)^2 \left| \langle \tilde{F}^n(V^n), V^n \rangle \right| + 2M(u_0)^{-1} \langle \tilde{F}^n(V^n), V^n \rangle^2 \tilde{H}^n(V^n).$$

Hence, by (5.6), (5.8) and (5.14), we obtain

$$2 \left| \langle \tilde{F}^n(V^n), \tilde{G}_2^n(V^n) \rangle \right| \leq C\tau^6.$$

Therefore, we conclude that

$$M(V^{n+1}) - M(V^n) \leq C\tau^6 + C\tau^2 |M(V^n) - M(u_0)|. \quad (5.18)$$

Then by the iteration, we get

$$|M(V^n) - M(u_0)| \leq C\tau^5. \quad (5.19)$$

This finishes the proof of Theorem 1.3. \square

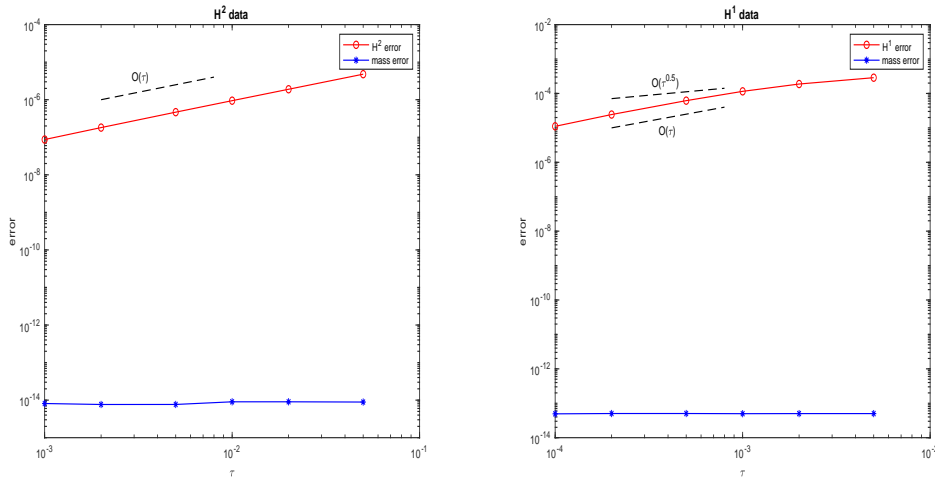


FIGURE 1. Convergence of (NLRI) (1.12): error $\|u - u^n\|_{H^\gamma}$ at $t_n = T = 2$ when $\gamma = 2$ (left) and when $\gamma = 1$ (right).

6. NUMERICAL EXPERIENCE

To set the initial data $u_0(x)$ with the desired regularity, we use the following strategy in [31]. Choose $N = 2^{10}$ and discrete the spatial domain \mathbb{T} with grid points $x_j = j\frac{2\pi}{N}$ for $j = 0, \dots, N$. Take a uniformly distributed random vectors $\text{rand}(N, 1) \in [0, 1]^N$ and define

$$u_0(x) := \frac{|\partial_{x,N}|^{-\gamma} \mathcal{U}^N}{\| |\partial_{x,N}|^{-\gamma} \mathcal{U}^N \|_{L^\infty}}, \quad x \in \mathbb{T}, \quad \mathcal{U}^N = \text{rand}(N, 1) + i \text{rand}(N, 1). \quad (6.1)$$

where the pseudo-differential operator $|\partial_{x,N}|^{-\gamma}$ for $\gamma \geq 0$ reads: for Fourier modes $l = -N/2, \dots, N/2 - 1$,

$$\left(|\partial_{x,N}|^{-\gamma} \right)_l = \begin{cases} |l|^{-\gamma} & \text{if } l \neq 0, \\ 0 & \text{if } l = 0. \end{cases}$$

Thus, we get $u_0 \in H^\gamma(\mathbb{T})$ for any $\gamma \geq 0$. Now we take $\tau = 10^{-5}$ and obtain Figure 1.

The numerical results imply that the scheme (1.12) has the first-order accuracy of $u(t_n) - u^n$ in H^2 -norm with initial data in H^2 , while the mass of the numerical solution is almost conserved (around 10^{-14}). Furthermore, for H^1 -data, $u(t_n) - u^n$ in H^1 -norm is a little bit better than 0.5 order accuracy.

7. CONCLUSION

In this work, we constructed a first-order Fourier integrator for solving the cubic nonlinear Schrödinger equation in one dimension. Our designation of the scheme is based on the exponential-type integration and the Phase-Space analysis of the nonlinear dynamics. The convergence theorem was established to prove that the first-order accuracy in H^γ with initial data in H^γ for any $\gamma > \frac{3}{2}$, where the regularity requirements are lower than existing methods so far. Further, we designed a modified numerical scheme to obtain the first-order convergence in H^γ with H^γ -data meanwhile keeps the fifth-order mass convergence. By our method, the scheme can be constructed to obtain the arbitrary high-order mass convergence. Numerical results were reported to justify the theoretical results.

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