ON BLOW-UP FORMULA OF INTEGRAL BOTT-CHERN COHOMOLOGY

YOUMING CHEN AND SONG YANG

Abstract. Recently, the blow-up formulae of cohomologies on complex manifolds have been extensively studied. The purpose of this paper is to give a proof for the blow-up formula of integral Bott-Chern cohomology on compact complex manifolds in terms of relative Dolbeault sheaves.

1. Introduction

As a fundamental tool, the blow-up transformation plays an important role in the development of complex geometry. Moreover, it is a basic problem to discover the so-called "blow-up formula" for any interesting cohomologies or invariants on (complex) manifolds. Recently, the blow-up formula that analogue of de Rham blow-up formula (cf. [14, Theorem 7.31]), for various complex cohomologies have been extensively studied in [1, 2, 6, 7, 8, 9, 12, 13, 16] etc. such as Bott-Chern cohomology, Dolbeault cohomology and twisted de Rham cohomology and so on.

Due to the Dolbeault resolution, the famous Dolbeault theorem says that the (p, q)-th Dolbeault cohomology is isomorphic to the q-th cohomology of the sheaf of holomorphic p-forms or the q-th hypercohomology of the sheaf complex of complex differential $(p, \cdot)$-forms. Likewise, the Bott-Chern cohomology can also be reinterpreted as a hypercohomology (see [5, 11]). Inspired by Bott-Chern hyercohomology and Deligne cohomology, Schweitzer [11, Section 4.e] introduced a notion of integral Bott-Chern cohomology in terms of hypercohomology. Let us recall the definition: for any fixed bi-degree (p, q) with $p \leq q$, the integral Bott-Chern complex $\mathfrak{I}_Y^{p,q}$ on a complex manifold Y is defined to be the sheaf complex

$$0 \to (2\pi i)^p \,\mathbb{Z} \xrightarrow{(+,-)} \mathcal{O}_Y \oplus \overline{\mathcal{O}}_Y \to \Omega_Y^1 \oplus \overline{\Omega}_Y^1 \to \cdots \to \Omega_Y^{p-1} \oplus \overline{\Omega}_Y^{p-1} \to \overline{\Omega}_Y^p \to \cdots \to \overline{\Omega}_Y^{q-1} \to 0,$$

where Ω_Y^s (resp. $\overline{\Omega}_Y^s$) is the sheaf of (resp. anti-) holomorphic s-forms and $\mathbb{Z}$ is the constant sheaf on Y.

Definition 1.1 ([11]). The (p, q)-th integral Bott-Chern cohomology of Y is the $(p+q)$-th hypercohomology

$$H_{BC}^{p,q}(Y, \mathbb{Z}) := \mathbb{H}^{p+q}(Y, \mathfrak{I}_Y^{p,q})$$

of the integral Bott-Chern complex $\mathfrak{I}_Y^{p,q}$.

The main goal of this paper is to give a proof of the following:

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Theorem 1.2. *Let Y be a compact complex manifold of complex dimension $n \geq 2$, $Z \subset Y$ be a closed complex submanifold of codimension $c \geq 2$, and $f : X \longrightarrow Y$ be the blow-up of Y with center Z. Then there exists an isomorphism*

$$H^{p,q}_{BC}(X,\mathbb{Z}) \cong H^{p,q}_{BC}(Y,\mathbb{Z}) \oplus \bigoplus_{i=1}^{c-1} H^{p-i,q-i}_{BC}(Z,\mathbb{Z})$$

for $p, q \leq n$.

This paper is devoted to the proof of the above theorem. Our basic strategy is the same as that in [2, 8, 9, 10, 16], which was essentially outlined in [2, Remark 11]. The structure of this paper is organized as follows. In Section 2, we briefly review the definition of relative Dolbeault sheaves and collect some basic facts concerning the relative Dolbeault sheaves under blow-ups. In Section 3, we present a proof of Theorem 1.2 by proving Lemma 3.3 and Lemma 3.4 with the language of relative Dolbeault sheaves. Moreover some remarks are proposed.

2. Relative Dolbeault sheaves

2.1. **Generalities.** Let Y be a complex manifold of complex dimension n. We denote by Ω^s_Y (resp. $\overline{\Omega}^s_Y$) the sheaf of (resp. anti-) holomorphic s-forms and $\mathcal{A}^{s,t}_Y$ the sheaf of complex differential (s,t)-forms. Note that $\mathcal{A}^{s,t}_Y$ is a fine sheaf and the Dolbeault resolution asserts that there is a fine resolution of Ω^s_Y:

$$0 \longrightarrow \Omega^s_Y \longrightarrow \mathcal{A}^{s,0}_Y \xrightarrow{\bar\partial} \mathcal{A}^{s,1}_Y \longrightarrow \cdots \xrightarrow{\bar\partial} \mathcal{A}^{s,n}_Y \longrightarrow 0$$

Next, we assume that $\iota : Z \hookrightarrow Y$ is a closed complex submanifold of Y. Then there exist two natural morphisms of sheaves

$$\Omega^s_Y \xrightarrow{\iota^\star} \iota_*\Omega^s_Z \quad \text{and} \quad \mathcal{A}^{s,t}_Y \xrightarrow{\iota^\star} \iota_*\mathcal{A}^{s,t}_Z,$$

on Y which are induced by the pullback $\iota^\star$ of the corresponding differential forms and ι_* is the direct image functor of sheaves.

Definition 2.1 ([9, 16]). Let Y be a complex manifold of complex dimension n and $\iota : Z \hookrightarrow Y$ be a closed complex submanifold. For any $0 \leq s, t \leq n$, the kernel sheaves

$$\mathcal{K}^s_{Y,Z} := \ker\left(\Omega^s_Y \xrightarrow{\iota^\star} \iota_*\Omega^s_Z\right) \quad \text{and} \quad \mathcal{K}^{s,t}_{Y,Z} := \ker\left(\mathcal{A}^{s,t}_Y \xrightarrow{\iota^\star} \iota_*\mathcal{A}^{s,t}_Z\right)$$

are called respectively the *s-th relative Dolbeault sheaf* and *(s,t)-th relative Dolbeault sheaf* of Y with respect to Z.

The basic properties of the relative Dolbeault sheaves are the following:

Lemma 2.2 ([9, 16]). *For any $s \geq 0$, the sheaf complex $\mathcal{K}^{s,\bullet}_{Y,Z}$ is a fine resolution of $\mathcal{K}^s_{Y,Z}$(called relative Dolbeault resolution). Moreover, there exist two short exact sequences*

$$0 \longrightarrow \mathcal{K}^s_{Y,Z} \longrightarrow \Omega^s_Y \xrightarrow{\iota^\star} \iota_*\Omega^s_Z \longrightarrow 0, \tag{2.1}$$

and

$$0 \longrightarrow \mathcal{K}^{s,t}_{Y,Z} \longrightarrow \mathcal{A}^{s,t}_Y \xrightarrow{\iota^\star} \iota_*\mathcal{A}^{s,t}_Z \longrightarrow 0, \tag{2.2}$$

for any $0 \leq s, t \leq n$.

2.2. **Direct images under blow-ups**. Let Y be a compact complex manifold of dimension $n \geq 2$ and $\iota : Z \hookrightarrow Y$ be a closed complex submanifold of codimension $c \geq 2$. Denote by $f : X \longrightarrow Y$ the blow-up of Y along Z and $\tilde{\iota} : E \hookrightarrow X$ the exceptional divisor of the blow-up; see for example [14, Section 3.3.3] for the construction of blow-ups. Consider the blow-up diagram

$$\begin{array}{ccc} E & \overset{\tilde{\iota}}{\hookrightarrow} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \overset{\iota}{\hookrightarrow} & Y. \end{array} \tag{2.3}$$

Then the sheaves of holomorphic forms satisfy the following relations:

Proposition 2.3. *For any $s \geq 0$, the following isomorphisms hold:*

(i) $f^\star : \Omega_Y^s \xrightarrow{\simeq} f_* \Omega_X^s$;
(ii) $g^\star : \Omega_Z^s \xrightarrow{\simeq} g_* \Omega_E^s$;
(iii) $\tilde{\iota}^\star : R^i f_* \Omega_X^s \xrightarrow{\simeq} \iota_* R^i g_* \Omega_E^s$ *for any* $i \geq 1$.

Proof. See for example [9, Lemma 4.1]. □

For our purpose, the key ingredient is the direct images of relative Dolbeault sheaves under blow-up morphisms. First of all, we can directly show that by Hartogs' extension theorem, the pullback of differential forms induces an isomorphism $f^\star : \mathcal{K}_{Y,Z}^s \xrightarrow{\simeq} f_* \mathcal{K}_{X,E}^s$ (see for example [9, Lemma 4.4]). Secondly, considering the higher direct images of a short exact sequence that analogous to (2.1) for the pair (X, E), we can obtain that the higher direct images $R^i f_* \mathcal{K}_{X,E}^s = 0$ for all $i \geq 1$ by using Proposition 2.3 (see for example [9, Lemma 4.4]). Moreover, due to the relative Dolbeault resolution as in Lemma 2.2, we have the following:

Lemma 2.4. *The pullback of differential forms induces a natural quasi-isomorphism*

$$f^\star : \mathcal{K}_{Y,Z}^s \xrightarrow{\sim} Rf_* \mathcal{K}_{X,E}^s$$

for any $s \in \mathbb{Z}$, where Rf_* is the derived direct image.

Proof. By Lemma 2.2, for the pair (X, E), the fact that $\mathcal{K}_{X,E}^{s,\bullet}$ is a fine resolution of $\mathcal{K}_{X,E}^s$ means the derived direct image $Rf_* \mathcal{K}_{X,E}^s = f_* \mathcal{K}_{X,E}^{s,\bullet}$. Since the higher direct images of $\mathcal{K}_{X,E}^s$ vanish, the sheaf complex $f_* \mathcal{K}_{X,E}^{s,\bullet}$ is exact except in degree 0. Moreover, note that the direct image $f_* \mathcal{K}_{X,E}^s$ is the kernel of the sheaf morphism $f_* \mathcal{K}_{X,E}^{s,0} \longrightarrow f_* \mathcal{K}_{X,E}^{s,1}$ since the direct image functor f_* is left exact. As a consequence, we have a canonical quasi-isomorphism

$$f_* \mathcal{K}_{X,E}^s \xrightarrow{\simeq} f_* \mathcal{K}_{X,E}^{s,\bullet} = Rf_* \mathcal{K}_{X,E}^s \,.$$

Furthermore, as mentioned above the pullback of differential forms induces an isomorphism $f^\star : \mathcal{K}_{Y,Z}^s \xrightarrow{\simeq} f_* \mathcal{K}_{X,E}^s$ of sheaves and then the lemma follows. □

3. Proof

3.1. **Proof of Theorem 1.2**. We start by recalling the definitions of the Deligne complex $\mathbb{Z}_D(p)$ (of level p) on a complex manifold Y,

$$0 \longrightarrow (2\pi i)^p \, \mathbb{Z} \longrightarrow \mathcal{O}_Y \longrightarrow \Omega_Y^1 \longrightarrow \cdots \longrightarrow \Omega_Y^{p-1} \longrightarrow 0 \;;$$

and the integral Bott-Chern complex $\mathfrak{I}_Y^{p,q}$

$$0 \to (2\pi i)^p\,\mathbb{Z} \xrightarrow{(+,-)} \mathcal{O}_Y \oplus \overline{\mathcal{O}}_Y \to \Omega_Y^1 \oplus \overline{\Omega}_Y^1 \to \cdots \to \Omega_Y^{p-1} \oplus \overline{\Omega}_Y^{p-1} \to \overline{\Omega}_Y^{p} \to \cdots \to \overline{\Omega}_Y^{q-1} \to 0\ .$$

Immediately, one has the following commutative diagram

$$\begin{array}{ccccccccc}
(2\pi i)^p\,\mathbb{Z} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \cdots & \longrightarrow & \Omega_Y^s & \longrightarrow & \cdots \\
\uparrow\big\downarrow id & & proj.\ \uparrow\big\downarrow \varphi & & & & proj.\ \uparrow\big\downarrow \varphi & & \\
(2\pi i)^p\,\mathbb{Z} & \longrightarrow & \mathcal{O}_Y \oplus \overline{\mathcal{O}}_Y & \longrightarrow & \cdots & \longrightarrow & \Omega_Y^s \oplus \overline{\Omega}_Y^s & \longrightarrow & \cdots
\end{array}$$

where $\varphi : \Omega_Y^s \to \Omega_Y^s \oplus \overline{\Omega}_Y^s$ is defined by $\varphi(\alpha) = \alpha + (-1)^{p-1}\overline{\alpha}$. Therefore the integral Bott-Chern complex $\mathfrak{I}_Y^{p,q}$ splits, i.e., there is a decomposition of sheaf complexes

$$\mathfrak{I}_Y^{p,q} \cong \mathbb{Z}_D(p) \oplus \overline{\Omega_Y^{[0\bullet q-1]}}[-1]. \tag{3.1}$$

where $\Omega_Y^{[s\bullet t]}$ is the truncated holomorphic de Rham complex

$$0 \longrightarrow \Omega_Y^s \xrightarrow{\partial} \Omega_Y^{s+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_Y^t \longrightarrow 0,$$

when $s \leq t$ while $\Omega_Y^{[s\bullet s]} := \Omega_Y^s$ and $\Omega_Y^{[s\bullet t]} := \Omega_Y^{[0\bullet t]}$ for $s \leq 0$. As a result, applying hypercohomology to the above decomposition, the integral Bott-Chern cohomology admits a decomposition as follows:

$$H_{BC}^{p,q}(Y,\mathbb{Z}) \cong H_D^{p+q}(Y,\mathbb{Z}(p)) \oplus \mathbb{H}^{p+q-1}(Y,\overline{\Omega_Y^{[0\bullet q-1]}}),$$

In the rest of this section, we always assume that Y is a compact complex manifold of complex dimension $n \geq 2$ and $\iota : Z \hookrightarrow Y$ is a closed complex submanifold of codimension $c \geq 2$. We denote by $f : X \longrightarrow Y$ the blow-up of Y along Z and $\tilde{\iota} : E \hookrightarrow X$ its exceptional divisor.

To finish the proof, we need the following blow-up formula of Deligne cohomology:

Proposition 3.1 (Barbieri-Viale [3]). *There is an isomorphism*

$$H_D^l(X,\mathbb{Z}(p)) \cong H_D^l(Y,\mathbb{Z}(p)) \oplus \bigoplus_{i=1}^{c-1} H_D^{l-2i}(Z,\mathbb{Z}(p-i))$$

for any $p \geq 0$ *and* $l \in \mathbb{Z}$.

Moreover, we have the following generalization of de Rham blow-up formula:

Proposition 3.2. *There is an isomorphism* [1]

$$\mathbb{H}^l(X,\Omega_X^{[p\bullet q]}) \cong \mathbb{H}^l(Y,\Omega_Y^{[p\bullet q]}) \oplus \bigoplus_{i=1}^{c-1} \mathbb{H}^{l-2i}(Z,\Omega_Y^{[p-i\bullet q-i]})$$

for any $p \geq 0$ *and* $l \in \mathbb{Z}$.

Consequently, basing on the decomposition (3.1), Theorem 1.2 follows directly from the above two Propositions:

$$H_{BC}^{p,q}(X,\mathbb{Z}) \quad \cong \quad H_D^{p+q}(X,\mathbb{Z}(p)) \oplus \mathbb{H}^{p+q-1}(X,\overline{\Omega_X^{[0\bullet q-1]}}),$$

[1]See Remark 3.7

$$\begin{aligned} &\cong H_D^{p+q}(Y,\mathbb{Z}(p))\oplus\bigoplus_{i=1}^{c-1}H_D^{p+q-2i}(Z,\mathbb{Z}(p-i))\oplus\mathbb{H}^{p+q-1}(Y,\overline{\Omega_Y^{[0\bullet q-1]}}) \\ &\quad\oplus\bigoplus_{i=1}^{c-1}\mathbb{H}^{p+q-1-2i}(Z,\overline{\Omega_Y^{[0\bullet q-1-i]}}) \\ &\cong H_{BC}^{p,q}(Y,\mathbb{Z})\oplus\bigoplus_{i=1}^{c-1}H_{BC}^{p-i,q-i}(Z,\mathbb{Z}). \end{aligned}$$

3.2. **Proof of Proposition 3.2**. For the pair (Y,Z), we consider the $[p,q]$-truncated double complexes $\mathbb{T}_Y^{\bullet;\bullet}$ and $\mathbb{K}_{Y,Z}^{\bullet;\bullet}$ of global sections endowed the natural differentials ∂ and $\overline{\partial}$:

$$\mathbb{T}_Y^{s,t}:=\Gamma(Y,\mathcal{A}_Y^{s,t})\ \text{ and }\ \mathbb{K}_{Y,Z}^{s,t}:=\Gamma(Y,\mathcal{K}_{Y,Z}^{s,t}),$$

for $p\le s\le q$ and $0\le t\le n$. Denote their associated $[p,q]$-*total complexes* as follows:

$$\mathbb{T}_Y^{l}:=\bigoplus_{s+t=l,\,p\le s\le q}\mathbb{T}_Y^{s,t}\ \text{ and }\ \mathbb{K}_{Y,Z}^{l}:=\bigoplus_{s+t=l,\,p\le s\le q}\mathbb{K}_{Y,Z}^{s,t}$$

By the Dolbeault resolution, we have

$$\mathbb{H}^l(Y,\Omega_Y^{[p\bullet q]})\cong H^l(Y,\mathbb{T}_Y^\bullet)\ \text{ and }\ \mathbb{H}^l(Y,\mathcal{K}_{Y,Z}^{[p\bullet q]})\cong H^l(Y,\mathbb{K}_{Y,Z}^\bullet) \tag{3.2}$$

for any $l\in\mathbb{Z}$.

Now we come to the key of the proof of Proposition 3.2. For two pairs (Y,Z) and (X,E), based on the short exact sequence (2.2) and that analogous to (2.2) for (X,E), we have two short exact sequences of complexes:

$$0\longrightarrow\mathbb{K}_{Y,Z}^\bullet\longrightarrow\mathbb{T}_Y^\bullet\xrightarrow{\iota^*}\mathbb{T}_Z^\bullet\longrightarrow 0$$

and

$$0\longrightarrow\mathbb{K}_{X,E}^\bullet\longrightarrow\mathbb{T}_X^\bullet\xrightarrow{\tilde{\iota}^*}\mathbb{T}_E^\bullet\longrightarrow 0\ .$$

Based on the blow-up diagram (2.3), the pullback of differential forms induces a commutative diagram of short exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{K}_{Y,Z}^\bullet & \longrightarrow & \mathbb{T}_Y^\bullet & \xrightarrow{\iota^*} & \mathbb{T}_Z^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \\ 0 & \longrightarrow & \mathbb{K}_{X,E}^\bullet & \longrightarrow & \mathbb{T}_X^\bullet & \xrightarrow{\iota^*} & \mathbb{T}_E^\bullet & \longrightarrow & 0. \end{array}$$

After taking the cohomologies of the above commutative diagram, the isomorphisms in (3.2) for the pairs (Y,Z) and (X,E) offer a long commutative ladder of cohomology groups

$$\begin{array}{ccccccccc} \cdots\to & \mathbb{H}^l(Y,\mathcal{K}_{Y,Z}^{[p\bullet q]}) & \to & \mathbb{H}^l(Y,\Omega_Y^{[p\bullet q]}) & \to & \mathbb{H}^l(Z,\Omega_Z^{[p\bullet q]}) & \to & \mathbb{H}^{l+1}(Y,\mathcal{K}_{Y,Z}^{[p\bullet q]}) & \to\cdots \\ & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle f^*} & \\ \cdots\to & \mathbb{H}^l(X,\mathcal{K}_{X,E}^{[p\bullet q]}) & \to & \mathbb{H}^l(X,\Omega_X^{[p\bullet q]}) & \to & \mathbb{H}^l(E,\Omega_E^{[p\bullet q]}) & \to & \mathbb{H}^{l+1}(X,\mathcal{K}_{X,E}^{[p\bullet q]}) & \to\cdots \end{array} \tag{3.3}$$

It is worthy of noticing that for every differential form α on Y, $f_\star f^\star\alpha = \alpha$; see for example the proof of [5, Theorem 12.9]. Therefore, by the Dolbeault resolution, the induced morphism by the pullback of differential forms

$$f^\star : \mathbb{H}^l(Y, \Omega_Y^{[p\bullet q]}) \longrightarrow \mathbb{H}^l(X, \Omega_X^{[p\bullet q]})$$

is injective for $l \in \mathbb{Z}$. According to the below Lemma 3.3, using the standard diagram-chasing in the commutative ladder (3.3), we obtain that the cokernel of $f^\star : \mathbb{H}^l(Y, \Omega_Y^{[p\bullet q]}) \longrightarrow \mathbb{H}^l(X, \Omega_X^{[p\bullet q]})$ is isomorphic to the cokernel of $g^\star : \mathbb{H}^l(Z, \Omega_Z^{[p\bullet q]}) \longrightarrow \mathbb{H}^l(E, \Omega_E^{[p\bullet q]})$. Consequently, the below Lemma 3.4 concludes the Proposition 3.2.

In the following, we prove the two necessary lemmas (Lemma 3.3 and Lemma 3.4):

Lemma 3.3. *The pullback of differential forms induces an isomorphism*

$$f^\star : \mathbb{H}^l(Y, \mathcal{K}_{Y,Z}^{[p\bullet q]}) \xrightarrow{\simeq} \mathbb{H}^l(X, \mathcal{K}_{X,E}^{[p\bullet q]})$$

for any $p, q \in \mathbb{Z}$ *and* $p < q$.

Proof. We consider the $[p,q]$-truncated double complexes $\mathbb{K}_{Y,Z}^{\bullet,\bullet}$ and $\mathbb{K}_{X,E}^{\bullet,\bullet}$ of $\mathcal{K}_{Y,Z}^{[p\bullet q]}$ and $\mathcal{K}_{X,E}^{[p\bullet q]}$ respectively. Denote by ${}^Z E_r^{\bullet,\bullet}$ and ${}^E E_r^{\bullet,\bullet}$ the associated spectral sequences of $\mathbb{K}_{Y,Z}^{\bullet,\bullet}$ and $\mathbb{K}_{X,E}^{\bullet,\bullet}$ respectively. Especially, at E_1-page, we have ${}^Z E_1^{s,t} = H^t(Z, \mathbb{K}_{Y,Z}^s)$ and ${}^E E_1^{p,q} = H^q(E, \mathbb{K}_{X,E}^p)$. Note that the pullback of differential forms induces a morphism of double complexes $f^\star : \mathbb{K}_{Y,Z}^{\bullet,\bullet} \longrightarrow \mathbb{K}_{X,E}^{\bullet,\bullet}$. Therefore, it induces a morphism of spectral sequences:

$$f^* : {}^Z E_r^{\bullet,\bullet} \longrightarrow {}^E E_r^{\bullet,\bullet}. \tag{3.4}$$

To show the lemma, by the standard theory of spectral sequence, it suffices to show that the morphism (3.4) is an isomorphism at E_1-page. In fact, by Lemma 2.4, we have

$$f^* : {}^Z E_1^{s,t} = H^t(Z, \mathbb{K}_{Y,Z}^s) \longrightarrow H^t(Z, Rf_* \mathbb{K}_{X,E}^s) \cong H^t(E, \mathbb{K}_{X,E}^s) = {}^E E_1^{s,t}$$

is isomorphic and then the lemma follows. □

Lemma 3.4. *The pullback of differential forms induces an isomorphism*

$$\sum_{i=0}^{c-1} h^i \wedge g^\star(-) : \bigoplus_{i=0}^{c-1} \mathbb{H}^{l-2i}(Z, \Omega_Z^{[p-i\bullet q-i]}) \xrightarrow{\simeq} \mathbb{H}^l(E, \Omega_E^{[p\bullet q]})$$

for any $p < q$ *and* $l \in \mathbb{Z}$*, where* $h := c_1(\mathcal{O}_E(1)) \in H^2(E, \mathbb{Z})$ *and* $\mathcal{O}_E(1)$ *is the relative tautological line bundle. In particular, there is an isomorphism* $H_{BC}^{p,q}(E, \mathbb{Z}) \cong \bigoplus_{i=0}^{c-1} H_{BC}^{p-i,q-i}(Z, \mathbb{Z})$.

Proof. Consider the $[p,q]$-truncated double complexes $\mathbb{T}_Z^{\bullet,\bullet}$ and $\mathbb{T}_E^{\bullet,\bullet}$ of Z and E respectively. For $0 \le i \le c-1$, we denote by $\mathbb{T}_Z^{\bullet-i,\bullet-i}$ the $[p-i, q-i]$-truncated double complex of Z and $\mathrm{Tot}(\mathbb{T}_Z^{\bullet-i,\bullet-i})$ its associated total complex. Considering the l-th cohomology of $\mathrm{Tot}(\mathbb{T}_Z^{\bullet-i,\bullet-i})$, we have

$$H^l(\bigoplus_{i=0}^{c-1} \mathrm{Tot}(\mathbb{T}_Z^{\bullet-i,\bullet-i})[-2i]) = \bigoplus_{i=0}^{c-1} H^{l-2i}(\mathrm{Tot}(\mathbb{T}_Z^{\bullet-i,\bullet-i})) = \bigoplus_{i=0}^{c-1} \mathbb{H}^{l-2i}(Z, \Omega_Z^{[p-i\bullet q-i]}).$$

Next we consider the morphism of the above double complexes which is induced by pullback of differential forms:

$$\sum_{i=0}^{c-1} h^i \wedge g^\star : \bigoplus_{i=0}^{c-1} \mathbb{T}_Z^{\bullet-i,\bullet-i}[-2i] \longrightarrow \mathbb{T}_E^{\bullet,\bullet},$$

where $[-2i]$ means shifting $-i$ for both degrees. Denote by ${}_ZE_r^{\bullet,\bullet}$ and ${}_EE_r^{\bullet,\bullet}$ the associated spectral sequences of the above two bounded double complexes. Then there is a natural morphism $g_r^\star : {}_ZE_r^{\bullet,\bullet} \longrightarrow {}_EE_r^{\bullet,\bullet}$ of spectral sequences induced by $g^\star$. Note that $p \leq s \leq q$, we have

$${}_ZE_1^{s,t} = \bigoplus_{i=0}^{c-1} H^{t-i}(Z, \Omega_Z^{s-i}) \quad \text{and} \quad {}_EE_1^{s,t} = H^t(E, \Omega_E^s).$$

According to Dolbeault projective bundle formula (cf. Lemma 3.5), we obtain that

$$g_1^\star = \sum_{i=0}^{c-1} h^i \wedge g^\star(-) : {}_ZE_1^{s,t} \longrightarrow {}_EE_1^{s,t}$$

is an isomorphism. Therefore, $g_r^\star$ is isomorphic for any $r \geq 1$, and hence the proposition follows by the spectral sequence theory. □

In the above lemma, we have used the projective bundle formula of Dolbeault cohomology. For convenience, we give a proof in terms of sheaf complexes:

Lemma 3.5 (cf. [8], [2, Remark 9]). *There is a natural quasi-isomorphism of sheaf complexes*

$$\sum_{i=0}^{c-1} h^i \wedge g^\star(-) : \bigoplus_{i=0}^{c-1} \mathcal{A}_Z^{p-i,\bullet}[-i] \xrightarrow{\sim} g_* \mathcal{A}_E^{p,\bullet} \tag{3.5}$$

which is induced by pullback of differential forms. In particular, there exists an isomorphism

$$\sum_{i=0}^{c-1} h^i \wedge g^\star(-) : \bigoplus_{i=0}^{c-1} H^{q-i}(Z, \Omega_Z^{p-i}) \xrightarrow{\sim} H^q(E, \Omega_E^p).$$

Proof. Consider the morphism of the cohomology sheaves of sheaf complexes in (3.5):

$$\mathcal{H}^j\big(\sum_{i=0}^{c-1} h^i \wedge g^\star(-)\big) : \bigoplus_{i=0}^{c-1} \mathcal{H}^j(\mathcal{A}_Z^{p-i,\bullet}[-i]) \longrightarrow \mathcal{H}^j(g_* \mathcal{A}_E^{p,\bullet})$$

for $j \in \mathbb{Z}$. As a matter of fact, since the sheaf complex $\mathcal{A}_Z^{p-i,\bullet}$ is exact except in degree 0, so the above morphism is an isomorphism by following isomorphism

$$g^\star : \Omega_Z^{p-j} \xrightarrow{\simeq} R^j g_* \Omega_E^p, \tag{3.6}$$

for $0 \leq j \leq c-1$. Note that the isomorphism (3.6) is local in Z, so we may assume that $E = U \times \mathbb{P}^{c-1}$ for a polydisk U. Then, by Künneth formula, we have

$$\begin{aligned} H^i(E, \Omega_E^p) &= H^i(U \times \mathbb{P}^{c-1}, \bigoplus_{j=0}^{p} \Omega_U^j \otimes \Omega_{\mathbb{P}^{c-1}}^{p-j}) \\ &\cong \bigoplus_{j=0}^{p} \bigoplus_{s=0}^{i} H^s(U, \Omega_U^j) \otimes H^{i-s}(\mathbb{P}^{c-1}, \Omega_{\mathbb{P}^{c-1}}^{p-j}) \end{aligned}$$

$$\begin{aligned} &\cong \bigoplus_{j=0}^{p} H^0(U,\Omega_U^j)\otimes H^i(\mathbb{P}^{c-1},\Omega_{\mathbb{P}^{c-1}}^{p-j}) \\ &\cong H^0(U,\Omega_U^{p-i})\otimes H^i(\mathbb{P}^{c-1},\Omega_{\mathbb{P}^{c-1}}^{i}) \\ &\cong H^0(U,\Omega_U^{p-i}) \end{aligned}$$

for any $0\leq i\leq c-1$. Note that h^i is the generator of $H^i(\mathbb{P}^{c-1},\Omega^i_{\mathbb{P}^{c-1}})$. Consequently, the isomorphism (3.5) follows directly from the local description of the higher direct images. □

As a by-product, the following Leray spectral sequence degenerates under the blow-up morphisms.

Corollary 3.6. *Consider the Leray spectral sequence for Ω_X^p associated to the blow-up morphism $f: X\longrightarrow Y$. Then the Leray spectral sequence*

$$E_2^{ij}:=H^i(Y,R^jf_*\Omega_X^p)\Longrightarrow H^{i+j}(X,\Omega_X^p)$$

degenerates at E_2.

Proof. Note that the Leray spectral sequence degenerates at E_2 if and only if the following equation holds

$$\sum_{j=0}^{q}\dim H^{q-j}(Y,R^jf_*\Omega_X^p)=\dim H^q(X,\Omega_X^p).$$

for any $q\geq 0$. By Proposition 2.3 (i), one has $H^q(Y,f_*\Omega_X^p)\cong H^q(Y,\Omega_Y^p)$. For $1\leq i\leq c-1$, by Proposition 2.3 (iii) and the isomorphism (3.6), we have

$$H^{q-j}(Y,R^if_*\Omega_X^p)\cong H^{q-j}(Y,\iota_*R^ig_*\Omega_E^p)\cong H^{q-j}(Y,\iota_*\Omega_Z^{p-i})\cong H^{q-j}(Z,\Omega_Z^{p-i}).$$

Hence, the Leray spectral sequence degenerates at E_2 if and only if

$$\dim\left(H^q(Y,\Omega_Y^p)\oplus\bigoplus_{i=1}^{c-1}H^{q-i}(Z,\Omega_Z^{p-i})\right)=\dim H^q(X,\Omega_X^p).$$

In fact, this is a consequence of the blow-up formula of Dolbeault cohomology in [8]. □

Remark 3.7. In [2, Question 10], we proposed a question on the blow-up formula (e.g., Proposition 3.2) of the hypercohomology of truncated (twisted) holomorphic de Rham complex, so the current work may be viewed as a sequel to [2]. Recently, in [7] Meng confirmed our question by a different approach; see [7] for a different proof of Proposition 3.2. Note that the proof of Proposition 3.2 we give here is our original idea proposed in [2, Remark 11]. Moreover, by using the Dolbeault blow-up formula in [8], Wu [15, Proposition 13] obtained a “partial blow-up formula" for the integral Bott-Chern cohomology. It is worthy of noticing that combining Lemma 3.4 with [15, Proposition 13], one can also obtain a proof of Theorem 1.2.

Remark 3.8. For $p\geq 0$, one can consider blow-up formula of the hypercohomology of sheaf complex $\mathbf{DR}_Y^{<p}$ on a complex manifold Y:

$$0\longrightarrow\mathbb{C}_Y\longrightarrow\mathcal{O}_Y\xrightarrow{\partial}\Omega_Y^1\xrightarrow{\partial}\cdots\xrightarrow{\partial}\Omega_Y^{p-1}\longrightarrow 0\ ,$$

and $\mathbf{DR}_Y^{<0}:=\mathbb{C}_Y$. Based on Proposition 3.2, the blow-up formula follows from the fact that $\mathbf{DR}_Y^{<p}$ is quasi-isomorphic to $\Omega_Y^{[p\bullet n]}[-p]$.

We conclude this section by giving some interesting questions. Analogous to Proposition 3.1, one may consider the smooth Deligne cohomology of complex manifolds. Let Y be a complex manifold. Recall that the smooth Deligne complex $\mathbb{R}(p)_D^\infty$ is the sheaf complex

$$0 \longrightarrow (2\pi i)^p\,\mathbb{R} \longrightarrow \mathcal{A}_Y^0 \longrightarrow \mathcal{A}_Y^1 \longrightarrow \cdots \longrightarrow \mathcal{A}_Y^{p-1} \longrightarrow 0\ ,$$

where $\mathcal{A}_Y^s$ is the sheaf of complex differential s-forms on Y. The q-th hypercohomology $\mathbb{H}^q(Y, \mathbb{R}(p)_D^\infty)$ of the smooth Deligne complex $\mathbb{R}(p)_D^\infty$ is called the *q-th smooth Deligne cohomology of* Y, and denote by $H_D^q(Y, \mathbb{R}(p)^\infty)$ (cf. [4, Definition 1.5.1]). Moreover, note that the blow-up formulae have been obtained for Bott-Chern cohomology and integral Bott-Chern cohomology. Naturally, one may also consider the real Bott-Chern cohomology. The real Bott-Chern complex $\mathfrak{R}_Y^{p,q}$ is the sheaf complex

$$0 \to (2\pi i)^p\,\mathbb{R} \overset{(+,-)}{\to} \mathcal{O}_Y \oplus \overline{\mathcal{O}}_Y \to \Omega_Y^1 \oplus \overline{\Omega}_Y^1 \to \cdots \to \Omega_Y^{p-1} \oplus \overline{\Omega}_Y^{p-1} \to \overline{\Omega}_Y^p \to \cdots \to \overline{\Omega}_Y^{q-1} \to 0\ .$$

Then the *(p,q)-th real Bott-Chern cohomology* of Y, denote by $H_{BC}^{p,q}(Y, \mathfrak{R}_Y^{p,q}) := \mathbb{H}^{q+p}(Y, \mathfrak{R}_Y^{p,q})$, is given by the q-th hypercohomology of the real Bott-Chern complex $\mathfrak{R}_Y^{p,q}$. Finally, we propose the following interesting question:

Question 3.9. Can one obtain the blow-up formulae of smooth Deligne cohomology and real Bott-Chern cohomology for compact complex manifolds?

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References

[1] Angella, D., Suwa, T., Tardini, N., Tomassini, A.: Note on Dolbeault cohomology and Hodge structures up to bimeromorphisms. Complex Manifolds **7**, 194-214 (2020).
[2] Chen, Y., Yang, S.: On the blow-up formula of twisted de Rham cohomology. Ann. Glob. Anal. Geom. **56**, 277-290 (2019).
[3] Barbieri-Viale, L.: $\mathscr{H}$-cohomologies versus algebraic cycles. Math. Nachr. **184**, 5-57 (1997).
[4] Brylinski, J.-L.: Loop spaces, characteristic classes and geometric quantizations. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, (2008).
[5] Demailly, J.-P.: Complex analytic and differential geometry. J.-P. Demailly's CADG e-book.
[6] Meng, L.: Leray-Hirsch theorem and blow-up formula for Dolbeault cohomology. Ann. Mat. Pura Appl. **199**, 1997-2014 (2020).
[7] Meng, L.: Hypercohomologies of truncated twisted holomorphic de Rham complexes. Ann. Glob. Anal. Geom. **57**, 519-535 (2020).
[8] Rao, S., Yang, S., Yang, X.: Dolbeault cohomologies of blowing up complex manifolds. J. Math. Pures Appl. **130**, 68-92 (2019).
[9] Rao, S., Yang, S., Yang, X.: Dolbeault cohomologies of blowing up complex manifolds II: bundle-valued case. J. Math. Pures Appl. **133**, 1-38 (2020).
[10] Rao, S., Yang, S., Yang, X., Yu, X.: Hodge cohomology on blow-ups along subvarieties. arXiv:1907.13281.
[11] Schweitzer, M.: Autour de la cohomologie de Bott-Chern. arXiv:0709.3528.
[12] Stelzig, J.: The double complex of a blow-up. Int. Math. Res. Not. 14, 10731-10744 (2021).

[13] Tardini, N.: Relative Čech-Dolbeault homology and applications. Ann. Mat. Pura Appl. **199**, 985-995 (2020).

[14] Voisin, C.: Hodge theory and complex algebraic geometry, I. Cambridge Studies in Advanced Mathematics, 76. Cambridge University Press, Cambridge (2002).

[15] Wu, X.: Intersection theory and Chern classes in Bott-Chern cohomology. arXiv:2011.13759

[16] Yang, S., Yang, X.: Bott-Chern blow-up formulae and the bimeromorphic invariance of the $\partial\overline{\partial}$-lemma for threefolds. Trans. Amer. Math. Soc. **373**, 8885-8909 (2020).

School of Science, Chongqing University of Technology, 400054 Chongqing, People's Republic of China

Email address: `youmingchen@cqut.edu.cn`

Center for Applied Mathematics, Tianjin University, Tianjin 300072, People's Republic of China

Email address: `syangmath@tju.edu.cn`