Poisson cohomology, Koszul duality, and Batalin-Vilkovisky algebras

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Abstract

We study the noncommutative Poincaré duality between the Poisson homology and cohomology of unimodular Poisson algebras, and show that Kontsevich's deformation quantization as well as Koszul duality preserve the corresponding Poincaré duality. As a corollary, the Batalin-Vilkovisky algebra structures that naturally arise in these cases are all isomorphic.
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1 Introduction

In this paper we study the noncommutative Poincaré duality between the Poisson homology and cohomology of unimodular Poisson algebras, and show that Kontsevich's deformation quantization as well as Koszul duality preserve the corresponding Poincaré duality.

Let $A = \mathbb{R}[x_1, \cdots, x_n]$ be the real polynomial algebra in n variables. A Poisson bivector on A, say π, is called *quadratic* if it is in the form

$$\pi = \sum_{i_i, i_2, j_1, j_2} c_{i_1 i_2}^{j_1 j_2} x_{i_1} x_{i_2} \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}, \quad c_{i_1 i_2}^{j_1 j_2} \in \mathbb{R}. \tag{1.1}$$

Several years ago, Shoikhet [28] observed that if π is quadratic, then the Koszul dual algebra $A^!$ of A, namely, the graded symmetric algebra $\mathbf{\Lambda}(\xi_1, \cdots, \xi_n)$ generated by n elements of degree -1, has a Poisson structure (let us call it the *Koszul dual* of π), given by

$$\pi^! = \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2}^{j_1 j_2} \xi_{j_1} \xi_{j_2} \frac{\partial}{\partial \xi_{i_1}} \wedge \frac{\partial}{\partial \xi_{i_2}}, \tag{1.2}$$

and proved that Kontsevich's deformation quantization preserves this type of Koszul duality. Shoikhet's result motivates us to study some other properties of a Poisson algebra under Koszul duality.

First, the following theorem is clear from Shoikhet's article, once we explicitly write down the corresponding complexes.

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Theorem 1.1. *Let $A = \mathbb{R}[x_1, \cdots, x_n]$ be a quadratic Poisson algebra. Denote by $A^!$ the Koszul dual Poisson algebra of A. Then there are isomorphisms*

$$\mathrm{HP}_\bullet(A) \cong \mathrm{HP}^{-\bullet}(A^!; A^¡) \quad \textit{and} \quad \mathrm{HP}^\bullet(A) \cong \mathrm{HP}^\bullet(A^!), \tag{1.3}$$

where $A^¡ := \mathrm{Hom}_{\mathbb{R}}(A^!, \mathbb{R})$ is the linear dual of $A^!$.

In the above theorem, $\mathrm{HP}_\bullet(-)$ is the Poisson homology, $\mathrm{HP}^\bullet(-)$ is the Poisson cohomology, and $\mathrm{HP}^\bullet(A^!; A^¡)$ is the Poisson cohomology of $A^!$ with values in its dual space.

Historically, the Poisson homology and cohomology were introduced by Koszul [19] and Lichnerowicz [23] respectively. In 1997 Weinstein [35] introduced the notion of *unimodular* Poisson manifolds, and two years later Xu [38] proved that in this case, there is a Poincaré duality between the Poisson cohomology and homology of M. A purely algebraic version of Weinstein's notion was later formulated by Dolgushev in [8] (see also [21, 25]), and in this case we also have

$$\mathrm{HP}^\bullet(A) \cong \mathrm{HP}_{n-\bullet}(A), \tag{1.4}$$

for some n depending on A.

For a *finite dimensional* algebra such as $A^!$ above, Zhu, Van Oystaeyen and Zhang introduced in [39] the notion of *Frobenius Poisson algebras* (in the rest of the paper, we shall use the word *symmetric* instead of *Frobenius*, just to be consistent with other references), and proved that if they are *unimodular* in some sense (to be recalled below), then there also exists a version of Poincaré duality:

$$\mathrm{HP}^\bullet(A^!) \cong \mathrm{HP}^{\bullet-n}(A^!; A^¡). \tag{1.5}$$

Combining the above two versions of Poincaré duality (1.4) and (1.5) as well as Theorem 1.1, we have the following:

Theorem 1.2. *Let $A = \mathbb{R}[x_1, \cdots, x_n]$ be a quadratic Poisson algebra. Then (A, π) is unimodular if and only if its Koszul dual $(A^!, \pi^!)$ is unimodular symmetric. In this case, we have the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{HP}^\bullet(A^!) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^!; A^¡). \end{array}$$

The main technique to prove the above theorem is the so-called "differential calculus", a notion introduced by Tamarkin and Tsygan in [29]. Later, Lambre [20] used the terminology "differential calculus with duality" to study the "noncommutative Poincaré duality" in these cases.

In the above-mentioned two references [38, 39], the authors also proved that the Poisson cohomology of a unimodular Poisson algebra (in both cases) has a Batalin-Vilkovisky algebra structure. The Batalin-Vilkovisky structure is a very important algebraic structure that has appeared in, for example, mathematical physics, Calabi-Yau geometry and string topology. For unimodular quadratic Poisson algebras, we have the following:

Theorem 1.3. *Suppose $A = \mathbb{R}[x_1, \cdots, x_n]$ is a unimodular quadratic Poisson algebra. Denote by $A^!$ its Koszul dual. Then*

$$\mathrm{HP}^\bullet(A) \cong \mathrm{HP}^\bullet(A^!)$$

is an isomorphism of Batalin-Vilkovisky algebras.

The above three theorems have some analogy to the case of Calabi-Yau algebras, which were introduced by Ginzburg [14] in 2006. Suppose a Calabi-Yau algebra, say A, is Koszul, then its Koszul dual is a symmetric algebra. In [14, §5.4] Ginzburg stated a conjecture, which he attributed to R. Rouquier, saying that for a Koszul Calabi-Yau algebra, say A, its Hochschild cohomology is isomorphic to the Hochschild cohomology of its Koszul dual $A^!$

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(A^!) \tag{1.6}$$

as Batalin-Vilkovisky algebras. This conjecture is recently proved by two authors of the current paper together with G. Zhou in [5]. In fact, Theorem 1.3 may be viewed as a generalization of Rouquier's conjecture in Poisson geometry, which has been a folklore for several years.

More than just being an analogy, in [8, Theorem 3], Dolgushev proved that for the coordinate ring A of an affine Calabi-Yau Poisson variety, its deformation quantization in the sense of Kontsevich, say $A_\hbar$, is Calabi-Yau if and only if A is unimodular. Similarly Felder and Shoikhet ([11]) and later Willwacher-Calaque ([37]) proved that, for a symmetric Poisson algebra, its deformation quantization is again symmetric if and only if it is unimodular. Based on these results, Dolgushev asked two questions in [8, §7] (see also [9]). The first question is whether there exists a relationship between the Poincaré duality of the Poisson (co)homology of A and the Poincaré duality of the Hochschild (co)homology of $A_\hbar$. The following theorem answers this question in the case of polynomials:

Theorem 1.4. (1) *Suppose $A = \mathbb{R}[x_1, \cdots, x_n]$ is a unimodular Poisson algebra. Let $A_\hbar$ be its deformation quantization. Then the following diagram*

$$\begin{array}{ccc}
\mathrm{HP}^\bullet(A[\![\hbar]\!]) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A[\![\hbar]\!]) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\mathrm{HH}^\bullet(A_\hbar) & \xrightarrow{\cong} & \mathrm{HH}_{n-\bullet}(A_\hbar)
\end{array}$$

commutes.

(2) *Similarly, suppose $A^! = \mathbf{\Lambda}(\xi_1, \cdots, \xi_n)$ is a unimodular symmetric Poisson algebra, and let $A^!_\hbar$ be its deformation quantization. Then the following diagram*

$$\begin{array}{ccc}
\mathrm{HP}^\bullet(A^![\![\hbar]\!]) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^![\![\hbar]\!]; A^{¡}[\![\hbar]\!]) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\mathrm{HH}^\bullet(A^!_\hbar) & \xrightarrow{\cong} & \mathrm{HH}^{\bullet-n}(A^!_\hbar; A^{¡}_\hbar)
\end{array}$$

commutes.

In other words, the two versions of Poincaré duality, one between the Poisson cohomology and homology, and the other between the Hochschild cohomology and homology, are preserved under Kontsevich's deformation quantization.

The second question is whether there is any relationship between the roles that the unimodularity plays in the above two types of deformation quantizations. The following theorem partially answers this question, although both cases that Dolgushev and Felder-Shoikhet considered are more general (i.e., not necessarily Koszul):

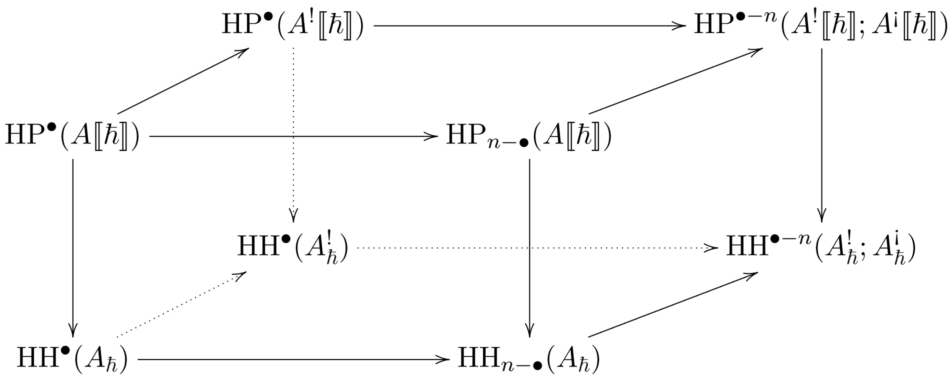
Theorem 1.5. *Suppose $A = \mathbb{R}[x_1, \cdots, x_n]$ is a quadratic Poisson algebra. Denote by $A^!$ the Koszul dual algebra of A, and by $A_\hbar$ and $A^!_\hbar$ the Kontsevich deformation quantization of A and $A^!$ respectively. If A is unimodular (and by Theorem 1.2 $A^!$ is unimodular symmetric), then $A_\hbar$ is Calabi-Yau and $A^!_\hbar$ is symmetric, and the following diagram*

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A[\![\hbar]\!]) & \xrightarrow{\cong} & \mathrm{HP}^\bullet(A^![\![\hbar]\!]) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathrm{HH}^\bullet(A_\hbar) & \xrightarrow{\cong} & \mathrm{HH}^\bullet(A^!_\hbar). \end{array} \tag{1.7}$$

is commutative as Batalin-Vilkovisky algebra isomorphisms, where $A[\![\hbar]\!]$ and $A^![\![\hbar]\!]$ are equipped with the Poisson bivectors $\hbar\pi$ and $\hbar\pi^!$ respectively.

In other words, the theorem says that, the unimodularity that appears in the deformation quantization of Poisson Calabi-Yau algebras and Poisson symmetric algebras are related by Koszul duality. Note that in the theorem, $A_\hbar$ and $A^!_\hbar$ are Koszul dual to each other by Shoikhet [28].

Thus as a corollary, one obtains that *if $A = \mathbb{R}[x_1, \cdots, x_n]$ is a unimodular quadratic Poisson algebra, then the homology and cohomology groups (Poisson and Hochschild) in Theorems 1.4 and 1.5 are all isomorphic.* That is, we have the following commutative diagram of isomorphisms:



where the horizontal arrows are the Poincaré duality, the vertical arrows are given by deformation quantization, and the slanted arrows are given by Koszul duality.

The rest of the paper is devoted to the proof of the above theorems. It is organized as follows: in §2 we collect several facts on Koszul algebras, and their application to quadratic Poisson polynomials; in §3 we first recall the definition of Poisson homology and cohomology, and then prove Theorem 1.1; in §4 we study unimodular quadratic Poisson algebras and their Koszul dual, and prove Theorem 1.2; in §5 we prove Theorem 1.3 by means of the so-called "differential calculus with duality"; in §6 we discuss Calabi-Yau algebras, their Koszul duality and the Batalin-Vilkovisky algebras associated to them; and at last, in §7 we discuss the deformation quantization of Poisson algebras and prove Theorems 1.4 and 1.5.

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Convention. Throughout the paper, k is a field of characteristic zero, which we may assume to be $\mathbb{R}$ as in §1. All tensors and morphisms are graded over k unless otherwise specified. For a chain complex, its homology is denoted by $\mathrm{H}_\bullet(-)$, and its cohomology is $\mathrm{H}^\bullet(-) := \mathrm{H}_{-\bullet}(-)$.

2 Preliminaries on Koszul algebras

In this section, we collect some necessary facts about Koszul algebras. The interested reader may refer to Loday-Vallette [24, Chapter 3] for some more details.

Let V be a finite-dimensional vector space over k. Denote by TV the free (tensor) algebra generated by V over k. Suppose R is a subspace of $V\otimes V$, and let (R) be the two-sided ideal generated by R in TV, then the quotient algebra $A := TV/(R)$ is called a *quadratic algebra*.

Consider the subspace

$$U=\bigoplus_{n=0}^{\infty}U_n:=\bigoplus_{n=0}^{\infty}\bigcap_{i+j+2=n}V^{\otimes i}\otimes R\otimes V^{\otimes j}$$

of TV, then U is a coalgebra whose coproduct is induced from the de-concatenation of the tensor products. The *Koszul dual coalgebra* of A, denoted by $A^{¡}$, is

$$A^{¡}=\bigoplus_{n=0}^{\infty}\Sigma^{\otimes n}(U_n),$$

where Σ is the degree shifting-up (suspension) functor. $A^{¡}$ has a graded coalgebra structure induced from that of U with

$$(A^{¡})_0=k,\quad (A^{¡})_1=\Sigma V,\quad (A^{¡})_2=(\Sigma\otimes\Sigma)(R),\quad \cdots\cdots$$

The *Koszul dual algebra* of A, denoted by $A^!$, is just the linear dual space of $A^{¡}$, which is then a graded algebra. More precisely, Let $V^*=\mathrm{Hom}(V,k)$ be the linear dual space of V, and let $R^\perp$ denote the space of annihilators of R in $V^*\otimes V^*$. Shift the grading of V^* down by one, denoted by $\Sigma^{-1}V^*$, then

$$A^!=T(\Sigma^{-1}V^*)/(\Sigma^{-1}\otimes\Sigma^{-1}\circ R^\perp).^1$$

Choose a set of basis $\{e_i\}$ for V, and let $\{e_i^*\}$ be their duals in V^*. There is a chain complex associated to A, called the *Koszul complex*:

$$\cdots\xrightarrow{\delta}A\otimes A^{¡}_{i+1}\xrightarrow{\delta}A\otimes A^{¡}_{i}\xrightarrow{\delta}\cdots\longrightarrow A\otimes A^{¡}_0\xrightarrow{\delta}k, \tag{2.1}$$

where for any $r\otimes f\in A\otimes A^{¡}$, $\delta(r\otimes f)=\sum_i e_ir\otimes\Sigma^{-1}e_i^*f$.

[1] In the literature such as [24], $A^!$ is defined to be $T(V^*)/R^\perp$, or equivalently, $(A^!)_i\cong\Sigma^i\mathrm{Hom}((A^{¡})_i,k)$ but not $\mathrm{Hom}((A^{¡})_i,k)$. This will cause some issues in our later calculations, so in this paper, we take $A^!$ as given above, or equivalently $A^!=\mathrm{Hom}(A^{¡},k)$.

Definition 2.1 (Koszul algebra)**.** A quadratic algebra $A = TV/(R)$ is called *Koszul* if the Koszul chain complex (2.1) is acyclic.

Example 2.2 (Polynomials)**.** Let $A = k[x_1, x_2, \cdots, x_n]$ be the space of polynomials (the symmetric tensor algebra) with n generators. Then A is a Koszul algebra, and its Koszul dual algebra $A^!$ is the graded symmetric algebra $\mathbf{\Lambda}(\xi_1, \xi_2, \cdots, \xi_n)$, with grading $|\xi_i| = -1$.

Lemma 2.3 (Shoikhet [28])**.** *Let $A = k[x_1, \cdots, x_n]$ with a bivector π in the form* (1.1)*. Then (A, π) is quadratic Poisson if and only if $(A^!, \pi^!)$ is quadratic Poisson, where $\pi^!$ is given by* (1.2)*.*

So far, we have assumed that V is a k-linear space. In §7, we will study the deformed algebras, which are algebras over $k[\![\hbar]\!]$. In [28], Shoikhet proved that the definitions and results in above subsections remain to hold for algebras over a discrete evaluation ring, such as $k[\![\hbar]\!]$. For example, $k[x_1, \cdots, x_n][\![\hbar]\!]$ is Koszul dual to $\mathbf{\Lambda}(\xi_1, \cdots, \xi_n)[\![\hbar]\!]$ as graded algebras over $k[\![\hbar]\!]$ (see [28, Theorem 0.3]).

3 Poisson homology and cohomology

The notions of Poisson homology and cohomology were introduced by Koszul [19] and Lichnerowicz [23] respectively. Later Huebschmann [15] studied both of them from purely algebraic perspective.

For an commutative algebra A, in the following we denote by $\Omega^p(A)$ the set of p-th Kähler differential forms of A, and by $\mathfrak{X}_A^{-p}(M)$ (or simply $\mathfrak{X}^{-p}(M)$ if A is clear from the context) the space of skew-symmetric multilinear maps $A^{\otimes p} \to M$ that are derivations in each argument. Note that from the universal property of Kähler differentials, there is an identity of left A-modules

$$\mathfrak{X}_A^{-p}(M) = \mathrm{Hom}_A(\Omega^p(A), M). \tag{3.1}$$

Definition 3.1 (Koszul [19])**.** Suppose (A, π) is a Poisson algebra. Then the *Poisson chain complex* of A, denoted by $\mathrm{CP}_\bullet(A)$, is

$$\cdots \longrightarrow \Omega^{p+1}(A) \xrightarrow{\partial} \Omega^p(A) \xrightarrow{\partial} \Omega^{p-1}(A) \xrightarrow{\partial} \cdots \longrightarrow \Omega^0(A) = A, \tag{3.2}$$

where ∂ is given by

$$\begin{aligned} \partial(f_0 df_1 \wedge \cdots \wedge df_p) &= \sum_{i=1}^{p} (-1)^{i-1} \{f_0, f_i\} df_1 \wedge \cdots \widehat{df_i} \cdots \wedge df_p \\ &+ \sum_{1 \le i < j \le p} (-1)^{j-i} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \cdots \widehat{df_i} \cdots \widehat{df_j} \cdots \wedge df_p. \end{aligned}$$

The associated homology is called the *Poisson homology* of A, and is denoted by $\mathrm{HP}_\bullet(A)$.

Definition 3.2 (Lichnerowicz [23])**.** Suppose (A, π) is a Poisson algebra and M is a left Poisson A-module. The *Poisson cochain complex* of A with values in M, denoted by $\mathrm{CP}^\bullet(A; M)$, is the cochain complex

$$M = \mathfrak{X}_A^0(M) \xrightarrow{\delta} \cdots \longrightarrow \mathfrak{X}_A^{-p+1}(M) \xrightarrow{\delta} \mathfrak{X}_A^{-p}(M) \xrightarrow{\delta} \mathfrak{X}_A^{-p-1}(M) \xrightarrow{\delta} \cdots$$

where δ is given by

$$\begin{aligned}\delta(P)(f_0, f_1, \cdots, f_p) &:= \sum_{0\le i\le p} (-1)^i \{f_i, P(f_0, \cdots, \widehat{f_i}, \cdots, f_p)\} \\ &+ \sum_{0\le i<j\le p} (-1)^{i+j} P(\{f_i, f_j\}, f_1, \cdots, \widehat{f_i}, \cdots, \widehat{f_j}, \cdots, f_p).\end{aligned}$$

The associated cohomology is called the *Poisson cohomology* of A with values in M, and is denoted by $\mathrm{HP}^\bullet(A; M)$. In particular, if $M = A$, then the cohomology is just called the *Poisson cohomology* of A, and is simply denoted by $\mathrm{HP}^\bullet(A)$.

Note that in the above definition, the Poisson cochain complex, viewed as a chain complex, is negatively graded, and the coboundary δ has degree -1. However, by our convention, the Poisson cohomology are positively graded.

Remark 3.3 (The graded case)**.** The Poisson homology and cohomology can be defined for graded Poisson algebras as well. In this case,

$$\Omega^p(A) = \bigoplus_{n\in\mathbb{Z}} \left\{ f_0 df_1 \wedge \cdots \wedge df_n \Big| f_i \in A, |f_0| + |f_1| + \cdots + |f_n| + n = p \right\}$$

and $\mathfrak{X}_A^{-p}(M)$ is again given by $\mathrm{Hom}_A(\Omega^p(A), M)$. The boundary maps are completely analogous to those of Poisson chain and cochain complexes (with Koszul's sign convention counted).

Proof of Theorem 1.1. (1) We first show the first isomorphism in (1.3). Since $A = k[x_1, \cdots, x_n]$, we have an explicit expression for $\Omega^\bullet(A)$, which is

$$\Omega^\bullet(A) = \mathbf{\Lambda}(x_1, \cdots, x_n, dx_1, \cdots, dx_n), \tag{3.3}$$

where $\mathbf{\Lambda}$ means the graded symmetric tensor product, and $|x_i| = 0$ and $|dx_i| = 1$, for $i = 1, \cdots, n$. Similarly,

$$\Omega^\bullet(A^!) = \mathbf{\Lambda}(\xi_1, \cdots, \xi_n, d\xi_1, \cdots, d\xi_n),$$

where $|\xi_i| = -1$ and $|d\xi_i| = 0$ for $i = 1, \cdots, n$, and therefore

$$\begin{aligned}\mathfrak{X}^\bullet_{A^!}(A^¡) &= \mathrm{Hom}_{A^!}(\Omega^\bullet(A^!), A^¡) \\ &= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1,\cdots,\xi_n)}(\mathbf{\Lambda}(\xi_1, \cdots, \xi_n, d\xi_1, \cdots, d\xi_n), \mathrm{Hom}(\mathbf{\Lambda}(\xi_1, \cdots, \xi_n), k)) \\ &= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1,\cdots,\xi_n)}(\mathbf{\Lambda}(\xi_1, \cdots, \xi_n) \otimes \mathbf{\Lambda}(d\xi_1, \cdots, d\xi_n), \mathrm{Hom}(\mathbf{\Lambda}(\xi_1, \cdots, \xi_n), k)) \\ &= \mathrm{Hom}(\mathbf{\Lambda}(d\xi_1, \cdots, d\xi_n), \mathrm{Hom}(\mathbf{\Lambda}(\xi_1, \cdots, \xi_n), k)) \\ &= \mathrm{Hom}(\mathbf{\Lambda}(d\xi_1, \cdots, d\xi_n) \otimes \mathbf{\Lambda}(\xi_1, \cdots, \xi_n), k) \\ &= \mathrm{Hom}(\mathbf{\Lambda}(d\xi_1, \cdots, d\xi_n, \xi_1, \cdots, \xi_n), k) \\ &= \mathbf{\Lambda}\Big(\frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_n}, \xi_1^*, \cdots, \xi_n^*\Big).\end{aligned} \tag{3.4}$$

Thus from (3.3) and (3.4) there is a canonical grading preserving isomorphism of vector spaces:

$$\begin{array}{rccl} \Phi: & \Omega^\bullet(A) & \longrightarrow & \mathfrak{X}^\bullet_{A^!}(A^¡) \\ & x_i & \longmapsto & \frac{\partial}{\partial \xi_i} \\ & dx_i & \longmapsto & \xi_i^*, \qquad i = 1, \cdots, n. \end{array} \tag{3.5}$$

It is a direct check that Φ is a chain map, and thus we obtain an isomorphism of Poisson complexes

$$\Phi : \mathrm{CP}_\bullet(A) \cong \mathrm{CP}^{-\bullet}(A^!; A^¡), \tag{3.6}$$

which then induces an isomorphism on the homology.

(2) We now show the second isomorphism in (1.3). Similarly to the above argument, we have

$$\begin{aligned}
\mathrm{CP}^\bullet(A) &= \mathrm{Hom}_A(\Omega^\bullet(A), A)\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(x_1,\cdots,x_n)}(\mathbf{\Lambda}(x_1,\cdots,x_n,dx_1,\cdots,dx_n), \mathbf{\Lambda}(x_1,\cdots,x_n))\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(x_1,\cdots,x_n)}(\mathbf{\Lambda}(x_1,\cdots,x_n)\otimes\mathbf{\Lambda}(dx_1,\cdots,dx_n), \mathbf{\Lambda}(x_1,\cdots,x_n))\\
&= \mathrm{Hom}(\mathbf{\Lambda}(dx_1,\cdots,dx_n), \mathbf{\Lambda}(x_1,\cdots,x_n))\\
&= \mathbf{\Lambda}\Big(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\Big)\otimes\mathbf{\Lambda}(x_1,\cdots,x_n)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\mathrm{CP}^\bullet(A^!) &= \mathrm{Hom}_{A^!}(\Omega^\bullet(A^!), A^!)\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1,\cdots,\xi_n)}(\mathbf{\Lambda}(\xi_1,\cdots,\xi_n,d\xi_1,\cdots,d\xi_n), \mathbf{\Lambda}(\xi_1,\cdots,\xi_n))\\
&= \mathrm{Hom}_{\mathbf{\Lambda}(\xi_1,\cdots,\xi_n)}(\mathbf{\Lambda}(\xi_1,\cdots,\xi_n)\otimes\mathbf{\Lambda}(d\xi_1,\cdots,d\xi_n), \mathbf{\Lambda}(\xi_1,\cdots,\xi_n))\\
&= \mathrm{Hom}(\mathbf{\Lambda}(d\xi_1,\cdots,d\xi_n), \mathbf{\Lambda}(\xi_1,\cdots,\xi_n))\\
&= \mathbf{\Lambda}\Big(\frac{\partial}{\partial \xi_1},\cdots,\frac{\partial}{\partial \xi_n}\Big)\otimes\mathbf{\Lambda}(\xi_1,\cdots,\xi_n).
\end{aligned} \tag{3.8}$$

Under the identity

$$x_i \mapsto \frac{\partial}{\partial \xi_i}, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i \tag{3.9}$$

we again obtain an isomorphism of chain complexes

$$\Psi : \mathrm{CP}^\bullet(A) \cong \mathrm{CP}^\bullet(A^!).$$

This completes the proof. □

4 Unimodular Poisson algebras and Koszul duality

In this section, we study *unimodular* Poisson algebras. We are particularly interested in the algebraic structures on their Poisson cohomology and homology groups, which are summarized by *differential calculus*, a notion introduced by Tamarkin and Tsygan in [29].

Definition 4.1 (Differential calculus; Tamarkin-Tsygan [29]). Let $\mathrm{H}^\bullet$ and $\mathrm{H}_\bullet$ be graded vector spaces. A *differential calculus* is the sextuple

$$(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, [-,-], d)$$

satisfying the following conditions:

(1) $(\mathrm{H}^\bullet, \cup, [-,-])$ is a Gerstenhaber algebra; that is, $(\mathrm{H}^\bullet, \cup)$ is a graded commutative algebra, $(\mathrm{H}^\bullet, [-,-])$ is a degree 1 or -1 graded Lie algebra, and the product and Lie bracket are compatible in the following sense

$$[P\cup Q, R] = P\cup[Q,R] + (-1)^{pq} Q\cup[P,R],$$

for homogeneous $P, Q, R \in V$ of degree p, q, r, respectively;

(2) $\mathrm{H}_\bullet$ is a graded (left) module over $(\mathrm{H}^\bullet, \cup)$ via the map

$$\iota : \mathrm{H}^n \otimes \mathrm{H}_m \to \mathrm{H}_{m-n},\ f \otimes \alpha \mapsto \iota_f \alpha,$$

for any $f \in \mathrm{H}^n$ and $\alpha \in \mathrm{H}_m$;

(3) There is a map $d : \mathrm{H}_\bullet \to \mathrm{H}_{\bullet+1}$ satisfying $d^2 = 0$, and

$$(-1)^{|f|+1}\iota_{[f,g]} = [L_f, \iota_g] := L_f \iota_g - (-1)^{|g|(|f|+1)}\iota_g L_f,$$

where $L_f := [d, \iota_f] = d\iota_f - (-1)^{|f|}\iota_f d$.

In the following, if $\cup$, ι, $[-,-]$ and d are clear from the context, we will simply write a differential calculus by $(\mathrm{H}^\bullet, \mathrm{H}_\bullet)$ for short.

4.1 Differential calculus on Poisson (co)homology

Suppose A is a commutative algebra. We have the following operations on $\mathfrak{X}^\bullet(A)$ and $\Omega^\bullet(A)$:

(1) Wedge (cup) product: suppose $P \in \mathfrak{X}^{-p}(A)$ and $Q \in \mathfrak{X}^{-q}(A)$, then the *wedge product* of P and Q, denoted by $P \cup Q$, is a polyvector in $\mathfrak{X}^{-p-q}(A)$ defined by

$$(P \cup Q)(f_1, f_2, \cdots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma) P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}) \cdot Q(f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q)}),$$

where σ runs over all (p,q)-shuffles of $(1, 2, \cdots, p+q)$.

(2) Schouten bracket: suppose $P \in \mathfrak{X}^{-p}(A)$ and $Q \in \mathfrak{X}^{-q}(A)$, then their *Schouten bracket*, denoted by $[P, Q]$, is an element in $\mathfrak{X}^{-p-q+1}(A)$ given by

$$\begin{aligned}[P,Q](f_1, f_2, \cdots, f_{p+q-1}) := &\sum_{\sigma \in S_{q,p-1}} \mathrm{sgn}(\sigma) P\big(Q(f_{\sigma(1)}, \cdots, f_{\sigma(q)}), f_{\sigma(q+1)}, \cdots, f_{\sigma(q+p-1)}\big)\\ &- (-1)^{(p-1)(q-1)} \sum_{\sigma \in S_{p,q-1}} \mathrm{sgn}(\sigma) Q\big(P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}), f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q-1)}\big).\end{aligned}$$

(3) Contraction (inner product): suppose $P \in \mathfrak{X}^{-p}(A)$ and $\omega = df_1 \wedge \cdots \wedge df_n \in \Omega^n(A)$, then the *contraction* of P with ω, denoted by $\iota_P(\omega)$, is an A-linear map with values in $\Omega^{n-p}(A)$ given by

$$\iota_P(\omega) = \begin{cases} \displaystyle\sum_{\sigma \in S_{p,n-p}} \mathrm{sgn}(\sigma) P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}) df_{\sigma(p+1)} \wedge \cdots \wedge df_{\sigma(n)}, & \text{if } n \geq p,\\ 0, & \text{otherwise.}\end{cases}$$

(4) Lie derivative: the *Lie derivative* is given by the Cartan formula, namely for $P \in \mathfrak{X}^{-p}(A)$ and $\omega \in \Omega^n(A)$, the Lie derivative of ω with respect to P is given by

$$L_P\omega := [\iota_P, d] = \iota_P(d\omega) - (-1)^p d(\iota_P \omega),$$

where d is the de Rham differential.

Theorem 4.2. *Suppose A is a Poisson algebra. Then*

$$\big(\mathrm{HP}^\bullet(A), \mathrm{HP}_\bullet(A), \cup, \iota, [-,-], d\big),$$

where d is the de Rham differential, is a differential calculus.

Proof. We only have to show the operations listed above respect the Poisson boundary and coboundary. It is a direct check and can be found in [22, Chapter 3]. □

In the following, we will give another differential calculus structure for a Poisson algebra, which will be used later:

(1) For any $P \in \mathfrak{X}^{-p}(A)$ and $\phi \in \mathfrak{X}^{-q}(A^*)$, let $\iota_P^*(\phi) \in \mathfrak{X}^{-p-q}(A^*)$ be given by

$$(\iota_P^*\phi)(f_1, \cdots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma) P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}) \cdot \phi(f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q)}). \tag{4.1}$$

It is clear that ι^* is associative, i.e., $\iota_Q^* \circ \iota_P^* = \iota_{P\cup Q}^*$. Also, ι^* respects the Poisson coboundary maps, which is completely analogous to the proof of that $\cup$ commutes with the Poisson coboundary map (*cf.* [22, §4.3]).

(2) Observe that

$$\begin{aligned} \mathfrak{X}^\bullet(A^*) &= \mathrm{Hom}_A(\Omega^\bullet(A), A^*) \\ &= \mathrm{Hom}_A\big(\Omega^\bullet(A), \mathrm{Hom}(A, k)\big) \\ &= \mathrm{Hom}_A(\Omega^\bullet(A) \otimes A, k) \\ &= \mathrm{Hom}(\Omega^\bullet(A), k). \end{aligned}$$

By dualizing the de Rham differential d on $\Omega^\bullet(A)$, we obtain a differential d^* on $\mathrm{Hom}(\Omega^\bullet(A), k)$, i.e., on $\mathfrak{X}^\bullet(A^*)$. It is proved in [39, Theorem 4.10] that d^* commutes with the Poisson boundary.

(3) For any $P \in \mathfrak{X}^\bullet(A)$ and $\omega \in \mathfrak{X}^\bullet(A^*)$, let $L_P\omega := [\iota_P^*, d^*](\omega)$; it is a direct check that

$$[L_P, \iota_Q^*] = \iota_{[P,Q]}^*.$$

By (1)-(3) listed above, we obtain the following.

Theorem 4.3. *Suppose A is a Poisson algebra, and denote A^* be its dual space. Then*

$$\big(\mathrm{HP}^\bullet(A), \mathrm{HP}^\bullet(A; A^*), \cup, \iota^*, [-,-], d^*\big)$$

is a differential calculus.

4.2 Unimodular Poisson algebras

Suppose A is a commutative algebra, and $\eta \in \Omega^n(A)$. We say η is a volume form if $\mathfrak{X}^\bullet(A) \stackrel{\iota_{(-)}\eta}{\longrightarrow} \Omega^{n+\bullet}(A)$ is an isomorphism of vector spaces. Now suppose A is Poisson, then we have the following diagram

$$\begin{array}{ccc} \mathfrak{X}^\bullet(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^{n+\bullet}(A) \\ \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\partial} \\ \mathfrak{X}^{\bullet+1}(A) & \xrightarrow{\iota_{(-)}\eta} & \Omega^{n+\bullet+1}(A), \end{array} \tag{4.2}$$

which may not be commutative, i.e., η may not be a Poisson cycle. We say A is *unimodular* if there exists a volume form η such that (4.2) commutes. The following is now immediate.

Theorem 4.4 (Xu)**.** *Suppose A is a unimodular Poisson algebra with the volume form of degree n. Then there exists an isomorphism (the Poincaré duality)*

$$\mathrm{HP}^\bullet(A) \cong \mathrm{HP}_{n-\bullet}(A).$$

4.3 Unimodular symmetric Poisson algebras

Now, we go to unimodular symmetric Poisson algebras, a notion introduced by Zhu, Van Oystaeyen and Zhang in [39].

Suppose $A^!$ is a finite dimensional graded not-necessarily commutative algebra. $A^!$ is called *symmetric*[2] if it is equipped with a bilinear, non-degenerate symmetric pairing

$$\langle -, - \rangle : A^! \otimes A^! \to k$$

of degree n which is cyclically invariant, that is, $\langle a, b\cdot c\rangle = (-1)^{(|a|+|b|)|c|}\langle c, a\cdot b\rangle$, for all homogeneous $a, b, c \in A^!$. This is equivalent to saying that there is an $A^!$-bimodule isomorphism

$$\eta^! : (A^!)^\bullet \longrightarrow (A^{¡})_{n+\bullet}, \quad \text{for some } n \in \mathbb{N},$$

where $A^{¡} = (A^!)^*$. In this case, we may view $\eta^!$ as an element in $\mathrm{Hom}_{A^!}(A^!, A^{¡}) \subset \mathfrak{X}^\bullet_{A^!}(A^{¡})$. Now assume $A^!$ is Poisson, then we have a diagram

$$\begin{array}{ccc} \mathfrak{X}^\bullet_{A^!}(A^!) & \xrightarrow{\iota^*_{(-)}\eta^!} & \mathfrak{X}^{n+\bullet}_{A^!}(A^{¡}) \\ \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\ \mathfrak{X}^{\bullet+1}_{A^!}(A^!) & \xrightarrow{\iota^*_{(-)}\eta^!} & \mathfrak{X}^{n+\bullet+1}_{A^!}(A^{¡}). \end{array} \tag{4.3}$$

According to Zhu-Van Oystaeyen-Zhang [39], if there exists $\eta^! \in \mathfrak{X}^\bullet_{A^!}(A^{¡})$ such that $\iota^*_{(-)}\eta^!$ is an isomorphism, then $\eta^!$ is called a *volume form*, and if furthermore, the digram (4.3) commutes, then $A^!$ is called a *unimodular symmetric Poisson algebra* of degree n (in [39] the authors call it *unimodular Fronbenius Poisson*). From the definition, we immediately have:

Theorem 4.5 (Zhu-Van Oystaeyen-Zhang [39])**.** *Suppose $A^!$ is a unimodular symmetric Poisson algebra with the volume form of degree n. Then there exists an isomorphism*

$$\mathrm{HP}^\bullet(A^!) \cong \mathrm{HP}^{\bullet-n}(A^!; A^{¡}).$$

In this paper, since we are interested in $A = k[x_1, \cdots, x_n]$ or $A^! = \mathbf{\Lambda}(\xi_1, \cdots, \xi_n)$, we always assume the volume form is constant.

Proof of Theorem 1.2. First, we show that a quadratic Poisson algebra $(A = k[x_1, \cdots, x_n], \pi)$ is unimodular if and only if $(A^!, \pi^!)$ is unimodular symmetric. In fact, recall that for $A = k[x_1, \cdots, x_n]$,

$$\mathfrak{X}^\bullet_A(A) = \mathbf{\Lambda}\Big(x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\Big), \quad \Omega^\bullet(A) = \mathbf{\Lambda}(x_1, \cdots, x_n, dx_1, \cdots, dx_n),$$
$$\mathfrak{X}^\bullet_{A^!}(A^!) = \mathbf{\Lambda}\Big(\xi_1, \cdots, \xi_n, \frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_n}\Big), \quad \mathfrak{X}^\bullet_{A^!}(A^{¡}) = \mathbf{\Lambda}\Big(\xi_1^*, \cdots, \xi_n^*, \frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_n}\Big).$$

Let

$$\eta = dx_1 dx_2 \cdots dx_n \quad \text{and} \quad \eta^! = \xi_1^*\xi_2^*\cdots\xi_n^*,$$

[2]In the literature some authors also use "symmetric Frobenius". For commutative algebras, there is no difference between being Frobenius and being symmetric Frobenius; in this paper we use "symmetric" just in order to be consistent with later discussions on associative algebras.

where $\eta^!$ is understood as contraction, namely,

$$\eta^!(\xi_{i_1}\cdots\xi_{i_p}) := \sum_{\sigma\in S_{p,n-p}} \langle \xi_{i_1}\cdots\xi_{i_p}, \xi^*_{\sigma(1)}\cdots\xi^*_{\sigma(p)}\rangle \cdot \xi^*_{\sigma(p+1)}\cdots\xi^*_{\sigma(n)},$$

then under the identification

$$x_i \mapsto \frac{\partial}{\partial\xi_i}, \quad dx_i \mapsto \xi_i^*, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i$$

the diagram

$$\begin{array}{ccc}
\mathfrak{X}^\bullet_A(A) = \mathbf{\Lambda}\left(x_1,\cdots,x_n,\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right) & \xrightarrow{\iota_{(-)}\eta} & \Omega^\bullet(A) = \mathbf{\Lambda}\left(x_1,\cdots,x_n,dx_1,\cdots,dx_n\right) \\
\downarrow\cong & & \downarrow\cong \\
\mathfrak{X}^\bullet_{A^!}(A^!) = \mathbf{\Lambda}\left(\xi_1,\cdots,\xi_n,\frac{\partial}{\partial\xi_1},\cdots,\frac{\partial}{\partial\xi_n}\right) & \xrightarrow{\iota^*_{(-)}\eta^!} & \mathfrak{X}^\bullet_{A^!}(A^¡) = \mathbf{\Lambda}\left(\xi^*_1,\cdots,\xi^*_n,\frac{\partial}{\partial\xi_1},\cdots,\frac{\partial}{\partial\xi_n}\right)
\end{array} \tag{4.4}$$

commutes. This means η is a Poisson cycle for A if and only if $\eta^!$ is a Poisson cocycle for $A^!$, which proves the claim.

Second, for A as above, we show the following diagram

$$\begin{array}{ccc}
\mathrm{HP}^\bullet(A) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A) \\
\downarrow\cong & & \downarrow\cong \\
\mathrm{HP}^\bullet(A^!) & \xrightarrow{\cong} & \mathrm{HP}^{\bullet-n}(A^!;A^¡).
\end{array} \tag{4.5}$$

commutes. In fact, the two vertical isomorphisms are given by Theorem 1.1, and the two horizontal isomorphisms are given by Theorems 4.4 and 4.5 respectively. The commutativity of the diagram (4.5) follows from the chain level commutative diagram (4.4). □

5 Poisson cohomology and the Batalin-Vilkovisky algebra

The purpose of this section is to show that for unimodular quadratic Poisson polynomial algebras, the horizontal isomorphisms in (4.5) naturally induce on $\mathrm{HP}^\bullet(A)$ and $\mathrm{HP}^\bullet(A^!)$ a Batalin-Vilkovisky algebra structure, and the vertical isomorphisms in (4.5) are isomorphisms of Batalin-Vilkovisky algebras. We start with the notion of *differential calculus with duality*.

Definition 5.1 (Lambre [20]). A differential calculus $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, [-,-], d)$ is called a *differential calculus with duality* if there exists an integer n and an element $\eta \in \mathrm{H}_n$ such that

(a) $\iota_1\eta = \eta$, where $1 \in \mathrm{H}^0$ is the unit, $d(\eta) = 0$, and

(b) for any $i \in \mathbb{Z}$,
$$\mathrm{PD}(-) := \iota_{(-)}\eta : \mathrm{H}^i \to \mathrm{H}_{n-i} \tag{5.1}$$
is an isomorphism.

Such isomorphism PD is called the *Van den Bergh duality* (also called *the noncommutative Poincaré duality*), and η is called the *volume form*.

Definition 5.2 (Batalin-Vilkovisky algebra)**.** Suppose $(V, \bullet)$ is an graded commutative algebra. A *Batalin-Vilkovisky algebra* structure on V is the triple $(V, \bullet, \Delta)$ such that

(1) $\Delta : V^i \to V^{i-1}$ is a differential, that is, $\Delta^2 = 0$; and

(2) Δ is second order operator, that is,

$$\begin{aligned}\Delta(a \bullet b \bullet c) &= \Delta(a \bullet b) \bullet c + (-1)^{|a|} a \bullet \Delta(b \bullet c) + (-1)^{(|a|-1)|b|} b \bullet \Delta(a \bullet c)\\ &\quad -(\Delta a) \bullet b \bullet c - (-1)^{|a|} a \bullet (\Delta b) \bullet c - (-1)^{|a|+|b|} a \bullet b \bullet (\Delta c).\end{aligned}$$

Equivalently, if we define the bracket

$$[a, b] := (-1)^{|a|+1}(\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)),$$

then $[-, -]$ is a derivation with respect to $\bullet$ for each component. In other words, a Batalin-Vilkovisky algebra is a Gerstenhaber algebra $(V, \bullet, [-, -])$ with a differential $\Delta : V^i \to V^{i-1}$ such that

$$[a, b] = (-1)^{|a|+1}(\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)), \tag{5.2}$$

for any $a, b \in V$ (*cf.* [13, Proposition 1.2]). Δ is also called the Batalin-Vilkovisky operator, or the generator (of the Gerstenhaber bracket).

Now suppose $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, [-, -], d, \eta)$ is a differential calculus with duality. Let $\Delta : \mathrm{H}^\bullet \to \mathrm{H}^{\bullet-1}$ be the linear operator such that

$$\begin{array}{ccc} \mathrm{H}^\bullet & \xrightarrow{\ \Delta\ } & \mathrm{H}^{\bullet-1} \\ \downarrow{\scriptstyle \mathrm{PD}} & & \downarrow{\scriptstyle \mathrm{PD}} \\ \mathrm{H}_{n-\bullet} & \xrightarrow{\ d\ } & \mathrm{H}_{n-\bullet+1} \end{array} \tag{5.3}$$

commutes. Then we have the following theorem:

Theorem 5.3 (Lambre [20])**.** *Let* $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, \iota, [-, -], d, \eta)$ *be a differential calculus with duality. Then the triple* $(\mathrm{H}^\bullet, \cup, \Delta)$ *is a Batalin-Vilkovisky algebra.*

The proof can be found in Lambre ([20, Théorème 1.6]); however, since some details in loc. cit. are omitted, we give a proof here for completeness.

Proof. Since $(\mathrm{H}^\bullet, \cup, [-, -])$ is a Gerstenhaber algebra, we only need to show that the Gerstenhaber bracket is compatible with the operator Δ in (5.3); that is, equation (5.2) holds. For any homogeneous elements $f, g \in \mathrm{H}^\bullet$, by the definition of Poincaré duality PD (5.1) and the Cartan formulae (Lemma 6.3), we have

$$\begin{aligned}
&(-1)^{|f|+1}\mathrm{PD}([f, g])\\
=\ &(-1)^{|f|+1}\iota_{[f,g]}(\eta) = [L_f, \iota_g](\eta) = L_f\iota_g(\eta) - (-1)^{|g|(|f|+1)}\iota_g L_f(\eta)\\
=\ &d\iota_f\iota_g(\eta) - (-1)^{|f|}\iota_f d\iota_g(\eta) - (-1)^{|g|(|f|+1)}\iota_g d\iota_f(\eta) + (-1)^{|g|(|f|+1)+|f|}\iota_g\iota_f d(\eta)\\
=\ &d \circ \mathrm{PD}(f \cup g) - (-1)^{|g|(|f|+1)}\iota_g d \circ \mathrm{PD}(f) - (-1)^{|f|}\iota_f d \circ \mathrm{PD}(g)\\
=\ &\mathrm{PD}(\Delta(f \cup g)) - (-1)^{|g|(|f|+1)}\iota_g \mathrm{PD}(\Delta(f)) - (-1)^{|f|}\iota_f \mathrm{PD}(\Delta(g))\\
=\ &\iota_{\Delta(f\cup g)}(\eta) - (-1)^{|g|(|f|+1)}\iota_g\iota_{\Delta(f)}(\eta) - (-1)^{|f|}\iota_f\iota_{\Delta(g)}(\eta))\\
=\ &(\iota_{\Delta(f\cup g)} - (-1)^{|g|(|f|+1)}\iota_{g\cup\Delta(f)} - (-1)^{|f|}\iota_{f\cup\Delta(g)})(\eta)
\end{aligned}$$

$$= \quad \mathrm{PD}(\Delta(f \cup g) - \Delta(f) \cup g - (-1)^{|f|} f \cup \Delta(g)).$$

Since PD is an isomorphism, we thus have

$$[f, g] = (-1)^{|f|+1}(\Delta(f \cup g) - \Delta(f) \cup g - (-1)^{|f|} f \cup \Delta(g)). \qquad \square$$

Corollary 5.4 (see also Xu [38] and Zhu-Van Oystaeyen-Zhang [39])**.** *Suppose A is a unimodular Poisson or unimodular symmetric Poisson algebra. Then* $\mathrm{HP}^\bullet(A)$ *admits a Batalin-Vilkovisky algebra structure.*

Proof. If A is unimodular Poisson, then Theorems 4.2 and 4.4 imply the pair $(\mathrm{HP}^\bullet(A), \mathrm{HP}_\bullet(A))$ is in fact a differential calculus with duality; similarly, if A is unimodular symmetric Poisson, Theorem 4.3 and 4.5 $(\mathrm{HP}^\bullet(A), \mathrm{HP}^\bullet(A; A^*))$ is a differential calculus with duality. The theorem then follows from Theorem 5.3. $\square$

Proof of Theorem 1.3. Note that in Theorem 1.2, the right vertical isomorphism preserves the Kähler differential as well as the volume form, that is, the two differential calculus with duality

$$\big(\mathrm{HP}^\bullet(A), \mathrm{HP}_\bullet(A)\big) \text{ and } \big(\mathrm{HP}^\bullet(A^!), \mathrm{HP}^\bullet(A^!; A^¡)\big)$$

are isomorphic. Combining with Corollary 5.4, the theorem follows. $\square$

Remark 5.5. Not all quadratic Poisson algebras are unimodular. For example, for $A = \mathbb{R}[x_1, x_2, x_3]$, Etingof-Ginzburg [10, Lemma 4.2.3 and Corollary 4.3.2] showed that any unimodular Poisson structure is of the form

$$\{x, y\} = \frac{\partial \phi}{\partial z}, \quad \{y, z\} = \frac{\partial \phi}{\partial x}, \quad \{z, x\} = \frac{\partial \phi}{\partial y},$$

for some $\phi \in A$ (taking ϕ to be cubic then the Poisson structure is quadratic); for $A = \mathbb{C}[x_1, x_2, x_3, x_4]$, Pym [26, §3] showed that any unimodular quadratic Poisson bracket on A may be written uniquely in the following form

$$\{f, g\} := \frac{df \wedge dg \wedge d\alpha}{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}, \ f, g \in A,$$

where $\alpha = \sum_{i=1}^4 \alpha_i dx_i \in \Omega^1(A)$ such that $\alpha \wedge d\alpha = 0$, and α_i's are homogeneous cubic polynomials satisfying $\sum_{i=1}^4 x_i \alpha_i = 0$.

6 Calabi-Yau algebras

At the end of §1 we sketched some analogy between unimodular Poisson algebras and Calabi-Yau algebras. In the following two sections, we study their relationships in more detail.

6.1 Calabi-Yau algebras and the Batalin-Vilkovisky algebra structure

Definition 6.1 (Calabi-Yau algebra; Ginzburg [14])**.** Let A be an associative algebra over k. A is called a *Calabi-Yau algebra of dimension n* if

(1) A is homologically smooth, that is, A, viewed as an A^e-module, has a bounded resolution of finitely generated projective A^e-modules, and

(2) there is an isomorphism
$$\mathrm{RHom}_{A^e}(A, A\otimes A)\cong \Sigma^{-n}A \tag{6.1}$$
in the derived category $D(A^e)$ of A^e-modules.

In the above definition, A^e is the enveloping algebra of A, namely $A^e := A\otimes A^{\mathrm{op}}$. There are a lot of examples of Calabi-Yau algebras, such as the universal enveloping algebra of semi-simple Lie algebras, the skew-product of complex polynomials with a finite subgroup of $\mathrm{SL}(n,\mathbb{R})$, the Yang-Mills algebras, etc.

We next study Van den Bergh's noncommutative Poincaré duality for Calabi-Yau algebras ([34]). To this end, we first recall the differential calculus structure for associative algebras.

For an associative algebra A, denote by $(\bar{\mathrm{C}}^\bullet(A;A),\delta)$ and $(\bar{\mathrm{C}}_\bullet(A;A), b)$ the reduced Hochschild cochain and chain complexes of A. Recall that the *Gerstenhaber cup product* and the *Gerstenhaber bracket* on $\bar{\mathrm{C}}^\bullet(A;A)$ are given as follows: for any $f\in\bar{\mathrm{C}}^n(A;A)$ and $g\in\bar{\mathrm{C}}^m(A;A)$,
$$f\cup g(\bar a_1,\dots,\bar a_{n+m}) := (-1)^{nm}f(\bar a_1,\dots,\bar a_n)g(\bar a_{n+1},\dots,\bar a_{n+m}),$$
and
$$\{f,g\} := f\circ g-(-1)^{(|f|+1)(|g|+1)}g\circ f,$$
where
$$f\circ g(\bar a_1,\dots,\bar a_{n+m-1}) := \sum_{i=0}^{n-1}(-1)^{(|g|+1)i}f(\bar a_1,\dots,\bar a_i,\overline{g(\bar a_{i+1},\dots,\bar a_{i+m})},\bar a_{i+m+1},\dots,\bar a_{n+m-1}).$$

Gerstenhaber proved in [12, Theorems 3-5] $\cup$ and $\{-,-\}$ are well-defined on the cohomology level, and moreover, $\cup$ is graded commutative. Therefore we obtain on the Hochschild cohomology $\mathrm{HH}^\bullet(A)$ a Gerstenhaber algebra structure.

Next, we consider the action of the Hochschild cochain complex on the Hochschild chain complex. Given any homogeneous elements $f\in\bar{\mathrm{C}}^n(A;A)$ and $\alpha=(a_0,\bar a_1,\dots,\bar a_m)\in\bar{\mathrm{C}}_m(A;A)$,

(1) the *cap product* $\cap:\bar{\mathrm{C}}^n(A;A)\times\bar{\mathrm{C}}_m(A;A)\to\bar{\mathrm{C}}_{m-n}(A;A)$ is given by
$$f\cap\alpha := \begin{cases}(a_0 f(\bar a_1,\dots,\bar a_n),\bar a_{n+1},\dots,\bar a_m), & \text{if } m\ge n\\ 0, & \text{otherwise.}\end{cases} \tag{6.2}$$
If we denote by $\iota_f(-) := f\cap -$ the contraction operator, then $\iota_f\iota_g=(-1)^{|f||g|}\iota_{g\cup f}=\iota_{f\cup g}$;

(2) the *Lie derivative* $L:\bar{\mathrm{C}}^n(A;A)\times\bar{\mathrm{C}}_m(A;A)\to\bar{\mathrm{C}}_{m-n}(A;A)$ is given as follows: for any $\alpha=(a_0,\bar a_1,\dots,\bar a_m)\in\bar{\mathrm{C}}_m(A,A)$, if $n\le m+1$, then
$$\begin{aligned}L_f(\alpha) &:= \sum_{i=0}^{m-n}(-1)^{(n+1)i}(a_0,\bar a_1\cdots,\bar a_i,\overline{f(\bar a_{i+1},\cdots,\bar a_{i+n})},\cdots,\bar a_m)\\ &+\sum_{i=m-n+1}^{m}(-1)^{m(i+1)+n+1}(f(\bar a_{i+1},\cdots,\bar a_m,\bar a_0,\dots,\bar a_{n-m+i-1}),\bar a_{n-m+i},\dots,\bar a_i),\end{aligned}$$
where the second sum is taken over all cyclic permutations such that a_0 is inside of f, and otherwise if $n>m+1$, $L_f(\alpha)=0$;

(3) the *Connes operator* $B : \bar{\mathrm{C}}_\bullet(A;A) \to \bar{\mathrm{C}}_{\bullet+1}(A;A)$ is given by

$$B(\alpha) := \sum_{i=0}^{m} (-1)^{mi}(1, \bar{a}_i, \cdots, \bar{a}_m, \bar{a}_0, \cdots, \bar{a}_{i-1}).$$

The following two lemmas first appeared in Daletskii-Gelfand-Tsygan [6], which we learned from Tamarkin-Tsygan in [29].

Lemma 6.2. *Keep the notations as in the above definition. Then*

(1) $(\bar{\mathrm{C}}_\bullet(A;A), b, \cap)$ *is a DG module over* $(\bar{\mathrm{C}}^\bullet(A;A), \delta, \cup)$*, that is,*

$$\iota_{\delta f} = (-1)^{|f|+1}[b, \iota_f], \quad \iota_f \iota_g = \iota_{f\cup g},$$

for any homogeneous elements $f, g \in \bar{\mathrm{C}}^\bullet(A;A)$*;*

(2) *for any homogeneous elements* $f, g \in \bar{\mathrm{C}}^\bullet(A;A)$*,*

$$[L_f, L_g] = L_{\{f,g\}},$$

and in particular $(-1)^{|f|+1}[b, L_f] + L_{\delta f} = 0$*.*

Lemma 6.3 (Homotopy Cartan formulae)**.** *Suppose* ι, L, B *are given as above and* $f, g \in \bar{\mathrm{C}}^\bullet(A;A)$ *are any homogeneous elements.*

(1) *Define an operation (cf. [29, Equ. (3.5)])*

$$S_f(\alpha) := \sum_{i=0}^{m-n} \sum_{j=i+n}^{m} (-1)^{\eta_{ij}}(1, \bar{a}_{j+1}, \cdots, \bar{a}_m, \bar{a}_0, \cdots, \bar{a}_i, \overline{f(\bar{a}_{i+1}, \cdots, \bar{a}_{i+n})}, \bar{a}_{i+n+1}, \cdots, \bar{a}_j)$$

for any $\alpha = (a_0, \bar{a}_1, \cdots, \bar{a}_m) \in \bar{\mathrm{C}}_m(A;A)$ *(the sum is taken over all cyclic permutations and* a_0 *always appears on the left of* f*), where* $\eta_{ij} := (n+1)m + (m-j)m + (n+1)(j-i)$*. Then we have*

$$L_f = [B, \iota_f] + [b, S_f] - S_{\delta f}. \tag{6.3}$$

(2) *Define*

$$T(f,g)(\alpha) := \sum_{i=l-n+2}^{l} \sum_{j=0}^{n+i-l-2} (-1)^{\theta_{ij}} (f(\bar{a}_{i+1}, \cdots, \bar{a}_l, \bar{a}_0, \cdots, \bar{a}_j, \overline{g(\bar{a}_{j+1}, \cdots, \bar{a}_{j+m})}, \cdots, \bar{a}_{n+m+i-l-2}), \cdots, \bar{a}_i)$$

for any $\alpha = (a_0, \bar{a}_1, \cdots, \bar{a}_l) \in \bar{\mathrm{C}}_l(A;A)$*, where* $\theta_{ij} = (m+1)(i+j+l) + l(i+1)$*. Then we have*

$$[L_f, \iota_g] - (-1)^{|f|+1}\iota_{\{f,g\}} = [b, T(f,g)] - T(\delta f, g) - T(f, \delta g). \tag{6.4}$$

The above two lemmas say that Definition 4.1 (2) (3) hold up homotopy on the chain level. Together with Gerstenhaber's theorem, we have the following.

Theorem 6.4 (Daletskii-Gelfand-Tsygan [6])**.** *Let* A *be an associative algebra. Then the following sextuple*

$$\big(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A), \cup, \iota, \{-,-\}, B\big)$$

is a differential calculus.

In [7, Proposition 5.5], de Thanhoffer de Völcsey and Van den Bergh proved that, for a Calabi-Yau algebra A of dimension n, there exists a class $\eta \in \mathrm{HH}_n(A)$ such that the contraction

$$\mathrm{HH}^\bullet(A) \xrightarrow{-\cap\eta} \mathrm{HH}_{n-\bullet}(A) \tag{6.5}$$

is an isomorphism. This immediately implies the following:

Theorem 6.5 ([14, 20])**.** *Suppose A is a Calabi-Yau algebra A of dimension n. Then*

$$(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A), \cup, \iota, \{-,-\}, B)$$

is a differential calculus with duality, and in particular, $(\mathrm{HH}^\bullet(A), \cup, \Delta)$ *is a Batalin-Vilkovisky algebra.*

6.2 Symmetric algebras and the Batalin-Vilkovisky algebra structure

We now recall a differential calculus structure on the Hochschild complexes of symmetric algebras.

First, for an associative algebra A, denote $A^* := \mathrm{Hom}(A, k)$, which is an A-bimodule. Denote by $\bar{\mathrm{C}}^\bullet(A; A^*)$ the reduced Hochschild cochain complex of A with values in A^*. Then under the identity

$$\bar{\mathrm{C}}^\bullet(A; A^*) = \bigoplus_{n\geq 0} \mathrm{Hom}(\bar{A}^{\otimes n}, A^*) = \bigoplus_{n\geq 0} \mathrm{Hom}(A \otimes \bar{A}^{\otimes n}, k),$$

one may equip on $\bar{\mathrm{C}}^\bullet(A; A^*)$ the dual Connes differential, which is denoted by B^*, i.e., $B^*(g) := (-1)^{|g|} g \circ B$ for homogeneous $g \in \bar{\mathrm{C}}^\bullet(A; A^*)$. B^* commutes with the Hochschild coboundary map δ, and thus is well-defined on the homology level.

Second, let

$$\begin{array}{ccc} \bar{\mathrm{C}}^\bullet(A; A) \times \bar{\mathrm{C}}^\bullet(A; A^*) & \xrightarrow{\cap^*} & \bar{\mathrm{C}}^\bullet(A; A^*) \\ (f, \quad \alpha) & \longmapsto & \iota_f^*(\alpha) := (-1)^{|f||\alpha|} \alpha \circ \iota_f, \end{array} \tag{6.6}$$

for any homogeneouss $f \in \bar{\mathrm{C}}^\bullet(A; A)$ and $\alpha \in \bar{\mathrm{C}}^\bullet(A; A^*)$. We have the following.

Theorem 6.6. *Let A be an associative algebra. Then*

$$(\mathrm{HH}^\bullet(A), \mathrm{HH}^\bullet(A; A^*), \cup, \iota^*, \{-,-\}, B^*)$$

is a differential calculus.

Proof. By the definition of differential calculus, we only need to show the last two equalities given in Definition 4.1.

(1) By the definition of ι^* and Lemma 6.2 (1), one has

$$\begin{aligned} \iota_f^* \iota_g^*(\alpha) &= (-1)^{|g||\alpha|} \iota_f^*(\alpha \circ \iota_g) = (-1)^{|g||\alpha| + |f|(|\alpha| + |g|)} (\alpha \circ \iota_g) \circ \iota_f \\ &= (-1)^{|g||\alpha| + |f|(|\alpha| + |g|)} \alpha \circ (\iota_{g \cup f}) = (-1)^{|f||g|} \iota_{g\cup f}^* \alpha = \iota_{f \cup g}^*(\alpha), \end{aligned}$$

for any homogenous elements $f, g \in \mathrm{HH}^\bullet(A)$ and $\alpha \in \mathrm{HH}^\bullet(A; A^*)$. This means that the cap product is a left module action.

(2) Given any homogenous elements $f \in \mathrm{HH}^\bullet(A)$ and $\alpha \in \mathrm{HH}^\bullet(A; A^*)$, define

$$L_f^*(\alpha) := (-1)^{|f||\alpha|+|\alpha|+1}\alpha \circ L_f (= [B^*, \iota_f^*](\alpha)), \tag{6.7}$$

and by Lemma 6.3 one has

$$\begin{aligned}
[L_f^*, \iota_g^*](\alpha) &= (L_f^*\iota_g^* - (-1)^{(|f|+1)|g|}\iota_g^* L_f^*)(\alpha) \\
&= (-1)^{(|f|+1)(|\alpha|+|g|)+|g||\alpha|+1}\alpha \circ (\iota_g L_f) - (-1)^{(|f|+|g|+1)|\alpha|+1}\alpha \circ (L_f \iota_g) \\
&= (-1)^{(|f|+|g|+1)|\alpha|}\alpha \circ ([L_f, \iota_g]) \\
&= (-1)^{(|f|+|g|+1)|\alpha|}\alpha \circ ((-1)^{|f|+1}\iota_{\{f,g\}}) \\
&= (-1)^{|f|+1}\iota_{\{f,g\}}^*(\alpha).
\end{aligned}$$

This completes the proof. □

Now suppose $A^!$ is symmetric. Recall that the existence of the degree n cyclic pairing is equivalent to an isomorphism

$$\eta : A^! \cong \Sigma^{-n} A^¡$$

as $A^!$-bimodules. Such η may be viewed as an element in $\bar{\mathrm{C}}^{-n}(A^!; A^¡)$, which is a cocycle, and hence represents a cohomology class. By abuse of notation, this class is also denoted by η. The following map

$$\begin{aligned}
- \cap^* \eta : \bar{\mathrm{C}}^\bullet(A^!; A^!) = \bigoplus_{q \geq 0} \mathrm{Hom}((\bar{A}^!)^{\otimes q}, A^!) & \\
\xrightarrow{\eta \circ -} \bigoplus_{q \geq 0} \mathrm{Hom}((\bar{A}^!)^{\otimes q}, \Sigma^{-n} A^¡) = \bar{\mathrm{C}}^{\bullet - n}(A^!; A^¡), &
\end{aligned} \tag{6.8}$$

where $\eta \circ -$ means composing with η, gives an isomorphism on the cohomology (due to Tradler [31]). Thus we have the following.

Theorem 6.7 ([20, 31]). *Suppose $A^!$ is a symmetric algebra of degree n.*

$$(\mathrm{HH}^\bullet(A), \mathrm{HH}^\bullet(A; A^*), \cup, \iota^*, \{-,-\}, B^*)$$

is a differential calculus with duality, and in particular, $\mathrm{HH}^\bullet(A^!)$ is a Batalin-Vilkovisky algebra.

6.3 Koszul Calabi-Yau algebras and Rouquier's conjecture

Analogously to the quadratic Poisson algebra case, the Koszul dual of a Koszul Calabi-Yau algebra is symmetric (chronologically the latter is discovered first), and we have the following theorem due to Van den Bergh (see [33, Theorem 9.2] or [5, Proposition 28] for a proof): Suppose A is a Koszul algebra and let $A^!$ be its Koszul dual algebra. Then A is Calabi-Yau of dimension n if and only if $A^!$ is symmetric of degree n.

It has been well-known that for a Koszul algebra, say A,

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(A^!),$$

as Gerstenhaber algebras, and Rouquier conjectured (it is stated in Ginzburg [14]) that, for a Koszul Calabi-Yau algebra, the above two Batalin-Vilkovisky are isomorphic, which turns out to be true (see [5, Theorem A] for a proof):

Theorem 6.8 (Rouquier's conjecture)**.** *Suppose A is a Koszul Calabi-Yau algebra. Denote by $A^!$ and by $A^{¡}$ the Koszul dual algebra and coalgebra of A respectively. Then*

$$\big(\mathrm{HH}^\bullet(A), \mathrm{HH}_\bullet(A)\big) \quad \textit{and} \quad \big(\mathrm{HH}^\bullet(A^!), \mathrm{HH}^\bullet(A^!; A^{¡})\big)$$

are isomorphic as differential calculus with duality. In particular, $\mathrm{HH}^\bullet(A)$ *and* $\mathrm{HH}^\bullet(A^!)$ *are isomorphic as Batalin-Vilkovisky algebras.*

The key point of the proof is that, with the differentials properly assigned on $A \otimes A^!$ and $A \otimes A^{¡}$ respectively, then

$$\bar{\mathrm{C}}^\bullet(A; A) \simeq A \otimes A^! \simeq \bar{\mathrm{C}}^\bullet(A^!; A^!) \quad \text{and} \quad \bar{\mathrm{C}}_\bullet(A; A) \simeq A \otimes A^{¡} \simeq \bar{\mathrm{C}}^\bullet(A^!; A^{¡}),$$

and via these quasi-isomorphisms, the volume forms as well as the contractions given by (6.2) and (6.6) are identical on the above middle terms (compare with the proof of Theorem 1.2).

Example 6.9 (The polynomial case)**.** Let $A = \mathbb{R}[x_1, x_2, \cdots, x_n]$, which is n-Calabi-Yau. Its Koszul dual algebra $A^! = \mathbf{\Lambda}(\xi_1, \xi_2, \cdots, \xi_n)$ is symmetric. As in the Poisson case, the volume forms on $\mathrm{HH}_\bullet(A)$ and $\mathrm{HH}^\bullet(A^!; A^{¡})$ are, via the above quasiisomorphisms, represented by $1 \otimes \xi_1^* \cdots \xi_n^*$ in $A \otimes A^{¡}$.

7 Deformation quantization

In this section, we take k to be a field containing $\mathbb{R}$. Dolgushev [8, Theorem 3] proved that for a Calabi-Yau algebra, if it is unimodular Poisson, then its deformation quantization is again Calabi-Yau. Analogously, Felder-Shoikhet [11, Corollary 1] and Willwacher-Calaque [37, Theorem 37] proved that for a symmetric algebra, if it is unimodular symmetric Poisson, then its deformation quantization is again symmetric. We use their results to prove Theorems 1.4 and 1.5.

7.1 The Maurer-Cartan formalism

In this subsection, we give an alternate description of the unimodular Poisson and Calabi-Yau algebras in terms of the negative cyclic homology, which we learned from de Thanhoffer de Völcsey and Van den Bergh [7].

Let us start with the notion of negative cyclic homology.

Definition 7.1 (Cyclic homology; *cf.* Jones [16] and Kassel [17])**.** Suppose $(\mathrm{C}_\bullet, b, B)$ is a mixed complex, with $|d| = -1$ and $|B| = 1$. Let u be a free variable of degree -2 which commutes with b and B. The *negative cyclic chain complex* of $\mathrm{C}_\bullet$ is the following complex

$$(\mathrm{C}_\bullet[\![u]\!], b + uB),$$

and is denoted by $\mathrm{CC}_\bullet^-(\mathrm{C}_\bullet)$. The associated homology is called the *negative cyclic homology* of $\mathrm{C}_\bullet$, and is denoted by $\mathrm{HC}_\bullet^-(\mathrm{C}_\bullet)$.

Remark 7.2 (Cyclic cohomology)**.** Suppose $(\mathrm{C}^\bullet, b, B)$ is a mixed cochain complex, namely $|b| = 1$ and $|B| = -1$. By negating the degrees of $\mathrm{C}^\bullet$, we obtain a mixed chain complex, denoted by $(\mathrm{C}_\bullet, b, B)$ with $|b| = -1$ and $|B| = 1$. By our convention, the *cyclic cohomology* of $(\mathrm{C}^\bullet, b, B)$, denoted by $\mathrm{HC}^\bullet(\mathrm{C}^\bullet)$, is the *cohomology* of the negative cyclic complex of $(\mathrm{C}_\bullet, b, B)$.

Example 7.3. (1) Suppose (A, π) is a Poisson algebra; then $(\Omega^\bullet(A), \partial, d)$, where ∂ and d are the Poisson boundary and de Rham differential respectively, is a mixed chain complex. Similarly, if $(A^!, \pi^!)$ is symmetric Poisson, then $(\Omega^\bullet(A^!, A^{¡}), \delta, d^*)$, where δ and d^* are the Poisson coboundary and dual de Rham differential respectively, is a mixed cochain complex.

(2) For an associative algebra A, $(\mathrm{CH}_\bullet(A), b, B)$ is a mixed chain complex. Similarly, for a symmetric algebra $A^!$, $(\mathrm{CH}^\bullet(A^!, A^{¡}), \delta, B^*)$ is a mixed cochain complex.

Equip $\mathfrak{X}^\bullet(A)$ with trivial differential, and $\Omega^\bullet(A)$ with mixed differential $(0, d)$, where d is the de Rham differential. Since $\Omega^\bullet(A)$ is a Lie module over $\mathfrak{X}^\bullet(A)$ whose action commutes with d, the negative cyclic complex $(\Omega^\bullet(A)[\![u]\!], ud)$ is a DG module over $\mathfrak{X}^\bullet(A)$. Consider the semi-direct product

$$\mathfrak{P}(A)^\# := \mathfrak{X}^\bullet(A) \ltimes \Sigma^d \Omega^\bullet(A)[\![u]\!], \tag{7.1}$$

which is a DG Lie algebra with differential $(0, ud)$.

Recall that for a DG Lie algebra (L, d), any solution, say a, to its Maurer-Cartan equation

$$d(a) + \frac{1}{2}[a, a] = 0$$

gives a new DG Lie algebra structure on L with differential $\tilde{d} = d + \frac{1}{2}[a, -]$. Denote this DG Lie algebra by L_a. Going back to the Poisson case, observe that the volume form η is a solution to the Maurer-Cartan equation $\mathfrak{P}(A)^\#$, and therefore we get a new DG Lie algebra

$$\mathfrak{P}(A, \eta) := \mathfrak{P}(A)^\#_\eta.$$

Proposition 7.4. *Let $A = k[x_1, \cdots, x_n]$ and $\hbar$ be a formal variable. For the algebra $A[\![\hbar]\!]$ over $k[\![\hbar]\!]$, a bivector*

$$\hbar\pi := \hbar \cdot \pi_0 + \hbar^2 \cdot \pi_1 + \cdots \in \hbar \cdot \mathfrak{X}^2(A[\![\hbar]\!])$$

and a volume form

$$\hbar\eta := \hbar \cdot \eta_1 + \hbar^1 \cdot \eta_2 + \cdots \in \hbar\Omega^n(A[\![\hbar]\!])$$

such that $(\hbar\pi, \eta + \hbar\eta)$ gives on $A[\![\hbar]\!]$ a unimodular Poisson structure if and only if $(\hbar\pi, \Sigma^n \hbar\eta)$ is a solution to the Maurer-Cartan equation of the DG Lie algebra $\mathfrak{P}(A[\![\hbar]\!], \eta)$.

Proof. Direct check. □

For a Maurer-Cartan element $(\hbar\pi, \eta + \hbar\eta)$ in the above theorem, the linear term with respect to the powers of $\hbar$ also implies that A with (π, η) is unimodular.

Completely analogously to Proposition 7.4, we have that:

Proposition 7.5. *Suppose $A^!$ is a symmetric algebra with volume form $\eta^!$. Then for a bivector $\hbar\pi^! \in \hbar \cdot \mathfrak{X}^2(A^!)$ and an n-form $\hbar\eta \in \hbar \cdot \mathfrak{X}^n(A^{¡})[\![u]\!]$, the pair $(\hbar\pi^!, \eta + \hbar\eta^!)$ gives a unimodular Poisson structure on $A^![\![\hbar]\!]$ if and only if $(\hbar\pi^!, \hbar\eta^!)$ is a Maurer-Cartan element of the DG Lie algebra*

$$\mathfrak{P}(A^!, \eta^!) := (\mathfrak{X}^\bullet(A^![\![\hbar]\!]) \ltimes \Sigma^{-n-1}\mathfrak{X}^\bullet(A^{¡}[\![\hbar]\!])[\![u]\!])_{\eta^!}.$$

For Calabi-Yau algebras and symmetric algebras, we have similar results, due to de Thanhoffer de Völcsey-Van den Bergh [7] and Terilla-Tradler [30] respectively (the interested reader may refer to these two works for proofs):

Proposition 7.6 ([7] Theorem 8.1)**.** *Suppose A is an n-Calabi-Yau algebra with volume form η. Then an element $\hbar\mu\in\hbar\cdot\mathrm{CH}^2(A[\![\hbar]\!])$ and an n-form $\hbar\eta\in\hbar\cdot\overline{\mathrm{CC}}_n(A[\![\hbar]\!])$ such that $(\mu+\hbar\mu,\eta+\hbar\eta)$ gives a Calabi-Yau structure on $A[\![\hbar]\!]$ if and only if $(\hbar\mu,\hbar\eta)$ is a Maurer-Cartan element of the DG Lie algebra*

$$\mathfrak{D}(A[\![\hbar]\!],\eta):=(\mathrm{CH}^\bullet(A[\![\hbar]\!])\ltimes\overline{\mathrm{CC}}_\bullet(A[\![\hbar]\!]))_\eta.$$

For a symmetric algebra, say $A^!$, the volume form lies in the Connes' cyclic cochain complex $\mathrm{CC}^\bullet_\lambda(A^!)$. The relationship between the Connes' cyclic cochain complex and the cyclic cochain complex given in Remark 7.2, viewed as DG Lie modules over $\bar{\mathrm{C}}^\bullet(A^!)$ is as follows:

Lemma 7.7. *For any symmetric algebra $A^!$, the classical quasiisomorphism of cochain complexes*

$$\mathrm{CC}^\bullet_\lambda(A^!)\xrightarrow{\simeq}\mathrm{CC}^\bullet(A^!) \tag{7.2}$$

is an quasiisomorphism of Lie modules over $\mathrm{CH}^\bullet(A^!)$.

Proof. We need to show that (7.2) respects the Lie derivative from the Hochschild cochain complex. This is true since $\mathrm{CC}^\bullet_\lambda(A^!)$ embeds into $\overline{\mathrm{CC}}^\bullet(A^!)$ and the Lie action of $\mathrm{CH}^\bullet(A^!)$ on the former is the restriction of the Lie action on the latter, given by (6.7). □

The following is the main result of Terilla-Tradler [30]:

Proposition 7.8 ([30] Theorem 3.7)**.** *Suppose $A^!$ is an n-symmetric algebra with volume form $\eta^!$. Then an element $\hbar\mu\in\hbar\cdot\mathrm{CH}^2(A^![\![\hbar]\!])$ and an n-form $\hbar\eta\in\hbar\cdot\overline{\mathrm{CC}}^n(A^![\![\hbar]\!])$ such that $(\mu+\hbar\mu,\eta+\hbar\eta)$ gives a symmetric algebra structure on $A^![\![\hbar]\!]$ if and only if $(\hbar\mu,\hbar\eta)$ is a Maurer-Cartan element of the DG Lie algebra*

$$\mathfrak{D}(A^![\![\hbar]\!],\eta^!):=(\mathrm{CH}^\bullet(A^![\![\hbar]\!])\ltimes\mathrm{CC}^\bullet_\lambda(A^![\![\hbar]\!]))_{\eta^!}.$$

Remark 7.9. (1) In Propositions 7.4 and 7.5, if A and $A^!$ are Koszul dual to each other (recall in this case the Poisson structure is quadratic), then under the correspondence (3.5) and (3.9), the two DG Lie algebras are isomorphic. Therefore the sets of Maurer-Cartan elements to these DG Lie algebras are isomorphic, too.

(2) Similarly, in Propositions 7.6 and 7.8 if A and $A^!$ are Koszul dual to each other, then the two DG Lie algebras are also quasi-isomorphic. This means the deformation theory of Koszul Calabi-Yau algebras is equivalent to the deformation theory of their Koszul dual algebras. However, in this case, it is a little complicated to show directly the quasi-isomorphism, and we plan to discuss it somewhere else.

Remark 7.10. It is proved by de Thanhoffer de Völcsey-Van den Bergh in [7] and Terilla-Tradler in [30] that the homology of the two DG Lie algebras in Propositions 7.6 and 7.8 are isomorphic to the negative cyclic homology and the cyclic cohomology respectively.

7.2 Deformation quantization of Calabi-Yau Poisson algebras

In this subsection we prove Theorem 1.4 (1).

Recall that for a Poisson algebra A with bracket $\{-,-\}$, its *deformation quantization*, denoted by $A_\hbar$, is a $k[\![\hbar]\!]$-linear associative product (called the *star-product*) on $A[\![\hbar]\!]$

$$a*b=a\cdot b+\mu_1(a,b)\hbar+\mu_2(a,b)\hbar^2+\cdots,$$

where $\hbar$ is the formal parameter and μ_i are bilinear operators, satisfying

$$\lim_{\hbar\to 0}\frac{1}{\hbar}(a*b-b*a)=\{a,b\},\quad \text{for all } a,b\in A.$$

In [18], Kontsevich constructed, for A being the algebra of smooth functions on a Poisson manifold, an explicit L_∞-quasiisomorphism from the space of polyvector fields to the Hochschild cochain complex of A, and therefore there is a one-to-one correspondence between the equivalence classes of star-products and the equivalence classes of Poisson algebra structures on $A[\![\hbar]\!]$. Thus via Kontsevich's map, the Poisson bivector $\hbar\pi$ on $A[\![\hbar]\!]$ gives a star-product on $A[\![\hbar]\!]$, which is called *Kontsevich's deformation quantization*.

Note that $\Omega^\bullet(A)$ and $\bar{\mathrm{C}}_\bullet(A;A)$ are modules over $\mathfrak{X}^\bullet(A)$ and over $\bar{\mathrm{C}}^\bullet(A;A)$ respectively, and in [32, Conjecture 5.3.2], Tsygan conjectured that Kontsevich's deformation quantization also gives an L_∞-quasiisomorphism of L_∞-modules between $\bar{\mathrm{C}}_\bullet(A;A)$ and $\Omega^\bullet(A)$. This is known as Tsygan's Formality Conjecture for chains, and is proved by Shoikhet in [27, Theorem 1.3.1]. Shoikhet also conjectured that such L_∞-morphism is also compatible with the cup product, which was later proved by Calaque and Rossi in [3, Theorem A].

Recall that on $\Omega^\bullet(A)$ and $\bar{\mathrm{C}}_\bullet(A;A)$, we have the de Rham differential operator and the Connes boundary operator respectively. One naturally expects the L_∞-quasiisomorphism constructed above respects these two operators. This is known as the Cyclic Formality Conjecture for chains, and is proved by Willwacher in [36, Theorem 1.3 and Corollary 1.4].

Proof of Theorem 1.4 (1). The works above, especially those of Kontsevich and Willwacher are equivalent to saying that there exists a roof of L_∞-quasiisomorphisms

$$\begin{array}{ccc} & \mathfrak{X}^\bullet(A[\![\hbar]\!])\ltimes\Sigma^{-d}\overline{\mathrm{CC}}_\bullet^-(A[\![\hbar]\!]) & \\ \swarrow & & \searrow \\ \mathfrak{X}^\bullet(A[\![\hbar]\!])\ltimes\Sigma^{-d}\Omega^\bullet(A[\![\hbar]\!])[\![u]\!] & & \overline{\mathrm{C}}^\bullet(A[\![\hbar]\!])\ltimes\Sigma^{-d}\overline{\mathrm{CC}}_\bullet^-(A[\![\hbar]\!]) \end{array}$$

of DG Lie algebras (see [7, §11.3] for a proof).

Twisting with the corresponding volume forms in the above roof we get a new roof of L_∞-quasiisomorphisms (we have to show that the volume form in the three DG Lie modules are the same on the homology level, but this is the case by Example 6.9). This then implies that we have an L_∞-quasiisomorphism of DG Lie algebras

$$\mathfrak{P}(A[\![\hbar]\!],\eta)\overset{\simeq}{\dashrightarrow}\mathfrak{D}(A[\![\hbar]\!],\eta)$$

given in Propositions 7.4 and 7.6, where the dotted arrow means the quasiisomorphism is given by a sequence of (roofs of) L_∞-morphisms.

As a corollary, the Maurer-Cartan elements of $\mathfrak{P}(A[\![\hbar]\!],\eta)$ (up to gauge equivalence) are in one-to-one correspondence, via the above L_∞-quasiisomorphisms, with the Maurer-Cartan elements of $\mathfrak{D}(A[\![\hbar]\!],\eta)$. In particular, if A is unimodular Poisson, then $A_\hbar$ is Calabi-Yau, and vice versa (*c.f.* Dolgushev's result [8, Theorem 3]).

Thus by Theorem 4.4 and the noncommutative Poincaré duality (6.5), we have a commutative diagram

$$\begin{array}{ccc} \mathrm{HP}^\bullet(A[\![\hbar]\!]) & \xrightarrow{\cong} & \mathrm{HP}_{n-\bullet}(A[\![\hbar]\!]) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{HH}^\bullet(A_\hbar) & \xrightarrow{\cong} & \mathrm{HH}_{n-\bullet}(A_\hbar) \end{array}$$

for A being unimodular. $\square$

7.3 Deformation quantization of symmetric Poisson algebras

We first rephrase Kontsevich's Cyclic Formality Conjecture for *cochains*, published in Felder-Shoihket [11, §1], in the case $k^{0|n}$. Note that in this case, $A^! = \mathscr{O}(k^{0|n})$, the space of functions on $k^{0|n}$. Kontsevich's L_∞-quasiisomorphism holds for the supermanifold case, as has been shown in Cattaneo and Felder [4, Appendix].

Fix a constant volume form Ω on $k^{0|n}$ (where in previous sections we use the notation $\eta^! = \xi_1^* \cdots \xi_n^*$), then via the pairing

$$\langle f, g\rangle := \Omega(f \cdot g)$$

one can identify $\bar{\mathrm{C}}^\bullet(A^!; A^!)$ with $\bar{\mathrm{C}}^\bullet(A^!; A^¡)$, which is the same as capping with the "volume form" as given in (6.8). Denote by $\mathscr{D}_{\mathrm{poly}}(k^{0|n})$ the polydifferential Hochschild cochain subspace of $\bar{\mathrm{C}}^\bullet(A^!; A^!)$, and denote the image of $\psi \in \mathscr{D}_{\mathrm{poly}}(k^{0|n})$ in $\bar{\mathrm{C}}^\bullet(A^!; A^¡)$ by $\tilde{\psi}$. Let

$$\widetilde{\mathrm{CC}}^\bullet(A^!) \;\; := \;\; \left\{\psi \in \mathscr{D}_{\mathrm{poly}}(k^{0|n}) \,\middle|\, \begin{array}{l} \tilde{\psi}(f_1, \cdots, f_{n+1}) \\ = (-1)^{|f_{n+1}|(|f_1|+\cdots+|f_n|)+n}\tilde{\psi}(f_{n+1}, f_1, \cdots, f_n) \end{array}\right\},$$

which is a subspace of the Connes cyclically invariant subspace of $\bar{\mathrm{C}}^\bullet(A^!; A^¡)$.

Given Kontsevich's L_∞-quasiisomorphism $U : \mathscr{T}^\bullet_{\mathrm{poly}}(k^{0|n}) \to \mathscr{D}_{\mathrm{poly}}(k^{0|n})$, the Cyclic Formality Conjecture for cochains can be rephrased as the existence of an L_∞-quasiisomorphism of Lie modules

$$\big(\mathfrak{X}^\bullet(A^¡)[\![u]\!], u \cdot d^*\big) \xrightarrow{\simeq} \big(\widetilde{\mathrm{CC}}^\bullet(A^!), \delta\big). \tag{7.3}$$

This conjecture is proved by Willwacher and Calaque in [37, Theorem 2] (see also Felder-Shoikhet [11] for some partial results). To relate these two DG Lie algebras with $\mathfrak{P}(A^!, \eta^!)$ and $\mathfrak{D}(A^!, \eta^!)$ given in §7.1, let us first notice the following.

Lemma 7.11. *For $A^!$ as above, we have quasi-isomorphisms of DG Lie algebras*

$$\mathfrak{X}^\bullet(A^!) \ltimes \mathfrak{X}^\bullet(A^¡)[\![u]\!] \xrightarrow{\simeq} \mathfrak{X}^\bullet(A^¡)[\![u]\!] \tag{7.4}$$

and

$$\mathrm{CH}^\bullet(A^!) \ltimes \mathrm{CC}^\bullet_\lambda(A^!) \xrightarrow{\simeq} [\widetilde{\mathscr{D}}_{\mathrm{poly}}(\mathbb{R}^{0|n})]_{\mathrm{cycl}}. \tag{7.5}$$

Proof. $\square$

Thus combining the above lemma with (7.3), we have an L_∞-quasiisomorphism of DG Lie algebras:

$$\mathfrak{X}^\bullet(A^!) \ltimes \mathfrak{X}^\bullet(A^¡)[\![u]\!] \xrightarrow{\simeq} \mathrm{CH}^\bullet(A^!) \ltimes \mathrm{CC}^\bullet_\lambda(A^!). \tag{7.6}$$

The following theorem is parallel to the result of Dolgushev for Calabi-Yau algebras, which is proved by Willwacher and Calaque in [37, Theorem 37]:

Theorem 7.12. *For $A^! = \mathbf{\Lambda}(\xi_1, \cdots, \xi_n)$, Kontsevich's deformation quantization of $A^!$, say $A^!_\hbar$, is symmetric if and only if $A^!$ is unimodular symmetric Poisson.*

Proof. By (7.6) and (7.2) we have an L_∞-quasiisomorphism of DG Lie algebras

$$\mathfrak{P}^\bullet(A^![\![\hbar]\!], \eta^!) \xrightarrow{\simeq} \mathfrak{D}^\bullet(A^![\![\hbar]\!], \eta^!).$$

The rest of the proof is completely analogous to the Calabi-Yau algebra case. Namely, applying the same argument as in the proof of Theorem 1.4 (1), we get the desired result by replacing Propositions 7.4 and 7.6 therein with Propositions 7.5 and 7.8. □

Proof of Theorem 1.4 (2). Combining the above theorem with Theorem 4.5 as well as the noncommutative Poincaré duality (6.8), we get the commutative diagram. □

Proof of Theorem 1.5. By Shoikhet [28, Theorem 0.3] (see also [2, Theorem 8.6]), $A_\hbar$ and $A^!_\hbar$ are Koszul dual algebras over $k[\![\hbar]\!]$, and hence the theorem follows from a combination of Theorems 1.3, 1.4, and Rouquier's conjecture (Theorem 6.8). □

7.4 Twisted Poincaré duality for Poisson algebras

For a general associative algebra, say A, it may not be Calabi-Yau, and therefore there may not exist any Poincaré duality between $\mathrm{HH}^\bullet(A)$ and $\mathrm{HH}_\bullet(A)$. In [1], Brown and Zhang introduced the so-called "twisted Poincaré duality" for associative algebras. That is, for such A, keeping its left A-module structure (the multiplication) as usual, the right A-module structure of A is the multiplication composed with an automorphism $\sigma : A \to A$. Denote such A-bimodule by A_σ, then Brown and Zhang showed that for a lot of algebras, there exists a twisted Poincaré duality $\mathrm{HH}^\bullet(A) \cong \mathrm{HH}_{n-\bullet}(A; A_\sigma)$ for some $n \in \mathbb{N}$ (cf. [1, Corollary 5.2]). In this case A is called a *twisted Calabi-Yau* algebra of dimension n.

Such phenomenon also occurs for Poisson algebras. Namely, not all Poisson algebras are unimodular, and hence there may not exist an isomorphism between $\mathrm{HP}^\bullet(A)$ and $\mathrm{HP}_\bullet(A)$. In [21, 25, 39, 40], the authors studied the so-called twisted Poincaré duality for Poisson algebras, similarly to that of associative algebras. They also studied some comparisons with twisted Calabi-Yau algebras. However, it would be very interesting to study the relationships between the deformation quantization of twisted unimodular Poisson algebras and twisted Calabi-Yau algebras, and obtain a theorem similar to Theorem 1.5 in this twisted case.

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