A FULLY DISCRETE LOW-REGULARITY INTEGRATOR FOR THE 1D PERIODIC CUBIC NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. A fully discrete and fully explicit low-regularity integrator is constructed for the one-dimensional periodic cubic nonlinear Schrödinger equation. The method can be implemented by using fast Fourier transform with $O(N\ln N)$ operations at every time level, and is proved to have an L^2-norm error bound of $O(\tau\sqrt{\ln(1/\tau)}+N^{-1})$ for H^1 initial data, without requiring any CFL condition, where τ and N denote the temporal stepsize and the degree of freedoms in the spatial discretisation, respectively.

1. Introduction

This article concerns the numerical solution of the cubic nonlinear Schrödinger (NLS) equation

$$\begin{cases} i\partial_t u(t,x)+\partial_{xx}u(t,x)=\lambda|u(t,x)|^2u(t,x) & \text{for } x\in\mathbb{T} \text{ and } t\in(0,T],\\ u(0,x)=u^0(x) & \text{for } x\in\mathbb{T}, \end{cases} \tag{1.1}$$

on the one-dimensional torus $\mathbb{T}=(-\pi,\pi)$ with a nonsmooth initial value $u^0\in H^1(\mathbb{T})$, where $\lambda=-1$ and 1 are referred to as the focusing and defocusing cases, respectively. It is known that problem (1.1) is globally well-posed in $H^s(\mathbb{T})$ for $s\geq 0$; see [2].

The construction of numerical methods for the NLS equation and related dispersive equations with nonsmooth initial data has attracted much attention recently since the pioneering work of Ostermann & Schratz [17], who introduced a low-regularity exponential-type integrator that could have first-order convergence in $H^\gamma(\mathbb{T}^d)$ for initial data $u^0\in H^{\gamma+1}(\mathbb{T}^d)$ and $\gamma>\frac{d}{2}$, where d denotes the dimension of space. Before their work, the traditional regularity assumption for the NLS equation for a time-stepping method to have first-order convergence in $H^\gamma(\mathbb{T}^d)$ is $u^0\in H^{\gamma+2}(\mathbb{T}^d)$ for $\gamma\geq 0$ (losing two derivatives). This includes the Strang splitting methods [6, 14], the Lie splitting method [10], and classical exponential integrators [8] (also see the discussion in [17, p. 733]). The finite difference methods [19, 21] generally require more regularity of the initial data (one temporal derivative on the solution generally requires the initial data to have two spatial derivatives to satisfy certain compatibility conditions).

The idea of Ostermann & Schratz [17] is to use twisted variable to reduce the consistency error in an exponential-type integrator, and to use harmonic analysis techniques to approximate the exponential integral. More recently, Wu & Yao [22] applied different harmonic analysis techniques to construct a time-stepping method for the one-dimensional NLS equation with first-order convergence in $H^\gamma(\mathbb{T})$ for initial data $u^0\in H^\gamma(\mathbb{T})$ and $\gamma>\frac{3}{2}$ (without losing any derivative). Ostermann, Rousset & Schratz furthermore weakened the regularity assumption of initial data to $u^0\in H^1(\mathbb{T})$ in [15] and $u^0\in H^s(\mathbb{T})$ with $s\in(0,1]$ in [16] by using estimates in the discrete Bourgain spaces. For $u^0\in H^1(\mathbb{T})$ these methods were proved to have L^2-norm error bounds of $O(\tau^{\frac{5}{6}})$ and $O(\tau^{\frac{7}{8}-\epsilon})$, respectively, for the one-dimensional NLS equation. A general framework of low-regularity integrators for nonlinear

2010 *Mathematics Subject Classification.* 65M12, 65M15, 35Q55.

Key words and phrases. Nonlinear Schrödinger equation, numerical solution, first-order convergence, low regularity, fast Fourier transform.

parabolic, dispersive and hyperbolic equations was introduced in [7], where the condition for the numerical solution of the NLS equation to have first-order convergence in $L^2(\mathbb{T})$ is $u^0 \in H^{\frac{5}{4}}(\mathbb{T})$.

Besides the NLS equation, the techniques of twisted variable and harmonic analysis techniques were also used in the construction of low-regularity integrators for other dispersive equations; see [9, 18, 20, 23, 24] and the references therein.

As far as we know, the analysis of all the low-regularity integrators for the NLS equation are limited to semidiscretisation in time (the error from spatial discretisation is unknown for nonsmooth initial data), and the regularity condition for the time-stepping method to have first-order convergence is $u^0 \in H^\gamma(\mathbb{T})$ for $\gamma \geq \frac{5}{4}$. We are only aware of a fully discrete Lawson-type exponential integrator for the Korteweg–de Vries equation [18], with first-order convergence in $L^2(\mathbb{T})$ in both time and space under a CFL condition $\tau = O(h)$ for solutions in $C(0, T; H^3(\mathbb{T}))$.

The objective of this article is to construct a fully discrete and fully explicit lower-regularity integrator that has first-order convergence (up to a logarithmic factor) in both time and space for H^1 initial data. The temporal low-regularity integrator is constructed using twisted variables and with different harmonic analysis techniques in approximating the low- and high-frequency parts of the functions in the exponential integral. The spatial discretisation is integrated in the temporal low-regularity integrator by repeatedly using frequency truncation and Fast Fourier transform (FFT) techniques in every nonlinear operation (i.e., computing the product of two functions). By using a $(4N + 1)$-point FFT for every product of two $(2N+1)$-term Fourier series in the numerical scheme and then truncating the obtained $(4N + 1)$-term product series to $(2N + 1)$-term again, we avoid generating trigonometric interpolation errors from using FFT. As a result, the spatial discretisation error of our method is purely due to frequency truncation and therefore can be analysed together with the temporal discretisation error in the frequency domain by using harmonic analysis techniques.

The rest of this article is organised as follows. The fully discrete low-regularity integrator and the main theorem on the convergence rates of the method are presented in section 2. Some technical tools of harmonic analysis are presented in section 3, which are used in section 4 in the construction of the numerical method and analysis of the consistency error. The error bound of proposed fully discrete low-regularity integrator is proved in section 5 by utilizing the consistency error bounds obtained in section 4 and the stability of the method, as well as the H^1-regularity of fully discrete numerical solution. The latter is proved to be bounded uniformly with respect to the temporal stepsize and the number of Fourier terms in the spatial discretisation. Numerical results are presented in section 6 to support the theoretical analysis in this article.

2. The numerical method and main theoretical result

It is known that the solution of the NLS equation satisfies the following two conservation laws (see e.g., [4]):

(1) Mass conservation:

$$\frac{1}{2\pi}\int_{\mathbb{T}} |u(t,x)|^2\, dx = \frac{1}{2\pi}\int_{\mathbb{T}} |u^0(x)|^2\, dx \quad \text{for } t > 0. \tag{2.1}$$

(2) Momentum conservation:

$$\frac{1}{2\pi}\int_{\mathbb{T}} u(t,x)\partial_x \bar{u}(t,x)\, dx = \frac{1}{2\pi}\int_{\mathbb{T}} u^0\, \partial_x \bar{u}^0\, dx \quad \text{for } t > 0. \tag{2.2}$$

These two conserved quantities will be approximated based on the initial data and utilized in the construction of the numerical method.

We denote by Π_0 and $\Pi_{\neq 0}$ the zero-mode and nonzero-mode operators, respectively, defined by

$$\Pi_0 f = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\,dx \quad \text{and} \quad \Pi_{\neq 0}(f) = \sum_{k\in\mathbb{Z},k\neq 0} e^{ikx}\hat{f}_k. \tag{2.3}$$

Then the conserved mass and momentum are denoted by

$$M = \frac{1}{2\pi}\int_{\mathbb{T}} |u^0(x)|^2\,dx = \Pi_0(|u^0|^2) \quad \text{and} \quad P = \frac{1}{2\pi}\int_{\mathbb{T}} u^0\partial_x\overline{u^0}\,dx = \Pi_0\big(u^0\partial_x\overline{u^0}\big), \tag{2.4}$$

respectively.

For any positive integer N, we denote by I_{2N} the $(4N+1)$-point trigonometric interpolation operator, which can be obtained through the discrete Fourier transform (see [5, 25])

$$I_{2N}f(x) = \sum_{k=-2N}^{2N} e^{ikx}\tilde{f}_k \quad \text{with} \quad \tilde{f}_k = \frac{1}{4N+1}\sum_{n=-2N}^{2N} e^{-ikx_n} f(x_n) \tag{2.5}$$

where

$$x_n = \frac{2\pi n}{4N+1} \quad \text{for } n = -2N, \cdots, 2N.$$

If the Fourier coefficient $\hat{f}_k$ of the function f satisfies that $\hat{f}_k = 0$ for $|k| > 2N$, then $I_{2N}f = f$ and therefore $\tilde{f}_k = \hat{f}_k$ in the formula (2.5). In this case, both

$$f(x_n) = \sum_{k=-2N}^{2N} e^{ikx_n}\hat{f}_k, \quad n = -2N, \cdots, 2N, \tag{2.6}$$

and

$$\hat{f}_k = \sum_{n=-2N}^{2N} e^{-ikx_n} f(x_n) \quad k = -2N, \cdots, 2N, \tag{2.7}$$

can be computed with cost $O(N\ln N)$ by using the fast Fourier transform (FFT); see [5].

Let S_N be the subspace of functions $f \in L^2(\mathbb{T})$ such that $\hat{f}_k = 0$ for $|k| > N$. If $w, v \in S_N$ and their Fourier coefficients $\hat{w}_k$ and $\hat{v}_k$, $k = -2N, \cdots, 2N$, are stored in the computer (with $\hat{w}_k = \hat{v}_k = 0$ for $N < |k| \le 2N$), then the values $w(x_n)$ and $v(x_n)$, $n = -2N, \ldots, 2N$, can be computed exactly by using (2.6) and FFT. Since $(wv)_k = 0$ for $|k| > 2N$, it follows that $wv = I_{2N}(wv)$. If we denote by $\mathcal{F}_k[v]$ the kth Fourier coefficient of the function v, then

$$\mathcal{F}_k[wv] = \sum_{n=-2N}^{2N} e^{-ikx_n} w(x_n)v(x_n), \quad k = -2N, \ldots, 2N,$$

which can also be computed exactly by using FFT. Therefore, if we denote by $\Pi_N : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ the projection operator defined by

$$\mathcal{F}_k[\Pi_N f] = \begin{cases} \hat{f}_k & \text{for } |k| \le N, \\ 0 & \text{for } |k| > N, \end{cases}$$

then the cost of computing the Fourier coefficients of $\Pi_N(wv) \in S_N$ from the Fourier coefficients of $w, v \in S_N$ is $O(N\ln N)$.

For any positive integer L, let $t_n = n\tau$, $n = 0, 1, \ldots, L$, be a partition of the time interval $[0, T]$ with stepsize $\tau = T/L$. The fully discrete low-regularity integrator for the NLS equation (1.1) to be constructed in this paper is: For given $u^n_{\tau,N} \in S_N$ compute $u^{n+1}_{\tau,N} \in S_N$ by

$$\begin{aligned} &u^{n+1}_{\tau,N} = \Psi(u^n_{\tau,N}) \quad \text{for } n = 0, 1 \ldots, L-1, \\ &\text{with } u^0_{\tau,N} = \Pi_N I_{2N} u^0 \in S_N, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}\Psi(f) := {} & \mathrm{e}^{i\tau(-2\lambda P_N\partial_x^{-1}-2\lambda M_N+\partial_x^2)} f + (1-\mathrm{e}^{-2i\lambda\tau M_N})\Pi_0 f - i\lambda\tau\Pi_0\big[\Pi_N(|f|^2)f\big] \\ & + \lambda\partial_x^{-1}\Pi_N\big[(\mathrm{e}^{i\tau\partial_x^2}f)\cdot\partial_x^{-1}\Pi_N(|\mathrm{e}^{i\tau\partial_x^2}f|^2)\big] - \lambda\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}\Pi_N\big[f\cdot\partial_x^{-1}\Pi_N(|f|^2)\big] \\ & - \frac{\lambda}{2}\Big[\partial_x^{-2}\Pi_N\Big((\mathrm{e}^{-i\tau\partial_x^2}\bar f\,)\,\mathrm{e}^{i\tau\partial_x^2}\Pi_N(f^2)\Big) - \mathrm{e}^{i\tau\partial_x^2}\partial_x^{-2}\Pi_N\Big(\bar f\,\Pi_N(f^2)\Big)\Big] \\ & - \frac{\lambda}{2}\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}\Pi_N\Big[\partial_x\bar f\Big(\,\mathrm{e}^{-i\tau\partial_x^2}\Pi_N\big[(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}f)^2\big] - \Pi_N\big[(\partial_x^{-1}f)^2\big]\Big)\Big] \\ & - i\lambda\tau\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}\Pi_N\big(\partial_x\bar f\,\Pi_N(f^2)\big) \\ & + 2i\lambda\tau\Pi_0 f\,\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}\Pi_N\left(\partial_x\bar f\,f\right) - i\lambda\tau(\Pi_0 f)^2\mathrm{e}^{i\tau\partial_x^2}\Pi_{\neq 0}\bar f \qquad \text{for } f\in S_N, \end{aligned} \tag{2.9}$$

and

$$M_N = \Pi_0(|u^0_{\tau,N}|^2) \quad \text{and} \quad P_N = \Pi_0\big(u^0_{\tau,N}\partial_x\overline{u^0_{\tau,N}}\big) \tag{2.10}$$

are the approximate mass and momentum, respectively. By using (2.5) with FFT, the initial value $u^0_{\tau,N} = \Pi_N I_{2N} u^0$ can be obtained with cost $O(N\ln N)$. Then, at every time level, the method only requires computing several functions in the following forms:

- $\mathrm{e}^{i\tau\left(-2M_N-2P_N\partial_x^{-1}+\partial_x^2\right)}f$, $\mathrm{e}^{\pm i\tau\partial_x^2}f$ and $\partial_x^{-1}f$ for some given function $f\in S_N$,
- $\Pi_N(fg)$ for some given functions $f,g\in S_N$,

where

$$\mathcal{F}_k[\mathrm{e}^{i\tau\left(-2M_N-2P_N\partial_x^{-1}+\partial_x^2\right)}f] = \begin{cases} \mathrm{e}^{-2M_N i\tau}\hat f_0 & \text{for } k=0,\\ \mathrm{e}^{i\tau\left(-2M_N-2P_N(ik)^{-1}-k^2\right)}\hat f_k & \text{for } k\neq 0.\end{cases}$$

$$\mathcal{F}_k[\mathrm{e}^{\pm i\tau\partial_x^2}f] = \mathrm{e}^{\mp i\tau k^2}\hat f_k \quad \text{and} \quad \partial_x^{-1}f = \begin{cases} 0 & \text{for } k=0,\\ (ik)^{-1}\hat f_k & \text{for } k\neq 0.\end{cases}$$

Hence, the computational cost is $O(N\ln N)$ at every time level.

The main theoretical result of this paper is the following theorem.

Theorem 2.1. *If $u^0\in H^1(\mathbb{T})$ then there exist positive constants τ_0, N_0 and C such that for $\tau\le\tau_0$ and $N\ge N_0$ the numerical solution given by (2.8)–(2.9) has the following error bound:*

$$\max_{1\le n\le L}\|u(t_n,\cdot)-u^n_{\tau,N}\|_{L^2} \le C\big(\tau\sqrt{\ln(1/\tau)}+N^{-1}\big), \tag{2.11}$$

where the constants τ_0, N_0 and C depend only on T and $\|u^0\|_{H^1}$.

The rest of this paper is devoted to the construction of the method (2.8)–(2.9) and the proof of Theorem 2.1.

Remark 2.2. The analysis in this article can be easily extended to proving higher-order convergence of the spatial discretisation method when the initial data is smoother. Namely, for $u^0\in H^s(\mathbb{T})$ with $s>1$, the error bound of of the proposed method should become

$$\max_{1\le n\le L}\|u(t_n,\cdot)-u^n_{\tau,N}\|_{L^2} \le C\big(\tau+N^{-s}\big). \tag{2.12}$$

The proof of this result (with smoother initial data) is easier than the proof of Theorem 2.1 and therefore omitted. The convergence results in (2.11) and (2.12) are illustrated by the numerical experiments in section 6 for $s=1$ and $s=2$, respectively.

3. Notation and technical tools

In this section we introduce the basic notation and technical lemmas to be used in analysing the error of the numerical method to be constructed.

3.1. Notation

The inner product and norm on $L^2(\mathbb{T})$ are denoted by

$$(f,g)=\int_{\mathbb{T}} f(x)\overline{g(x)}\,dx \quad \text{and} \quad \|f\|_{L^2}=\sqrt{(f,f)}, \text{ respectively.}$$

The norm on the Sobolev space $H^s(\mathbb{T})$, $s\in\mathbb{R}$, is denoted by

$$\|f\|_{H^s}^2 = 2\pi\sum_{k\in\mathbb{Z}}(1+k^2)^s|\hat{f}_k|^2.$$

For a function $f:[0,T]\times\mathbb{T}\to\mathbb{C}$ we denote by $\|f\|_{L^p(0,T;H^s)}$ its space-time Sobolev norm, defined by

$$\|f\|_{L^p(0,T;H^s)}=\begin{cases}\left(\int_0^T \|f(t)\|_{H^s}^p\,dt\right)^{\frac{1}{p}} & \text{for } p\in[1,\infty),\\ \operatorname*{ess\,sup}_{t\in[0,T]}\|f(t)\|_{H^s} & \text{for } p=\infty.\end{cases}$$

The Fourier coefficients of a function f on $\mathbb{T}$ are denoted by $\mathcal{F}_k[f]$ or simply $\hat{f}_k$, defined by

$$\hat{f}_k=\frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx}f(x)\,dx \quad \text{for } k\in\mathbb{Z}.$$

The Fourier inversion formula is given by

$$f(x)=\sum_{k\in\mathbb{Z}} e^{ikx}\hat{f}_k.$$

The Fourier coefficients are known to have the following properties:

$$\|f\|_{L^2}^2 = 2\pi\sum_{k\in\mathbb{Z}}\left|\hat{f}_k\right|^2 \qquad \text{(Plancherel identity);}$$

$$\mathcal{F}_k[fg]=\sum_{k_1\in\mathbb{Z}}\hat{f}_{k-k_1}\hat{g}_{k_1} \quad \text{(Convolution).}$$

For any function $\sigma:\mathbb{Z}\to\mathbb{C}$ such that $|\sigma(k)|\le C_\sigma(1+|k|)^m$ for some constants C_σ and $m\ge 0$, we denote by $\sigma(i^{-1}\partial_x):H^s(\mathbb{T})\to H^{s-m}(\mathbb{T})$ the operator defined by

$$\sigma(i^{-1}\partial_x)f=\sum_{k\in\mathbb{Z}}\sigma(k)\hat{f}_k e^{ikx}.$$

For abbreviation, we denote

$$\langle k\rangle=(1+k^2)^{\frac{1}{2}} \quad \text{and} \quad J^s=\langle i^{-1}\partial_x\rangle^s,$$

which imply that

$$\|f\|_{H^s}^2=\|J^sf\|_{L^2}^2 \quad \text{and} \quad \widehat{J^sf}_k=\langle k\rangle^s\hat{f}_k.$$

Moreover, we denote by $\partial_x^{-1}:H^s(\mathbb{T})\to H^{s+1}(\mathbb{T})$, $s\in\mathbb{R}$, the operator such that

$$\mathcal{F}_k[\partial_x^{-1}f]=\begin{cases}(ik)^{-1}\hat{f}_k, & \text{when } k\neq 0,\\ 0, & \text{when } k=0.\end{cases} \tag{3.1}$$

We denote by $A\lesssim B$ or $B\gtrsim A$ the statement $A\le CB$ for some constant $C>0$. The value of C may depend on T and $\|u^0\|_{H^1}$, and may be different at different occurrences, but is always independent of τ, N and n. The notation $A\sim B$ means that $A\lesssim B\lesssim A$.

We denote by $O(Y)$ any quantity X such that $X\lesssim Y$. For any function $\sigma:\mathbb{Z}^{m+1}\to\mathbb{C}$ and $w\in H^1(\mathbb{T})$ we denote by $\mathcal{T}_m(\sigma;w)$ the class of functions $f\in L^2(\mathbb{T})$ such that

$$\hat{f}_k\lesssim\sum_{k_1+\cdots+k_m=k}|\sigma(k,k_1,\cdots,k_m)|\,|\hat{w}_{k_1}|\cdots|\hat{w}_{k_m}| \quad \forall\, f\in\mathcal{T}_m(\sigma;w). \tag{3.2}$$

If $F = \int_{t_1}^{t_2} f(t)dt$ for some function $f(t) \in \mathcal{T}_m(\sigma; v(t))$, then we simply denote

$$F \in \int_{t_1}^{t_2} \mathcal{T}_m(\sigma; v(t))dt. \tag{3.3}$$

3.2. Two technical lemmas

We will use the following version of the Kato–Ponce inequalities, which was originally proved in [12] and subsequently improved to cover the endpoint case in [3, 13].

Lemma 3.1 (The Kato–Ponce inequalities)**.**

(i) *If $s > \frac{1}{2}$ and $f, g \in H^s(\mathbb{T})$ then*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}.$$

(ii) *If $s \geq 0, s_1 > \frac{1}{2}$, $f \in H^{s+s_1}(\mathbb{T})$ and $g \in H^s(\mathbb{T})$, then*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^{s+s_1}}\|g\|_{H^s}.$$

In addition to Lemma 3.1 we also need the following results, which are consequences of the Kato–Ponce inequalities.

Lemma 3.2.

(i) *If $s > \frac{1}{2}$ and $f, g \in H^s(\mathbb{T})$ then*

$$\|J^{-1}(Jf\, g)\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}.$$

(ii) *If $f, g \in H^1(\mathbb{T})$ then*

$$\|J^{-1}(Jf\, g)\|_{L^2} \lesssim \min\left\{\|f\|_{L^2}\|g\|_{H^1}, \|g\|_{L^2}\|f\|_{H^1}\right\}.$$

Proof. (i) The desired inequality is equivalent to $\|J^{s-1}(Jf\, g)\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{H^s}$. By the duality between $L^2(\mathbb{T})$ and itself, it suffices to prove

$$(J^{s-1}(Jf\, g), h) \lesssim \|f\|_{H^s}\|g\|_{H^s}\|h\|_{L^2} \quad \forall\, h \in L^2(\mathbb{T}),$$

which is equivalent to

$$\sum_k \sum_{k_1+k_2=k} \langle k\rangle^{s-1}\langle k_1\rangle \hat{f}_{k_1}\hat{g}_{k_2}\overline{\hat{h}_k} \lesssim \|f\|_{H^s}\|g\|_{H^s}\|h\|_{L^2}.$$

Since the term corresponding to $k = 0$ satisfies

$$\begin{aligned}
\sum_{k_1+k_2=0} \langle k_1\rangle \hat{f}_{k_1}\hat{g}_{k_2}\overline{\hat{h}_0} &= \sum_{k_1} \langle k_1\rangle^{\frac{1}{2}}\hat{f}_{k_1}\langle -k_1\rangle^{\frac{1}{2}}\hat{g}_{-k_1}\overline{\hat{h}_0} \\
&\lesssim \|(\langle k_1\rangle^{\frac{1}{2}}\hat{f}_{k_1})_{k_1\in\mathbb{Z}}\|_{l^2}\|(\langle -k_1\rangle^{\frac{1}{2}}\hat{g}_{-k_1})_{k_1\in\mathbb{Z}}\|_{l^2}|\hat{h}_0| \\
&\lesssim \|f\|_{H^{\frac{1}{2}}}\|g\|_{H^{\frac{1}{2}}}\|h\|_{L^1} \\
&\lesssim \|f\|_{H^s}\|g\|_{H^s}\|h\|_{L^2} \quad \text{when } s > \frac{1}{2},
\end{aligned}$$

we only need to prove the following result:

$$\sum_{k\neq 0} \sum_{k_1+k_2=k} |k|^{s-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\overline{\hat{h}_k} \lesssim \|f\|_{H^s}\|g\|_{H^s}\|h\|_{L^2}.$$

To this end, we decompose the left-hand side of the inequality above into two parts, i.e.,

$$\begin{aligned}
&\sum_{k\neq 0} \sum_{k_1+k_2=k} |k|^{s-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\overline{\hat{h}_k} \\
&\lesssim \sum_{k\neq 0} \sum_{\substack{k_1+k_2=k \\ |k_1|\leq 10|k|}} |k|^{s-1}|k_1||\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_k| + \sum_{k\neq 0} \sum_{\substack{k_1+k_2=k \\ |k_1|>10|k|}} |k|^{s-1}|k_1||\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_k|.
\end{aligned} \tag{3.4}$$

The first term on the right-hand side of (3.4) can be estimated by using Plancherel's identity and Lemma 3.1 as follows:

$$\begin{aligned}\sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|\le 10|k|}}|k|^{-1+s}|k_1||\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_k| &\lesssim \sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|\le 10|k|}}|k|^s|\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_k|\\ &\lesssim (J^s(\tilde{f}\tilde{g}),\tilde{h})\\ &\lesssim \big\|\tilde{f}\tilde{g}\big\|_{H^s}\|\tilde{h}\|_{L^2}\lesssim \|\tilde{f}\|_{H^s}\|\tilde{g}\|_{H^s}\|\tilde{h}\|_{L^2},\end{aligned}$$

where $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ are functions with Fourier coefficients $|\hat{f}_k|$, $|\hat{g}_k|$ and $|\hat{h}_k|$, respectively. Since

$$\|\tilde{f}\|_{H^s}\sim\|f\|_{H^s},\quad \|\tilde{g}\|_{H^s}\sim\|g\|_{H^s}\quad\text{and}\quad \|\tilde{h}\|_{L^2}\sim\|h\|_{L^2},$$

it follows that

$$\sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|\le 10|k|}}|k|^{-1+s}|k_1||\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_k|\lesssim \|f\|_{H^s}\|g\|_{H^s}\|h\|_{L^2}.$$

In the second term on the right-hand side of (3.4), we have $|k_1|\sim|k_2|>|k|$. For $s>\frac{1}{2}$ we have

$$|k|^{s-1}|k_1|=|k|^{-s}|k|^{2s-1}|k_1|\le|k|^{-s}|k_1|^{2s}\sim|k|^{-s}|k_1|^s|k_2|^s$$

and therefore

$$\begin{aligned}\sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|>10|k|}}|k|^{s-1}|k_1||\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_k| &\lesssim \sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|>10|k|}}|k|^{-s}|k_1|^s|k_2|^s|\hat{f}_{k_1}||\hat{g}_{k_2}||\hat{h}_k|\\ &\lesssim \sum_{k\neq 0}\mathcal{F}_k[J^s\tilde{f}\,J^s\tilde{g}]|k|^{-s}|\hat{h}_k|\\ &\lesssim \max_k|\mathcal{F}_k[J^s\tilde{f}\,J^s\tilde{g}]|\sum_{k\neq 0}|k|^{-s}|\hat{h}_k|\\ &\lesssim \|J^s\tilde{f}\,J^s\tilde{g}\|_{L^1}\|(|k|^{-s})_{0\neq k\in\mathbb{Z}}\|_{l^2}\|(|\hat{h}_k|)_{0\neq k\in\mathbb{Z}}\|_{l^2}\\ &\lesssim \|J^s\tilde{f}\|_{L^2}\|J^s\tilde{g}\|_{L^2}\|\tilde{h}\|_{L^2}\\ &\lesssim \|f\|_{H^s}\|g\|_{H^s}\|h\|_{L^2}.\end{aligned}$$

This completes the proof of (i).

(ii) Similarly as (i), it suffices to prove

$$\sum_{k\neq 0}\sum_{k_1+k_2=k}|k|^{-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\overline{\hat{h}_k}\lesssim\min(\|f\|_{L^2}\|g\|_{H^1},\|f\|_{H^1}\|g\|_{L^2})\|h\|_{L^2}\quad\forall\,h\in L^2(\mathbb{T}).$$

In view of the proof of (i), we can assume $\hat{f}_k\ge 0$, $\hat{g}_k\ge 0$ and $\hat{h}_k\ge 0$ without loss of generality (otherwise we can replace f, g and h by $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$, respectively, in the estimates below). Then

$$\begin{aligned}&\sum_{k\neq 0}\sum_{k_1+k_2=k}|k|^{-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\hat{h}_k\\ &\lesssim\sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|\le 10|k|}}|k|^{-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\hat{h}_k+\sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|>10|k|}}|k|^{-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\hat{h}_k.\end{aligned}\tag{3.5}$$

The first term on the right-hand side of (3.5) can be estimated by using Plancherel's identity and Lemma 3.1:

$$\sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|\le 10|k|}}|k|^{-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\hat{h}_k\lesssim\sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\|k_1|\le 10|k|}}\hat{f}_{k_1}\hat{g}_{k_2}\hat{h}_k$$

$$
\begin{aligned}
&\lesssim \sum_{k\neq 0} \mathcal{F}_k[fg]\hat{h}_k \\
&\lesssim \|(\mathcal{F}_k[fg])_{0\neq k\in\mathbb{Z}}\|_{l^2}\|(\hat{h}_k)_{0\neq k\in\mathbb{Z}}\|_{l^2} \\
&\lesssim \|fg\|_{L^2}\|h\|_{L^2} \\
&\lesssim \min(\|f\|_{L^2}\|g\|_{L^\infty}, \|f\|_{L^\infty}\|g\|_{L^2})\|h\|_{L^2} \\
&\lesssim \min(\|f\|_{L^2}\|g\|_{H^1}, \|f\|_{H^1}\|g\|_{L^2})\|h\|_{L^2}.
\end{aligned}
$$

In the second term on the right-hand side of (3.5) we have $|k_1| \sim |k_2| > k$. On the one hand, we have

$$
\begin{aligned}
\sum_{k\neq 0}\sum_{\substack{k_1+k_2=k\\ |k_1|>10|k|}} |k|^{-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\hat{h}_k &\lesssim \sum_{k\neq 0}\sum_{k_2} |k|^{-1}|k_2|\hat{f}_{k-k_2}\hat{g}_{k_2}\hat{h}_k \\
&\lesssim \Big(\sup_k \sum_{k_2} |\hat{f}_{k-k_2}|^2\Big)^{\frac{1}{2}} \Big(\sum_{k_2} |k_2|^2|\hat{g}_{k_2}|^2\Big)^{\frac{1}{2}} \sum_{k\neq 0} |k|^{-1}\hat{h}_k \\
&\lesssim \|f\|_{L^2}\|g\|_{H^1}\|h\|_{L^2}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\sum_{k\neq 0}\sum_{\substack{k=k_1+k_2\\ |k_1|>10|k|}} |k|^{-1}|k_1|\hat{f}_{k_1}\hat{g}_{k_2}\hat{h}_k &\lesssim \sum_{k\neq 0}\sum_{k_1} |k|^{-1}|k_1|\hat{f}_{k_1}\hat{g}_{k-k_1}\hat{h}_k \\
&\lesssim \Big(\sum_{k_1} |k_1|^2|\hat{f}_{k_1}|^2\Big)^{\frac{1}{2}} \Big(\sup_k \sum_{k_1} |\hat{g}_{k-k_1}|^2\Big)^{\frac{1}{2}} \sum_{k\neq 0} |k|^{-1}\hat{h}_k \\
&\lesssim \|f\|_{H^1}\|g\|_{L^2}\|h\|_{L^2}.
\end{aligned}
$$

This completes the proof of (ii). □

4. Construction of the method through analysing consistency error

In this section we construct the numerical method based on twisted variables and Duhamel's formula through analysing the consistency errors in approximating the exponential integrals using harmonic analysis techniques. For readers' convenience, we present the derivation of the numerical method in subsection 4.1 and defer the technical estimates to subsection 4.2.

4.1. Construction of the numerical method

As mentioned in the introduction section and the beginning of section 2, the NLS equation (1.1) has a unique solution $u \in C([0,T];H^1(\mathbb{T}))$ satisfying the Duhamel's formula:

$$
u(t_{n+1}) = \mathrm{e}^{i\tau\partial_x^2}u(t_n) - i\lambda\int_0^\tau \mathrm{e}^{i(t_{n+1}-(t_n+s))\partial_x^2}|u(t_n+s)|^2u(t_n+s)\,ds, \tag{4.1}
$$

as well as the mass and momentum conservations (2.1)–(2.2). The norm $\|u\|_{C([0,T];H^1(\mathbb{T}))}$ is bounded by a constant depending on $\|u^0\|_{H^1}$; see [2].

Let $v(t) := \mathrm{e}^{-it\partial_x^2}u(t)$ be the twisted variable. Then $v \in C([0,T];H^1(\mathbb{T}))$ satisfies $\|v\|_{C([0,T];H^1(\mathbb{T}))} = \|u\|_{C([0,T];H^1(\mathbb{T}))}$ and the following conservation laws simiarly as u, i.e.,

(1) Mass conservation:

$$
\frac{1}{2\pi}\int_{\mathbb{T}} |v(t,x)|^2\,dx = \frac{1}{2\pi}\int_{\mathbb{T}} |u(t,x)|^2\,dx = M \quad \text{for } t>0. \tag{4.2}
$$

(2) Momentum conservation:

$$\frac{1}{2\pi}\int_{\mathbb{T}} v(t,x)\partial_x\bar{v}(t,x)\,dx = \frac{1}{2\pi}\int_{\mathbb{T}} u(t,x)\partial_x\bar{u}(t,x)\,dx = P \quad \text{for } t>0. \tag{4.3}$$

Applying the operator $\mathrm{e}^{-it_{n+1}\partial_x^2}$ to the identity (4.1), we obtain

$$v(t_{n+1}) = v(t_n) - i\lambda\int_0^\tau \mathrm{e}^{-i(t_n+s)\partial_x^2}\big[|\mathrm{e}^{i(t_n+s)\partial_x^2}v(t_n+s)|^2\,\mathrm{e}^{i(t_n+s)\partial_x^2}v(t_n+s)\big]\,ds. \tag{4.4}$$

The Fourier coefficients of both sides of (4.4) should be equal, i.e.,

$$\hat{v}_k(t_{n+1}) = \hat{v}_k(t_n) - i\lambda\int_0^\tau \sum_{k_1+k_2+k_3=k} \mathrm{e}^{i(t_n+s)\phi}\,\hat{\bar{v}}_{k_1}(t_n+s)\hat{v}_{k_2}(t_n+s)\hat{v}_{k_3}(t_n+s)\,ds, \tag{4.5}$$

with a phase function

$$\phi = \phi(k,k_1,k_2,k_3) = k^2 + k_1^2 - k_2^2 - k_3^2.$$

Replacing τ and s in (4.5) by s and σ, respectively, we have

$$\hat{v}_k(t_n+s) = \hat{v}_k(t_n) - i\lambda\int_0^s \sum_{k_1+k_2+k_3=k} \mathrm{e}^{i(t_n+\sigma)\phi}\,\hat{\bar{v}}_{k_1}(t_n+s)\hat{v}_{k_2}(t_n+\sigma)\hat{v}_{k_3}(t_n+\sigma)\,d\sigma. \tag{4.6}$$

In view of (4.6) and the definition of $\mathcal{T}_m(M;v)$ in (3.2), we have

$$v(t_n+s) - v(t_n) \in \int_0^s \mathcal{T}_3(1; v(t_n+\sigma))d\sigma. \tag{4.7}$$

As a result, (4.5) can be written as

$$\hat{v}_k(t_{n+1}) = \hat{v}_k(t_n) - i\lambda\sum_{k_1+k_2+k_3=k} \hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\int_0^\tau \mathrm{e}^{i(t_n+s)\phi}\,ds + \hat{\mathcal{R}}_{1,k}, \tag{4.8}$$

with

$$\begin{aligned}
&\hat{\mathcal{R}}_{1,k}\\
&= -i\lambda\int_0^\tau \sum_{k_1+k_2+k_3=k} \mathrm{e}^{i(t_n+s)\phi}\,\big(\hat{\bar{v}}_{k_1}(t_n+s)\hat{v}_{k_2}(t_n+s)\hat{v}_{k_3}(t_n+s) - \hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\big)\,ds\\
&\in \int_0^\tau\int_0^s \mathcal{T}_5(1; v(t_n+\sigma))d\sigma ds,
\end{aligned}$$

where the last inclusion is based on the definition in (3.3). If $\mathcal{R}_1$ denotes the function with Fourier coefficients $\hat{\mathcal{R}}_{1,k}$, then the relation above implies that (according to Lemma 4.1 (i) of the next subsection)

$$\big\|\mathcal{R}_1\big\|_{H^1} \lesssim \tau^2\|v\|^5_{L_t^\infty H_x^1}. \tag{4.9}$$

This term will be dropped in our numerical scheme.

In the following, we approximate the second term on the right-hand side of (4.8) by expressions that can be evaluated efficiently with FFT. To this end, we consider the three cases $k=0$, $|k|>N$ and $0\neq|k|\le N$, separately.

Case 1: $k=0$. In this case, (4.8) reduces to

$$\begin{aligned}
\hat{v}_0(t_{n+1}) =&\hat{v}_0(t_n) - i\lambda\sum_{k_1+k_2+k_3=0} \hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\int_0^\tau \mathrm{e}^{i(t_n+s)(k_1^2-k_2^2-k_3^2)}\,ds + \hat{\mathcal{R}}_{1,0}\\
=&\hat{v}_0(t_n) - i\lambda\tau\sum_{k_1+k_2+k_3=0} \mathrm{e}^{it_n(k_1^2-k_2^2-k_3^2)}\,\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) + \hat{\mathcal{R}}_{1,0} + \hat{\mathcal{R}}_{2,0}\\
=&\hat{v}_0(t_n) - i\lambda\tau\Pi_0\big(\big|\mathrm{e}^{it_n\partial_x^2}v(t_n)\big|^2\mathrm{e}^{it_n\partial_x^2}v(t_n)\big) + \hat{\mathcal{R}}_{1,0} + \hat{\mathcal{R}}_{2,0}
\end{aligned} \tag{4.10}$$

$$=\hat{v}_0(t_n) - i\lambda\tau\Pi_0\big[\Pi_N\big(\big|e^{it_n\partial_x^2}v(t_n)\big|^2\big)e^{it_n\partial_x^2}v(t_n)\big] + \hat{\mathcal{R}}_{1,0} + \hat{\mathcal{R}}_{2,0} + \hat{\mathcal{R}}^*_{2,0}, \qquad (4.11)$$

with

$$\hat{\mathcal{R}}_{2,k} = -i\lambda \sum_{k_1+k_2+k_3=k} \hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\int_0^\tau (e^{i(t_n+s)(k_1^2-k_2^2-k_3^2)} - e^{it_n(k_1^2-k_2^2-k_3^2)})\,ds,$$

$$\mathcal{R}_2^* = -i\lambda\tau\big[(1-\Pi_N)\big(\big|e^{it_n\partial_x^2}v(t_n)\big|^2\big)\big]e^{it_n\partial_x^2}v(t_n) \in \tau e^{it_n\partial_x^2}v(t_n)\mathcal{T}_2(1_{>N};v(t_n)),$$

where

$$(1_{>N})_k = 1_{|k|>N} = \begin{cases} 0 & \text{for } |k| \le N, \\ 1 & \text{for } |k| > N. \end{cases} \qquad (4.12)$$

Since $k_1+k_2+k_3=0$, it follows that there holds $k_1^2-k_2^2-k_3^2 = 2k_2k_3$ and therefore

$$\int_0^\tau \Big(e^{i(t_n+s)(k_1^2-k_2^2-k_3^2)} - e^{it_n(k_1^2-k_2^2-k_3^2)}\Big)\,ds = \tau^2 O(k_2k_3).$$

As a result, the function $\mathcal{R}_2$ (with Fourier coefficients $\hat{\mathcal{R}}_{2,k}$) satisfies that $\mathcal{R}_2 \in \tau^2\mathcal{T}_3(k_2k_3;v(t_n))$ in view of the definition in (3.2). According to Lemma 4.1 (i)–(ii) of the next subsection, $\mathcal{R}_2$ and $\mathcal{R}_2^*$ satisfy the following estimates:

$$|\hat{\mathcal{R}}_{2,0}| \lesssim \tau^2\|v\|^3_{L_t^\infty H_x^1}, \qquad (4.13)$$

$$|\hat{\mathcal{R}}^*_{2,0}| \lesssim \|\mathcal{R}_2^*\|_{L^1} \lesssim \tau\|e^{it_n\partial_x^2}v(t_n)\|_{L^\infty}\|(1-\Pi_N)(|e^{it_n\partial_x^2}v(t_n)|^2)\|_{L^2} \lesssim \tau N^{-1}\|v\|^3_{L_t^\infty H_x^1}. \qquad (4.14)$$

The two terms $\hat{\mathcal{R}}_{2,0}$ and $\hat{\mathcal{R}}^*_{2,0}$ will be dropped in our numerical scheme.

Case 2: $|k| > N$. Let $\mathcal{R}_3$ be the function with Fourier coefficients

$$\hat{\mathcal{R}}_{3,k} = -1_{|k|>N}\, i\lambda \sum_{k_1+k_2+k_3=k} \hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\int_0^\tau e^{i(t_n+s)\phi}\,ds.$$

Then

$$\mathcal{R}_3 \in \tau\,\mathcal{T}_3\left(1_{>N};v(t_n)\right).$$

Lemma 4.1 (i) of the next subsection implies that

$$\big\|\mathcal{R}_3\big\|_{H^s} \lesssim \tau N^{-1+s}\|v\|^3_{L_t^\infty H_x^1} \quad \text{for } s\in[0,1]. \qquad (4.15)$$

This term will be dropped in the numerical scheme.

Case 3: $0 \ne |k| \le N$. By using the identity

$$1 = \frac{(k_1+k_2)+(k_1+k_3)-k_1}{k}$$

and symmetry between k_2 and k_3, we can decompose the second term on the right-hand side of (4.8) into two parts, i.e.,

$$\hat{v}_k(t_{n+1}) = \hat{v}_k(t_n) - 2i\lambda \sum_{k_1+k_2+k_3=k} \hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\int_0^\tau \frac{k_1+k_2}{k}e^{i(t_n+s)\phi}\,ds \qquad (4.16a)$$

$$+\, i\lambda \sum_{k_1+k_2+k_3=k} \hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\int_0^\tau \frac{k_1}{k}e^{i(t_n+s)\phi}\,ds \qquad (4.16b)$$

$$+\, \hat{\mathcal{R}}_{1,k}. \qquad (4.16c)$$

We furthermore truncate (4.16a) to the frequency domain $|k_1+k_3| \le N$, i.e.,

$$(4.16a) = \hat{v}_k(t_n) - 2i\lambda \sum_{\substack{k_1+k_2+k_3=k \\ |k_1+k_3|\le N}} \left(\int_0^\tau \frac{k_1+k_2}{k}e^{i(t_n+s)\phi}\,ds\right)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) + \hat{\mathcal{R}}_{4,k}, \qquad (4.17)$$

with

$$\hat{\mathcal{R}}_{4,k} = \begin{cases} -2i\lambda \displaystyle\sum_{\substack{k_1+k_2+k_3=k \\ |k_1+k_3|>N}} \left(\int_0^\tau \frac{k_1+k_2}{k} \mathrm{e}^{i(t_n+s)\phi}\, ds \right) \hat{\bar{v}}_{k_1}(t_n) \hat{v}_{k_2}(t_n) \hat{v}_{k_3}(t_n) & \text{for } 0 \neq |k| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding function $\mathcal{R}_4$ with Fourier coefficients $\hat{\mathcal{R}}_{4,k}$ satisfies that

$$\mathcal{R}_4 \in \tau\, \mathcal{T}_3 \left(\frac{k_1+k_2}{k} 1_{0\neq|k|\leq N} 1_{|k_1+k_3|>N}; v(t_n) \right).$$

By Lemma 4.1 (iii) in the next subsection and symmetry between k_2 and k_3, and we have

$$\left\| \mathcal{R}_4 \right\|_{H^s} \lesssim \tau N^{-1+s} \|v\|^3_{L^\infty_t H^1_x} \quad \text{for } s \in [0,1]. \tag{4.18}$$

Since $k_1 + k_2 + k_3 = k$, it is straightforward to verify that $\phi = 2(k_1 + k_2)(k_1 + k_3)$. As a result, if $k_1 + k_3 \neq 0$ then

$$\int_0^\tau \frac{k_1+k_2}{k} \mathrm{e}^{i(t_n+s)\phi}\, ds = \frac{1}{2ik(k_1+k_3)} \left(\mathrm{e}^{it_{n+1}\phi} - \mathrm{e}^{it_n\phi} \right); \tag{4.19}$$

If $k_1 + k_3 = 0$ then $\phi = 0$ and $k = k_2$, and therefore

$$\int_0^\tau \frac{k_1+k_2}{k} \mathrm{e}^{i(t_n+s)\phi}\, ds = \tau \Big(\frac{k_1}{k} + 1 \Big). \tag{4.20}$$

Substituting the two relations (4.19)–(4.20) into (4.17), we obtain

$$\begin{aligned} \text{(4.16a)} = \hat{v}_k(t_n) - \lambda & \sum_{\substack{k_1+k_2+k_3=k \\ 0\neq|k_1+k_3|\leq N}} \frac{1}{k(k_1+k_3)} \left(\mathrm{e}^{it_{n+1}\phi} - \mathrm{e}^{it_n\phi} \right) \hat{\bar{v}}_{k_1}(t_n) \hat{v}_{k_2}(t_n) \hat{v}_{k_3}(t_n) \\ & - 2i\lambda\tau \sum_{k_1+k_3=0} \left(\frac{k_1}{k} + 1 \right) \hat{\bar{v}}_{k_1}(t_n) \hat{v}_k(t_n) \hat{v}_{k_3}(t_n) + \hat{\mathcal{R}}_{4,k}. \end{aligned}$$

Then we apply the mass and momentum conservations in (4.2)–(4.3), which imply that

$$-2i\lambda\tau \sum_{k_1+k_3=0} \left(\frac{k_1}{k} + 1 \right) \hat{\bar{v}}_{k_1}(t_n) \hat{v}_k(t_n) \hat{v}_{k_3}(t_n) = -2i\lambda\tau P\, (ik)^{-1} \hat{v}_k(t_n) - 2i\lambda\tau M\, \hat{v}_k(t_n).$$

Therefore,

$$\begin{aligned} \text{(4.16a)} =& \hat{v}_k(t_n) - \lambda \sum_{\substack{k_1+k_2+k_3=k \\ 0\neq|k_1+k_3|\leq N}} \frac{1}{k(k_1+k_3)} \left(\mathrm{e}^{it_{n+1}\phi} - \mathrm{e}^{it_n\phi} \right) \hat{\bar{v}}_{k_1}(t_n) \hat{v}_{k_2}(t_n) \hat{v}_{k_3}(t_n) \\ & - 2i\lambda\tau P\, (ik)^{-1} \hat{v}_k(t_n) - 2i\lambda\tau M\, \hat{v}_k(t_n) + \hat{\mathcal{R}}_{4,k} \\ =& e^{-2i\lambda\tau P\, (ik)^{-1} - 2i\lambda\tau M}\, \hat{v}_k(t_n) \\ & - \lambda \sum_{\substack{k_1+k_2+k_3=k \\ 0\neq|k_1+k_3|\leq N}} \frac{1}{k(k_1+k_3)} \left(\mathrm{e}^{it_{n+1}\phi} - \mathrm{e}^{it_n\phi} \right) \hat{\bar{v}}_{k_1}(t_n) \hat{v}_{k_2}(t_n) \hat{v}_{k_3}(t_n) \\ & + \hat{\mathcal{R}}_{4,k} + \hat{\mathcal{R}}^*_{4,k}, \end{aligned} \tag{4.21}$$

where

$$\hat{\mathcal{R}}^*_{4,k} = \begin{cases} \left(1 - 2i\lambda\tau P (ik)^{-1} - 2i\lambda\tau M - e^{-2i\lambda\tau P (ik)^{-1} - 2i\lambda\tau M} \right) \hat{v}_k(t_n) & \text{for } 0 \neq |k| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

From this expression we see that the function $\mathcal{R}^*_4$ with Fourier coefficients $\hat{\mathcal{R}}^*_{4,k}$ satisfies that

$$\mathcal{R}^*_4 \in \tau^2 T_1(1; v(t_n)). \tag{4.22}$$

Note that for $k_1+k_2+k_3=k$ the following equalities hold:

$$\phi(k,k_1,k_2,k_3)=2kk_1+2k_2k_3, \tag{4.23a}$$

$$2kk_1=k^2+k_1^2-(k_2+k_3)^2, \tag{4.23b}$$

$$2k_2k_3=(k_2+k_3)^2-k_2^2-k_3^2. \tag{4.23c}$$

By using these relations, we have

$$e^{i(t_n+s)\phi}=e^{it_n\phi}e^{2iskk_1}e^{2isk_2k_3}=e^{it_n\phi}[e^{2iskk_1}+(e^{2isk_2k_3}-1)+(e^{2iskk_1}-1)(e^{2isk_2k_3}-1)],$$

and therefore (4.16b) can be decomposed into the following three terms:

$$\begin{aligned}(4.16\text{b}) =& i\lambda\sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\le N}}\left(\int_0^\tau\frac{k_1}{k}\mathrm{e}^{it_n\phi}\mathrm{e}^{2iskk_1}\,ds\right)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) && (4.24\text{-}1)\\ &+i\lambda\sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\le N}}\left(\int_0^\tau\frac{k_1}{k}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{2isk_2k_3}-1\big)\,ds\right)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) && (4.24\text{-}2)\\ &+\widehat{\mathcal{R}}_{5,k}+\widehat{\mathcal{R}}^*_{5,k}, && (4.24\text{-}3)\end{aligned}$$

where

$$\widehat{\mathcal{R}}_{5,k}=i\lambda\sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\le N}}\left(\int_0^\tau\frac{k_1}{k}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{2iskk_1}-1\big)\big(\mathrm{e}^{2isk_2k_3}-1\big)\,ds\right)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n), \tag{4.25}$$

$$\widehat{\mathcal{R}}^*_{5,k}=i\lambda\sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|>N}}\left(\int_0^\tau\frac{k_1}{k}\mathrm{e}^{i(t_n+s)\phi}\,ds\right)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n), \tag{4.26}$$

for $0\ne|k|\le N$, and $\widehat{\mathcal{R}}_{5,k}=\widehat{\mathcal{R}}^*_{5,k}=0$ for $k=0$ and $|k|>N$. Lemma 4.2 of the next subsection implies that

$$\big\|\mathcal{R}_5\big\|_{H^s}\lesssim\tau^{\frac32}\|v\|^3_{L^\infty_tH^1_x}\quad\text{for }\ s\in(\tfrac12,1), \tag{4.27a}$$

$$\big\|\mathcal{R}_5\big\|_{L^2}\lesssim\tau^2\sqrt{\ln\tau^{-1}}\|v\|^3_{L^\infty_tH^1_x}. \tag{4.27b}$$

Obviously,

$$\mathcal{R}^*_5\in\tau\mathcal{T}_3\left(\sigma;v(t_n)\right)\quad\text{ with some }\ |\sigma(k,k_1,k_2,k_3)|\le|k|^{-1}|k_1|1_{0\ne|k|\le N}1_{|k_2+k_3|>N}.$$

By Lemma 4.1 (iii) and symmetry, and we have that for any $s\in[0,1]$,

$$\big\|\mathcal{R}^*_5\big\|_{H^s}\lesssim\tau N^{-1+s}\|v\|^3_{L^\infty_tH^1_x}. \tag{4.28}$$

Note that

$$(4.24\text{-}1)=\sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\le N}}\frac{\lambda}{2k^2}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{i\tau(k^2+k_1^2-(k_2+k_3)^2)}-1\big)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n), \tag{4.29}$$

$$\begin{aligned}(4.24\text{-}2)&=i\lambda\sum_{\substack{k_1+k_2+k_3=k\\k_2\ne0,k_3\ne0\\|k_2+k_3|\le N}}\left(\int_0^\tau\frac{k_1}{k}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{2isk_2k_3}-1\big)\,ds\right)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\\&=\lambda\sum_{\substack{k_1+k_2+k_3=k\\k_2\ne0,k_3\ne0\\|k_2+k_3|\le N}}\frac{k_1}{2kk_2k_3}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{2i\tau k_2k_3}-1\big)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\end{aligned}$$

$$
\begin{aligned}
&- i\lambda\tau \sum_{\substack{k_1+k_2+k_3=k\\ k_2\neq 0,k_3\neq 0\\ |k_2+k_3|\leq N}} \frac{k_1}{k} \mathrm{e}^{it_n\phi}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\\
=&\ \lambda \sum_{\substack{k_1+k_2+k_3=k\\ k_2\neq 0,k_3\neq 0\\ |k_2+k_3|\leq N}} \frac{k_1}{2kk_2k_3}\mathrm{e}^{it_n\phi}\Big(\mathrm{e}^{i\tau\big((k_2+k_3)^2-k_2^2-k_3^2\big)}-1\Big)\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\\
&- i\lambda\tau \sum_{\substack{k_1+k_2+k_3=k\\ |k_2+k_3|\leq N}} \frac{k_1}{k} \mathrm{e}^{it_n\phi}\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\\
&+ 2i\lambda\tau \sum_{\substack{k_1+k_2=k\\ |k_2|\leq N}} \frac{k_1}{k} \mathrm{e}^{it_n\phi}\,\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{0}(t_n)\\
&- i\lambda\tau\mathrm{e}^{it_n\phi}\,\hat{\bar{v}}_{k}(t_n)\hat{v}_{0}(t_n)\hat{v}_{0}(t_n). \qquad (4.30)
\end{aligned}
$$

Substituting (4.29)–(4.30) into (4.24), and then substituting (4.21) and (4.24) into (4.16), we obtain

$$
\begin{aligned}
\hat{v}_k(t_{n+1}) =&\mathrm{e}^{-2i\lambda\tau P\,(ik)^{-1}-2i\lambda\tau M}\ \hat{v}_k(t_n)\\
&+ \lambda \sum_{\substack{k_1+k_2+k_3=k\\ 0\neq|k_1+k_3|\leq N}} \frac{1}{ik(ik_1+ik_3)}\big(\mathrm{e}^{it_{n+1}\phi}-\mathrm{e}^{it_n\phi}\big)\,\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\\
&- \lambda \sum_{\substack{k_1+k_2+k_3=k\\ |k_2+k_3|\leq N}} \frac{1}{2(ik)^2}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{i\tau(k^2+k_1^2-(k_2+k_3)^2)}-1\big)\,\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\\
&- \lambda \sum_{\substack{k_1+k_2+k_3=k\\ k_2\neq 0,k_3\neq 0\\ |k_2+k_3|\leq N}} \frac{ik_1}{2(ik)(ik_2)(ik_3)}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{i\tau((k_2+k_3)^2-k_2^2-k_3^2)}-1\big)\,\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\\
&- i\lambda\tau \sum_{\substack{k_1+k_2+k_3=k\\ |k_2+k_3|\leq N}} \frac{k_1}{k} \mathrm{e}^{it_n\phi}\,\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\\
&+ 2i\lambda\tau \sum_{\substack{k_1+k_2=k\\ |k_2|\leq N}} \frac{k_1}{k} \mathrm{e}^{it_n\phi}\,\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{0}(t_n)\\
&- i\lambda\tau\mathrm{e}^{it_n\phi}\,\hat{\bar{v}}_{k}(t_n)\hat{v}_{0}(t_n)\hat{v}_{0}(t_n)\\
&+ \hat{\mathcal{R}}_{1,k}+\hat{\mathcal{R}}_{4,k}+\hat{\mathcal{R}}^*_{4,k}+\hat{\mathcal{R}}_{5,k}+\hat{\mathcal{R}}^*_{5,k} \quad \text{for } k\neq 0 \text{ and } |k|\leq N. \qquad (4.31)
\end{aligned}
$$

Then substituting (4.10) and (4.31) into the expression $v(t_{n+1})=\sum_{k\in\mathbb{Z}}\hat{v}_k(t_{n+1})e^{ikx}$ yields

$$
v(t_{n+1}) =\Phi^n(v(t_n);M,P)+\mathcal{R}_1+\hat{\mathcal{R}}_{2,0}+\hat{\mathcal{R}}^*_{2,0}+\mathcal{R}_3+\mathcal{R}_4+\mathcal{R}^*_4+\mathcal{R}_5+\mathcal{R}^*_5, \qquad (4.32)
$$

where

$$
\begin{aligned}
\Phi^n(f;M,P) :=&\ \mathrm{e}^{-2i\lambda\tau P\partial_x^{-1}-2i\lambda\tau M}f+(1-\mathrm{e}^{-2i\lambda\tau M})\Pi_0 f\\
&- i\lambda\tau\Pi_0\big[\Pi_N\big(\big|\mathrm{e}^{it_n\partial_x^2}f\big|^2\big)\mathrm{e}^{it_n\partial_x^2}f\big]\\
&+ \lambda\mathrm{e}^{-it_{n+1}\partial_x^2}\partial_x^{-1}\Pi_N\Big[\big(\mathrm{e}^{it_{n+1}\partial_x^2}f\big)\cdot\partial_x^{-1}\Pi_N\big(|\mathrm{e}^{it_{n+1}\partial_x^2}f|^2\big)\Big]\\
&- \lambda\mathrm{e}^{-it_n\partial_x^2}\partial_x^{-1}\Pi_N\Big[\big(\mathrm{e}^{it_n\partial_x^2}f\big)\cdot\partial_x^{-1}\Pi_N\big(|\mathrm{e}^{it_n\partial_x^2}f|^2\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\lambda}{2}\Big[\mathrm{e}^{-it_{n+1}\partial_x^2}\partial_x^{-2}\Pi_N\Big(\big(\mathrm{e}^{-it_{n+1}\partial_x^2}\bar{f}\big)\cdot \mathrm{e}^{i\tau\partial_x^2}\Pi_N\big(\mathrm{e}^{it_n\partial_x^2}f\big)^2\Big)\\
&\qquad -\mathrm{e}^{-it_n\partial_x^2}\partial_x^{-2}\Pi_N\Big(\mathrm{e}^{-it_n\partial_x^2}\bar{f}\Pi_N\big(\mathrm{e}^{it_n\partial_x^2}f\big)^2\Big)\Big]\\
&-\frac{\lambda}{2}\Big[\mathrm{e}^{-it_n\partial_x^2}\partial_x^{-1}\Pi_N\Big(\big(\mathrm{e}^{-it_n\partial_x^2}\partial_x\bar{f}\big)\cdot \mathrm{e}^{-i\tau\partial_x^2}\Pi_N\big(\mathrm{e}^{it_{n+1}\partial_x^2}\partial_x^{-1}f\big)^2\Big)\\
&\qquad -\mathrm{e}^{-it_n\partial_x^2}\partial_x^{-1}\Pi_N\Big(\big(\mathrm{e}^{-it_n\partial_x^2}\partial_x\bar{f}\big)\cdot \Pi_N\big(\mathrm{e}^{it_n\partial_x^2}\partial_x^{-1}f\big)^2\Big)\Big]\\
&-i\lambda\tau\mathrm{e}^{-it_n\partial_x^2}\partial_x^{-1}\Pi_N\Big(\mathrm{e}^{-it_n\partial_x^2}\partial_x\bar{f}\Pi_N\big(\mathrm{e}^{it_n\partial_x^2}f\big)^2\Big)\\
&+2i\lambda\tau\Pi_0(f)\mathrm{e}^{-it_n\partial_x^2}\partial_x^{-1}\Big(\mathrm{e}^{-it_n\partial_x^2}\partial_x\bar{f}\,\mathrm{e}^{it_n\partial_x^2}f\Big)\\
&-i\lambda\tau(\Pi_0 f)^2\Pi_{\neq 0}\big(\mathrm{e}^{-it_n\partial_x^2}\bar{f}\big). \qquad (4.33)
\end{aligned}
$$

The numerical scheme can be defined by dropping the defect terms $\mathcal{R}_j$ and $\mathcal{R}_j^*$ in (4.32) and replacing the numbers M and P by their approximations M_N and P_N defined in (2.10), respectively. Namely, for given $v_n \in S_N$ compute $v_{n+1} \in S_N$ by

$$
v^{n+1} = \Phi^n(v^n; M_N, P_N), \quad n = 0, 1\ldots, L-1; \quad \text{with } v^0 = u^0. \qquad (4.34)
$$

Then, replacing v^n and v^{n+1} by $\mathrm{e}^{-it_n\partial_x^2}u^n$ and $\mathrm{e}^{-it_{n+1}\partial_x^2}u^n$ in (4.34), we obtain the numerical scheme (2.8)–(2.9).

4.2. Technical lemmas for analysing the consistency errors

In this subsection, we present two technical lemmas, which are used in estimating the defect terms $\mathcal{R}_j$ and $\mathcal{R}_j^*$ in the previous subsection.

Lemma 4.1. *For any given $v \in H^1(\mathbb{T})$ and $s \in [0,1]$, the following results hold.*

(i) *Let $m \geq 1, N \in \mathbb{Z}^+$. Then, for any $f \in \mathcal{T}_m(1; v)$ and any $g \in \mathcal{T}_m(1_{>N}; v)$,*

$$
\begin{aligned}
\|f\|_{H^1} &\lesssim \|v\|_{H^1}^m;\\
\|g\|_{H^s} &\lesssim N^{-1+s}\|v\|_{H^1}^m.
\end{aligned}
$$

(ii) *For any $f \in \mathcal{T}_3(k_2k_3; v)$ there holds*

$$
|\Pi_0 f| \lesssim \|v\|_{H^1}^3.
$$

(iii) *Let $N \in \mathbb{Z}^+$, $N \geq 10$ and $f \in \mathcal{T}_3(\sigma; v)$. If*

$$
|\sigma(k,k_1,k_2,k_3)| \lesssim |k|^{-1}|k_j|\, 1_{0\neq|k|\leq N}\, 1_{|k_1+k_2|>N}, \quad \textit{for some } j \in \{1,2,3\},
$$

then

$$
\|f\|_{H^s} \lesssim N^{-1+s}\|v\|_{H^1}^3.
$$

Proof. Without loss of generality, we can assume that $\hat{v}_{k_j}, j = 1, \cdots, m$ are positive for any $t \in [0,T]$. Otherwise we replace $\hat{v}_{k_j}$ by $|\hat{v}_{k_j}|$ as we did in the proof of Lemma 3.2.

(i) By the definition of $\mathcal{T}_m(\sigma; v)$ in (3.2), $f \in \mathcal{T}_m(1; v)$ implies that

$$
|\hat{f}_k| \lesssim \sum_{k_1+\cdots+k_m=k} \hat{v}_{k_1}\cdots\hat{v}_{k_m} \sim \mathcal{F}_k[v^m].
$$

By Plancherel's identity and Lemma 3.1 (i), we obtain that

$$
\|f\|_{H^1} \lesssim \|v^m\|_{H^1} \lesssim \|v\|_{H^1}^m.
$$

For $g \in \mathcal{T}_m(1_{>N}; v)$, we use the inequality $\|g\|_{H^s} \lesssim N^{-1+s}\|g\|_{H^1}$ together with the inequality above, which implies that $\|g\|_{H^1} \lesssim \|v\|_{H^1}^m$. This yields the desired inequality for g, i.e.,

$$
\|g\|_{H^s} \lesssim N^{-1+s}\|v\|_{H^1}^m.
$$

(ii) For any $f \in \mathcal{T}_3(k_2k_3; v)$ we have that

$$
\begin{aligned}
|\Pi_0 f| &\lesssim \sum_{k_1+k_2+k_3=0} \hat{v}_{k_1}(t)\, |k_2|\hat{v}_{k_2}(t)\, |k_3|\hat{v}_{k_3}(t) \\
&\lesssim \sum_{k_1+k_1'=0} \hat{v}_{k_1}(t)\mathcal{F}_{k_1'}[(|\nabla| v(t))^2] \\
&\lesssim \int_{\mathbb{T}} v\,(|\nabla| v)^2\, dx \lesssim \|v\|_{L^\infty}\||\nabla| v\|_{L^2}^2 \lesssim \|v\|_{H^1}^3 .
\end{aligned}
$$

(iii) We only consider the case when $j = 1$, since the other cases can be treated in the same way. Since the Fourier coefficients of $J^s f$ satisfies

$$
\begin{aligned}
\mathcal{F}_k[J^s f] = \langle k\rangle^s \hat{f}_k \lesssim & \sum_{\substack{k_1+k_2+k_3=k \\ |k_1+k_2|>N}} 1_{0\neq|k|\leq N}\langle k\rangle^{-1+s}\langle k_1\rangle\, \hat{v}_{k_1}(t)\hat{v}_{k_2}(t)\hat{v}_{k_3}(t) \\
\lesssim & N^{-1+s} \sum_{\substack{k_1+k_2+k_3=k \\ |k_1+k_2|>N}} \langle k\rangle^{-1}\langle k_1+k_2\rangle\langle k_1\rangle\hat{v}_{k_1}(t)\hat{v}_{k_2}(t)\hat{v}_{k_3}(t) \\
\lesssim & N^{-1+s}\mathcal{F}_k[J^{-1}(vJ(vJv))],
\end{aligned}
$$

it follows from Lemma 3.2 (ii) that

$$
\|J^s f\|_{L^2} \lesssim N^{-1+s}\|vJv\|_{L^2}\|v\|_{H^1} \lesssim N^{-1+s}\|v\|_{L^\infty}\|Jv\|_{L^2}\|v\|_{H^1} \lesssim N^{-1+s}\|v\|_{H^1}^3 .
$$

This proves the desired results in Lemma 4.1.

□

Lemma 4.2. *If $v \in L^\infty(0,T;H^1(\mathbb{T}))$ then*

$$
\big\|\mathcal{R}_5\big\|_{L^2} \lesssim \tau^2\sqrt{\ln\tau^{-1}}\|v\|_{L^\infty(0,T;H^1)}^3 . \tag{4.35}
$$

Moreover, for any $s \in (\frac{1}{2}, 1)$,

$$
\big\|\mathcal{R}_5\big\|_{H^s} \lesssim \tau^{\frac{3}{2}}\|v\|_{L^\infty(0,T;H^1)}^3 . \tag{4.36}
$$

Proof. For $k_1 + k_2 + k_3 = k$ and $|k_2| \geq |k_3|$ we claim that the following inequality holds:

$$
\left|\frac{k_1}{k}\big(\mathrm{e}^{2iskk_1}-1\big)\big(\mathrm{e}^{2isk_2k_3}-1\big)\right| \lesssim \tau|k|^{-\alpha}|k_1||k_2||k_3|^\alpha \quad \forall\, s \in [0,\tau],\ \forall\, \alpha \in [0,1]. \tag{4.37}
$$

In order to prove (4.37), we consider the following two cases: $|k| \geq |k_3|$ and $|k| < |k_3|$.

Case 1: $|k| \geq |k_3|$. In this case, we use the following inequalities:

$$
\big|\mathrm{e}^{2iskk_1}-1\big| \leq 2 \quad \text{and} \quad \big|\mathrm{e}^{2isk_2k_3}-1\big| \leq 2\tau|k_2||k_3|,
$$

it follows that

$$
\left|\frac{k_1}{k}\big(\mathrm{e}^{2iskk_1}-1\big)\big(\mathrm{e}^{2isk_2k_3}-1\big)\right| \leq 4\tau|k|^{-1}|k_1||k_2||k_3| \lesssim \tau|k|^{-\alpha}|k_1||k_2||k_3|^\alpha .
$$

Case 2: $|k| < |k_3|$. In this case $k_1 + k_2 + k_3 = k$ and $|k_2| \geq |k_3|$ imply

$$
|k_1| \leq |k_2| + |k_3| + |k| \lesssim |k_2|.
$$

We use the following inequalities:

$$
\big|\mathrm{e}^{2iskk_1}-1\big| \leq 2\tau|k||k_1| \quad \text{and} \quad \big|\mathrm{e}^{2isk_2k_3}-1\big| \leq 2.
$$

Then we obtain

$$
\left|\frac{k_1}{k}\big(\mathrm{e}^{2iskk_1}-1\big)\big(\mathrm{e}^{2isk_2k_3}-1\big)\right| \leq 4\tau|k_1|^2 .
$$

Since $|k_1| \lesssim |k_2|$, it follows that

$$|k_1|^2 \lesssim |k_1||k_2| \lesssim |k_1||k_2| \left(\frac{|k_3|}{|k|}\right)^\alpha .$$

This proves (4.37).

By using the symmetry between k_2 and k_3 in the expression of $|\hat{\mathcal{R}}_{5,k}|$ in (4.25) and applying (4.37) with $\alpha = 1$ in the case $|k| \geq |k_3|$ and $\alpha = 0$ in the case $|k| < |k_3|$, we obtain for any $k \neq 0$,

$$\begin{aligned}|\hat{\mathcal{R}}_{5,k}| \lesssim \tau^2 &\sum_{\substack{k_1+k_2+k_3=k \\ |k_2|\geq|k_3|,|k|\geq|k_3|}} |k|^{-1}|k_1||k_2||k_3||\hat{\bar{v}}_{k_1}(t_n)||\hat{v}_{k_2}(t_n)||\hat{v}_{k_3}(t_n)| \\ &+ \tau^2 \sum_{\substack{k_1+k_2+k_3=k \\ |k_2|\geq|k_3|,|k|<|k_3|}} |k_1||k_2||\hat{\bar{v}}_{k_1}(t_n)||\hat{v}_{k_2}(t_n)||\hat{v}_{k_3}(t_n)|. \end{aligned} \tag{4.38}$$

Without loss of generality, we may assume that $\hat{\bar{v}}_{k_1}(t_n), \hat{v}_{k_2}(t_n)$ and $\hat{v}_{k_3}(t_n)$ are nonnegative. Otherwise we replace them by their absolute values as we did in the proof of Lemma 3.2.

By the duality between $L^2(\mathbb{T})$ and itself, it is sufficient to prove the following result to obtain (4.35):

$$|\langle \mathcal{R}_5, f\rangle| \lesssim \tau^2\sqrt{\ln(\tau^{-1})}\|v\|^3_{L^\infty(0,T;H^1)}\|f\|_{L^2} \quad \forall\, f \in L^2(\mathbb{T}). \tag{4.39}$$

From the definition below (4.26) we see that $\mathcal{R}_{5,0} = 0$. As a result, we have

$$|\langle \mathcal{R}_5, f\rangle| \lesssim \sum_{k\neq 0} |\hat{\mathcal{R}}_{5,k}|\,|\hat{f}_k| \lesssim \sum_{|k|>\tau^{-1}} |\hat{\mathcal{R}}_{5,k}|\,|\hat{f}_k| + \sum_{0\neq|k|\leq\tau^{-1}} |\hat{\mathcal{R}}_{5,k}|\,|\hat{f}_k|. \tag{4.40}$$

From the expression of $\mathcal{R}_{5,k}$ in (4.25) we see that for $|k| > \tau^{-1}$ there holds

$$|\hat{\mathcal{R}}_{5,k}| \leq \tau^2 \sum_{k_1+k_2+k_3=k} |k_1|\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n).$$

Hence, by the Cauchy–Schwartz inequality and Plancherel's identity, we have

$$\begin{aligned}\sum_{|k|>\tau^{-1}} |\hat{\mathcal{R}}_{5,k}|\,|\hat{f}_k| \leq&\tau^2 \sum_k \sum_{k_1+k_2+k_3=k} |k_1|\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\,|\hat{f}_k| \\ =&\tau^2 \sum_{k_2,k_3}\sum_k |k-k_2-k_3|\hat{\bar{v}}_{k-k_2-k_3}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n)\,|\hat{f}_k| \\ \lesssim&\tau^2\|(\hat{f}_k)_{k\in\mathbb{Z}}\|_{l^2}\|(k_1\hat{\bar{v}}_{k_1}(t_n))_{k_1\in\mathbb{Z}}\|_{l^2}\|(\hat{v}_{k_2}(t_n))_{k_2\in\mathbb{Z}}\|_{l^1}\|(\hat{v}_{k_3}(t_n))_{k_3\in\mathbb{Z}}\|_{l^1} \\ \lesssim&\tau^2\|f\|_{L^2}\|v\|^3_{H^1}, \end{aligned} \tag{4.41}$$

where the last inequality uses the following result:

$$\|(\hat{v}_{k_2}(t_n))_{k_2\in\mathbb{Z}}\|_{l^1} \lesssim \|(\langle k_2\rangle^{-1})_{k_2\in\mathbb{Z}}\|_{l^2}\|(\langle k_2\rangle\hat{v}_{k_2}(t_n))_{k_2\in\mathbb{Z}}\|_{l^2} \lesssim \|v\|_{H^1}.$$

The second term in (4.40) can be estimated by using (4.38), i.e.,

$$\begin{aligned}&\sum_{0\neq|k|\leq\tau^{-1}} |\hat{\mathcal{R}}_{5,k}|\,|\hat{f}_k| \qquad (4.42)\\ \lesssim&\tau^2 \sum_{0\neq|k|\leq\tau^{-1}} \sum_{\substack{k_1+k_2+k_3=k \\ |k_2|\geq|k_3|,|k|\geq|k_3|}} |k|^{-1}|k_1||k_2||k_3||\hat{f}_k|\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) \\ &+ \tau^2 \sum_{0\neq|k|\leq\tau^{-1}} \sum_{\substack{k_1+k_2+k_3=k \\ |k_2|\geq|k_3|,|k|<|k_3|}} |k_1||k_2||\hat{f}_k|\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) \\ \lesssim&\tau^2 \sum_{0\neq|k|\leq\tau^{-1}} \sum_{|k_3|\leq|k|} \sum_{k_1} |k|^{-1}|\hat{f}_k||k_1|\hat{\bar{v}}_{k_1}(t_n)|k-k_1-k_3|\hat{v}_{k-k_1-k_3}(t_n)|k_3|\hat{v}_{k_3}(t_n) \end{aligned}$$

$$
\begin{aligned}
&+\tau^2 \sum_{0\neq|k|\le\tau^{-1}} \sum_{|k_3|>|k|} \sum_{k_1} |\hat f_k||k_1|\hat{\bar v}_{k_1}(t_n)|k-k_1-k_3|\hat v_{k-k_1-k_3}(t_n)\hat v_{k_3}(t_n)\\
\lesssim&\tau^2\|(k_1\hat v_{k_1}(t_n))_{k_1\in\mathbb{Z}}\|_{l^2}\|(k_2\hat v_{k_2}(t_n))_{k_2\in\mathbb{Z}}\|_{l^2} \sum_{0\neq|k|\le\tau^{-1}} |k|^{-1}|\hat f_k| \sum_{|k_3|\le|k|} |k_3|\hat v_{k_3}(t_n)\\
&+\tau^2\|(k_1\hat v_{k_1}(t_n))_{k_1\in\mathbb{Z}}\|_{l^2}\|(k_2\hat v_{k_2}(t_n))_{k_2\in\mathbb{Z}}\|_{l^2} \sum_{0\neq|k|\le\tau^{-1}} |\hat f_k| \sum_{|k_3|>|k|} \hat v_{k_3}(t_n)\\
\lesssim&\tau^2\|v(t_n)\|_{H^1}^2 \sum_{0\neq|k|\le\tau^{-1}} |k|^{-\frac12}|\hat f_k|\|(k_3\hat v_{k_3}(t_n))_{k_3\in\mathbb{Z}}\|_{l^2}\\
&+\tau^2\|v(t_n)\|_{H^1}^2 \sum_{0\neq|k|\le\tau^{-1}} |\hat f_k|\|(\langle k_3\rangle^{-1})_{|k_3|>k}\|_{l^2}\|(\langle k_3\rangle\hat v_{k_3}(t_n))_{|k_3|>k}\|_{l^2}\\
\lesssim&\tau^2\|v(t_n)\|_{H^1}^2 \sum_{0\neq|k|\le\tau^{-1}} |k|^{-\frac12}|\hat f_k|\|(k_3\hat v_{k_3}(t_n))_{k_3\in\mathbb{Z}}\|_{l^2}\\
\lesssim&\tau^2\|v(t_n)\|_{H^1}^3\|(|k|^{-\frac12})_{0\neq|k|\le\tau^{-1}}\|_{l^2}\|(\hat f_k)_{|k|\le\tau^{-1}}\|_{l^2}\\
\lesssim&\tau^2\|v(t_n)\|_{H^1}^3\sqrt{\ln(\tau^{-1})}\|f\|_{L^2}.
\end{aligned}
\tag{4.43}
$$

Substituting (4.41)–(4.43) into (4.40) yields (4.39), which implies the desired result in (4.35).

It remains to prove (4.36). To this end, we use the following inequalities:

$$
|\mathrm{e}^{2iskk_1}-1|\le 2 \quad\text{and}\quad |\mathrm{e}^{2isk_2k_3}-1|\lesssim s^{\frac12}||k_2|^{\frac12}|k_3|^{\frac12},
$$

which imply that

$$
\left|\frac{k_1}{k}\big(\mathrm{e}^{2iskk_1}-1\big)\big(\mathrm{e}^{2isk_2k_3}-1\big)\right|\lesssim \tau^{\frac12}|k|^{-1}|k_1||k_2|^{\frac12}|k_3|^{\frac12}\quad \forall\, s\in[0,\tau].
$$

By substituting this into the expression of $\hat{\mathcal{R}}_{5,k}$ in (4.25), and using Plancherel's identity, we obtain

$$
\big\|\mathcal{R}_5\big\|_{H^s}\lesssim\tau^{\frac32}\left\||\nabla|^{-1+s}\left(|\nabla|\bar v\,\big(|\nabla|^{\frac12}v\big)^2\right)\right\|_{L^2}.
$$

Then using the Sobolev inequality, we get that for any $s\in(\frac12,1)$,

$$
\begin{aligned}
\big\|\mathcal{R}_5\big\|_{H^s}\lesssim&\tau^{\frac32}\left\||\nabla|\bar v\,\big(|\nabla|^{\frac12}v\big)^2\right\|_{L^{\frac{2}{3-2s}}}\\
\lesssim&\tau^{\frac32}\big\||\nabla|\bar v\big\|_{L^2}\,\big\||\nabla|^{\frac12}v\big\|^2_{L^{\frac{2}{1-s}}}\lesssim\tau^{\frac32}\|v\|^3_{H^1}.
\end{aligned}
$$

This completes the proof of Lemma 4.2. □

5. Proof of Theorem 2.1

The proof of Theorem 2.1 is divided into two parts. In subsection 5.1, we present an error estimate for the numerical solution in $H^s(\mathbb{T})$ with $s\in(\frac12,1)$, and then use this result to prove the boundedness of the numerical solution in $H^1(\mathbb{T})$ uniformly with respect to τ and N. In subsection 5.2, we utilize the H^1-boundedness of the numerical solution to prove the desired error estimate in $L^2(\mathbb{T})$.

5.1. Boundedness of the numerical solution in $H^1(\mathbb{T})$

Lemma 5.1. *Let $u^0\in H^1(\mathbb{T})$, and let $u^n_{\tau,N}$, $n=0,1,\dots,L$, be the numerical solution given by (2.8)–(2.9). Then there exist positive constants τ_s and N_s such that for $\tau\in(0,\tau_s]$ and $N\ge N_s$ the following error bound holds:*

$$
\max_{0\le n\le L}\|u(t_n,\cdot)-u^n_{\tau,N}\|_{H^s}\lesssim_s\tau^{\frac12}+N^{-1+s}\quad\forall\, s\in(\tfrac12,1),
\tag{5.1}
$$

where τ_s and N_s depend only on $\|u^0\|_{H^1}$, T and s.

Proof. Let $v^n = e^{-it_n\partial_x^2}u^n_{\tau,N}$. Then $v^{n+1} = \Phi^n(v^n; M_N, P_N)$ as shown in (4.34). By using this identity we have

$$\begin{aligned} v(t_{n+1}) - v^{n+1} =& v(t_{n+1}) - \Phi^n(v(t_n); M, P) + \Phi^n(v(t_n); M, P) - \Phi^n(v^n; M_N, P_N) \\ =&: \mathcal{L}^n + \Phi^n(v(t_n); M, P) - \Phi^n(v^n; M_N, P_N), \end{aligned} \tag{5.2}$$

where

$$\mathcal{L}^n = v(t_{n+1}) - \Phi^n(v(t_n); M, P) = \mathcal{R}_1 + \hat{\mathcal{R}}_{2,0} + \hat{\mathcal{R}}^*_{2,0} + \hat{\mathcal{R}}_{3,0} + \mathcal{R}_4 + \mathcal{R}^*_4 + \mathcal{R}_5 + \mathcal{R}^*_5,$$

which is shown in (4.32). From (4.9), (4.13), (4.15), (4.18), (4.27) and (4.28) we see that

$$\big\|\mathcal{L}^n\big\|_{H^s} \lesssim \tau^{\frac{3}{2}} + \tau N^{-1+s} \quad \forall\, s \in [0, 1). \tag{5.3}$$

Note that the functional $\Phi^n(f; M, P)$ defined in (4.33) can be rewritten into the following form:

$$\begin{aligned} \Phi^n(f; M, P) = f &+ \big(\mathrm{e}^{-2i\lambda\tau P\partial_x^{-1} - 2i\lambda\tau M} - 1 + 2i\lambda\tau P\partial_x^{-1} + 2i\lambda\tau M\big)f + (1 - \mathrm{e}^{-2i\lambda\tau M})\Pi_0 f \\ &- i\lambda\tau\Pi_0\big[\Pi_N\big(\big|\mathrm{e}^{it_n\partial_x^2}f\big|^2\big)\mathrm{e}^{it_n\partial_x^2}f\big] \\ &- 2i\lambda \sum_{0\neq|k|\leq N} \mathrm{e}^{ikx}\Bigg(\sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\leq N}} \int_0^\tau \frac{k_1+k_2}{k}\mathrm{e}^{i(t_n+s)\phi}\,ds\Bigg)\,\hat{\bar{f}}_{k_1}\hat{f}_{k_2}\hat{f}_{k_3} \\ &+ i\lambda \sum_{0\neq|k|\leq N} \mathrm{e}^{ikx} \sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\leq N}} \Bigg(\int_0^\tau \frac{k_1}{k}\mathrm{e}^{it_n\phi}\mathrm{e}^{2iskk_1}\,ds\Bigg)\,\hat{\bar{f}}_{k_1}\hat{f}_{k_2}\hat{f}_{k_3} \\ &+ i\lambda \sum_{0\neq|k|\leq N} \mathrm{e}^{ikx} \sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\leq N}} \Bigg(\int_0^\tau \frac{k_1}{k}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{2isk_2k_3} - 1\big)\,ds\Bigg)\,\hat{\bar{f}}_{k_1}\hat{f}_{k_2}\hat{f}_{k_3}. \end{aligned} \tag{5.4}$$

For example, the third line of (5.4) comes from (4.21), which can be rewritten back into (4.17). This is how we obtain the third line in the expression above. The other terms are obtained similarly.

From (5.4) we furthermore derive that

$$\Phi^n(v(t_n); M, P) - \Phi^n(v^n; M_N, P_N) = v(t_n) - v^n + \Phi^n_1 + \Phi^n_2 + \Phi^n_3 + \Phi^n_4 + \Phi^n_5, \tag{5.5}$$

where

$$\begin{aligned} \Phi^n_1 =& \big(\mathrm{e}^{-2i\lambda\tau P\partial_x^{-1} - 2i\lambda\tau M} - 1 + 2i\lambda\tau P\partial_x^{-1} + 2i\lambda\tau M + (1 - \mathrm{e}^{-2i\lambda\tau M})\Pi_0\big)v(t_n) \\ &- \big(\mathrm{e}^{-2i\lambda\tau P_N\partial_x^{-1} - 2i\lambda\tau M_N} - 1 + 2i\lambda\tau P_N\partial_x^{-1} + 2i\lambda\tau M_N + (1 - \mathrm{e}^{-2i\lambda\tau M_N})\Pi_0\big)v^n, \\ \Phi^n_2 =& - i\lambda\tau\Pi_0\Big(|\mathrm{e}^{it_n\partial_x^2}v(t_n)|^2\mathrm{e}^{it_n\partial_x^2}v(t_n) - |\mathrm{e}^{it_n\partial_x^2}v^n|^2\mathrm{e}^{it_n\partial_x^2}v^n\Big), \\ \Phi^n_3 =& - 2i\lambda \sum_{0\neq|k|\leq N} \mathrm{e}^{ikx} \sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\leq N}} \Bigg(\int_0^\tau \frac{k_1+k_2}{k}\mathrm{e}^{i(t_n+s)\phi}\,ds\Bigg)\big(\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) - \hat{\bar{v}}^n_{k_1}\hat{v}^n_{k_2}\hat{v}^n_{k_3}\big), \\ \Phi^n_4 =& i\lambda \sum_{0\neq|k|\leq N} \mathrm{e}^{ikx} \sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\leq N}} \Bigg(\int_0^\tau \frac{k_1}{k}\mathrm{e}^{it_n\phi}\mathrm{e}^{2iskk_1}\,ds\Bigg)\big(\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) - \hat{\bar{v}}^n_{k_1}\hat{v}^n_{k_2}\hat{v}^n_{k_3}\big), \\ \Phi^n_5 =& i\lambda \sum_{0\neq|k|\leq N} \mathrm{e}^{ikx} \sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\leq N}} \Bigg(\int_0^\tau \frac{k_1}{k}\mathrm{e}^{it_n\phi}\big(\mathrm{e}^{2isk_2k_3} - 1\big)\,ds\Bigg)\big(\hat{\bar{v}}_{k_1}(t_n)\hat{v}_{k_2}(t_n)\hat{v}_{k_3}(t_n) - \hat{\bar{v}}^n_{k_1}\hat{v}^n_{k_2}\hat{v}^n_{k_3}\big). \end{aligned}$$

Note that P, M, P_N and M_N defined in (2.4) and (2.10) are all bounded numbers, with bounds depending on $\|u^0\|_{H^1}$. In particular,

$$\begin{aligned}|M-M_N| &= \left|\frac{1}{2\pi}\int_{\mathbb{T}}(|u^0|^2-|u^0_{\tau,N}|^2)\,dx\right| \\ &\lesssim \left|\frac{1}{2\pi}\int_{\mathbb{T}}\Big[(u^0-u^0_{\tau,N})\overline{u^0}+u^0_{\tau,N}\overline{(u^0-u^0_{\tau,N})}\Big]\,dx\right| \\ &\lesssim \|u^0-u^0_{\tau,N}\|_{L^2}(\|u^0\|_{L^2}+\|u^0_{\tau,N}\|_{L^2}) \\ &\lesssim N^{-1}\|u^0\|^2_{H^1} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned}|P-P_N| &= \left|\frac{1}{2\pi}\int_{\mathbb{T}}(u^0\partial_x\overline{u^0}-u^0_{\tau,N}\partial_x\overline{u}^0_{\tau,N})\,dx\right| \\ &\lesssim \left|\frac{1}{2\pi}\int_{\mathbb{T}}\Big[(u^0-u^0_{\tau,N})\partial_x\overline{u^0}+u^0_{\tau,N}\partial_x\overline{(u^0-u^0_{\tau,N})}\Big]\,dx\right| \\ &= \left|\frac{1}{2\pi}\int_{\mathbb{T}}\Big[(u^0-u^0_{\tau,N})\partial_x\overline{u^0}-\partial_x u^0_{\tau,N}\overline{(u^0-u^0_{\tau,N})}\Big]\,dx\right| \\ &\lesssim \|u^0-u^0_{\tau,N}\|_{L^2}(\|\partial_x u^0\|_{L^2}+\|\partial_x u^0_{\tau,N}\|_{L^2}) \\ &\lesssim N^{-1}\|u^0\|^2_{H^1}. \end{aligned} \tag{5.7}$$

From the expression of Φ_1^n we see that its Fourier coefficients can be written as

$$\mathcal{F}_k[\Phi_1^n] = F(M,P;k)\hat{v}_k(t_n) - F(M_N,P_N;k)\hat{v}_k^n,$$

with

$$F(M,P;k) := \mathrm{e}^{-2i\lambda\tau Pk^{-1}1_{k\neq 0}-2i\lambda\tau M} - 1 + 2i\lambda\tau Pk^{-1}1_{k\neq 0} + 2i\lambda\tau M + (1-\mathrm{e}^{-2i\lambda\tau M})1_{k=0}.$$

By using Taylor's expansion and mean value theorem, it is straightforward to verify that

$$|F(M,P;k) - F(M_N,P_N;k)| \lesssim \tau(|P-P_N| + |M-M_N|).$$

As a result, we have

$$\begin{aligned}\|\Phi_1^n\|_{H^s} \lesssim& \|(\langle k\rangle^s\mathcal{F}_k[\Phi_1^n])_{k\in\mathbb{Z}}\|_{l^2} \\ \lesssim& \tau(|P-P_N|+|M-M_N|)\|(\langle k\rangle^s\hat{v}_k(t_n))_{k\in\mathbb{Z}}\|_{l^2} + \|(\langle k\rangle^s(\hat{v}_k(t_n)-\hat{v}_k))_{k\in\mathbb{Z}}\|_{l^2} \\ \lesssim& \tau(|P-P_N|+|M-M_N|)\|v(t_n)\|_{H^s} + \tau\|v(t_n)-v^n\|_{H^s} \\ \lesssim& \tau N^{-1}\|v\|^3_{L^\infty(0,T;H^1)} + \tau\|v(t_n)-v^n\|_{H^s}, \end{aligned} \tag{5.8}$$

where the last inequality follows from (5.6)–(5.7).

Since Φ_2^n is a constant, it is straightforward to show that (similarly as (5.6))

$$\begin{aligned}|\Phi_2^n| \lesssim& \tau\big(\|v^n-v(t_n)\|_{L^2}(\|e^{it_n\partial_x^2}v(t_n)\|^2_{L^\infty}+\|e^{it_n\partial_x^2}v^n\|^2_{L^\infty}\big) \\ \lesssim& \tau\big(\|v^n-v(t_n)\|_{L^2}(\|v(t_n)\|^2_{H^s}+\|v^n\|^2_{H^s}\big) \qquad \text{(this holds for } s>\tfrac12) \\ \lesssim& \tau\big(\|v^n-v(t_n)\|_{L^2}(\|v(t_n)\|^2_{H^s}+\|v^n-v(t_n)\|^2_{H^s}\big). \end{aligned} \tag{5.9}$$

Similarly, Φ_3^n can be decomposed into several functions of the following form:

$$\Phi_3^n = -2i\sum_{0\neq|k|\leq N}\mathrm{e}^{ikx}\sum_{\substack{k_1+k_2+k_3=k\\|k_2+k_3|\leq N}}\left(\int_0^\tau\frac{k_1+k_2}{k}\mathrm{e}^{i(t_n+s)\phi}\,ds\right)\hat{f}_{1,k_1}\hat{f}_{2,k_2}\hat{f}_{3,k_3},$$

where $\hat{f}_{j,k}$ denotes the kth Fourier coefficient of the functions f_j, and one of the three functions $f_j, j=1,2,3$, is

$$v^n - v(t_n) \text{ or its conjugate;}$$

the other two of the three functions $f_j, j=1,2,3$, are either v^n or $v(t_n)$ or their conjugates. We assume that $\hat{f}_{j,k}, k\in\mathbb{Z}$ are nonnegative; otherwise we consider functions with Fourier coefficients $|\hat{f}_{j,k}|$ as we did in the proof of Lemma 3.2 (ii). Then

$$\big|\big(\widehat{\Phi_3^n}\big)_k\big| \lesssim \tau \sum_{k_1+k_2+k_3=k} \frac{|k_1+k_2|}{|k|}\hat{f}_{1,k_1}\hat{f}_{2,k_2}\hat{f}_{3,k_3} = \mathcal{F}_k[\tau J^{-1}(f_1 J(f_2 f_3))].$$

As a result, by Plancherel's identity and Lemma 3.2 (i), we have

$$\begin{aligned}\|\Phi_3^n\|_{H^s} \lesssim& \|\tau J^{-1}(f_3 J(f_1 f_2))\|_{H^s}\\ \lesssim& \tau\|f_3\|_{H^s}\|f_1 f_2\|_{H^s} \qquad \text{(this requires } s>\tfrac12)\\ \lesssim& \tau\|f_3\|_{H^s}\|f_1\|_{H^s}\|f_2\|_{H^s}\\ \lesssim& \tau\|v^n-v(t_n)\|_{H^s}(\|v^n\|_{H^s}^2+\|v(t_n)\|_{H^s}^2)\\ \lesssim& \tau\|v^n-v(t_n)\|_{H^s}(\|v^n-v(t_n)\|_{H^s}^2+\|v(t_n)\|_{H^s}^2).\end{aligned} \tag{5.10}$$

Φ_4^n and Φ_5^n can be estimated similarly, i.e.,

$$\|\Phi_4^n\|_{H^s}+\|\Phi_5^n\|_{H^s} \lesssim \tau\|v^n-v(t_n)\|_{H^s}(\|v^n-v(t_n)\|_{H^s}^2+\|v(t_n)\|_{H^s}^2).$$

Hence, combining with the estimates of Φ_j^n, $j=1,\dots,5$, we have

$$\begin{aligned}&\|\Phi^n(v(t_n);M,P)-\Phi^n(v^n;M_N,P_N)\|_{H^s}\\ &\le (1+C\tau)\|v^n-v(t_n)\|_{H^s}+C\tau\|v^n-v(t_n)\|_{H^s}^3+C\tau N^{-1},\end{aligned}$$

which holds for any given $s\in(\frac12,1)$. Substituting this and (5.3) into (5.2) yields that

$$\|v(t_{n+1})-v^{n+1}\|_{H^s} \le C\big(\tau^{\frac32}+\tau N^{-1+s}\big)+(1+C\tau)\|v^n-v(t_n)\|_{H^s}+C\tau\|v^n-v(t_n)\|_{H^s}^3.$$

By using the discrete Gronwall's inequality with induction assumption on $\|v^n-v(t_n)\|_{H^s}\le 1$, we obtain (for sufficiently small τ)

$$\max_{0\le n\le L}\big\|v(t_n)-v^n\big\|_{H^s} \lesssim \tau^{\frac12}+N^{-1+s}.$$

This proves the desired result in Lemma 5.1. □

Lemma 5.1 implies that $\|v(t_n)-v^n\|_{H^s}\lesssim 1$. Then, by using the triangle inequality and boundedness of the exact solution in H^1, we have

$$\|v^n\|_{H^s} \lesssim \|v(t_n)-v^n\|_{H^s}+\|v(t_n)\|_{H^s} \lesssim 1.$$

This result can be furthermore improved to the H^1 norm, as shown in the following lemma.

Lemma 5.2. *Let $u^0\in H^1(\mathbb{T})$, and let $u_{\tau,N}^n$, $n=0,1,\dots,L$, be the numerical solution given by* (2.8)–(2.9). *Then there exists a constant $\tau_0>0$ such that for $\tau\in(0,\tau_0]$ the following estimate holds:*

$$\max_{0\le n\le L}\|u_{\tau,N}^n\|_{H^1} \lesssim 1. \tag{5.11}$$

Proof. Let $v^n=e^{-it_n\partial_x^2}u_{\tau,N}^n$. By using the expression of Φ^n in (5.4), we immediately obtain that

$$\|\Phi^n(v^n;M_N,P_N)\|_{H^1} \le \|v^n\|_{H^1}+C\tau\|v^n\|_{H^1}+C\tau\|v^n\|_{H^1}\|v^n\|_{H^s}^2, \tag{5.12}$$

which holds for any fixed $s\in(\frac12,1)$. Since $\|v^n\|_{H^s}\lesssim 1$ is already proved in Lemma 5.1, substituting this into (4.34) yields

$$\|v^{n+1}\|_{H^1} \le \|v^n\|_{H^1}+C\tau\|v^n\|_{H^1}, \tag{5.13}$$

which implies $\max_{0\le n\le L}\|v^n\|_{H^1}\lesssim 1$ after iteration in n. The desired result follows from the relation $\|v^n\|_{H^1}=\|u_{\tau,N}^n\|_{H^1}$. □

5.2. Error estimation in $L^2(\mathbb{T})$

From (4.9), (4.13), (4.15), (4.18), (4.27) and (4.28) we conclude that

$$\big\|\mathcal{L}^n\big\|_{L^2} \le C\big(\tau^2\sqrt{\ln\tau^{-1}} + \tau N^{-1}\big). \tag{5.14}$$

By choosing $s = 0$ in (5.8) and choosing a fixed $s \in (\frac{1}{2}, 1)$ in (5.9), we have

$$\|\Phi_1^n\|_{L^2} + \|\Phi_2^n\|_{L^2} \lesssim \tau N^{-1} + \tau\|v^n - v(t_n)\|_{L^2}.$$

Instead of (5.10), we need to use the following estimate for Φ_3^n:

$$\|\Phi_3^n\|_{L^2} \lesssim \|\tau J^{-1}(f_3 J(f_1 f_2))\|_{L^2} \lesssim \tau\min(\|f_3\|_{H^1}\|f_1 f_2\|_{L^2}, \|f_3\|_{L^2}\|f_1 f_2\|_{H^1}).$$

which is a consequence of Lemma 3.2 (ii). Recall that one of the three functions $f_j, j = 1, 2, 3$, is $v^n - v(t_n)$ or its conjugate, and the other two functions are either v^n or $v(t_n)$ (or their conjugates). If f_1 is $v^n - v(t_n)$ or its conjugate, then we choose L^2 norm on f_1; otherwise we choose L^2 norm on $f_2 f_3$. In either case we obtain

$$\|\Phi_3^n\|_{L^2} \lesssim \tau\|v^n - v(t_n)\|_{L^2}(\|v(t_n)\|_{H^1}^2 + \|v^n\|_{H^1}^2) \lesssim \tau\|v^n - v(t_n)\|_{L^2}.$$

The two terms Φ_4^n and Φ_5^n can be estimated similarly, i.e.,

$$\|\Phi_4^n\|_{L^2} + \|\Phi_5^n\|_{L^2} \lesssim \tau\|v^n - v(t_n)\|_{L^2}.$$

Substituting the estimates of $\|\Phi_j^n\|_{L^2}$, $j = 1, \dots, 5$, into (5.5), we have

$$\|\Phi^n(v(t_n); M, P) - \Phi^n(v^n; M_N, P_N)\|_{L^2} \lesssim \tau N^{-1} + \tau\|v^n - v(t_n)\|_{L^2}.$$

Then, substituting this into (5.2) and using estimate (5.14), we obtain

$$\|v(t_{n+1}) - v^{n+1}\|_{L^2} \le C\big(\tau^2\sqrt{\ln\tau^{-1}} + \tau N^{-1}\big) + (1 + C\tau)\|v^n - v(t_n)\|_{L^2}. \tag{5.15}$$

Iterating this inequality yields

$$\max_{1\le n\le L}\|v(t_n) - v^n\|_{L^2} \lesssim \|v(t_0) - v^0\|_{L^2} + \tau\sqrt{\ln\tau^{-1}} + N^{-1} \lesssim \tau\sqrt{\ln\tau^{-1}} + N^{-1}.$$

This completes the proof of Theorem 2.1 in view of $\|v(t_n) - v^n\|_{L^2} = \|u(t_n) - u_{\tau,N}^n\|_{L^2}$. □

6. Numerical experiments

In this section we present numerical experiments to support the theoretical analysis presented in Theorem 2.1. We consider the NLS equation (1.1) with $\lambda = -1$ and initial value

$$u^0(x) = \frac{1}{10}\sum_{0\ne k\in\mathbb{Z}} |k|^{-0.51-\alpha} e^{ikx}, \tag{6.1}$$

which satisfies that $u^0 \in H^\alpha(\mathbb{T})$ and $u^0 \notin H^{\alpha-0.01}(\mathbb{T})$.

We solve the problem by the proposed method (2.8)–(2.9) for $\alpha = 2$ and $\alpha = 1$, respectively, and present the time discretisation errors $\|u_{\tau,N} - u_{\tau/2,N}\|_{L^2}$ in Tables 1–2 for several sufficiently large N. From the numerical results we can see that the error from spatial discretisation is negligibly small in observing the temporal convergence rates, i.e., almost first-order convergent as $\tau \to 0$. This is consistent with the theoretical result proved in Theorem 2.1.

We present the spatial discretisation errors $\|u_{\tau,N} - u_{\tau,2N}\|_{L^2}$ for $\alpha = 2$ and $\alpha = 1$ in Tables 3–4 for several sufficiently small stepsize τ. From the numerical results we can see that the error from temporal discretisation is negligibly small in observing the spatial convergence rates, i.e., αth-order convergence for H^α initial data. This is consistent with the result proved in Theorem 2.1 and the comments in Remark 2.2.

Table 1. Temporal discretisation error $\|u_{\tau,N} - u_{\tau/2,N}\|_{L^2}$ at $T = 1$ with $\alpha = 2$ in (6.1) (for H^2 initial data).

	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
$\tau = 2^{-6}$	7.662E–06	7.662E–06	7.662E–06
$\tau = 2^{-7}$	3.829E–06	3.829E–06	3.829E–06
$\tau = 2^{-8}$	1.915E–06	1.915E–06	1.915E–06
convergence rate	$O(\tau^{1.00})$	$O(\tau^{1.00})$	$O(\tau^{1.00})$

Table 2. Temporal discretisation error $\|u_{\tau,N} - u_{\tau/2,N}\|_{L^2}$ at $T = 1$ with $\alpha = 1$ in (6.1) (for H^1 initial data).

	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
$\tau = 2^{-6}$	2.144E–05	2.146E–05	2.146E–05
$\tau = 2^{-7}$	1.023E–05	1.021E–05	1.021E–05
$\tau = 2^{-8}$	5.057E–06	5.067E–06	5.066E–06
convergence rate	$O(\tau^{1.02})$	$O(\tau^{1.02})$	$O(\tau^{1.02})$

Table 3. Spatial discretisation error $\|u_{\tau,N} - u_{\tau,2N}\|_{L^2}$ at $T = 1$ with $\alpha = 2$ in (6.1) (for H^2 initial data).

	$\tau = 2^{-8}$	$\tau = 2^{-9}$	$\tau = 2^{-10}$
$N = 16$	2.430E–04	2.430E–03	2.430E–03
$N = 32$	6.237E–05	6.237E–05	6.237E–05
$N = 64$	1.574E–05	1.574E–05	1.574E–05
convergence rate	$O(N^{-1.99})$	$O(N^{-1.99})$	$O(N^{-1.99})$

Table 4. Spatial discretisation error $\|u_{\tau,N} - u_{\tau,2N}\|_{L^2}$ at $T = 1$ with $\alpha = 1$ in (6.1) (for H^1 initial data).

	$\tau = 2^{-8}$	$\tau = 2^{-9}$	$\tau = 2^{-10}$
$N = 16$	5.056E–03	5.056E–03	5.056E–03
$N = 32$	2.559E–03	2.559E–03	2.559E–03
$N = 64$	1.283E–03	1.283E–03	1.283E–03
convergence rate	$O(N^{-1.00})$	$O(N^{-1.00})$	$O(N^{-1.00})$

7. Conclusion

We have constructed a fast fully discrete low-regularity integrator for solving the NLS equation with nonsmooth initial data in one dimension. The method can be implemented by using FFT with $O(N \ln N)$ operations at every time level, and is proved to have an error bound of $O(\tau\sqrt{\ln(1/\tau)} + N^{-1})$ when the initial data is in $H^1(\mathbb{T})$. For initial data in $H^s(\mathbb{T})$ with $s > 1$, the numerical results show that the proposed method can have an error bound of $O(\tau + N^{-s})$. We expect that the techniques for constructing and analysing the spatial discretisation method in combination with the temporal low-regularity integrator may also be extended to other dispersive equations with nonsmooth data.

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