MULTILINEAR COMMUTATORS IN THE TWO-WEIGHT SETTING

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ABSTRACT. We extend the recently much-studied two-weight commutator estimates to the multilinear setting. In contrast to previous results, our result respects the multilinear nature of the problem fully and is formulated with the genuinely multilinear weights.

1. INTRODUCTION

The characterization of BMO spaces via the boundedness of commutators has attracted a lot of attention recently. Recall that given a linear operator T and a locally integrable function b, the commutator of T and b is defined as

$$[b,T]f = bT(f) - T(bf).$$

This research line initiated from the work of Nehari [26], who studied the case when the singular integral operator T is the Hilbert transform, by complex analysis methods. Starting from the celebrated work of Coifman-Rochberg-Weiss [5], commutators have been a central part of harmonic analysis. In this work, they showed that the boundedness of $[b, R_j]$ characterizes the BMO membership of b, where R_j is the j-th Riesz transform and BMO stands for the usual bounded mean oscillation function space.

Later, in 1976, Muckenhoupt and Wheeden [25] introduced the weighted BMO space BMO_ν with $\nu \in A_\infty$. Although some interesting estimates related with the Hilbert transform are formulated in [25], it was until 1985, Bloom [4] finally found the connection between BMO_ν and the two-weight boundedness of $[b, H]$:

$$\|b\|_{\mathrm{BMO}_\nu} \lesssim \|[b,H]\|_{L^p(\mu)\to L^p(\lambda)} \lesssim \|b\|_{\mathrm{BMO}_\nu}, \qquad 1<p<\infty,$$

where $\mu, \lambda \in A_p$ and $\nu = \mu^{\frac{1}{p}}\lambda^{-\frac{1}{p}}$. Bloom's result extended the commutator estimates to the two-weight context, and from then on Bloom type estimates have been at the focus of harmonic analysts. For the upper bound, Segovia–Torrea [30] first extended Bloom's result to general Calderón-Zygmund operators, see also Holmes–Lacey–Wick [11] for a modern proof and Lerner–Ombrosi–Rivera-Ríos [19] for the quantitative upper bounds. The k-order iterated case, which is defined inductively by

$$C_b^k(T) = [b, C_b^{k-1}(T)], \qquad C_b^1(T) = [b,T]$$

was considered initially with the assumption $b \in \mathrm{BMO} \cap \mathrm{BMO}_\nu$ (see [12, 14]), and was refined to $\mathrm{BMO}_{\nu^{1/k}}$ later in [20]. For the lower bound, the influential paper by Holmes–Lacey–Wick [11] solved the problem for the Riesz transforms. The general case, i.e. non-degenerate singular integrals, were solved only recently by Hytönen [15] through the median method. This also improves an earlier result in [20].

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Recently, Lerner–Ombrosi–Rivera-Ríos [20] and Hytönen [15] also studied the case when T is the rough homogeneous singular integrals T_Ω. However, both of them did not provide the upper bound. We will address this problem in Section 5 as a by-product of our new method. For more about the linear theory, we refer the readers to [1, 13, 21] and the references therein.

Now it is natural to ask whether one can establish the corresponding multilinear theory. Given an n-linear operator T and a locally integrable function b, we define

$$[b,T]_i(f_1,\dots,f_n) = bT(f_1,\dots,f_n) - T(f_1,\dots,bf_i,\dots,f_n), \qquad 1\le i\le n.$$

Notice that by the above definition, $[b_2,[b_1,T]_i]_j = [b_1,[b_2,T]_j]_i$, so given $\mathcal{I}\subset\{1,\dots,n\}$ we may define the general iterated commutator inductively as

$$C^{k_{\mathcal{I}}}_{b_{\mathcal{I}}}(T) := C^{k_i}_{b_{\{i\}}}(C^{k_{\mathcal{I}\setminus\{i\}}}_{b_{\mathcal{I}\setminus\{i\}}}(T)), \quad C^{k_i}_{b_{\{i\}}}(T) := [b^i_{k_i},\cdots,[b^i_1,T]_i,\cdots,]_i, \quad i\in\mathcal{I}. \tag{1.1}$$

We simply denote $C^{k_i}_{b_{\{i\}}}(T)$ by $C^{k_i}_b(T)$ when $b=b^i_1=\cdots=b^i_{k_i}$.

In the multilinear setting the most interesting phenomena occurs when the genuinely multilinear weights are used. The genuinely multilinear weights were introduced by Lerner, Ombrosi, Pérez, Torres and Trujillo-González in the very influential work [18]. The point is, one only needs to assume a weaker joint condition on the tuple of weights $(w_1,\cdots,w_n)$ rather than to assume individual conditions on w_i. Commutator estimates involving genuinely multilinear weights in the one-weight situation already appear in the literature, see e.g. [18], where they proved

$$[b,T]_j : L^{p_1}(w_1)\times\cdots\times L^{p_n}(w_n)\to L^p\big(\prod_{i=1}^n w_i^{\frac{p}{p_i}}\big) \qquad 1\le j\le n. \tag{1.2}$$

Multilinear commutator estimates in the two-weight setting have also been studied by Kunwar and Ou in [17]. However, they did not use the genuinely multilinear weights. To be more precise, when considering e.g. $[b,T]_1$, they need to assume $w_1,\lambda_1\in A_{p_1}$ in addition to natural assumptions. Here, for the first time, we work in the simultaneous presence of both complications.

1.3. **Theorem.** *Let T be an n-linear Calderón-Zygmund operator, $1\le i\le n$, $1\le k_i<\infty$ and $C^{k_i}_{b_{\{i\}}}(T)$ be defined as in (1.1). Given $\theta_1,\dots,\theta_{k_i}\in[0,1]$ such that $\sum_{\ell=1}^{k_i}\theta_\ell=1$, and let $\theta=\max\{\theta_\ell\}_{1\le\ell\le k_i}$. Let $1<p_1,\dots,p_n<\infty$ and $\frac1p=\sum_{i=1}^n\frac{1}{p_i}$. Assume that $(w_1,\dots,w_n),(w_1,\dots,w_{i-1},\lambda_i,w_{i+1},\dots,w_n)\in A_{\vec p}$ with $\nu_i^\theta := w_i^{\frac{\theta}{p_i}}\lambda_i^{-\frac{\theta}{p_i}}\in A_\infty$ and $b^i_\ell\in \mathrm{BMO}_{\nu_i^{\theta_\ell}}$ for every $1\le\ell\le k_i$. Then*

$$\Big\|C^{k_i}_{b_{\{i\}}}(T) : L^{p_1}(w_1)\times\cdots\times L^{p_n}(w_n)\to L^p\big(\lambda_i^{\frac{p}{p_i}}\prod_{j\ne i}w_j^{\frac{p}{p_j}}\big)\Big\| \lesssim \prod_{\ell=1}^{k_i}\|b^i_\ell\|_{\mathrm{BMO}_{\nu_i^{\theta_\ell}}}.$$

1.4. *Remark.* Theorem 1.3 is the upper bound of the iterated commutator when $\mathcal{I}$ contains only one element. For general $\mathcal{I}$, our method also works, but it is more technical and we will record it in Section 3, where we also comment on the quantitative upper bounds and a comparison between our bounds and the bounds via the Cauchy integral trick in [7, 3].

1.5. *Remark.* The assumption $\nu_i^\theta \in A_\infty$ ensures that $\nu_i^{\theta\ell} \in A_\infty$ for all $1 \leq \ell \leq k_i$ so that the weighted BMO spaces are well-defined. However, if $\theta \leq \frac{1}{n}$, then there is no need to assume $\nu_i^\theta \in A_\infty$ as it is automatically true. More details are provided in Subsection 2.4.

Unlike the two-weight situation, the one-weight situation has many tools such as sharp maximal function estimates [28, 29], Cauchy integral trick [7, 3] and also sparse domination. In the two-weight setting it seems that only sparse domination survives. However, if one simply follows the known strategy, one will meet a term which involves a composition of multilinear sparse operator and linear sparse operator, and this is the reason why the genuinely multilinear weights were not able to appear in the two-weight setting before. See [17] for the details. We overcome this difficulty by establishing a Muckenhoupt-Wheeden type result, then we are able to reduce the problem to estimating the composition of multilinear sparse operator and *weighted* linear sparse operator. The idea behind is based on two facts:

(1) for any genuinely multilinear weight $(w_1, \cdots, w_n)$, $w_i^{1-p_i'} \in A_\infty$;
(2) weighted linear sparse operator is bounded if the underlying weight belongs to A_∞.

To complete the multilinear commutator theory in the two-weight setting we still need to provide the lower bounds. It is worth mentioning that, for the lower bound the genuinely multilinear weights were not used even in the one-weight situation in previous results. For example, in [10] Guo, Lian and Wu achieved the lower bound when T is certain non-degenerate multilinear Calderón-Zygmund operator. Kunwar and Ou [17] also obtained a similar result for the multilinear Haar multipliers. However, both of them need to assume individual conditions on w_i.

In contrast to previous results, we are able to use genuinely multilinear weights in the two-weight setting.

1.6. **Theorem.** *Let T be an n-linear non-degenerate Calderón-Zygmund operator (see Subsection 2.2 for the definition), $1 \leq i \leq n$, $1 \leq k_i < \infty$, $b \in L^{k_i}_{\text{loc}}$ and $C_b^{k_i}(T)$ be defined as in* (1.1). *Let $1 < p_1, \ldots, p_n < \infty$ and $\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i}$. Assume that*

$$(w_1, \ldots, w_n), (w_1, \ldots, w_{i-1}, \lambda_i, w_{i+1}, \ldots, w_n) \in A_{\vec{p}}$$

with $\nu_i^{\frac{1}{k_i}} := w_i^{\frac{1}{k_i p_i}} \lambda_i^{-\frac{1}{k_i p_i}} \in A_\infty$. Then

$$\|b\|^{k_i}_{\text{BMO}_{\nu_i^{1/k_i}}} \lesssim \Big\| C_b^{k_i}(T) : L^{p_1}(w_1) \times \cdots \times L^{p_n}(w_n) \to L^p\big(\lambda_i^{\frac{p}{p_i}} \prod_{j \neq i} w_j^{\frac{p}{p_j}}\big) \Big\|.$$

1.7. *Remark.* Similar as Remark 1.5, if $k_i \geq n$ then we do not need to assume $\nu_i^{\frac{1}{k_i}} \in A_\infty$ as it is automatically true. Moreover, in Section 4 we will actually prove the above result with a weaker boundedness assumption.

This paper is organized as the following: in Section 2 we provide necessary notations and auxiliary results. Section 3 is devoted to proving Theorem 1.3. The lower bound is handled is Section 4. In Section 5 we provide a short discussion about the upper bound of the commutator of rough homogeneous singular integrals, and general sparse operators involving weighted BMO functions.

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2. Definitions and auxiliary lemmata

2.1. **Basic notations.** We denote $A \lesssim B$ if $A \le CB$ for some constant that can depend on the dimension, Lebesgue exponents, weight constants, and on various other constants appearing in the assumptions. We denote $A \sim B$ if $B \lesssim A \lesssim B$.

Given a cube Q, a measure μ and a locally integrable function f, we denote the average $\mu(Q)^{-1} \int_Q f \,\mathrm{d}\mu = f_Q^\mu = \langle f \rangle_Q^\mu$. When μ is Lebesgue measure we simply write $|Q|^{-1} \int_Q f = ⨍_Q f = f_Q = \langle f \rangle_Q$.

2.2. **Multilinear Calderón-Zygmund operators.** The multilinear Calderón-Zygmund theory was systematically formulated by Grafakos and Torres [9]. Let us begin with the definition of multilinear Calderón-Zygmund operators (CZOs). Let $\Delta := \{(x, y_1, \cdots, y_n) \in (\mathbb{R}^d)^{n+1} : x = y_1 = \cdots = y_n\}$ be the diagonal in $(\mathbb{R}^d)^{n+1}$. We say $K : (\mathbb{R}^d)^{n+1} \setminus \Delta \to \mathbb{C}$ is a multilinear Calderón-Zygmund kernel if

$$|K(x,y)| \le \frac{C}{(\sum_{i=1}^n |x-y_i|)^{nd}}, \tag{2.1}$$

$$\begin{aligned} |K(x+h,y) - K(x,y)| &+ \sum_{i=1}^n |K(x,\cdots,y_i+h,\cdots) - K(x,y)| \\ &\le \frac{C}{(\sum_{i=1}^n |x-y_i|)^{nd}} \omega\left(\frac{h}{\sum_{i=1}^n |x-y_i|}\right) \end{aligned} \tag{2.2}$$

whenever $h \le \frac{1}{2} \max_i |x - y_i|$, where ω is an increasing subadditive function with $\omega(0) = 0$ and

$$\|\omega\|_{\mathrm{Dini}} = \int_0^1 \omega(t) \frac{\mathrm{d}t}{t} < \infty.$$

Then we say T is a multilinear CZO if T is initially bounded from $L^{q_1}(\mathbb{R}^d) \times \cdots \times L^{q_n}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ with $q_i \in (1,\infty]$, $\frac{1}{q} = \sum_{i=1}^n \frac{1}{q_i} > 0$ and there exists a multilinear Calderón-Zygmund kernel K such that for all $f_1, \ldots, f_n \in C_c^\infty(\mathbb{R}^d)$,

$$T(f_1,\ldots,f_n)(x) = \int_{\mathbb{R}^{nd}} K(x, y_1, \cdots, y_n) \prod_{i=1}^n f_i(y_i) \,\mathrm{d}y, \qquad x \notin \bigcap_{i=1}^n \operatorname{spt} f_i. \tag{2.3}$$

In [7], the author, Damián and Hormozi obtained a pointwise sparse bound for multilinear CZOs introduced in the above. That is,

$$|T(f_1,\ldots,f_n)(x)| \lesssim \sum_{j=1}^{3^d} \mathcal{A}_{\mathcal{S}_j}(f_1,\ldots,f_n)(x) := \sum_{j=1}^{3^d} \sum_{Q\in\mathcal{S}_j} \prod_{i=1}^n \langle |f_i| \rangle_Q 1_Q(x),$$

where $\mathcal{S}_j$ is a sparse collection for every $1 \le j \le 3^d$. Recall that we say a collection of cubes $\mathcal{S}$ is ρ-sparse if for every $S \in \mathcal{S}$, there exists $E_S \subset S$ with $|E_S| \ge \rho |S|$ and $\{E_S\}$ are pairwise disjoint. Usually we choose $\rho = \frac{1}{2}$.

Now we define the non-degenerate Calderón-Zygmund operators. We say T is a non-degenerate multilinear CZO if there is a function K such that (2.1), (2.2) and (2.3) holds with $\omega(0) \to 0$ when $t \to 0$, and in addition, for every $y \in \mathbb{R}^d$ and $r > 0$, there exists $x \notin B(y,r)$ with

$$|K(x, y, \dots, y)| \gtrsim \frac{1}{r^{nd}}. \tag{2.4}$$

Note that (2.1) and (2.4) imply that $|x - y| \sim r$. The multilinear Riesz transforms are typical examples of non-degenerate Calderón-Zygmund operators.

2.3. **Weights.** By weights we mean positive locally integrable functions. Recall that we say a weight $w \in A_p$ if

$$[w]_{A_p} := \sup_Q \fint_Q w \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty, \qquad 1 < p < \infty.$$

And we say $w \in A_\infty$ if

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < \infty.$$

An important property of A_∞ weights is recorded as the following lemma.

2.5. **Lemma.** *Let $t_1, \dots, t_n \in (0, \infty)$ and $w_1, \dots, w_n \in A_\infty$. Then there exists a constant $\mathrm{RH}_{\vec{w},\vec{t}}$ such that for every cube Q,*

$$\prod_{i=1}^n \left(\fint_Q w_i \right)^{t_i} \le \mathrm{RH}_{\vec{w},\vec{t}} \fint_Q \prod_{i=1}^n w_i^{t_i}. \tag{2.6}$$

We may abuse notation a little that we still denote by $\mathrm{RH}_{\vec{w},\vec{t}}$ the best constant such that (2.6) *holds.*

2.7. *Remark.* The case when $\sum_i t_i \le 1$ was obtained first by Xue and Yan [32], and rediscovered recently by Cruz-Uribe and Moen [6]. They also referred it as the multilinear Reverse Hölder property. The case $\sum_i t_i > 1$ is actually a simple consequence of the aforementioned result and Hölder's inequality. However, for sake of completeness we provide a direct proof in below.

Proof of Lemma 2.5. By embedding of A_∞ weights, there exist $q_1, \dots, q_n \in (1, \infty)$ such that $w_i \in A_{q_i}$, $1 \le i \le n$. Fix a cube Q, let

$$E_i := \left\{ x \in Q : w_i(x) < \frac{\langle w_i \rangle_Q}{(2n)^{q_i - 1} [w_i]_{A_{q_i}}} \right\}.$$

Then by the definition of A_{q_i} weights, we have

$$\frac{|E_i|}{|Q|} \le \frac{1}{2n [w_i]_{A_{q_i}}^{\frac{1}{q_i - 1}}} \langle w_i \rangle_Q^{\frac{1}{q_i - 1}} \left(\fint_Q w_i^{-\frac{1}{q_i - 1}} \right) \le \frac{1}{2n}. \tag{2.8}$$

Let $E := \cup_{i=1}^n E_i$ and $F = Q \setminus E$, we have by (2.8) that $|F| \ge \frac{1}{2}|Q|$. Thus

$$\prod_{i=1}^n \left(\fint_Q w_i \right)^{t_i} \le \left(\prod_{i=1}^n (2n)^{(q_i - 1)t_i} [w_i]_{A_{q_i}}^{t_i} \right) \fint_F \prod_{i=1}^n w_i^{t_i}$$

$$\le 2\left(\prod_{i=1}^n (2n)^{(q_i-1)t_i}[w_i]_{A_{q_i}}^{t_i}\right) \fint_Q \prod_{i=1}^n w_i^{t_i}.$$

□

When $\sum_{i=1}^n t_i > 1$, the converse of (2.6) cannot be obtained through Hölder's inequality. Nevertheless, we record the following lemma, which is useful in the proof of our main results.

2.9. **Lemma.** *Let $t_1, \dots, t_n \in (0,\infty)$ and $w_1, \dots, w_n$ be weights such that $\prod_{i=1}^n w_i^{t_i} \in A_\infty$. Then there is a constant K depending on $[\prod_{i=1}^n w_i^{t_i}]_{A_\infty}$ such that for every cube Q,*

$$\fint_Q \prod_{i=1}^n w_i^{t_i} \le K \prod_{i=1}^n \left(\fint_Q w_i\right)^{t_i}.$$

Proof. Fix a cube Q. Let $E_i := \{x \in Q : w_i(x) > 2n\langle w_i\rangle_Q\}$. Then by Chebyshev it is easy to see that $|E_i| \le \frac{1}{2n}|Q|$. Let $F = Q \setminus \cup_{i=1}^n E_i$, then $|F| \ge \frac{1}{2}|Q|$ and therefore,

$$\begin{aligned}\int_Q \prod_{i=1}^n w_i^{t_i} &\lesssim_{[\prod_{i=1}^n w_i^{t_i}]_{A_\infty}} \int_F \prod_{i=1}^n w_i^{t_i}\\ &\le (2n)^{\sum_{i=1}^n t_i}|F| \prod_{i=1}^n \left(\fint_Q w_i\right)^{t_i} \le (2n)^{\sum_{i=1}^n t_i}|Q| \prod_{i=1}^n \left(\fint_Q w_i\right)^{t_i}.\end{aligned}$$

We are done. □

In the multilinear setting, Lerner, Ombrosi, Pérez, Torres and Trujillo-González [18] introduced the multiple $A_{\vec{p}}$ weights: we say $\vec{w} := (w_1, \dots, w_n) \in A_{\vec{p}}$ if

$$[\vec{w}]_{A_{\vec{p}}} := \sup_Q \left(\fint_Q \prod_{i=1}^n w_i^{\frac{p}{p_i}}\right) \prod_{i=1}^n \left(\fint_Q w_i^{1-p_i'}\right)^{\frac{p}{p_i'}} < \infty.$$

It is proved in [18] that $\vec{w} \in A_{\vec{p}}$ if and only if

$$w := \prod_{i=1}^n w_i^{\frac{p}{p_i}} \in A_{np} \qquad \text{and} \qquad w_i^{1-p_i'} \in A_{np_i'}, \qquad 1 \le i \le n. \tag{2.10}$$

We shall use (2.10) frequently.

For our purpose we also record the following result obtained in [23] by the author, Moen and Sun.

2.11. **Proposition.** *Let $\mathcal{A}_\mathcal{S}$ be a multilinear sparse operator, $1 < p_1, \dots, p_n < \infty$ with $1/p = \sum_{i=1}^n 1/p_i$ and $\vec{w} \in A_{\vec{p}}$. Then*

$$\|\mathcal{A}_\mathcal{S}\|_{L^{p_1}(w_1)\times\cdots\times L^{p_n}(w_n)\to L^p(w)} \le C_{n,d,\vec{p},T}[\vec{w}]_{A_{\vec{p}}}^{\max\{1,\frac{p_1'}{p},\cdots,\frac{p_n'}{p}\}}.$$

In particular, when $p \le 1$, the following stronger estimate holds

$$\Big\|\Big(\sum_{Q\in\mathcal{S}}\prod_{i=1}^n \langle |f_i|\rangle_Q^p 1_Q\Big)^{\frac{1}{p}}\Big\|_{L^p(w)} \le C_{n,d,\vec{p},T}[\vec{w}]_{A_{\vec{p}}}^{\max\{\frac{p_1'}{p},\cdots,\frac{p_n'}{p}\}} \prod_{i=1}^n \|f_i\|_{L^{p_i}(w_i)}.$$

2.4. **BMO spaces.** Let $\nu \in A_\infty$, we say $b \in \mathrm{BMO}_\nu$ if

$$\|b\|_{\mathrm{BMO}_\nu} := \sup_Q \frac{1}{\nu(Q)} \int_Q |b - b_Q| < \infty.$$

In practical cases, $\nu = \mu^{\frac{1}{p}} \lambda^{-\frac{1}{p}}$. In the linear case, there is no need to a priori assume that $\nu \in A_\infty$ because we always have $\nu \in A_2$, which is easy to verify since $\mu, \lambda \in A_p$. However, in the multilinear case, by (2.10) we only have $\mu^{1-p'}, \lambda^{1-p'} \in A_{np'}$, which implies that $\nu^{\frac{1}{n}} \in A_\infty$, but in general one cannot deduce that $\nu \in A_\infty$– it can be even not locally integrable. This is easily seen by the following example.

2.12. **Example.** For simplicity we provide an example for $n = 2$ and $d = 1$. Let $(p_1, p_2) = (2, 4)$ and $(w_1, w_2) = (|x|^{-2}, |x|^2)$, $(\lambda_1, w_2) = (1, |x|^2)$. It is easy to check that

$$w_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}} = |x|^{-\frac{2}{3}} \in A_{2p},\ w_1^{1-p_1'} = |x|^2 \in A_{2p_1'},\ w_2^{1-p_2'} = |x|^{-\frac{2}{3}} \in A_{2p_2'}$$

and $\lambda_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}} = |x|^{\frac{2}{3}} \in A_{2p}$. With these facts and (2.10) we know that $(w_1, w_2), (\lambda_1, w_2) \in A_{\vec{p}}$. However, $\nu = w_1^{\frac{1}{p_1}} \lambda_1^{-\frac{1}{p_1}} = |x|^{-1} \notin L^1_{\mathrm{loc}}(\mathbb{R})$.

Example 2.12 shows that it is reasonable to a priori assume that $\nu \in A_\infty$. In fact, only if $\nu \in A_\infty$ something interesting can happen, for example, Wu [31] showed that when $\nu \in A_\infty$ the dual space of BMO_ν is the weighted Hardy space $H^1(\nu)$. Moreover, it was proved by Muckenhoupt and Wheeden that for several interesting estimates related with $b \in \mathrm{BMO}_\nu$, it is necessary to assume $\nu \in A_\infty$ (see [25]).

We also define the weighted Bloom BMO: we say $b \in \mathrm{BMO}_\nu(\sigma)$ if

$$\|b\|_{\mathrm{BMO}_\nu(\sigma)} := \sup_Q \frac{1}{\nu\sigma(Q)} \int_Q |b - b_Q|\sigma < \infty.$$

Recall that when $\nu = 1$, it is a classical result by Muckenhoupt and Wheeden [25] that for any $\sigma \in A_\infty$, one has

$$\mathrm{BMO}_1 = \mathrm{BMO}_1(\sigma).$$

It would be natural to expect that in the Bloom BMO setting, one has similar result. We shall show that this is indeed the case, with some natual assumption on the weights.

2.13. **Lemma.** *Let $\nu, \sigma \in A_\infty$. If $\nu\sigma \in A_\infty$, then $\mathrm{BMO}_\nu = \mathrm{BMO}_\nu(\sigma)$.*

We will see that in practical cases, we always have $\nu\sigma \in A_\infty$ for free. To prove this lemma, we shall need the following proposition, whose proof is standard, but for sake of completeness, we give the full details.

2.14. **Proposition.** *Let $\sigma \in A_\infty$. Then there holds*

$$|b - b^\sigma_{Q_0}| 1_{Q_0} \lesssim \sum_{Q \in \mathcal{S}(Q_0)} \langle |b - b^\sigma_Q| \rangle^\sigma_Q 1_Q, \tag{2.15}$$

where $\mathcal{S}(Q_0)$ is a sparse collection with respect to σ and with all elements contained in Q_0.

Proof. Let $\alpha = 2\langle |b - b^\sigma_{Q_0}| \rangle^\sigma_{Q_0}$. Form the Calderón-Zygmund decomposition of $|b - b^\sigma_{Q_0}| 1_{Q_0}$ at height α with respect to σ, we get a collection of maximal cubes $\{Q_j\}$ in $\mathcal{D}(Q_0)$ with the property that

$$\langle |b - b^\sigma_{Q_0}| \rangle^\sigma_{Q_j} > \alpha.$$

Denote $E := Q_0 \setminus \cup_j Q_j$, we have $|b - b^\sigma_{Q_0}| 1_E \le \alpha$. By maximality,

$$\langle |b - b^\sigma_{Q_0}| \rangle^\sigma_{Q_j^{(1)}} \le \alpha. \tag{2.16}$$

Hence

$$\sum_j \sigma(Q_j) < \alpha^{-1} \sum_j \int_{Q_j} |b - b^\sigma_{Q_0}| \sigma \le \frac{1}{2} \sigma(Q_0).$$

Finally we are able to write

$$\begin{aligned} |b - b^\sigma_{Q_0}| 1_{Q_0} &\le |b - b^\sigma_{Q_0}| 1_E + \sum_j |b - b^\sigma_{Q_0}| 1_{Q_j} \\ &\le \alpha 1_E + \sum_j |b^\sigma_{Q_j} - b^\sigma_{Q_0}| 1_{Q_j} + \sum_j |b - b^\sigma_{Q_j}| 1_{Q_j} \\ &\le (D_\sigma + 1) \alpha 1_{Q_0} + \sum_j |b - b^\sigma_{Q_j}| 1_{Q_j}, \end{aligned}$$

where in the last step we have used (2.16) and the doubling property of σ (the doubling constant depends on $[\sigma]_{A_\infty}$). Then (2.15) is concluded by the above recursive inequality. □

2.17. *Remark.* Proposition 2.14 provides an alternative definition of $\mathrm{BMO}_\nu(\sigma)$, that is, we may define

$$\|b\|_{\widetilde{\mathrm{BMO}}_\nu(\sigma)} := \sup_Q \frac{1}{\nu\sigma(Q)} \int_Q |b - b^\sigma_Q| \sigma.$$

Indeed, $\|b\|_{\widetilde{\mathrm{BMO}}_\nu(\sigma)} \lesssim \|b\|_{\mathrm{BMO}_\nu(\sigma)}$ is trivial. So we only show the other direction. We have

$$\frac{1}{\nu\sigma(Q)} \int_Q |b - b_Q| \sigma \le \frac{1}{\nu\sigma(Q)} \int_Q |b - b^\sigma_Q| \sigma + \frac{\sigma(Q)}{\nu\sigma(Q)} |b_Q - b^\sigma_Q|.$$

By Proposition 2.14 we have

$$\begin{aligned} |b_Q - b^\sigma_Q| \le \frac{1}{|Q|} \int_Q |b - b^\sigma_Q| &\lesssim \|b\|_{\widetilde{\mathrm{BMO}}_\nu(\sigma)} \frac{1}{|Q|} \sum_{P \in \mathcal{S}(Q)} \frac{\nu\sigma(P)}{\sigma(P)} |P| \\ &\lesssim \|b\|_{\widetilde{\mathrm{BMO}}_\nu(\sigma)} \frac{1}{|Q|} \sum_{P \in \mathcal{S}(Q)} \nu(P) \lesssim \|b\|_{\widetilde{\mathrm{BMO}}_\nu(\sigma)} \langle \nu \rangle_Q, \end{aligned}$$

where we have used Lemma 2.9 in the third inequality. The argument is concluded now by using Lemma 2.5.

Proof of Lemma 2.13. We first show that $\mathrm{BMO}_\nu \subset \mathrm{BMO}_\nu(\sigma)$. Let $b \in \mathrm{BMO}_\nu$. Fix a cube Q_0, by Proposition 2.14,

$$|b - b_{Q_0}| 1_{Q_0} \lesssim \sum_{Q \in \mathcal{S}(Q_0)} \langle |b - b_Q| \rangle_Q 1_Q,$$

where $\mathcal{S}(Q_0)$ is a sparse family contained in Q_0. Then we have

$$\frac{1}{\nu\sigma(Q_0)} \int_{Q_0} |b - b_{Q_0}| \sigma \lesssim \|b\|_{\mathrm{BMO}_\nu} \frac{1}{\nu\sigma(Q_0)} \sum_{Q \in \mathcal{S}(Q_0)} \langle \nu \rangle_Q \sigma(Q)$$

$$\lesssim \|b\|_{\mathrm{BMO}_\nu} \frac{1}{\nu\sigma(Q_0)} \sum_{Q\in\mathcal{S}(Q_0)} \nu\sigma(Q) \lesssim \|b\|_{\mathrm{BMO}_\nu},$$

where in the second inequality we have used Lemma 2.5, and in the last step we have used $\nu\sigma \in A_\infty$. It remains to prove the converse direction. Let $b \in \mathrm{BMO}_\nu(\sigma)$ and fix a cube Q_0. By (2.15), we have

$$\begin{aligned}\frac{1}{\nu(Q_0)}\int_{Q_0}|b-b_{Q_0}| &\le \frac{1}{\nu(Q_0)}\int_{Q_0}|b-b^\sigma_{Q_0}| + \frac{|Q_0|}{\nu(Q_0)}|b_{Q_0}-b^\sigma_{Q_0}|\\ &\le \frac{2}{\nu(Q_0)}\int_{Q_0}|b-b^\sigma_{Q_0}|\\ &\lesssim \frac{2}{\nu(Q_0)}\sum_{Q\in\mathcal{S}(Q_0)}\langle|b-b^\sigma_Q|\rangle^\sigma_Q|Q|\\ &\le \|b\|_{\mathrm{BMO}_\nu(\sigma)}\frac{4}{\nu(Q_0)}\sum_{Q\in\mathcal{S}(Q_0)}\frac{\nu\sigma(Q)}{\sigma(Q)}|Q|,\end{aligned}$$

where we have used the fact that

$$\langle|b-b^\sigma_Q|\rangle^\sigma_Q \le \langle|b-b_Q|\rangle^\sigma_Q + \langle|b_Q-b^\sigma_Q|\rangle^\sigma_Q \le 2\langle|b-b_Q|\rangle^\sigma_Q.$$

Now since $\nu\sigma \in A_\infty$, by Lemma 2.9 we have

$$\langle\nu\sigma\rangle_Q \lesssim \langle\nu\rangle_Q\langle\sigma\rangle_Q.$$

Hence

$$\sum_{Q\in\mathcal{S}(Q_0)}\frac{\nu\sigma(Q)}{\sigma(Q)}|Q| \lesssim \sum_{Q\in\mathcal{S}(Q_0)}\nu(Q) \lesssim \nu(Q_0)$$

and we are done. □

John-Nirenberg inequality is a key feature of BMO spaces, which has independent interest. We record the John-Nirenberg inequality in our setting as the following:

2.18. **Theorem.** *Given μ, λ and $1 < q < \infty$ such that $\eta := \lambda^{1-q'}, \sigma := \mu^{1-q'} \in A_\infty$. Let $\nu = \mu^{\frac{1}{q}}\lambda^{-\frac{1}{q}}$. Then*

$$\sup_Q\left(\frac{1}{\eta(Q)}\int_Q|b-b_Q|^{q'}\sigma\right)^{\frac{1}{q'}} \sim \|b\|_{\mathrm{BMO}_\nu}.$$

Proof. We first prove the $\gtrsim$ direction. Indeed, by Hölder's inequality, we see that

$$\frac{1}{\nu\sigma(Q)}\int_Q|b-b_Q|\sigma \le \frac{1}{\nu\sigma(Q)}\Big(\int_Q|b-b_Q|^{q'}\sigma\Big)^{\frac{1}{q'}}\sigma(Q)^{\frac{1}{q}} \lesssim \left(\frac{1}{\eta(Q)}\int_Q|b-b_Q|^{q'}\sigma\right)^{\frac{1}{q'}},$$

where in the last step we have used by Lemma 2.5 that

$$\langle\sigma\rangle_Q^{\frac{1}{q}}\langle\eta\rangle_Q^{\frac{1}{q'}} \lesssim \big\langle\sigma^{\frac{1}{q}}\eta^{\frac{1}{q'}}\big\rangle_Q = \langle\nu\sigma\rangle_Q.$$

Next we prove the $\lesssim$ direction. Again, we use

$$|b-b_Q|1_Q \lesssim \sum_{I\in\mathcal{S}(Q)}\langle|b-b_I|\rangle_I 1_I \le \|b\|_{\mathrm{BMO}_\nu}\sum_{I\in\mathcal{S}(Q)}\langle\nu\rangle_I 1_I \lesssim \|b\|_{\mathrm{BMO}_\nu}\sum_{I\in\mathcal{S}(Q)}\langle\nu\rangle^\sigma_I 1_I,$$

where in the last step we have used

$$\langle \nu \rangle_I \lesssim \frac{\langle \nu\sigma \rangle_I}{\langle \sigma \rangle_I},$$

which is again, due to Lemma 2.5. Since $\sigma \in A_\infty$, the sparseness with respect to Lebesgue measure is equivalent with the sparseness with respect to σ. Thus using the boundedness of sparse operator we have

$$\int_Q |b-b_Q|^{q'}\sigma \lesssim \|b\|_{\mathrm{BMO}_\nu}^{q'} \int_Q \nu^{q'}\sigma = \|b\|_{\mathrm{BMO}_\nu}^{q'} \eta(Q)$$

and we are done. □

3. Proof of Theorem 1.3

Our proof is based on sparse domination technique. For this purpose we record the following result, which is more general than [17, Proposition 2.1].

3.1. **Proposition.** *Let T be an n-linear Calderón-Zygmund operator, $\mathcal{I} \subset \{1,\dots,n\}$, $1 \le k_i < \infty$ and $C_{b_\mathcal{I}}^{k_\mathcal{I}}(T)$ be defined as in* (1.1). *Let $f_1,\dots,f_n$ be compactly supported functions. Then there exist 3^d sparse collections $\mathcal{S}_m$, $1 \le m \le 3^d$, such that*

$$|C_{b_\mathcal{I}}^{k_\mathcal{I}}(T)(f_1,\dots,f_n)| \lesssim \sum_{m=1}^{3^d} \sum_{\vec\gamma \in \{1,2\}^L} \mathcal{A}_{\mathcal{S}_m,b_\mathcal{I}}^{\vec\gamma}(f_1,\dots,f_n),$$

where $L = \sum_{i\in\mathcal{I}} k_i$ and for fixed $\vec\gamma$,

$$\begin{aligned}&\mathcal{A}_{\mathcal{S}_m,b_\mathcal{I}}^{\vec\gamma}(f_1,\dots,f_n)\\ &= \sum_{Q\in\mathcal{S}_m} \prod_{i\in\mathcal{I}} \Big\langle |f_i| \prod_{\ell_i\in B_i} |b_{\ell_i}^i - \langle b_{\ell_i}^i\rangle_Q| \Big\rangle_Q \Big(\prod_{j\notin\mathcal{I}} \langle |f_j|\rangle_Q\Big) \prod_{i\in\mathcal{I}}\prod_{\ell_i\in A_i} |b_{\ell_i}^i - \langle b_{\ell_i}^i\rangle_Q| 1_Q,\end{aligned}$$

where

$$A_i = \{\ell_i : \gamma_{\ell_i} = 1\}, \qquad B_i = \{\ell_i : \gamma_{\ell_i} = 2\}.$$

The proof of Proposition 3.1 is similar as [17], by employing the idea in [16]. Thus we omit the details. Before we begin the proof, we also need the following result stated in [19, Lemma 5.1], which is a stronger version than Proposition 2.14 in the unweighted case.

3.2. **Lemma.** *Let $\mathcal{S}$ be a γ-sparse collection of dyadic cubes and $b \in L^1_{\mathrm{loc}}$. Then there exists a $\frac{\gamma}{2(1+\gamma)}$-sparse family $\widetilde{\mathcal{S}}$ such that $\widetilde{\mathcal{S}} \supset \mathcal{S}$ and for all $Q \in \widetilde{\mathcal{S}}$,*

$$|b-b_Q|1_Q \lesssim \sum_{P\in\widetilde{\mathcal{S}}, P\subset Q} \langle |b-b_P|\rangle_P 1_P,$$

Now we start the proof of Theorem 1.3. Fix $\vec\gamma$. By Proposition 3.1 it suffices to bound

$$\Big\| \sum_{Q\in\mathcal{S}} \Big\langle |f_i| \prod_{\ell\in B_i} |b_\ell^i - \langle b_\ell^i\rangle_Q| \Big\rangle_Q \Big(\prod_{j\ne i}\langle |f_j|\rangle_Q\Big) \prod_{\ell\in A_i} |b_\ell^i - \langle b_\ell^i\rangle_Q| 1_Q \Big\|_{L^p(\lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}})}.$$

Let us first consider the case $p \le 1$. By Lemma 3.2, we have

$$|b_\ell^i - \langle b_\ell^i\rangle_Q| 1_Q \lesssim \sum_{P_\ell \in \mathcal{S}_\ell, P_\ell \subset Q} \langle |b_\ell^i - \langle b_\ell^i\rangle_{P_\ell}|\rangle_{P_\ell} 1_{P_\ell} \le \|b_\ell^i\|_{\mathrm{BMO}_{\nu_i^{\theta_\ell}}} \sum_{P_\ell \in \mathcal{S}_\ell, P_\ell \subset Q} \langle \nu_i^{\theta_\ell}\rangle_{P_\ell} 1_{P_\ell},$$

where we may assume (by splitting $\mathcal{S}$ if necessary) that $\mathcal{S}_\ell$ is ρ-sparse with $\rho < k_i^{-1}$. Now that $\widetilde{\mathcal{S}} = \cup_{1\le \ell \le k_i} \mathcal{S}_\ell$ is still sparse, we have

$$\begin{aligned} &\int_Q \Big(\prod_{\ell \in A_i} |b_\ell^i - \langle b_\ell^i\rangle_Q|^p\Big) \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}} \\ &\qquad \lesssim \prod_{\ell\in A_i} \|b_\ell^i\|^p_{\mathrm{BMO}_{\nu_i^{\theta_\ell}}} \sum_{\substack{P_\ell \in \widetilde{\mathcal{S}}, P_\ell \subset Q \\ \ell \in A_i}} \Big(\prod_{\ell\in A_i} \langle \nu_i^{\theta_\ell}\rangle_{P_\ell}^p\Big) \int_{\cap_{\ell\in A_i} P_\ell} \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}}. \end{aligned}$$

We may assume $\cap_{\ell\in A_i} P_\ell = P_{\ell_0}$. Then by Lemma 2.5 we see that

$$\langle \nu_i^{\theta_{\ell_0}}\rangle_{P_{\ell_0}}^p \int_{P_{\ell_0}} \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}} \lesssim \int_{P_{\ell_0}} \nu_i^{p\theta_{\ell_0}} \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}}.$$

Since $0 \le \theta_{\ell_0} \le 1$ and

$$\nu_i^p \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}} = \prod_j w_j^{\frac{p}{p_j}} \in A_\infty, \qquad \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}} \in A_\infty,$$

we know that

$$\nu_i^{p\theta_{\ell_0}} \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}} = \Big(\nu_i^p \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}}\Big)^{\theta_{\ell_0}} \Big(\lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}}\Big)^{1-\theta_{\ell_0}} \in A_\infty.$$

Therefore, we are able to sum over $P_{\ell_0} \subset \cap_{\ell \in A_i \setminus\{\ell_0\}} P_\ell$ and we arrive at

$$\sum_{\substack{P_\ell \in \widetilde{\mathcal{S}}, P_\ell \subset Q \\ \ell \in A_i\setminus\{\ell_0\}}} \prod_{\ell\in A_i\setminus\{\ell_0\}} \langle \nu_i^{\theta_\ell}\rangle_{P_\ell}^p \int_{\cap_{\ell\in A_i\setminus\{\ell_0\}} P_\ell} \nu_i^{p\theta_{\ell_0}} \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}}.$$

Iterating this process we get

$$\int_Q \Big(\prod_{\ell \in A_i} |b_\ell^i - \langle b_\ell^i\rangle_Q|^p\Big) \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}} \lesssim \prod_{\ell\in A_i} \|b_\ell^i\|^p_{\mathrm{BMO}_{\nu_i^{\theta_\ell}}} \int_Q \nu_i^{p\sum_{\ell\in A_i}\theta_\ell} \lambda_i^{\frac{p}{p_i}} \prod_{j\ne i} w_j^{\frac{p}{p_j}}. \tag{3.3}$$

The idea is to use Proposition 2.11 as a blackbox. It is not difficult to check that

$$(w_1, \dots, w_{i-1}, \nu_i^{p_i\theta_{A_i}} \lambda_i, w_{i+1}, \dots, w_n) \in A_{\vec{p}}, \qquad \theta_{A_i} := \sum_{\ell\in A_i} \theta_\ell.$$

With this observation in mind, if $\#B_i = 0$, then we are done. So we may assume $0 < \#B_i \le k_i$. Similarly as the above, we write

$$\int_Q |f_i| \prod_{\ell\in B_i} |b_\ell^i - \langle b_\ell^i\rangle_Q| \lesssim \prod_{\ell\in B_i} \|b_\ell^i\|_{\mathrm{BMO}_{\nu_i^{\theta_\ell}}} \sum_{\substack{P_\ell \in \widetilde{\mathcal{S}}, P_\ell\subset Q \\ \ell\in B_i}} \prod_{\ell\in B_i} \langle \nu_i^{\theta_\ell}\rangle_{P_\ell} \int_{\cap_{\ell\in B_i} P_\ell} |f_i|.$$

We may assume $\cap_{\ell\in B_i}P_\ell = P_{\ell_1}$, and $\cap_{\ell\in B_i\setminus\{\ell_1\}}P_\ell = P_{\ell_2}$ (if $\#B_i = 1$ we take $P_{\ell_2} = Q$). Let $\zeta_1 \in A_\infty$ which will be determined later. The point now is we can write

$$\sum_{\substack{P_{\ell_1}\in\widetilde{\mathcal{S}}\\ P_{\ell_1}\subset P_{\ell_2}}} \langle \nu_i^{\theta_{\ell_1}}\rangle_{P_{\ell_1}} \int_{P_{\ell_1}} |f_i| = \sum_{\substack{P_{\ell_1}\in\widetilde{\mathcal{S}}\\ P_{\ell_1}\subset P_{\ell_2}}} \langle \nu_i^{\theta_{\ell_1}}\rangle_{P_{\ell_1}} \zeta_1(P_{\ell_1}) \langle |f_i|\zeta_1^{-1}\rangle_{P_{\ell_1}}^{\zeta_1}$$
$$\lesssim \int_{P_{\ell_2}} \mathcal{A}_{\widetilde{\mathcal{S}}}^{\zeta_1}(f_i\zeta_1^{-1})\nu_i^{\theta_{\ell_1}}\zeta_1,$$

where we have used Lemma 2.5 and

$$\mathcal{A}_{\widetilde{\mathcal{S}}}^{\sigma}(h) := \sum_{P\in\widetilde{\mathcal{S}}} \langle h\rangle_Q^\sigma 1_Q.$$

Let $h_{\ell_1} = \mathcal{A}_{\widetilde{\mathcal{S}}}^{\zeta_1}(f_i\zeta_1^{-1})\nu_i^{\theta_{\ell_1}}\zeta_1$ and $h_{\ell_k} = \mathcal{A}_{\widetilde{\mathcal{S}}}^{\zeta_k}(h_{\ell_{k-1}}\zeta_k^{-1})\nu_i^{\theta_{\ell_k}}\zeta_k$, where again, $\zeta_2,\dots,\zeta_{\#B_i} \in A_\infty$ weights which will be determined later. By iterating this process we are able to write

$$\int_Q |f_i| \prod_{\ell\in B_i} |b_\ell^i - \langle b_\ell^i\rangle_Q| \lesssim \sum_{(\ell_s)} \int_Q h_{\ell_{\#B_i}},$$

where the sum is taken over all the permutations of B_i. It remains to define ζ_k, $1\le k\le \#B_i$. In fact, to bound $\|h_{\ell_{\#B_i}}\|_{L^{p_i}(\nu_i^{p_i\theta_{A_i}}\lambda_i)}$ correctly we need

(1) $(\nu_i^{\theta_{\ell_{\#B_i}}}\zeta_{\#B_i})^{p_i}\nu_i^{p_i\theta_{A_i}}\lambda_i = \zeta_{\#B_i}$;
(2) $(\nu_i^{\theta_{\ell_{k-1}}}\zeta_{k-1})^{p_i}\zeta_k^{1-p_i} = \zeta_{k-1}$;
(3) $\zeta_1^{1-p_i} = w_i$.

These give us that $\zeta_1 = w_i^{1-p_i'} \in A_\infty$ and

$$\zeta_k = w_i^{1-p_i'}\nu_i^{p_i'\sum_{s=1}^{k-1}\theta_{\ell_s}} = (\lambda_i^{1-p_i'})^{\sum_{s=1}^{k-1}\theta_{\ell_s}}(w_i^{1-p_i'})^{1-\sum_{s=1}^{k-1}\theta_{\ell_s}} \in A_\infty, \quad 2\le k\le \#B_i.$$

The point of these conditions are that, each time they allow us to use the $L^{p_i}(\zeta_k)$ boundedness of $A_{\widetilde{\mathcal{S}}}^{\zeta_k}$, which is guaranteed by $\zeta_k\in A_\infty$. In particular, we have

$$\|h_{\ell_{\#B_i}}\|_{L^{p_i}(\nu_i^{p_i\theta_{A_i}}\lambda_i)} \lesssim \|f_i\|_{L^{p_i}(w_i)}.$$

Combining the arguments in the above, by Proposition 2.11 we get

$$\Big\|\sum_{Q\in\mathcal{S}}\Big\langle |f_i|\prod_{\ell\in B_i}|b_\ell^i - \langle b_\ell^i\rangle_Q|\Big\rangle_Q\Big(\prod_{j\ne i}\langle |f_j|\rangle_Q\Big)\prod_{\ell\in A_i}|b_\ell^i - \langle b_\ell^i\rangle_Q|1_Q\Big\|_{L^p(\lambda_i^{\frac{p}{p_i}}\prod_{j\ne i} w_j^{\frac{p}{p_j}})}$$
$$\lesssim \prod_{\ell=1}^{k_i}\|b_\ell^i\|_{\mathrm{BMO}_{\nu_i^{\theta_\ell}}}\prod_{i=1}^n\|f_i\|_{L^{p_i}(w_i)}.$$

It remains to prove the case of $p>1$. In this case, by duality we reduce the problem to estimate

$$\sum_{Q\in\mathcal{S}}\Big\langle |f_i|\prod_{\ell\in B_i}|b_\ell^i - \langle b_\ell^i\rangle_Q|\Big\rangle_Q\Big(\prod_{j\ne i}\langle |f_j|\rangle_Q\Big)\Big\langle |f_{n+1}|\prod_{\ell\in A_i}|b_\ell^i - \langle b_\ell^i\rangle_Q|\Big\rangle_Q|Q|,$$

where $f_{n+1} \in L^{p'}\big((\lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}})^{1-p'}\big)$. Then one may deal with

$$\Big\langle |f_{n+1}| \prod_{\ell\in A_i} |b_\ell^i - \langle b_\ell^i\rangle_Q| \Big\rangle_Q$$

similarly as the above. In fact, if $\#A_i = 0$ then we are done. If $\#A_i > 0$ then the point now is we can write

$$\Big\langle |f_{n+1}| \prod_{\ell\in A_i} |b_\ell^i - \langle b_\ell^i\rangle_Q| \Big\rangle_Q \lesssim \sum_{(\ell_t)} \langle g_{\ell_{\#A_i}}\rangle_Q,$$

where the sum is taken over all permutations of A_i and

$$g_{\ell_1} = \mathcal{A}_{\widetilde{\mathcal{S}}}^{\eta_1}(f_{n+1}\eta_1^{-1})\nu_i^{\theta_{\ell_1}}\eta_1, \qquad g_{\ell_k} = \mathcal{A}_{\widetilde{\mathcal{S}}}^{\eta_k}(g_{\ell_{k-1}}\eta_k^{-1})\nu_i^{\theta_{\ell_k}}\eta_k$$

with $\eta_1 = \lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}} \in A_\infty$ and for $2 \le k \le \#A_i$

$$\eta_k = \nu_i^{p\sum_{t=1}^{k-1}\theta_{\ell_t}} \lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}} = \Big(\prod_{i=1}^n w_i^{\frac{p}{p_i}}\Big)^{\sum_{t=1}^{k-1}\theta_{\ell_t}} \Big(\lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}}\Big)^{1-\sum_{t=1}^{k-1}\theta_{\ell_t}} \in A_\infty.$$

In particular,

$$(\nu_i^{\theta_{\ell_{\#A_i}}}\eta_{\#A_i})^{p'} \Big(\nu_i^{p\sum_{\ell\in A_i}\theta_\ell} \lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}}\Big)^{1-p'} = \eta_{\#A_i}$$

and

$$\|g_{\ell_{\#A_i}}\|_{L^{p'}\Big(\big(\nu_i^{p\sum_{\ell\in A_i}\theta_\ell}\lambda_i^{\frac{p}{p_i}}\prod_{j\neq i} w_j^{\frac{p}{p_j}}\big)^{1-p'}\Big)} \lesssim \|f_{n+1}\|_{L^{p'}\big((\lambda_i^{\frac{p}{p_i}}\prod_{j\neq i} w_j^{\frac{p}{p_j}})^{1-p'}\big)}.$$

Hence by applying Hölder's inequality and Proposition 2.11 with the multiple weight

$$(w_1, \dots, w_{i-1}, \nu_i^{p_i\theta_{A_i}}\lambda_i, w_{i+1}, \dots, w_n) \in A_{\vec{p}}$$

concludes the proof.

3.4. *Remark.* The reason why we do not track the constant above is that, we use Lemma 2.5 frequently which makes the presentation of the constant already very complicated, taking into account that there is even permutation involved. However, we would like to comment that the method used in the above improves the known bound obtained through the Cauchy integral trick if $\nu_i = 1$. In fact, if $\nu_i = 1$, for simplicity denote

$$w = \prod_{i=1}^n w_i^{\frac{p}{p_i}} = \lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}}.$$

Then apart from the constant from Proposition 2.11, the constant produces in the case $p \le 1$ is

$$[w]_{A_\infty}^{\#A_i} \|A_{\widetilde{\mathcal{S}}}^{w_i^{1-p_i'}}\|_{L^{p_i}(w_i^{1-p_i'})}^{\#B_i} \le C[w]_{A_\infty}^{\#A_i}[w_i^{1-p_i'}]_{A_\infty}^{\#B_i} \le C([w]_{A_\infty} + [w_i^{1-p_i'}]_{A_\infty})^{k_i}.$$

And for the case $p > 1$, we can track that the related constant is

$$\|A_{\widetilde{\mathcal{S}}}^{w_i^{1-p_i'}}\|_{L^{p_i}(w_i^{1-p_i'})}^{\#B_i} \|A_{\widetilde{\mathcal{S}}}^{w}\|_{L^{p'}(w)}^{\#A_i} \le C([w]_{A_\infty} + [w_i^{1-p_i'}]_{A_\infty})^{k_i}.$$

In both cases we have used the fact that

$$\|A_{\mathcal{S}}^{\sigma}\|_{L^p(\sigma)} \leq C[\sigma]_{A_\infty}, \qquad 1<p<\infty.$$

Indeed, for $f, g\geq 0$

$$\langle A_{\mathcal{S}}^{\sigma} f, g\sigma\rangle = \sum_{Q\in\mathcal{S}} \langle f\rangle_Q^{\sigma}\langle g\rangle_Q^{\sigma}\sigma(Q) \leq \sum_{Q\in\mathcal{S}}\Big(\big\langle (M_d^{\sigma} f M_d^{\sigma} g)^{\frac12}\big\rangle_Q^{\sigma}\Big)^2\sigma(Q)$$
$$\leq c[\sigma]_{A_\infty}\|M_d^{\sigma} f M_d^{\sigma} g\|_{L^1(\sigma)} \leq c[\sigma]_{A_\infty}\|f\|_{L^p(\sigma)}\|g\|_{L^{p'}(\sigma)},$$

where we have used Carleson's embedding theorem in the second inequality due to the fact that

$$\sum_{Q\in\mathcal{S}, Q\subset R}\sigma(Q)\leq c[\sigma]_{A_\infty}\sigma(R).$$

Now we see that both cases improve the bounds obtained through the Cauchy integral trick (see [7, Theorem 5.1] and [3, Theorem 5.6]) since

$$[w]_{A_\infty} + [w_i^{1-p_i'}]_{A_\infty} \leq \begin{cases} [w]_{A_\infty} + \sum_{i=1}^n [w_i^{1-p_i'}]_{A_\infty}, \\ c[\vec{w}]_{A_{\vec{p}}}^{\max\{1, \frac{p_1'}{p}, \cdots, \frac{p_n'}{p}\}}. \end{cases}$$

The first inequality is trivial, for the second, it is recorded in [8, Lemma 3.1] that

$$[w_i^{1-p_i'}]_{A_\infty} \leq [\vec{w}]_{A_{\vec{p}}}^{\frac{p_i'}{p}},$$

together with [23, Lemma 3.1] we have $[w]_{A_\infty} \leq [\vec{w}]_{A_{\vec{p}}}$ (actually this can be checked directly). The general logic is, our method only involves two A_∞ constants which connect to the definition of the commutator, while the Cauchy integral trick needs to involve all A_∞ constants.

We complete this section by the following general version of Theorem 1.3, whose proof is similar as the above and hence we omit the details.

3.5. **Theorem.** *Let T be an n-linear Calderón-Zygmund operator and $\mathcal{I}\subset\{1,\dots,n\}$. Let $C_{b_{\mathcal{I}}}^{k_{\mathcal{I}}}(T)$ be defined as in (1.1). Assume that for any $i\in\mathcal{I}$, $\theta_1^i,\dots,\theta_{k_i}^i\in[0,1]$ such that $\sum_{\ell=1}^{k_i}\theta_\ell^i = 1$, and let $\theta^i = \max\{\theta_\ell^i\}_{1\leq \ell\leq k_i}$. Let $1<p_1,\dots,p_n<\infty$ and $\frac1p = \sum_{i=1}^n\frac{1}{p_i}$. Assume that $(u_1,\dots,u_n)\in A_{\vec{p}}$, where*

$$u_i = \begin{cases} w_i, & \text{if } i\notin\mathcal{I} \\ w_i \text{ or } \lambda_i & \text{if } i\in\mathcal{I}. \end{cases}$$

If $\nu_i^{\theta^i} := w_i^{\frac{\theta^i}{p_i}}\lambda_i^{-\frac{\theta^i}{p_i}}\in A_\infty$ and $b_\ell^i\in \mathrm{BMO}_{\nu_i^{\theta_\ell^i}}$ for every $1\leq\ell\leq k_i$ and every $i\in\mathcal{I}$, then

$$\Big\|C_{b_{\mathcal{I}}}^{k_{\mathcal{I}}}(T): L^{p_1}(w_1)\times\cdots\times L^{p_n}(w_n)\to L^p\big((\prod_{i\in\mathcal{I}}\lambda_i^{\frac{p}{p_i}})\prod_{j\notin\mathcal{I}}w_j^{\frac{p}{p_j}}\big)\Big\| \lesssim \prod_{i\in\mathcal{I}}\prod_{\ell=1}^{k_i}\|b_\ell^i\|_{\mathrm{BMO}_{\nu_i^{\theta_\ell^i}}}.$$

4. PROOF OF THEOREM 1.6

This section is devoted to proving Theorem 1.6. The idea is to follow the median method in [15] and developed in [22]. Notice that the non-degeneracy of the kernel (2.4) implies that, for any cube Q, we may find $\widetilde{Q}$ such that $\ell(Q) = \ell(\widetilde{Q})$ and $\operatorname{dist}(Q, \widetilde{Q}) \geq C_0\ell(Q)$ and there exists some $\sigma \in \mathbb{C}$ with $|\sigma| = 1$ such that for all $x \in \widetilde{Q}$ and $y_1, \dots, y_n \in Q$ there holds

$$\operatorname{Re} \sigma K(x, y_1, \dots, y_n) \gtrsim |Q|^{-n}.$$

As we have mentioned in Remark 1.7, we will prove the same result under a weaker boundedness assumption. That is, instead of assuming

$$\Big\| C_b^{k_i}(T) : L^{p_1}(w_1) \times \cdots \times L^{p_n}(w_n) \to L^p\big(\lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}}\big)\Big\| < \infty,$$

we assume

$$\sup_{A\subset Q} \frac{1}{\prod_{i=1}^n \sigma_i(Q)^{\frac{1}{p_i}}} \Big\| 1_{\widetilde{Q}} C_b^{k_i}(T)(1_Q\sigma_1, \dots, 1_A\sigma_i, \dots, 1_Q\sigma_n)\Big\|_{L^{p,\infty}(\lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}})}, \tag{4.1}$$

where $\sigma_i = w_i^{1-p_i'}$, $1 \leq i \leq n$. For arbitrary $\alpha \in \mathbb{R}$ and $x \in \widetilde{Q} \cap \{b \geq \alpha\}$, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q (\alpha - b)_+^{k_i} \sigma_i \prod_{j\neq i} \frac{\sigma_j(Q)}{|Q|} \\
&\qquad \leq \operatorname{Re}\sigma \int_{Q\cap\{b\leq\alpha\}} \int_{Q^{n-1}} (b(x) - b(y_i))^{k_i} K(x, y_1, \dots, y_n) \prod_{j=1}^n \sigma_j(y_j) dy_j.
\end{aligned}$$

Then let α be a median of b on $\widetilde{Q}$, since $w \in A_\infty$ we have

$$\big|\widetilde{Q} \cap \{b \geq \alpha\}\big| \sim |\widetilde{Q}| \Rightarrow v(\widetilde{Q} \cap \{b \geq \alpha\}) \sim v(\widetilde{Q}) \sim v(Q),$$

where the last $\sim$ holds since $v = \lambda_i^{\frac{p}{p_i}} \prod_{j\neq i} w_j^{\frac{p}{p_j}} \in A_\infty$ is doubling. Hence we have

$$\begin{aligned}
&v(Q)^{\frac{1}{p}} \frac{1}{|Q|}\int_Q (\alpha - b)_+^{k_i} \sigma_i \prod_{j\neq i} \frac{\sigma_j(Q)}{|Q|} \\
&\lesssim \Bigg\| \operatorname{Re}\sigma \int_{Q\cap\{b\leq\alpha\}} \int_{Q^{n-1}} (b(x) - b(y_i))^{k_i} K(x, y_1, \dots, y_n) \prod_{j=1}^n \sigma_j(y_j) dy_j \Bigg\|_{L^{p,\infty}(\widetilde{Q}\cap\{b\geq\alpha\}; v)} \\
&\lesssim \prod_{i=1}^n \sigma_i(Q)^{\frac{1}{p_i}}.
\end{aligned}$$

Plugging the fact that

$$\Big(\frac{v(Q)}{|Q|}\Big)^{\frac{1}{p}} \Big(\frac{\eta_i(Q)}{|Q|}\Big)^{\frac{1}{p_i'}} \prod_{j\neq i} \Big(\frac{\sigma_j(Q)}{|Q|}\Big)^{\frac{1}{p_j'}} \geq 1,$$

where $\eta_i = \lambda_i^{1-p_i'}$, we arrive at

$$\frac{1}{\eta_i(Q)^{\frac{1}{p_i'}} \sigma_i(Q)^{\frac{1}{p_i}}} \int_Q (\alpha - b)_+^{k_i} \sigma_i \lesssim 1.$$

By Hölder's inequality we then get

$$\fint_Q (\alpha - b)_+ \sigma_i^{\frac{1}{k_i}} \lesssim \langle \eta_i \rangle_Q^{\frac{1}{k_i p_i'}} \langle \sigma_i \rangle_Q^{\frac{1}{k_i p_i}} \lesssim \langle \nu_i^{\frac{1}{k_i}} \sigma_i^{\frac{1}{k_i}} \rangle_Q,$$

where in the last inequality we have used Lemma 2.5. Likewise, we also have the symmetrical estimate

$$\fint_Q (b - \alpha)_+ \sigma_i^{\frac{1}{k_i}} \lesssim \langle \nu_i^{\frac{1}{k_i}} \sigma_i^{\frac{1}{k_i}} \rangle_Q.$$

Hence

$$\frac{1}{\nu_i^{\frac{1}{k_i}} \sigma_i^{\frac{1}{k_i}}(Q)} \int_Q |b - \alpha| \sigma_i^{\frac{1}{k_i}} \lesssim 1.$$

This is almost done, however, we need to take some care here. We first notice that the above estimate gives us that $b \in \widetilde{\mathrm{BMO}}_{\nu_i^{1/k_i}}(\sigma_i^{1/k_i})$. This step is simply seen by using triangle inequality. Then we conclude that $b \in \mathrm{BMO}_{\nu_i^{1/k_i}}$ by Lemma 2.13 and Remark 2.17.

5. Further discussions

In the last section we provide a useful idea to deal with the sparse form related with commutators. Our start point is, given $\mu, \lambda \in A_p$ and $\nu = \mu^{\frac{1}{p}} \lambda^{-\frac{1}{p}}$, the upper bound of

$$\sum_{Q \in \mathcal{S}} \left\langle |b - b_Q|^s |f|^s \right\rangle_Q^{\frac{1}{s}} \langle |g| \rangle_Q |Q| \le C \|b\|_{\mathrm{BMO}_\nu} \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\lambda^{1-p'})}, \qquad s > 1,$$

which is left open in [20]. The main point is, the average of $|b - b_Q| f$ is not L^1 but rather some L^s with $s > 1$. In the first sight this may produce some trouble, as if we use

$$|b - b_Q| 1_Q \lesssim \sum_{P \in \mathcal{S}, P \subset Q} \langle |b - b_P| \rangle_P 1_P \le \|b\|_{\mathrm{BMO}_\nu} \sum_{P \in \mathcal{S}, P \subset Q} \langle \nu \rangle_P 1_P,$$

one can not handle it as before due to the power s. However, the new idea here is to use the Fefferman-Stein type inequality. In the classical case, the optimal result is due to Pérez [27]. Here we use the corresponding result for sparse operators, see [2, 24]. For our purpose we record it in the following way:

$$\|A_{\mathcal{S}} f\|_{L^p(w)} \le C p' (r')^{\frac{1}{p'}} \|f\|_{L^p(M_r w)}. \tag{5.1}$$

Applying (5.1) we have

$$\left\langle |b - b_Q|^s |f|^s \right\rangle_Q^{\frac{1}{s}} \lesssim s' (r')^{\frac{1}{s'}} \|b\|_{\mathrm{BMO}_\nu} \langle \nu^s M_r(|f|^s) \rangle_Q^{\frac{1}{s}} = s'(r')^{\frac{1}{s'}} \|b\|_{\mathrm{BMO}_\nu} \langle \nu^s M_{rs}(f)^s \rangle_Q^{\frac{1}{s}}.$$

Then the proof is done by using the well-known bound of $(s, 1)$-sparse form (see [24]) and the boundedness of M_{rs} on $L^p(\mu)$ provided that rs is close to 1 enough so that $\mu \in A_{p/sr}$. Without providing more details we record the following result.

5.2. **Theorem.** *Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy $\int_{\mathbb{S}^{n-1}} \Omega = 0$. Suppose that $\mu, \lambda \in A_p$ and let $\nu = \mu^{\frac{1}{p}} \lambda^{-\frac{1}{p}}$. If $b \in \mathrm{BMO}_{\nu^{1/k}}$ for some $k \in \mathbb{Z}_+$, then*

$$\|C_b^k(T_\Omega)\|_{L^p(\mu) \to L^p(\lambda)} \lesssim \|b\|_{\mathrm{BMO}_{\nu^{1/k}}}^k.$$

In the multilinear case, the idea is essentially by combining the idea in Section 3 and the above. However, we are not able to prove the case when different BMO functions are paired with the same function yet. Nevertheless, we show that our method works for, e.g.

$$\sum_{Q\in\mathcal{S}} \big\langle |b-b_Q|^{ks}|f|^s\big\rangle_Q^{\frac1s}\langle |g|^s\rangle_Q^{\frac1s}\langle |h|^s\rangle_Q^{\frac1s}|Q|$$
$$\lesssim \|b\|_{\mathrm{BMO}_{\nu^{1/k}}}^k \|f\|_{L^{p_1}(w_1)}\|g\|_{L^{p_2}(w_2)}\|h\|_{L^{p'}((\lambda_1^{\frac{p}{p_1}}w_2^{\frac{p}{p_2}})^{1-p'})},$$

where $(w_1,w_2)\in A_{\vec p}$, $\nu^{\frac1k}=w_1^{\frac{1}{kp_1}}\lambda^{-\frac{1}{kp_1}}\in A_\infty$ and again, $s>1$ can be taken to be very close to 1. Now the idea is, we write

$$|b-b_Q|^k 1_Q \lesssim \|b\|_{\mathrm{BMO}_{\nu^{1/k}}}^k\Big(\sum_{P\in\mathcal{S},P\subset Q}\langle \nu^{\frac1k}\rangle_P 1_P\Big)^k$$
$$\lesssim \|b\|_{\mathrm{BMO}_{\nu^{1/k}}}^k\Big(\sum_{P\in\mathcal{S},P\subset Q}\langle \nu^{\frac1k}\rangle_P^\sigma 1_P\Big)^k,$$

where $\sigma\in A_\infty$ will be determined later and we have used Lemma 2.5 in the second inequality. Then it is a matter to formulate a weighted version of (5.1), say

$$\|A_{\mathcal{S}}^\sigma f\|_{L^p(w\sigma)}\le C\|f\|_{L^p(\sigma M^d_{r,\sigma}w)}. \tag{5.3}$$

The proof of (5.3) is quite similar as the proof of (5.1) stated in [24] and hence we omit the details. Now applying (5.3) with $\sigma=w_1^{1-\left(\frac{p_1}{s}\right)'}$, which is in A_∞ with suitable s, we have

$$\big\langle |b-b_Q|^{ks}|f|^s\big\rangle_Q^{\frac1s}\lesssim \|b\|_{\mathrm{BMO}_{\nu^{1/k}}}^k\big\langle \nu^s M^d_{r,\sigma}(|f|^s\sigma^{-1})\sigma\big\rangle_Q^{\frac1s}$$
$$=\|b\|_{\mathrm{BMO}_{\nu^{1/k}}}^k\big\langle \nu^s M^d_{rs,\sigma}(|f|\sigma^{-\frac1s})^s\sigma\big\rangle_Q^{\frac1s}.$$

Then it is a matter of using the weighted estimates for (s,s,s) sparse form (this is easily seen by the open property of $A_{\vec p}$) and the boundedness of $M^d_{rs,\sigma}$ on $L^{p_1}(\sigma)$, together with the following two facts:

$$\nu^{p_1}\sigma^{\frac{p_1}{s}}\lambda_1=\sigma,\qquad \text{and}\qquad \sigma^{1-\frac{p_1}{s}}=w_1.$$

In the above we do not track the dependence on the weight constants, and we encourage the interested readers to do so.

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