Cramér moderate deviations for the elephant random walk

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Abstract. We establish some limit theorems for the elephant random walk, including Berry-Esseen's bounds, Cramér moderate deviations and local limit theorems. These limit theorems can be regarded as refinements of the central limit theorem for the elephant random walk. Moreover, by these limit theorems, we conclude that the convergence rate of normal approximations and the domain of attraction of normal distribution mainly depend on a memory parameter p which lies between 0 and 3/4.

Keywords: elephant random walk, normal approximations, Berry-Esseen's bounds, Cramér moderate deviations, local limit theorems

1. Introduction

The elephant random walk (ERW) was introduced by Schütz and Trimper [10] in order to study the memory effects in the non-Markovian random walk. The model has a link to a famous saying that elephants can remember where they have been. Since the seminal work of Schütz and Trimper [10], the ERW has recently attracted a lot of attentions. A wide range of literature is available for the asymptotic behavior of the ERW and its extensions, see [1]-[5] and [11]. Baur and Bertoin [1] derived the functional limit theorem via a method of connection to Pólya-type urns. Coletti, Gava and Schütz [4, 5] proved the central limit theorem (CLT) and a strong invariance principle for $p \in [0, 3/4]$ and a law of large numbers for $p \in [0, 1)$. They also showed that if $p \in (3/4, 1]$, then the ERW converges to a non-degenerate random variable which is not normal. Vázquez Guevara [11] gave the almost sure central limit theorems. For the multi-dimensional ERW, we refer to [3], where Bercu and Lucile have established the CLT.

The one-dimensional ERW can be defined as follows. It starts at time $n = 0$, with position $S_0 = 0$. At time $n = 1$, the elephant moves to 1 with probability q and to -1 with probability $1 - q$, where $q \in [0, 1]$. So the position of the elephant at time $n = 1$ is given by $S_1 = X_1$, with X_1 a Rademacher $\mathcal{R}(q)$ random variable. At time $n + 1$, for

$n \geq 1$, an integer n' is chosen from the set $\{1, 2, \ldots, n\}$ uniformly at random. Then X_{n+1} is determined stochastically by the following rule. If $X_{n'} = 1$, then

$$X_{n+1} = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p. \end{cases}$$

If $X_{n'} = -1$, then

$$X_{n+1} = \begin{cases} 1 & \text{with probability } 1-p \\ -1 & \text{with probability } p. \end{cases}$$

Thus, for $n \geq 1$, the position of the elephant at time $n+1$ is

$$S_{n+1} = \sum_{i=1}^{n+1} X_i,$$

where

$$X_{n+1} = \alpha_n X_{\beta_n},$$

with α_n has a Rademacher distribution $\mathcal{R}(p)$, $p \in [0,1]$, and β_n is uniformly distributed over the integers $\{1, 2, \ldots, n\}$. Moreover, α_n is independent of $X_1, ..., X_n$. Here p is called the memory parameter. The ERW is respectively called diffusive, critical and superdiffusive according to $p \in [0, 3/4)$, $p = 3/4$ and $p \in (3/4, 1]$.

Recently, Bercu [2] presented the following CLT for the ERW: if $p \in [0, 3/4)$, then

$$\frac{S_n}{\sqrt{n/(3-4p)}} \xrightarrow{\mathbf{D}} \mathcal{N}(0,1), \quad n \to \infty; \tag{1}$$

and if $p = 3/4$, then

$$\frac{S_n}{\sqrt{n \log n}} \xrightarrow{\mathbf{D}} \mathcal{N}(0,1), \quad n \to \infty, \tag{2}$$

where $\xrightarrow{\mathbf{D}}$ stands for convergence in distribution. He also showed that when $p \in (3/4, 1]$, the ERW converges to a non-normal random variable.

In this paper we are interested in the absolute and relative errors of the normal approximations (1) and (2), and therefore we focus on the case where the memory parameter p lies between 0 and 3/4. For the absolute errors of normal approximations, we establish Berry-Esseen's bounds, which show that convergence rate of the absolute errors depends on the memory parameter p. For the relative errors of normal approximations, we obtain the Cramér moderate deviations, which conclude that the domain of attraction of normal distribution mainly depends on the memory parameter p. Moreover, the local limit theorems for the ERW are also established. Notice that when $p = 0$ or $1/2$, the ERW reduces to the classical symmetric random walk. Thus we do not consider the cases $p = 0$ and $1/2$.

The paper is organized as follows. In Section 2, we present our main results, including Berry-Esseen's bounds, Cramér moderate deviations and the local limit theorems. Section 3 contains a simulation study and applications of our results. The proofs of our results are given in Section 4. At the last section, we give a conclusion for the paper.

Throughout the paper, C and C_p, probably supplied with some indices, denote respectively a generic positive absolute constant and a generic positive constant depending only on p. For two sequences of positive numbers (a_n) and (b_n), write $a_n \asymp b_n$ if there exists an absolute constant $C > 0$ such that $a_n/C \leq b_n \leq Ca_n$ for all sufficiently large n. We also write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. Moreover, θ stands for values satisfying $|\theta| \leq 1$.

2. Main results

2.1. Berry-Esseen's bounds

Denote by $\Phi(t)$ the standard normal distribution function, and

$$D\,(X) = \sup_{t\in\mathbf{R}}\Big|\mathbf{P}(X \leq t) - \Phi(t)\Big|$$

the absolute error of the normal approximation for X. The upper bounds of $D(X)$ are called as Berry-Esseen's bounds. For $p \in (0, 3/4]$, denote

$$a_n = \frac{\Gamma(n)\Gamma(2p)}{\Gamma(n+2p-1)} \qquad \text{and} \qquad v_n = \sum_{i=1}^{n} a_i^2.$$

By Stirling's formula

$$\log\Gamma(x) = (x-\frac{1}{2})\log x - x + \frac{1}{2}\log 2\pi + O(\frac{1}{x}) \qquad \text{as} \quad x\to\infty,$$

we deduce that

$$\lim_{n\to\infty} a_n n^{2p-1} = \Gamma(2p). \tag{3}$$

Moreover, in the diffusive regime ($p \in (0, 3/4)$), we have

$$\lim_{n\to\infty}\frac{v_n}{n^{3-4p}} = \frac{\Gamma(2p)^2}{3-4p}, \tag{4}$$

and, in the critical regime ($p = 3/4$), it holds

$$\lim_{n\to\infty}\frac{v_n}{\log n} = \frac{\pi}{4}. \tag{5}$$

Our first result concerns with Berry-Esseen's bounds for the ERW, which shows that the absolute errors of normal approximations mainly depend on the memory parameter p.

Theorem 1 *Assume $p \in (0, 3/4]$ and $p \neq 1/2$. The following Berry-Esseen's bounds hold.*

[i] *If $p \in (0, 1/2)$, then*

$$D\Big(\frac{a_n S_n}{\sqrt{v_n}}\Big) \leq C\frac{\log n}{\sqrt{n}}. \tag{6}$$

[ii] *If* $p \in (1/2, 3/4)$*, then*

$$D\Big(\frac{a_n S_n}{\sqrt{v_n}}\Big) \leq C_p \frac{\log n}{n^{(3-4p)/2}}. \tag{7}$$

[iii] *If* $p = 3/4$*, then*

$$D\Big(\frac{a_n S_n}{\sqrt{v_n}}\Big) \leq C \frac{\log\log n}{\sqrt{\log n}}. \tag{8}$$

The following corollary is a simple consequence of Theorem 1, which bridges Theorem 1 and the CLT of Bercu [2].

Corollary 1 *Assume* $p \in (0, 3/4]$ *and* $p \neq 1/2$*. The following Berry-Esseen's bounds hold.*

[i] *If* $p \in (0, 1/2)$*, then*

$$D\Big(\frac{S_n}{\sqrt{n/(3-4p)}}\Big) \leq C\Big(\frac{\log n}{\sqrt{n}} + \Big|\frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}} - 1\Big|\Big). \tag{9}$$

[ii] *If* $p \in (1/2, 3/4)$*, then*

$$D\Big(\frac{S_n}{\sqrt{n/(3-4p)}}\Big) \leq C_p\Big(\frac{\log n}{n^{(3-4p)/2}} + \Big|\frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}} - 1\Big|\Big). \tag{10}$$

[iii] *If* $p = 3/4$*, then*

$$D\Big(\frac{S_n}{\sqrt{n\log n}}\Big) \leq C\Big(\frac{\log\log n}{\sqrt{\log n}} + \Big|\frac{\sqrt{v_n}}{a_n\sqrt{n\log n}} - 1\Big|\Big). \tag{11}$$

By the equalities (3) and (4), it is easy to verify that for $0 < p < 3/4$, it holds

$$\lim_{n\to\infty} \frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}} = 1. \tag{12}$$

Similarly, for $p = 3/4$, by the equalities (3) and (5) and the fact $\Gamma(3/2) = \sqrt{\pi}/2$, it holds $\lim_{n\to\infty} \frac{\sqrt{v_n}}{a_n\sqrt{n\log n}} = 1$. Thus the results of Corollary 1 coincide with the CLT of Bercu [2], that is, (1) and (2).

2.2. Cramér moderate deviations

The following theorem gives some Cramér moderate deviations for the ERW.

Theorem 2 *Assume* $p \in (0, 3/4]$ *and* $p \neq 1/2$*. We have the following equalities.*

[i] *If* $p \in (0, 1/2)$*, then there is an absolute constant* $\alpha_0 > 0$ *such that for all* $0 \leq x \leq \alpha_0 \sqrt{n}$*,*

$$\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1 - \Phi(x)} = \exp\Big\{\theta C_{\alpha_0}\Big(\frac{x^3}{\sqrt{n}} + (1+x)\frac{\log n}{\sqrt{n}}\Big)\Big\}. \tag{13}$$

[ii] *If* $p \in (1/2, 3/4)$*, then there is an absolute constant* $\alpha_0 > 0$ *such that for all* $0 \leq x \leq \alpha_0 \, n^{(3-4p)/2}$*,*

$$\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1 - \Phi(x)} = \exp\Big\{\theta C_{\alpha_0,p}\Big(\frac{x^3}{n^{(3-4p)/2}} + (1+x)\frac{\log n}{n^{(3-4p)/2}}\Big)\Big\}. \tag{14}$$

[iii] *If $p = 3/4$, then there is an absolute constant $\alpha_0 > 0$ such that for all $0 \leq x \leq \alpha_0 \sqrt{\log n}$,*

$$\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1-\Phi(x)} = \exp\Big\{\theta C_{\alpha_0}\Big(\frac{x^3}{\sqrt{\log n}} + (1+x)\frac{\log\log n}{\sqrt{\log n}}\Big)\Big\}. \tag{15}$$

Moreover, the same equalities hold when $\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1-\Phi(x)}$ is replaced by $\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \leq -x)}{\Phi(-x)}$.

It is easy to see that $|e^x - 1| \leq e|x|$ for all $|x| \leq 1$. By Theorem 2, we have the following corollary, which shows that the domain of attraction of normal distribution mainly depends on the memory parameter p.

Corollary 2 *Assume $p \in (0, 3/4]$ and $p \neq 1/2$. The following results hold.*

[i] *If $p \in (0, 1/2)$, then for all $0 \leq x \leq n^{1/6}$,*

$$\Big|\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1-\Phi(x)} - 1\Big| \leq C\Big(\frac{x^3}{\sqrt{n}} + (1+x)\frac{\log n}{\sqrt{n}}\Big). \tag{16}$$

In particular, the last inequality implies that

$$\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1-\Phi(x)} = 1 + o(1) \tag{17}$$

uniformly for $0 \leq x = o(n^{1/6})$ as $n \to \infty$.

[ii] *If $p \in (1/2, 3/4)$, then for all $0 \leq x \leq n^{(3-4p)/6}$,*

$$\Big|\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1-\Phi(x)} - 1\Big| \leq C_p\Big(\frac{x^3}{n^{(3-4p)/2}} + (1+x)\frac{\log n}{n^{(3-4p)/2}}\Big). \tag{18}$$

In particular, the last inequality implies that (17) holds uniformly for $0 \leq x = o(n^{(3-4p)/6})$ as $n \to \infty$.

[iii] *If $p = 3/4$, then for all $0 \leq x \leq (\log n)^{1/6}$,*

$$\Big|\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1-\Phi(x)} - 1\Big| \leq C\Big(\frac{x^3}{\sqrt{\log n}} + (1+x)\frac{\log\log n}{\sqrt{\log n}}\Big). \tag{19}$$

In particular, the last inequality implies that (17) holds uniformly for $0 \leq x = o((\log n)^{1/6})$ as $n \to \infty$.

Moreover, the same equalities hold when $\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1-\Phi(x)}$ is replaced by $\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \leq -x)}{\Phi(-x)}$.

From Theorem 2, we can deduce the following moderate deviation principles (MDP) for the ERW.

Corollary 3 *Assume $p \in (0, 3/4]$ and $p \neq 1/2$. The following MDP hold.*

[i] *Assume $p \in (0, 1/2)$. Let b_n be any sequence of real numbers satisfying $b_n \to \infty$ and $b_n/\sqrt{n} \to 0$ as $n \to \infty$. Then for each Borel set B,*

$$\begin{aligned} -\inf_{x \in B^o} \frac{x^2}{2} &\leq \liminf_{n\to\infty} \frac{1}{b_n^2} \ln \mathbf{P}\Big(\frac{a_n S_n}{b_n\sqrt{v_n}} \in B\Big) \\ &\leq \limsup_{n\to\infty} \frac{1}{b_n^2} \ln \mathbf{P}\Big(\frac{a_n S_n}{b_n\sqrt{v_n}} \in B\Big) \leq -\inf_{x \in \overline{B}} \frac{x^2}{2}, \end{aligned} \tag{20}$$

where B^o and $\overline{B}$ denote the interior and the closure of B, respectively.

[ii] *Assume* $p \in (1/2, 3/4)$. *Let* b_n *be any sequence of real numbers satisfying* $b_n \to \infty$ *and* $b_n/n^{(3-4p)/2} \to 0$ *as* $n \to \infty$. *Then (20) holds.*

[iii] *Assume* $p = 3/4$. *Let* b_n *be any sequence of real numbers satisfying* $b_n \to \infty$ *and* $b_n/\sqrt{\ln n} \to 0$ *as* $n \to \infty$. *Then (20) holds.*

2.3. Local limit theorems

The following theorem gives asymptotic expansions for the probability $\mathbf{P}(S_n = k)$.

Theorem 3 *Assume* $p \in (0, 3/4]$ *and* $p \neq 1/2$. *The following results hold.*

[i] *If* $p \in (0, 1/2)$, *then for all* $|k| = o(n^{2/3})$,

$$\left| \frac{\mathbf{P}(S_n = k)}{\frac{a_n}{\sqrt{2\pi v_n}} \exp\{-\frac{(a_n k)^2}{2v_n}\}} - 1 \right| \leq C\Big(\frac{|k|}{n} + \frac{k^2}{n^{3/2}} + \frac{\log n}{\sqrt{n}}\Big). \tag{21}$$

In particular, the last inequality implies the following local limit theorem:

$$\frac{\mathbf{P}(S_n = k)}{\frac{a_n}{\sqrt{2\pi v_n}} \exp\{-\frac{(a_n k)^2}{2v_n}\}} = 1 + o(1) \tag{22}$$

uniformly for $|k| = o(n^{2/3})$ *as* $n \to \infty$.

[ii] *If* $p \in (1/2, 3/4)$, *then for all* $|k| = o(n^{(3-2p)/3})$,

$$\left| \frac{\mathbf{P}(S_n = k)}{\frac{a_n}{\sqrt{2\pi v_n}} \exp\{-\frac{(a_n k)^2}{2v_n}\}} - 1 \right| \leq C_p\Big(\frac{|k|}{n} + \frac{k^2}{n^{(5-4p)/2}} + \frac{\log n}{n^{(3-4p)/2}}\Big). \tag{23}$$

In particular, the last inequality implies that (22) holds uniformly for $|k| = o(n^{(3-2p)/3})$ *as* $n \to \infty$.

[ii] *If* $p = 3/4$, *then for all* $|k| = o((n \log n)^{1/2})$,

$$\left| \frac{\mathbf{P}(S_n = k)}{\frac{a_n}{\sqrt{2\pi v_n}} \exp\{-\frac{(a_n k)^2}{2v_n}\}} - 1 \right| \leq C\Big(\frac{|k|}{n \log n} + \frac{k^2}{n(\log n)^{3/2}} + \frac{\log \log n}{\sqrt{\log n}}\Big). \tag{24}$$

Thus (22) holds uniformly for $|k| = o((n \log n)^{1/2})$ *as* $n \to \infty$.

Denote by

$$L(S_n) = \sup_{k \in \mathbf{Z}} \left| \mathbf{P}(S_n = k) - \frac{a_n}{\sqrt{2\pi v_n}} \exp\Big\{ - \frac{(a_n k)^2}{2v_n} \Big\} \right|$$

the absolute error for the local limit theorem.

Corollary 4 *Assume* $p \in (0, 3/4)$ *and* $p \neq 1/2$. *The following inequalities hold for* $L(S_n)$.

[i] *If* $p \in (0, 1/2)$, *then*

$$L(S_n) \leq C \frac{\log n}{n}. \tag{25}$$

[ii] *If* $p \in (1/2, 3/4)$, *then*

$$L(S_n) \leq C_p \frac{\log n}{n^{2-2p}}. \tag{26}$$

From the last corollary, it is easy to see that the absolute errors for the local limit theorem are much smaller than that for the CLT (cf. Berry-Esseen's bounds in Theorem 1).

3. A simulation study and applications

3.1. A simulation study

In this section, we study the normal approximation accuracies for the ERW. We let $n = 10^4$, and choose 8 levels of p: $p = 0.1, 0.2, ..., 0.7, 0.75$. Figures 1 and 2 show the Cramér moderate deviation ratios $\frac{\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)}{1-\Phi(x)}$ and $\frac{\mathbf{P}(S_n/\sqrt{n \log n} \geq x)}{1-\Phi(x)}$, where the probabilities $\mathbf{P}(a_n S_n/\sqrt{v_n} \geq x)$ and $\mathbf{P}(S_n/\sqrt{n \log n} \geq x)$ are approximated by simulating 10^5 realizations of the ERW. When $p = 0.1, 0.2, ..., 0.5$, we see that the Cramér moderate deviation ratios have comparable performance; see Figure 1. When $p = 0.6, 0.7, 0.75$, the normal approximations are worse than the case of $p = 0.1, 0.2, ..., 0.5$; see Figure 2. Moreover, as x moves away from 0, the moderate deviation approximations become worse. These simulation results coincide with Corollary 2.

3.2. Applications

Berry-Esseen's bounds and Cramér moderate deviations for the ERW can be applied to construct confidence lower limit for the memory parameter p. Assume $p \in (0, 3/4)$ and $\kappa \in (0, 1)$. By the inequalities (9) and (10), then we have

$$\mathbf{P}\Big(|S_n/\sqrt{n/(3-4p)}| \leq \Phi^{-1}(1-\kappa/2) \Big) \to 1-\kappa, \qquad n \to \infty. \tag{27}$$

Clearly, $|S_n/\sqrt{n/(3-4p)}| \leq \Phi^{-1}(1-\kappa/2)$ means that $p \geq \frac{1}{4}(3 - n(\frac{\Phi^{-1}(1-\kappa/2)}{S_n})^2)$. Thus (27) implies that $\frac{1}{4}(3 - n(\frac{\Phi^{-1}(1-\kappa/2)}{S_n})^2)$ is a $1-\kappa$ lower confidence limit for p, for n large enough. When κ is replaced by κ_n such that $\kappa_n \to 0$ and $|\log \kappa_n| = o(\sqrt{n})$ as $n \to \infty$, similar results are allowed to be established via Cramér moderate deviations for the ERW. Conversely, if p is known, Cramér moderate deviations can also be used to interval estimations for the position of the elephant in the ERW model.

4. Proofs of Theorems

4.1. Preliminary lemmas

Set

$$\gamma_n = 1 + \frac{2p-1}{n}$$

and put $a_1 = 1$. For $n \geq 2$, it is easy to see that

$$a_n = \frac{\Gamma(n)\Gamma(2p)}{\Gamma(n+2p-1)} = \prod_{i=1}^{n-1} \frac{1}{\gamma_i}.$$

Define the filtration $\mathcal{F}_n = \sigma\{X_i : 1 \leq i \leq n\}$ and

$$M_n = a_n S_n \qquad a.s. \tag{28}$$

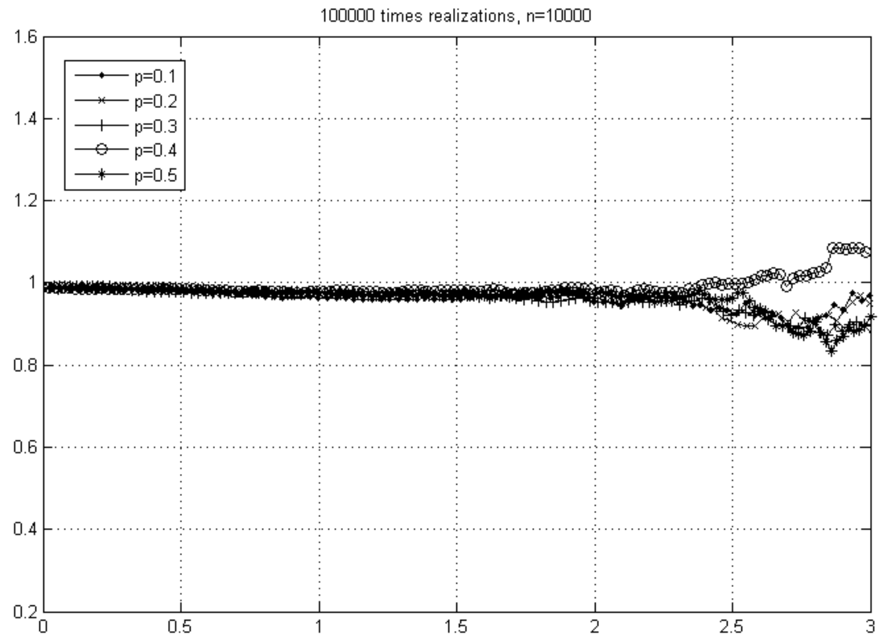

Figure 1. Ratios $\frac{\mathbf{P}(a_nS_n/\sqrt{v_n}\geq x)}{1-\Phi(x)}$ for ERW with $n=10^4$ and $p=0.1,0.2,...,0.5$.

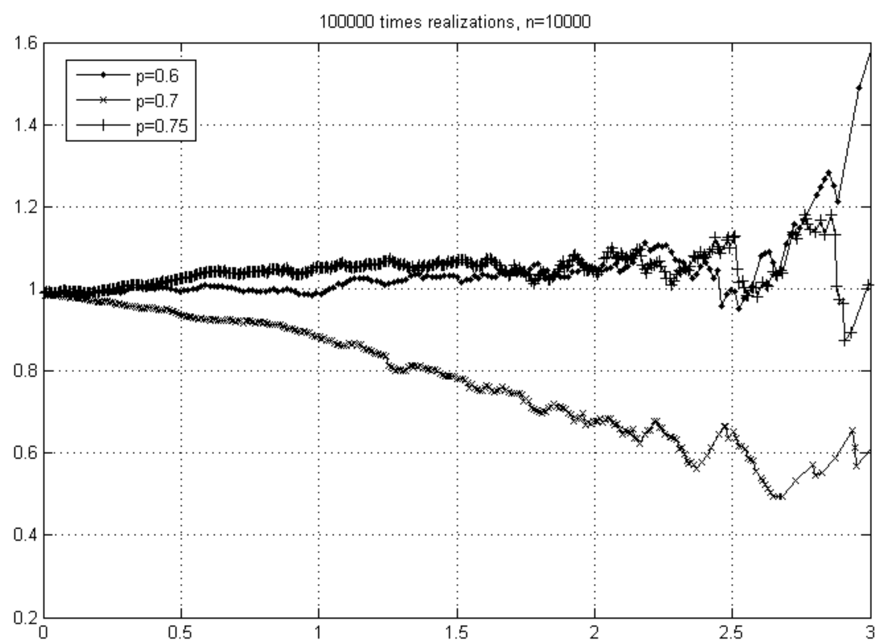

Figure 2. Ratios $\frac{\mathbf{P}(a_nS_n/\sqrt{v_n}\geq x)}{1-\Phi(x)}$ for ERW with $n=10^4$ and $p=0.6$, 0.7. For $p=0.75$, we adopt the ratio $\frac{\mathbf{P}(S_n/\sqrt{n\log n}\geq x)}{1-\Phi(x)}$.

It is easy to verify that $(M_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale. Indeed, for all $n \geq 1$, we have

$$\begin{aligned}\mathbf{E}[M_{n+1}|\mathcal{F}_n] &= \mathbf{E}[a_{n+1}(S_n + \alpha_n X_{\beta_n})|\mathcal{F}_n] \\ &= a_{n+1}S_n + a_{n+1}\mathbf{E}[\alpha_n]\mathbf{E}[X_{\beta_n}|\mathcal{F}_n] \\ &= a_{n+1}\Big(S_n + (2p-1)\frac{S_n}{n}\Big) \\ &= a_n S_n \\ &= M_n \qquad a.s. \end{aligned} \tag{29}$$

Moreover, we can rewrite $(M_n)_{n\geq 1}$ in the following additive form

$$M_n = \sum_{i=1}^n a_i \varepsilon_i, \tag{30}$$

where $\varepsilon_i = S_i - \gamma_{i-1}S_{i-1}$ with $S_0 = 0$. Let $(\Delta M_n)_{n\geq 1}$ be the martingale differences defined by $\Delta M_1 = M_1$ and for $n \geq 2$,

$$\Delta M_n = M_n - M_{n-1}.$$

Denote by $\langle M\rangle_n$ the quadratic variation

$$\langle M\rangle_n = \sum_{i=1}^n \mathbf{E}[\Delta M_i^2|\mathcal{F}_{i-1}].$$

In the proofs of theorems, we need the following two lemmas for the boundness of martingale differences and the convergence of quadratic variation.

Lemma 1 *For each $n \geq 1$ and $p \in [0,1]$, it holds*

$$|\Delta M_n| \leq 2a_n.$$

Proof. It holds obviously for $n = 1$. Observe that

$$\Delta M_n = a_n S_n - a_{n-1}S_{n-1} = a_n X_n - a_n \frac{S_{n-1}}{n-1}(2p-1).$$

Since $|X_n| \leq 1$, we have $|S_{n-1}| \leq n-1$. It is easy to see that $|\Delta M_n| \leq 2a_n$.

Lemma 2 *Assume $p \in (0, 3/4]$. For all $1 \leq i \leq n$, we have*

$$\Big\|\frac{\Delta M_i}{\sqrt{v_n}}\Big\|_\infty \leq \frac{2\,a_i}{\sqrt{v_n}}$$

and

$$\Big\|\frac{\langle M\rangle_n}{v_n} - 1\Big\|_\infty \leq \begin{cases} \dfrac{C}{3-4p}\dfrac{1}{n}, & \text{if } 0 < p < 3/4, \\ \\ \dfrac{C}{\log n}, & \text{if } p = 3/4. \end{cases} \tag{31}$$

Proof. Clearly, from Lemma 1, we have

$$\|\Delta M_i/\sqrt{v_n}\|_\infty \leq 2a_i/\sqrt{v_n},$$

which gives the first desired inequality. Next we give an estimation of $\|\langle M\rangle_n/v_n - 1\|_\infty$ for $0 < p \leq 3/4$. From (30), we get $\Delta M_k = a_k\varepsilon_k = a_k(S_k - \gamma_{k-1}S_{k-1})$. Thus, it holds

$$\begin{aligned}\mathbf{E}[(\Delta M_k)^2|\mathcal{F}_{k-1}] &= a_k^2\mathbf{E}[(S_k - \gamma_{k-1}S_{k-1})^2|\mathcal{F}_{k-1}] \\ &= a_k^2(\mathbf{E}[S_k^2|\mathcal{F}_{k-1}] - 2\gamma_{k-1}S_{k-1}\mathbf{E}[S_k|\mathcal{F}_{k-1}] + \gamma_{k-1}^2S_{k-1}^2).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\mathbf{E}[S_k^2|\mathcal{F}_{k-1}] &= \mathbf{E}[(S_{k-1} + \alpha_k X_{\beta_k})^2|\mathcal{F}_{k-1}] \\ &= S_{k-1}^2 + 2S_{k-1}\mathbf{E}[\alpha_k X_{\beta_k}|\mathcal{F}_{k-1}] + 1 \\ &= S_{k-1}^2 + 2\frac{2p-1}{k-1}S_{k-1}^2 + 1 \\ &= (2\gamma_{k-1} - 1)S_{k-1}^2 + 1\end{aligned}$$

and

$$\mathbf{E}[S_k|\mathcal{F}_{k-1}] = \mathbf{E}[S_{k-1} + \alpha_k X_{\beta_k}|\mathcal{F}_{k-1}] = S_{k-1} + \frac{2p-1}{k-1}S_{k-1} = \gamma_{k-1}S_{k-1}.$$

Thus, we have $\mathbf{E}[(\Delta M_1)^2] = 1 = a_1^2$ and for $k \geq 2$,

$$\begin{aligned}\mathbf{E}[(\Delta M_k)^2|\mathcal{F}_{k-1}] &= a_k^2((2\gamma_{k-1} - 1)S_{k-1}^2 + 1 - 2\gamma_{k-1}^2S_{k-1}^2 + \gamma_{k-1}^2S_{k-1}^2) \\ &= a_k^2(1 - (\gamma_{k-1} - 1)^2S_{k-1}^2) \\ &= a_k^2 - (2p-1)^2a_k^2(\frac{S_{k-1}}{k-1})^2.\end{aligned}$$

Hence, by the definition of v_n and M_k, we obtain

$$\langle M\rangle_n = v_n - (2p-1)^2\Big(\sum_{k=1}^{n-1}\Big(\frac{a_{k+1}}{a_k}\Big)^2\Big(\frac{M_k}{k}\Big)^2\Big).$$

Since $\frac{a_{n+1}}{a_n} \sim 1$ as $n \to \infty$ (cf. (3)), we have

$$\|\langle M\rangle_n - v_n\|_\infty \leq C_1(2p-1)^2\Big\|\sum_{k=1}^{n-1}(\frac{M_k}{k})^2\Big\|_\infty \leq C_2\sum_{k=1}^{n-1}\frac{1}{k^2}\|M_k\|_\infty^2. \tag{32}$$

Using Lemma 1, we derive that

$$\|M_k\|_\infty^2 \leq \sum_{l=1}^{k}\|\Delta M_l\|_\infty^2 \leq 4v_k.$$

In the diffusive regime $0 < p < 3/4$, by (4), we get

$$\|\langle M\rangle_n - v_n\|_\infty \leq 4C_2\sum_{k=1}^{n-1}\frac{1}{k^2}v_k \leq C_3\frac{\Gamma(2p)^2}{3-4p}\sum_{k=1}^{n-1}k^{1-4p} \leq \frac{C_4}{3-4p}n^{2-4p}. \tag{33}$$

In the critical regime $p = 3/4$, by (5), we have

$$\|\langle M\rangle_n - v_n\|_\infty \leq C_5\sum_{k=1}^{n-1}\frac{\log k}{k^2} \leq C_6. \tag{34}$$

Consequently, again by (4) and (5), we obtain (31). This completes the proof of lemma.

4.2. Proof of Theorem 1

In the proof of Theorem 1, we will make use of the following lemma of Fan [7], which gives an exact Berry-Esseen's bound for martingales.

Lemma 3 *Assume that there exist positive numbers $\epsilon_n > 0$ and $\delta_n \geq 0$, such that for all $1 \leq i \leq n$,*

$$\|\Delta M_i/\sqrt{v_n}\|_\infty \leq \epsilon_n \tag{35}$$

and

$$\|\langle M\rangle_n/v_n - 1\|_\infty \leq \delta_n^2 \qquad a.s. \tag{36}$$

If $\epsilon_n, \delta_n \to 0$ as $n \to \infty$, then

$$D\Big(\frac{M_n}{\sqrt{v_n}}\Big) \leq C\left(\epsilon_n|\log \epsilon_n| + \delta_n\right), \tag{37}$$

where C is a positive absolute constant.

Now we are in position to prove Theorem 1. Clearly, we have

$$\frac{a_n S_n}{\sqrt{v_n}} = \frac{M_n}{\sqrt{v_n}} = \sum_{i=1}^{n} \frac{\Delta M_i}{\sqrt{v_n}},$$

and $(\Delta M_i/\sqrt{v_n}, \mathcal{F}_i)_{i=1,\ldots,n}$ is a finite sequence of martingale differences. From Lemma 2, we have

$$\|\Delta M_i/\sqrt{v_n}\|_\infty \leq 2 \max_{1\leq i\leq n} a_i/\sqrt{v_n} =: \epsilon_n.$$

Using the inequalities (3)-(5), we deduce that

$$\epsilon_n \asymp \begin{cases} n^{-1/2}, & \text{if } 0 < p < 1/2, \\ \sqrt{3-4p}\; n^{-(3-4p)/2}, & \text{if } 1/2 < p < 3/4, \\ (\log n)^{-1/2}, & \text{if } p = 3/4. \end{cases} \tag{38}$$

Moreover, from Lemma 2, we have

$$\Big\|\frac{\langle M\rangle_n}{v_n} - 1\Big\|_\infty \leq \delta_n^2 := \begin{cases} \dfrac{C}{3-4p} n^{-1}, & \text{if } 0 < p < 3/4, \\ C(\log n)^{-1}, & \text{if } p = 3/4. \end{cases} \tag{39}$$

Applying Lemma 3 to $M_n/\sqrt{v_n}$, we obtain the desired results.

Remark 1 *Under the conditions of Lemma 3, El Machkouri and Ouchti [6] had obtained the following Berry-Esseen bound*

$$D(\frac{M_n}{\sqrt{v_n}}) \leq C\Big(\epsilon_n \log n + \delta_n\Big).$$

For $0 < p < 1/2$, we have $\epsilon_n \asymp 1/\sqrt{n}$, and thus for $0 < p < 1/2$, Theorem 1 can also be obtained via the Berry-Esseen bound of El Machkouri and Ouchti [6] stated before.

4.3. Proof of Corollary 1

We only present the proof of Corollary 1 for $0 < p < 1/2$. For $1/2 < p \leq 3/4$, Corollary 1 can be proved in a similar way. Clearly, it holds

$$
\begin{aligned}
& D\Big(\frac{S_n}{\sqrt{n/(3-4p)}}\Big) \\
&= \sup_{t\in\mathbf{R}}\Big|\mathbf{P}\Big(\frac{S_n}{\sqrt{n/(3-4p)}} \leq t\Big) - \Phi(t)\Big| \\
&= \sup_{t\in\mathbf{R}}\Big|\mathbf{P}\Big(\frac{S_n}{\sqrt{n/(3-4p)}} \leq \frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}}t\Big) - \Phi\Big(\frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}}t\Big)\Big| \\
&\leq \sup_{t\in\mathbf{R}}\Big|\mathbf{P}\Big(\frac{S_n}{\sqrt{n/(3-4p)}} \leq \frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}}t\Big) - \Phi(t)\Big| \\
&+ \sup_{t\in\mathbf{R}}\Big|\Phi(t) - \Phi\Big(\frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}}t\Big)\Big| \\
&= \sup_{t\in\mathbf{R}}\Big|\mathbf{P}\Big(\frac{a_nS_n}{\sqrt{v_n}} \leq t\Big) - \Phi(t)\Big| + \sup_{t\in\mathbf{R}}\Big|\Phi(t) - \Phi\Big(\frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}}t\Big)\Big|.
\end{aligned}
$$

By inequality (6), we get

$$
\begin{aligned}
D\Big(\frac{S_n}{\sqrt{n/(3-4p)}}\Big) &\leq C_1\frac{\log n}{\sqrt{n}} + C_2\Big|\frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}} - 1\Big| \\
&\leq C\Big(\frac{\log n}{\sqrt{n}} + \Big|\frac{\sqrt{v_n}}{a_n\sqrt{n/(3-4p)}} - 1\Big|\Big),
\end{aligned}
$$

which gives the desired inequality.

4.4. Proof of Theorem 2

In the proof of Theorem 2, we shall make use of the following lemma of Fan, Grama and Liu [8], which gives a Cramér type moderate deviation for martingales. The lemma is a simple consequence of Theorem 1 of [8], with $\xi_i = \Delta M_i/\sqrt{v_n}$, $\rho = 1$ and $\varepsilon_n = e\epsilon_n$. See also Grama and Haeusler [9] for an earlier result.

Lemma 4 *Assume the conditions of Lemma 3. Then there is an absolute constant* $\alpha_0 > 0$ *such that for all* $0 \leq x \leq \alpha_0\, \epsilon_n^{-1}$,

$$
\frac{\mathbf{P}(M_n/\sqrt{v_n} \geq x)}{1-\Phi\,(x)} = \exp\Big\{\theta C_{\alpha_0}\Big(x^3\epsilon_n + x^2\delta_n^2 + (1+x)\,(\epsilon_n\,|\log \epsilon_n| + \delta_n)\Big)\Big\}
$$

and

$$
\frac{\mathbf{P}(M_n/\sqrt{v_n} \leq -x)}{\Phi\,(-x)} = \exp\Big\{\theta C_{\alpha_0}\Big(x^3\epsilon_n + x^2\delta_n^2 + (1+x)\,(\epsilon_n\,|\log \epsilon_n| + \delta_n)\Big)\Big\}.
$$

Now we are in position to prove Theorem 2. Recall that

$$\frac{a_n S_n}{\sqrt{v_n}} = \frac{M_n}{\sqrt{v_n}} = \sum_{i=1}^{n} \frac{\Delta M_i}{\sqrt{v_n}}.$$

According to the proof of Theorem 1, the conditions of Lemma 3 are satisfied with ϵ_n and δ_n^2 satisfying the inequalities (38) and (39) respectively. Notice that for all $x \geq 0$, the following three inequalities hold

$$\frac{x^3}{\sqrt{n}} + \frac{x^2}{n} + (1+x)\frac{\log n}{\sqrt{n}} \leq 2\Big(\frac{x^3}{\sqrt{n}} + (1+x)\frac{\log n}{\sqrt{n}}\Big), \quad 0 < p < 1/2,$$

$$\frac{x^3}{n^{(3-4p)/2}} + \frac{x^2}{n} + (1+x)\frac{\log n}{n^{(3-4p)/2}} \leq 2\Big(\frac{x^3}{n^{(3-4p)/2}} + (1+x)\frac{\log n}{n^{(3-4p)/2}}\Big), \quad 1/2 < p < 3/4,$$

and

$$\frac{x^3}{\sqrt{\log n}} + \frac{x^2}{\log n} + (1+x)\frac{\log\log n}{\sqrt{\log n}} \leq 2\Big(\frac{x^3}{\sqrt{\log n}} + (1+x)\frac{\log\log n}{\sqrt{\log n}}\Big), \quad p = 3/4.$$

Applying Lemma 4 to $M_n/\sqrt{v_n}$, we obtain the desired results of Theorem 2.

4.5. Proof of Corollary 3

We only present the proof of Corollary 3 for $0 < p < 1/2$. For $1/2 < p \leq 3/4$, the proof of Corollary 3 is similar. To prove Corollary 3, we need the following inequalities for the normal distribution function:

$$\frac{1}{\sqrt{2\pi}(1+x)} e^{-x^2/2} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{\pi}(1+x)} e^{-x^2/2}, \qquad x \geq 0. \tag{40}$$

First we show that

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \ln \mathbf{P}\Big(\frac{a_n S_n}{b_n\sqrt{v_n}} \in B\Big) \leq -\inf_{x\in \overline{B}} \frac{x^2}{2}. \tag{41}$$

When $B = \emptyset$, the last inequality holds obviously, with $-\inf_{x\in\emptyset} \frac{x^2}{2} = -\infty$. Thus, we may assume that $B \neq \emptyset$. For a given Borel set $B \subset \mathbf{R}$, denote $x_0 = \inf_{x\in B}|x|$. Notice that $\overline{B} \supset B$, which leads to $x_0 \geq \inf_{x\in\overline{B}}|x|$ and $x_0^2/2 \geq \inf_{x\in\overline{B}} x^2/2$. Therefore, from Theorem 2, it follows that

$$\begin{aligned}\mathbf{P}\Big(\frac{a_n S_n}{b_n\sqrt{v_n}} \in B\Big) &\leq \mathbf{P}\Big(\Big|\frac{a_n S_n}{\sqrt{v_n}}\Big| \geq b_n x_0\Big) \\ &\leq 2\Big(1-\Phi\left(b_n x_0\right)\Big) \exp\Big\{C_p\Big(\frac{(b_n x_0)^3}{\sqrt{n}} + (1+b_n x_0)\frac{\log n}{\sqrt{n}}\Big)\Big\}.\end{aligned}$$

Using (40), we get

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \ln \mathbf{P}\Big(\frac{a_n S_n}{b_n\sqrt{v_n}} \in B\Big) \ \leq\ -\frac{x_0^2}{2} \ \leq\ -\inf_{x\in\overline{B}} \frac{x^2}{2},$$

which gives (41).

Next we prove that

$$\liminf_{n\to\infty} \frac{1}{b_n^2} \ln \mathbf{P}\Big(\frac{a_n S_n}{b_n \sqrt{v_n}} \in B \Big) \geq - \inf_{x \in B^o} \frac{x^2}{2}. \tag{42}$$

The last inequality holds obviously for $B^o = \emptyset$, with $-\inf_{x\in\emptyset} \frac{x^2}{2} = -\infty$. So, we assume that $B^o \neq \emptyset$. Notice that B^o is an open set. Therefore, for any given $\varepsilon_1 > 0$, there exists an $x_0 \in B^o$, such that

$$0 < \frac{x_0^2}{2} \leq \inf_{x\in B^o} \frac{x^2}{2} + \varepsilon_1. \tag{43}$$

Without loss of generality, we assume that $x_0 > 0$. For all small enough $\varepsilon_2 \in (0, x_0)$, it holds $(x_0 - \varepsilon_2, x_0 + \varepsilon_2] \subset B^o \subset B$. Clearly, we have

$$\begin{aligned}\mathbf{P}\Big(\frac{a_n S_n}{b_n \sqrt{v_n}} \in B \Big) &\geq \mathbf{P}\Big(\frac{a_n S_n}{\sqrt{v_n}} \in (b_n(x_0 - \varepsilon_2), b_n(x_0 + \varepsilon_2)] \Big) \\ &= \mathbf{P}\Big(\frac{a_n S_n}{\sqrt{v_n}} \geq b_n(x_0 - \varepsilon_2) \Big) - \mathbf{P}\Big(\frac{a_n S_n}{\sqrt{v_n}} \geq b_n(x_0 + \varepsilon_2) \Big).\end{aligned}$$

From Theorem 2, it is easy to see that

$$\lim_{n\to\infty} \frac{\mathbf{P}\Big(\frac{a_n S_n}{\sqrt{v_n}} \geq b_n(x_0 + \varepsilon_2) \Big)}{\mathbf{P}\Big(\frac{a_n S_n}{\sqrt{v_n}} \geq b_n(x_0 - \varepsilon_2) \Big)} = 0.$$

Using (40), we get

$$\liminf_{n\to\infty} \frac{1}{b_n^2} \ln \mathbf{P}\Big(\frac{a_n S_n}{b_n \sqrt{v_n}} \in B \Big) \geq -\frac{1}{2}(x_0 - \varepsilon_2)^2.$$

Letting $\varepsilon_2 \to 0$, by (43), we have

$$\liminf_{n\to\infty} \frac{1}{b_n^2} \ln \mathbf{P}\Big(\frac{a_n S_n}{b_n \sqrt{v_n}} \in B \Big) \ \geq \ -\frac{x_0^2}{2} \ \geq \ - \inf_{x\in B^o} \frac{x^2}{2} - \varepsilon_1.$$

Because ε_1 can be arbitrarily small, we obtain (42). Combining (41) and (42) together, we complete the proof of Corollary 3 for $0 < p < 1/2$.

4.6. Proof of Theorem 3

We only present the proof of point [i]. Points [ii] and [iii] can be proved in a similar way. We first consider the case of $1 \leq k = o(n^{2/3})$. It is easy to see that

$$\begin{aligned}\mathbf{P}(S_n = k) &= \mathbf{P}(k - 1 < S_n \leq k) \\ &= \mathbf{P}(a_n(k-1)/\sqrt{v_n} < a_n S_n/\sqrt{v_n} \leq a_n k/\sqrt{v_n}) \\ &= \mathbf{P}(a_n S_n/\sqrt{v_n} \leq a_n k/\sqrt{v_n}) - \mathbf{P}(a_n S_n/\sqrt{v_n} \leq a_n(k-1)/\sqrt{v_n}).\end{aligned}$$

For simplicity of notation, denote

$$x_k = a_n k/\sqrt{v_n}, \qquad 0 \leq k = o(n^{2/3}).$$

By (3) and (4), we have $x_k = o(n^{1/6})$ for all $0 \leq k = o(n^{2/3})$. From Corollary 2, it is easy to see that for all $0 \leq k = o(n^{2/3})$,

$$\begin{aligned}
&\mathbf{P}(a_n S_n/\sqrt{v_n} \leq x_k) - \Phi(x_k) \\
&= (1-\Phi(x_k)) - \mathbf{P}(a_n S_n/\sqrt{v_n} > x_k) \\
&\leq \Big(1-\Phi(x_k)\Big)\Big(1 + C\Big(\frac{x_k^3}{\sqrt{n}} + (1+x_k)\frac{\log n}{\sqrt{n}}\Big)\Big) \\
&\leq \Big(1-\Phi(x_{k-1})\Big)\Big(1 + C\Big(\frac{x_k^3}{\sqrt{n}} + (1+x_k)\frac{\log n}{\sqrt{n}}\Big)\Big) \qquad (44)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{P}(a_n S_n/\sqrt{v_n} \leq x_{k-1}) - \Phi(x_{k-1}) \\
&\geq \Big(1-\Phi(x_{k-1})\Big)\Big(1 - C\Big(\frac{x_{k-1}^3}{\sqrt{n}} + (1+x_{k-1})\frac{\log n}{\sqrt{n}}\Big)\Big) \\
&\geq \Big(1-\Phi(x_{k-1})\Big)\Big(1 - C\Big(\frac{x_k^3}{\sqrt{n}} + (1+x_k)\frac{\log n}{\sqrt{n}}\Big)\Big). \qquad (45)
\end{aligned}$$

Using the inequalities (44) and (45), we deduce that for all $1 \leq k = o(n^{2/3})$,

$$\begin{aligned}
&\mathbf{P}(S_n = k) \\
&= \mathbf{P}(a_n S_n/\sqrt{v_n} \leq x_k) - \mathbf{P}(a_n S_n/\sqrt{v_n} \leq x_{k-1}) \\
&\leq \Phi(x_k) - \Phi(x_{k-1}) + 2C\Big(1-\Phi(x_{k-1})\Big)\Big(\frac{x_k^3}{\sqrt{n}} + (1+x_k)\frac{\log n}{\sqrt{n}}\Big) \\
&= \int_{x_{k-1}}^{x_k} \frac{1}{\sqrt{2\pi}} \exp\Big\{-\frac{1}{2}t^2\Big\}dt + 2C\Big(1-\Phi(x_{k-1})\Big)\Big(\frac{x_k^3}{\sqrt{n}} + (1+x_k)\frac{\log n}{\sqrt{n}}\Big) \\
&\leq \frac{1}{\sqrt{2\pi}} \exp\Big\{-\frac{1}{2}x_{k-1}^2\Big\}|x_k - x_{k-1}| \\
&\quad + 2C\Big(1-\Phi(x_{k-1})\Big)\Big(\frac{x_k^3}{\sqrt{n}} + (1+x_k)\frac{\log n}{\sqrt{n}}\Big) \\
&\leq \frac{a_n}{\sqrt{2\pi v_n}} \exp\Big\{-\frac{1}{2}x_{k-1}^2\Big\} + 2C\Big(1-\Phi(x_{k-1})\Big)\Big(\frac{x_k^3}{\sqrt{n}} + (1+x_k)\frac{\log n}{\sqrt{n}}\Big). \qquad (46)
\end{aligned}$$

Applying (40) to (46), we get for all $1 \leq k = o(n^{2/3})$,

$$\begin{aligned}
\frac{\mathbf{P}(S_n = k)}{\frac{a_n}{\sqrt{2\pi v_n}}\exp\{-\frac{1}{2}x_{k-1}^2\}} &\leq 1 + \frac{2C}{\sqrt{\pi}(1+x_{k-1})}\Big(\frac{x_k^3}{\sqrt{n}} + (1+x_k)\frac{\log n}{\sqrt{n}}\Big) \\
&\leq 1 + 2C\Big(\frac{x_k^2}{\sqrt{n}} + \frac{\log n}{\sqrt{n}}\Big).
\end{aligned}$$

Hence, we have for all $1 \leq k = o(n^{2/3})$,

$$\begin{aligned}
\frac{\mathbf{P}(S_n = k)}{\frac{a_n}{\sqrt{2\pi v_n}}\exp\{-\frac{1}{2}x_k^2\}} &\leq \exp\Big\{\frac{1}{2}(x_k^2 - x_{k-1}^2)\Big\}\Big(1 + 2C\Big(\frac{x_k^2}{\sqrt{n}} + \frac{\log n}{\sqrt{n}}\Big)\Big) \\
&\leq \exp\Big\{\frac{k a_n^2}{v_n}\Big\}\Big(1 + 2C\Big(\frac{x_k^2}{\sqrt{n}} + \frac{\log n}{\sqrt{n}}\Big)\Big).
\end{aligned}$$

By (12), we have $a_n^2/v_n \asymp 1/n$ as $n \to \infty$, and $\exp\big\{\frac{ka_n^2}{v_n}\big\} \leq 1 + C\frac{k}{n}$ for all $1 \leq k = o(n)$. Thus, it holds for all $1 \leq k = o(n^{2/3})$,

$$\frac{\mathbf{P}(S_n = k)}{\frac{a_n}{\sqrt{2\pi v_n}} \exp\big\{-\frac{1}{2}x_k^2\big\}} \leq \Big(1 + C\frac{k}{n}\Big)\Big(1 + 2C\Big(\frac{x_k^2}{\sqrt{n}} + \frac{\log n}{\sqrt{n}}\Big)\Big)$$

$$\leq 1 + 2C'\Big(\frac{k}{n} + \frac{k^2}{n^{3/2}} + \frac{\log n}{\sqrt{n}}\Big). \tag{47}$$

Similarly, we can prove that for all $1 \leq k = o(n^{2/3})$,

$$\frac{\mathbf{P}(S_n = k)}{\frac{a_n}{\sqrt{2\pi v_n}} \exp\big\{-\frac{1}{2}x_k^2\big\}} \geq 1 - 2C\Big(\frac{k}{n} + \frac{k^2}{n^{3/2}} + \frac{\log n}{\sqrt{n}}\Big). \tag{48}$$

Combining (47) and (48) together, we get the desired inequality of point [i] and $1 \leq k = o(n^{2/3})$. For the case of $0 \leq -k = o(n^{2/3})$, the proof is similar.

4.7. Proof of Corollary 4

We first prove the point [i] of Corollary 4. By (3) and (4), we have $\frac{a_n^2}{v_n} \asymp \frac{1}{n}$. Thus, from Theorem 3, it is easy to see that

$$\sup_{|k| \leq n^{5/8}} \Big|\mathbf{P}(S_n = k) - \frac{a_n}{\sqrt{2\pi v_n}} \exp\Big\{-\frac{(a_n k)^2}{2v_n}\Big\}\Big|$$

$$\leq \sup_{|k| \leq n^{5/8}} C_p \frac{a_n}{\sqrt{2\pi v_n}} \exp\Big\{-\frac{(a_n k)^2}{2v_n}\Big\}\Big(\frac{|k|}{n} + \frac{k^2}{n^{3/2}} + \frac{\log n}{\sqrt{n}}\Big)$$

$$\leq \sup_{|k| \leq n^{5/8}} C_p' \frac{1}{\sqrt{n}} \exp\Big\{-\frac{k^2}{nC_p'}\Big\}\Big(\frac{|k|}{n} + \frac{k^2}{n^{3/2}} + \frac{\log n}{\sqrt{n}}\Big)$$

$$= C_p' \frac{1}{n} \sup_{|k| \leq n^{5/8}} \exp\Big\{-\frac{1}{C_p'}\frac{k^2}{n}\Big\}\Big(\frac{|k|}{\sqrt{n}} + \frac{k^2}{n} + \log n\Big)$$

$$\leq C_p'' \frac{\log n}{n}. \tag{49}$$

Notice that

$$\sup_{|k| \geq n^{5/8}} \Big|\mathbf{P}(S_n = k) - \frac{a_n}{\sqrt{2\pi v_n}} \exp\Big\{-\frac{(a_n k)^2}{2v_n}\Big\}\Big|$$

$$\leq \sup_{|k| \geq n^{5/8}} \mathbf{P}(|S_n| \geq |k|) + \sup_{|k| \geq n^{5/8}} \frac{a_n}{\sqrt{2\pi v_n}} \exp\Big\{-\frac{(a_n k)^2}{2v_n}\Big\}$$

$$\leq \mathbf{P}(|S_n| \geq n^{5/8}) + \sup_{|k| \geq n^{5/8}} C_p \frac{1}{\sqrt{n}} \exp\Big\{-\frac{k^2}{nC_p}\Big\}$$

$$\leq \mathbf{P}(|S_n| \geq n^{5/8}) + C_p' \frac{1}{n}.$$

Using Theorem 2, we can deduce that $\mathbf{P}(|S_n| \geq n^{5/8}) \leq C_p \frac{1}{n}$. Thus, it holds

$$\sup_{|k| \geq n^{5/8}} \Big|\mathbf{P}(S_n = k) - \frac{a_n}{\sqrt{2\pi v_n}} \exp\Big\{-\frac{(a_n k)^2}{2v_n}\Big\}\Big| \leq C_p \frac{1}{n}. \tag{50}$$

Combining (49) and (50) together, we obtain the desired inequality of point [i]. Point [ii] of Corollary 4 can be proved by a similar argument, with $n^{(9-4p)/12}$ replacing $n^{5/8}$.

5. Conclusion

The ERW was introduced in order to study the memory effects in the non-Markovian random walk. In this paper, we study the normal approxiamtions for the ERW, including Berry-Esseen's bounds, Cramér moderate deviations and the local limit theorems. These results can be regarded as refinements of the CLT. Berry-Esseen's bounds (cf. Theorem 1) make us better understand how the memory parameter p effects the convergence rates for the absolute error of normal approximations. From Cramér moderate deviations, we find the domains of attraction of normal distribution for various $p \in (0, 3/4]$. See Corollary 2. The local limit theorem (cf. Theorem 3) gives the asymptotic probability for the position of elephant. As applications, our results can be applied to statistical inferences of the memory parameter p and interval estimations for the position of the elephant.

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