Harnack Inequality and Applications for SDEs Driven by G-Brownian motion

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August 28, 2018

Abstract

We establish Harnack inequality and shift Harnack inequality for stochastic differential equation driven by G-Brownian motion. As applications, the uniqueness of invariant linear expectations and estimates on the sup-kernel are investigated, where the sup-kernel is introduced in this paper for the first time.

Keywords: Harnack inequality; shift Harnack inequality; stochastic differential equations; G-Brownian motion; G-expectation.

1 Introduction

Since Wang [19] introduced dimensional-free Harnack inequality for diffusions on Riemannian manifold, his Harnack inequality has been extensively investigated. His type Harnack inequality as a powerful tool in the study of functional inequalities (see [1, 15, 16, 20, 21]), heat kernel estimates (see [6]), high order eigenvalues (see [17, 9]), transportation cost inequalities (see [4]), and short-time behavior of transition probabilities (see [2, 3, 9]). To establish Wang's Harnack inequality, Wang and co-authors introduced the coupling by change of measures, see Wang [18] and references within for details.

On the other hand, for the potential applications in uncertainty problems, risk measures and the superhedging in finance, the theory of nonlinear expectation has been developed. Especially, Peng [12, 13] established the fundamental theory of G-expectation theory, G-Brownian motion and stochastic differential equations driven by G-Brownian motion (G-SDEs, in short).

To establish Wang's Harnack inequality using coupling by change of measures in the linear probability setting, the Girsanov transform plays a crucial role. In [7, 11, 22], the

Girsanov's theorem has been extended to the G-framework, and the Girsanov's formula has been derived for G-Brownian motion. Recently, Hu et al. [8] studied the invariant and ergodic nonlinear expectations for G-diffusion processes.

In this paper, we investigate Wang's Harnack inequality and applications for the following G-SDE

$$dX_t = b(X_t)d\langle B\rangle_t + dB_t, \tag{1.1}$$

where B_t is a G-Brownian motion, and $\langle B\rangle_t$ is the quadratic variation process associated with B_t. Moreover, we study shift Harnack inequality and applications for the following G-SDE

$$dX_t = b(X_t)dt + dB_t, \tag{1.2}$$

where B_t is a G-Brownian motion,

The paper is organized as follows. In Section 2, we recall some preliminaries on G-Brownian motion, related stochastic calculus and transformation for G-Expection. In Section 3, Wang's Harnack inequality and shift Harnack inequality are established for the nonlinear Markov operator associated with (1.1) and (1.2) respectively. As applications, the sup-kernel and invariant linear expectation for nonlinear Markov operator are investigated.

2 Preliminaries

2.1 Sublinear expectation spaces

In this section, we propose some preliminaries and notations which appeared in Peng [12, 14]. Let Ω be a given set and $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω, namely $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables.

Definition 2.1. (Sublinear expectation space) A sublinear expectation $\overline{E}$ on $\mathcal{H}$ is a functional $\overline{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, it holds that

(a) Monotonicity: if $X \geq Y$ then $\overline{E}[X] \geq \overline{E}[Y]$,

(b) Constant preservation: $\overline{E}[c] = c$,

(c) Sub-additivity: $\overline{E}[X + Y] \leq \overline{E}[X] + \overline{E}[Y]$,

(d) Positive homogeneity: $\overline{E}[\lambda X] = \lambda \overline{E}[X]$ for each $\lambda \geq 0$.

Then, $(\Omega, \mathcal{H}, \overline{E})$ is called a sublinear expectation space.

Let $\mathbb{S}^d$ be the collection of all $d \times d$ symmetric matrices, X be a G-normal distributed random vector, and $G : \mathbb{S}^d \to R$ is defined by

$$G(A) := \frac{1}{2}\overline{E}[\langle AX, X\rangle] = \sup_{\gamma\in\Theta} \frac{1}{2}tr[\gamma\gamma^* A], \ A \in \mathbb{S}^d.$$

Then the distribution of X is characterized by

$$u(t,x)=\overline{E}[\varphi(x+\sqrt{t}X)],\ \varphi\in C_{l,Lip}(\mathbb{R}^d),$$

where $C_{l,lip}(\mathbb{R}^n)$ be the space of all real functions φ on $\mathbb{R}^n$ satisfying

$$|\varphi(x)-\varphi(y)|\leq C(1+|x|^m+|y|^m)|x-y|,\quad x,y\in\mathbb{R}^n,$$

for some $C>0, m\in\mathbb{N}$ depending on φ.

In particular, $\overline{E}[\varphi(X)]=u(1,0)$, where u is the solution of the following parabolic PDE defined on $[0,\infty)\times\mathbb{R}^d$:

$$\begin{cases}\frac{\partial u}{\partial t}-G(\frac{\partial^2 u}{\partial x^2})=0,\\ u(0,x)=\varphi(x).\end{cases}$$

This parabolic PDE is called a G-heat equation.

2.2 G-expectation and G-Brownian motion

Let $\Omega=C_0^d(\mathbb{R}^+)$ the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0=0$, equipped with the distance

$$\rho(\omega^1,\omega^2):=\sum_{i=1}^{\infty}2^{-i}\left[(\max_{t\in[0,i]}|\omega_t^1-\omega_t^2|)\wedge 1\right].$$

Consider the canonical process $B_t(\omega)=\omega_t, t\in[0,\infty)$, for $\omega\in\Omega$. For each $T\in[0,\infty)$, let $\Omega_T=\{\omega_{\cdot\wedge T}:\omega\in\Omega\}$, and set

$$L_{ip}(\Omega_T)=\{\varphi(B_{t_1\wedge T},\cdots,B_{t_n\wedge T}):n\in\mathbb{N},t_1,\cdots,t_n\in[0,\infty),\varphi\in C_{l,lip}(\mathbb{R}^{d\times n})\}.$$

Then $L_{ip}(\Omega_t)\subseteq L_{ip}(\Omega_T)$, $t\leq T$. Set

$$L_{ip}(\Omega)=\cup_{n=1}^{\infty}L_{ip}(\Omega_n).$$

Let $(\xi_i)_{i=1}^{\infty}$ a sequence of d-dimensional random vectors on a sublinear expectation space $(\tilde{\Omega},\tilde{\mathcal{H}},\tilde{E})$ such that ξ_i is G-normal distributed and ξ_{i+1} is independent from $(\xi_1,\cdots,\xi_i)$ for each $i=1,2,\cdots$. We now introduce a sublinear expectation $\overline{E}$ defined on $L_{ip}(\Omega)$ via the following procedure: for each $X\in L_{ip}(\Omega)$ with

$$X=\varphi(B_{t_1}-B_{t_0},B_{t_2}-B_{t_1},\cdots,B_{t_n}-B_{t_{n-1}}),\ \varphi\in C_{l,lip}(\mathbb{R}^{d\times n}),$$

let

$$\overline{E}[\varphi(B_{t_1}-B_{t_0},B_{t_2}-B_{t_1},\cdots,B_{t_n}-B_{t_{n-1}})]=\tilde{E}[\varphi(\sqrt{t_1-t_0}\xi_1,\sqrt{t_2-t_1}\xi_1,\cdots,\sqrt{t_n-t_{n-1}}\xi_n)].$$

Remark 2.1. Let $\langle B\rangle_t:=(\langle B^i,B^j\rangle_t)_{1\leq i,j\leq d}, 0\leq t\leq T$ be the quadratic variation of B_t. In the 1-dimensional case, it holds that $\underline{\sigma}^2(t-s)\leq\langle B\rangle_t-\langle B\rangle_s\leq\overline{\sigma}^2(t-s),\ 0\leq s\leq t\leq T$.

Definition 2.2. (G-expectation and G-Brownian motion) The sublinear expectation $\overline{E} : L_{ip}(\Omega) \to \mathbb{R}$ defined through the above procedure is called a G-expectation. The corresponding canonical process $(B_t)_{t\geq 0}$ on the sublinear expectation space $(\Omega, L_{ip}(\Omega), \overline{E})$ is called a G-Brownian motion.

Remark 2.2. Let $L_G^p(\Omega_T)$ (respectively $L_G^p(\Omega)$) be the completion of $L_{ip}(\Omega_T)$ (respectively Ω_t) under the norm $(\overline{E}[|\cdot|^p])^{\frac{1}{p}}$. Then $\overline{E}[\cdot]$ can be continuously extends to a sublinear expectation on $(\Omega, L_G^1(\Omega))$, which is still denoted by $\overline{E}$.

Let

$$M_G^{p,0}([0,T]) := \left\{ \eta_t := \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})};\ \xi_j \in L_G^p(\Omega_{t_j}), N \in \mathbb{N},\ 0 = t_0 < t_1 < \cdots < t_N = T \right\}.$$

For $p \geq 1$, let $M_G^p([0,T])$ be the completion of $M_G^{p,0}([0,T])$ under the following norm

$$\|\eta\|_{M_G^p([0,T])} = \left[\overline{E}\left(\int_0^T |\eta_t|^p dt \right) \right]^{\frac{1}{p}}.$$

2.3 Capacity and Quasi-Sure Analysis for G-Brownian Paths

Denis et al. [5] proved that there exists a weakly compact family $\{E_\theta : \theta \in \Theta\}$ of expectations introduced by probability measures$\{P_\theta : \theta \in \Theta\}$ defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\overline{E}[X] = \sup_{\theta \in \Theta} E_\theta[X],\ X \in L_{ip}(\Omega).$$

Then the associated Choquet capacity is given by

$$c(A) = \sup_{\theta \in \Theta} P_\theta(A),\ A \in \mathcal{B}(\Omega).$$

Definition 2.3. (quasi-surely) A set $A \in \mathcal{B}(\Omega)$ is called polar if $c(A) = 0$ and a property holds quasi-surely (q.s.) if it holds outside a polar set.

Remark 2.3. Let X and Y be two random variables, we say that X is a version of Y, if $X = Y$ q.s..

2.4 Stopping times

In the sequel, we introduce stopping times under G-expectation framework.

Definition 2.4. Let $\mathcal{F}_t := \mathcal{B}(\Omega)$, a stopping time τ relative to the filtration $(\mathcal{F}_t)$ is a map on Ω with values in $[0,T]$, such that

$$\{\tau \leq t\} \in \mathcal{F}_t,\ t \leq T.$$

Lemma 2.4. ([10]) *For each stopping time τ and $\eta \in M_G^p([0,T])$, we have $I_{[0,\tau]}(\cdot)\eta \in M_G^p([0,T])$, and*

$$\int_0^{t\wedge\tau} \eta_s dB_s = \int_0^{t\wedge\tau} \eta_s I_{[0,\tau]}(s) dB_s.$$

2.5 Transformation for G-Expection

To introduce the Girsanov type theorem under G-framework presented in [11, 22], using the G-capacity of [5], we need the G-Novikov's condition. For $h \in (M_G^2([0,T]))^d$, let

$$M_t := \exp\left\{ \int_0^t h_s \cdot dB_s - \frac{1}{2}\int_0^t h_s \cdot (d\langle B\rangle_s h_s) \right\},$$
$$\widehat{B}_t := B_t - \int_0^t (d\langle B\rangle_s h_s),\ t \in [0,T].$$

Set $\hat{L}_{ip}(\Omega_T) := \{\varphi(\hat{B}_{t_1\wedge T}, \cdots, \hat{B}_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0,\infty), \varphi \in C_{l,lip}(\mathbb{R}^{d\times n}\}$. Let $\hat{L}_G^1(\Omega)$ be the completion of $\hat{L}_{ip}(\Omega_T)$ under the norm $\hat{E}[|\cdot|]$, and extend $\hat{E}$ to a unique sublinear expectation on $\hat{L}_G^1(\Omega)$.

Lemma 2.5. ([11, 22]) *If $h \in (M_G^2([0,T]))^d$ satisfies G-Novikov's condition, for some $\epsilon_0 > 0$,*

$$\overline{E}\left[\exp\left\{\left(\frac{1}{2}+\epsilon_0\right)\int_0^T h_s \cdot (d\langle B\rangle_s h_s)\right\}\right] < \infty, \tag{2.1}$$

then the process M is a symmetric G-martingale.

We propose Girsanov's formula for G-Brownian motion as follows.

Lemma 2.6. ([11]) (G-Girsanov's formula) *Assume that there exists $\sigma_0 > 0$ such that*

$$\gamma\gamma^* \geq \sigma_0 I_d \quad for \quad all \quad \gamma \in \Theta,$$

and that M is a symmetric G-martingale on $(\Omega, L_G^1(\Omega), \overline{E})$. Define a sublinear expectation $\widehat{E}$ by

$$\widehat{E}[X] := \overline{E}[XM_T] \quad for \quad X \in \hat{L}_{ip}^1(\Omega).$$

Then $\widehat{B}_t$ is a G-Brownian motion on the sublinear expectation space $(\Omega, \hat{L}_G^1(\Omega), \widehat{E})$.

Moreover, Hu et al. [7] construct an auxiliary extended $\widetilde{G}$-expectation space $(\widetilde{\Omega}_T, L_{\widetilde{G}}^1, \overline{E}^{\widetilde{G}})$ with $\widetilde{\Omega}_T = C_0([0,T],\mathbb{R}^2)$ and

$$\widetilde{G}(A) = \frac{1}{2}\sup_{\underline{\sigma}^2 \leq v \leq \overline{\sigma}^2} tr\left[A\begin{pmatrix} v & 1 \\ 1 & v^{-1}\end{pmatrix}\right],\ A \in \mathbb{S}^2.$$

Let $(B_t, \bar{B}_t)$ be the canonical process in the extended space. Then $\langle B_t, \bar{B}_t\rangle = t$, and

$$\overline{E}^{\widetilde{G}}[\xi] = \overline{E}[\xi],\ \xi \in L_G^1(\Omega_T).$$

Lemma 2.7. ([7]) *Let $(h_t)_{t\geq 0}$ be a bounded process, then the process $\widetilde{B}_t := B_t + \int_0^t h_s ds$ is a G-Brownian motion under $\widetilde{E}$ for*

$$\widetilde{E}[X] = \overline{E}^{\widetilde{G}}\left[X\exp\left(\int_0^T h_s d\bar{B}_s - \frac{1}{2}\int_0^T h_s^2 d\langle \bar{B}\rangle_s\right)\right],\ X \in L_G^1(\Omega_T).$$

Remark 2.8. We should remark that the $\bar{B}_t$ is a $\hat{G}$-Brownian motion under $\overline{E}^{\widetilde{G}}$ with $\hat{G}(A) = \frac{1}{2}\sup_{\overline{\sigma}^{-2}\leq v \leq \underline{\sigma}^{-2}} tr[Av],\ A \in \mathbb{S}^1$.

3 Main Results

For a family of probability measures $\{\mu_{x,\theta} : x \in \mathbb{R}^d, \theta \in \Theta\}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$, define

$$\bar{P}f(x) = \sup_{\theta\in\Theta} P_\theta f(x) = \sup_{\theta\in\Theta} \int_{\mathbb{R}^d} f(y)\mu_{x,\theta}(dy),\ \ f \in \mathcal{B}_b(\mathbb{R}^d),$$

where P_θ is a linear Markov operator.

We aim to establish the following Harnack-type inequality introduced by Feng-Yu Wang:

$$\Phi(\bar{P}f(x))) \leq \bar{P}\Phi(f(y))e^{\Psi(x,y)},\ \ x, y \in \mathbb{R}^d, f \in \mathcal{B}_b^+(\mathbb{R}^d),$$

where Φ is a nonnegative convex function on $[0, \infty)$ and Ψ is a nonnegative function on $\mathbb{R}^d \times \mathbb{R}^d$.

In the setting of G-SDEs, we establish this type inequality for the associated Markov operator $\bar{P}_T$. For simplicity, we consider the 1-dimensional G-Brownian motion case, but our results and methods still hold for the case $d > 1$. More precisely, consider the following G-SDE

$$dX_t = b(X_t)d\langle B\rangle_t + dB_t, \tag{3.1}$$

where B_t is a G-Brownian motion, $\langle B\rangle_t$ is the quadratic variation process associated with B_t for $\overline{\sigma}^2 = \overline{E}[B_1^2] \geq -\overline{E}[-B_1^2] = \underline{\sigma}^2 > 0$, and $b : \mathbb{R} \to \mathbb{R}$ satisfies

(H1) $|b(x) - b(y)| \leq K|x - y|,\ \ x, y \in \mathbb{R}$

for some constant $K > 0$. From [13], under (H1) the G-SDE (3.1) has a unique solution for any initial value.

In what follows, for $T > 0$, we define

$$\bar{P}_T f(x) = \overline{E} f(X_T^x),\ \ f \in \mathcal{B}_b^+(\mathbb{R}), \tag{3.2}$$

where X_T^x solves (3.1) with initial value x. We have the following result.

Theorem 3.1. *Under* (H1), *for any nonnegative* $f \in \mathcal{B}_b^+(\mathbb{R})$ *and* $T > 0, x, y \in \mathbb{R}$, *it holds that*

$$(\bar{P}_T f)^p(y) \leq \bar{P}_T f^p(x) \exp\left\{\frac{pK\overline{\sigma}^4(1 - e^{-2\underline{\sigma}^2 KT})}{(p-1)^2\underline{\sigma}^6(1 - e^{-2\overline{\sigma}^2 KT})^2}|x - y|^2\right\}. \tag{3.3}$$

Proof. We use the coupling by change of measures as explained in [18], consider the following coupled stochastic differential equations

$$\begin{aligned} dY_t &= b(Y_t)d\langle B\rangle_t + dB_t + u_t d\langle B\rangle_t,\ Y_0 = y, \\ dX_t &= b(X_t)d\langle B\rangle_t + dB_t,\ X_0 = x, \end{aligned} \tag{3.4}$$

where $u_t = \eta_t \cdot \frac{X_t - Y_t}{|X_t - Y_t|} 1_{t \le \tau}$, τ is the coupling time of X and Y defined by

$$\tau = \inf\{t \ge 0 : X_t = Y_t\},$$

and $u.$ is a force can make the two processes X and Y move together before time T .

From assumption (H1) and the expression of η_t, (3.4) has a unique solution.

By (H1), we have

$$d|X_t - Y_t| \le K|X_t - Y_t| d\langle B\rangle_t - \eta_t d\langle B\rangle_t, \ t < \tau.$$

Then

$$e^{-K\langle B\rangle_{T\wedge\tau}} |X_{T\wedge\tau} - Y_{T\wedge\tau}| \le |x - y| - \int_0^{T\wedge\tau} e^{-K\langle B\rangle_t} \eta_t d\langle B\rangle_t.$$

Taking

$$\eta_t = \frac{|x - y| e^{-K\langle B\rangle_t}}{\int_0^T e^{-2K\langle B\rangle_t} d\langle B\rangle_t}, \quad t \in [0, T],$$

we have

$$|x - y| - \int_0^T e^{-K\langle B\rangle_t} \eta_t d\langle B\rangle_t = 0,$$

which implies $\tau \le T$, and thus $X_T = Y_T$.

By Remark (2.1), we have

$$\begin{aligned}
&\exp\left\{\int_0^T |u_s|^2 d\langle B\rangle_s\right\} \\
= \ &\exp\left\{\overline{\sigma}^2 \int_0^T \left|\frac{|x - y| e^{-K\langle B\rangle_s}}{\int_0^T e^{-2K\langle B\rangle_s} d\langle B\rangle_s}\right|^2 ds\right\} \\
\le \ &\exp\left\{\frac{\overline{\sigma}^2}{\underline{\sigma}^4} \frac{\int_0^T \left||x - y| e^{-K\langle B\rangle_s}\right|^2 ds}{\left|\int_0^T e^{-2K\langle B\rangle_s} ds\right|^2}\right\} \\
\le \ &\exp\left\{\frac{2K\overline{\sigma}^4 (1 - e^{-2\underline{\sigma}^2 KT})}{\underline{\sigma}^6 (1 - e^{-2\overline{\sigma}^2 KT})^2} |x - y|^2\right\}. \qquad (3.5)
\end{aligned}$$

Letting $\epsilon_0 = \frac{1}{2}$, we have

$$\overline{E}\left[\exp\left\{\int_0^T |u_s|^2 d\langle B\rangle_s\right\}\right] \le \exp\left\{\frac{2K\overline{\sigma}^4 (1 - e^{-2\underline{\sigma}^2 KT})}{\underline{\sigma}^6 (1 - e^{-2\overline{\sigma}^2 KT})^2} |x - y|^2\right\} < \infty,$$

which satisfies G-Novikov's condition.

Let

$$M_T = \exp\left\{-\int_0^T u_s dB_s - \frac{1}{2}\int_0^T |u_s|^2 d\langle B\rangle_s\right\}.$$

Define a sublinear expectation $\widehat{E}$ by $\widehat{E}[\xi] := \overline{E}[\xi M_T]$. By Lemma 2.5 and Lemma 2.6, the process

$$\widehat{B}_t := B_t + \int_0^t u_s d\langle B\rangle_s, \quad t \geq 0$$

is a G-Brownian motion under $\widehat{E}$.

Moreover, Girsanov's formula also implies that

$$\langle \widehat{B}\rangle_{\widehat{E}} = \langle B\rangle_{\overline{E}}.$$

Then, Y_t can be reformulated by

$$dY_t = b(Y_t)d\langle \widehat{B}\rangle_t + d\widehat{B}_t.$$

So, $\bar{P}_T f(y) = \overline{E} f(X_T^y) = \widehat{E} f(Y_T^y) = \widehat{E} f(X_T^x) = \overline{E}(M_T f(X_T^x))$.

Using Hölder's inequality, we have

$$\begin{aligned} (\bar{P}_T f)^p(y) &= (\overline{E}[M_T f(X_T^x)])^p \\ &\leq (\overline{E}[f^p(X_T^x)]) \left(\overline{E}\left[M_T^{\frac{p}{p-1}}\right]\right)^{\frac{p-1}{p}}. \end{aligned} \tag{3.6}$$

Now we estimate the moment of M_T. It holds that

$$\begin{aligned} \overline{E}\left[M_T^{\frac{p}{p-1}}\right] &= \overline{E}\exp\left\{-\frac{p}{p-1}\int_0^T u_s dB_s - \frac{p}{2(p-1)}\int_0^T |u_s|^2 d\langle B\rangle_s\right\} \\ &= \overline{E}\exp\Bigg\{-\frac{p}{p-1}\int_0^T u_s dB_s - \frac{p^2}{2(p-1)^2}\int_0^T |u_s|^2 d\langle B\rangle_s \\ &\quad + \frac{p}{2(p-1)^2}\int_0^T |u_s|^2 d\langle B\rangle_s\Bigg\}. \end{aligned} \tag{3.7}$$

From (3.5), we have

$$\exp\left\{\frac{p}{2(p-1)^2}\int_0^T |\eta_s|^2 d\langle B\rangle_s\right\} \leq \exp\left\{\frac{pK\overline{\sigma}^4(1-e^{-2\underline{\sigma}^2 KT})}{(p-1)^2\underline{\sigma}^6(1-e^{-2\overline{\sigma}^2 KT})^2}|x-y|^2\right\}.$$

Substituting this into (3.7), we have

$$\overline{E}\left[M_T^{\frac{p}{p-1}}\right] \leq \exp\left\{\frac{pK\overline{\sigma}^4(1-e^{-2\underline{\sigma}^2 KT})}{(p-1)^2\underline{\sigma}^6(1-e^{-2\overline{\sigma}^2 KT})^2}|x-y|^2\right\}. \tag{3.8}$$

Combing (3.6) and (3.8), we prove (3.3).

To obtain the shift Harnack inequality, we consider the following G-SDE

$$dX_t = b(X_t)\mathrm{d}t + dB_t, \ X_0 = x, \tag{3.9}$$

where B_t is a G-Brownian motion, and $b : \mathbb{R} \to \mathbb{R}$ satisfies the assumption (H1). Then, the G-SDE (3.9) has a unique solution for any initial value.

For $T > 0$, we define

$$\bar{P}_T f(x) = \overline{E} f(X_T^x), \ f \in \mathcal{B}_b^+(\mathbb{R}),$$

where X_T^x solves (3.9) with initial value x. By the definition of $\overline{E}^{\widetilde{G}}$ (in section 2.6), we have

$$\overline{E}[f(X_T^x)] = \overline{E}^{\widetilde{G}}[f(X_T^x)] =: \bar{P}_T^{\widetilde{G}} f(x).$$

We have the following result.

Theorem 3.2. *Under* (H1)*, for any nonnegative* $f \in \mathcal{B}_b^+(\mathbb{R})$ *and* $T > 0, x, y \in \mathbb{R}$*, the following shift Harnack inequality holds*

$$(\bar{P}_T f(x))^p \le (\bar{P}_T f^p(v + \cdot))(x) \exp\left\{ \frac{pv^2}{2\underline{\sigma}^2(p-1)} \left(\frac{1}{T} + K + \frac{K^2 T}{3} \right) \right\}. \tag{3.10}$$

Proof. Let $Y_t = X_t + \frac{t}{T} v$ with $Y_0 = X_0 = x$ and

$$R_t = \exp\left(-\int_0^t \frac{v}{T} + b(X_s) - b(Y_s) d\bar{B}_s - \frac{1}{2} \int_0^t (\frac{v}{T} + b(X_s) - b(Y_s))^2 d\langle \bar{B} \rangle_s \right),$$

where $\bar{B}_t$ is G-Brownian motion under $\overline{E}^{\widetilde{G}}$, which is an auxiliary process, one can see in section 2.6 for details.

From (H2), we have

$$\left| \frac{v}{T} + b(X_s) - b(Y_s) \right| \le \frac{1 + Ks}{T} |v|. \tag{3.11}$$

By Lemma2.7,

$$\widetilde{B}_t := B_t + \int_0^t \left(\frac{v}{T} + b(X_s) - b(Y_s) \right) ds$$

is a G-Brownian motion under $\widetilde{E}$ with

$$\widetilde{E}[\xi] = \overline{E}^{\widetilde{G}} [\xi R_T], \ \xi \in L_G^1(\Omega_T).$$

Then

$$\mathrm{d}Y_t = b(Y_t)dt + d\widetilde{B}_t, \ Y_0 = x.$$

That is $Y_T = X_T + v$ under $\widetilde{E}$.

Then for $f \in \mathcal{B}_b^+(\mathbb{R})$, $p \ge 1$, by Hölder inequality, we have

$$\begin{aligned} \left(\bar{P}_T f\right)^p(x) &= \left(\widetilde{E}[f(Y_T^x)]\right)^p \\ &= \left(\widetilde{E}[f(X_T^x + v)]\right)^p \end{aligned}$$

$$
\begin{aligned}
&= \left(\overline{E}^{\widetilde{G}}[R_T f(X_T^x+v)]\right)^p \\
&\leq \left(\bar{P}_T^{\widetilde{G}} f^p(v+\cdot)\right)(x)\left(\overline{E}^{\widetilde{G}}[R_T^{\frac{p}{p-1}}]\right)^{p-1} \\
&= \left(\bar{P}_T f^p(v+\cdot)\right)(x)\left(\overline{E}^{\widetilde{G}}\left[R_T^{\frac{p}{p-1}}\right]\right)^{p-1}. \qquad (3.12)
\end{aligned}
$$

Letting $h_s = \frac{v}{T} + b(X_s) - b(Y_s)$, by Remark 2.8 and (3.11), we have

$$
\begin{aligned}
\exp\left\{\frac{p}{2(p-1)^2}\int_0^T |h_s|^2 d\langle \bar{B}\rangle_s\right\} &\leq \exp\left\{\frac{p\underline{\sigma}^{-2}}{2(p-1)^2}\int_0^T |h_s|^2 ds\right\} \\
&= \exp\left\{\frac{p\underline{\sigma}^{-2}v^2}{2(p-1)^2}\left(\frac{1}{T}+K+\frac{K^2T}{3}\right)\right\}.
\end{aligned}
$$

Similarly with the proof of Theorem 3.1, we have

$$
\overline{E}^{\widetilde{G}}\left[R_T^{\frac{p}{p-1}}\right] \leq \exp\left\{\frac{p\underline{\sigma}^{-2}v^2}{2(p-1)^2}\left(\frac{1}{T}+K+\frac{K^2T}{3}\right)\right\}. \qquad (3.13)
$$

It follows from (3.12) and (3.13) that (3.10) holds.

3.1 Applications of Harnack and shift Harnack Inequalities

In this subsection, we give some applications of Harnack and shift Harnack inequalities for invariant linear expectation and sup-kernel estimates. Before that, due to technical difficulties, we need the following invariant linear expectation, let us define the (quasi) invariant linear (nonlinear) expectation and sup-kernel of the operator $\bar{P}$.

Definition 3.1. Let E be a linear (nonlinear) expectation, and $\bar{P}$ be a nonlinear operator defined on $\mathcal{B}_b^+(\mathbb{R}^d)$.

(1) E is called a quasi-invariant linear (nonlinear) expectation of $\bar{P}$, if there exists a function $0 \leq g \in \mathcal{B}_b(\mathbb{R}^d)$ with $E[g] < \infty$, such that

$$
E[\bar{P}f] \leq E[gf], \quad 0 \leq f \in \mathcal{B}_b(\mathbb{R}^d).
$$

Moreover, if

$$
E[(\bar{P}f)] = E[f], \quad 0 \leq f \in \mathcal{B}_b(\mathbb{R}^d),
$$

then E is called an invariant linear (nonlinear) expectation of $\bar{P}$.

(2) A function p on $\mathbb{R}^d \times \mathbb{R}^d$ is called the sup-kernel (or sup-density) of $\bar{P}$ with respect to E, if

$$
\bar{P}f(x) \leq E[p(x,\cdot)f(\cdot)], \quad 0 \leq f \in \mathcal{B}_b(\mathbb{R}^d),\ x \in \mathbb{R}^d.
$$

To illustrate the above definition, we consider an example as follows.

Example 3.3. We consider the following Ornstein-Uhlenbeck process driven by G-Brownian motion: for each $x \in \mathbb{R}^d$,

$$Y_t^x = x - \alpha \int_0^t Y_s^x ds + B_t,\ t \geq 0,$$

where B_t is a d-dimensional G-Brownian motion. Hu et al. [8] proved the unique invariant expectation for G-Ornstein-Uhlenbeck process Y is the G-normal distribution of $\sqrt{\frac{1}{2\alpha}}B_1$.

Example 3.4. For $\theta \in \left[\frac{1}{2}, 1\right]$, let W_t be the stand 1-dimensional Brownian motion, consider the following SDE,

$$dX_t = -\theta X_t dt + \sqrt{2\theta} dW_t,\ X_0 = x.$$

Then, $X_t = e^{-\theta t}x + \int_0^t \sqrt{2\theta} e^{-\theta(t-s)} dW_s$, $X_t \to N(0,1)$ in distribution as $t \to \infty$.

Let

$$P_\theta f(x) = \int_R \frac{1}{\sqrt{2\pi(1-e^{-2\theta})}} \exp\left\{-\frac{(z-e^\theta x)^2}{2(1-e^{-2\theta})}\right\} f(z)dz,\ \theta \in \left[\frac{1}{2}, 1\right].$$

Therefore,

$$\begin{aligned} P_\theta f(x) &= E_0\left[\frac{\frac{1}{\sqrt{2\pi(1-e^{-2\theta})}} \exp\left\{-\frac{(\cdot - e^\theta x)^2}{2(1-e^{-2\theta})}\right\} f(\cdot)}{\frac{1}{\sqrt{2\pi}} e^{-\frac{(\cdot)^2}{2}}}\right] \\ &= E_0\left[\frac{\frac{1}{\sqrt{1-e^{-2\theta}}} \exp\left\{-\frac{(\cdot - e^\theta x)^2}{2(1-e^{-2\theta})}\right\} f(\cdot)}{e^{-\frac{(\cdot)^2}{2}}}\right], \end{aligned}$$

where $E_0[f] = \int_{\mathbb{R}} f d\mu_0, \mu_0 = N(0,1)$.

Moreover,

$$\frac{\frac{1}{\sqrt{1-e^{-2\theta}}} \exp\left\{-\frac{(z-e^\theta x)^2}{2(1-e^{-2\theta})}\right\}}{e^{-\frac{z^2}{2}}} \leq \frac{e^{\frac{z^2}{2}}}{\sqrt{(1-e^{-1})}}.$$

Then, $p(x,y) = \frac{e^{\frac{y^2}{2}}}{\sqrt{1-e^{-1}}}$ is a sup-kernel of $\bar{P}$ with respect to E_0, where $\bar{P}f = \sup_{\theta \in \Theta} P_\theta f$.

3.1.1 Applications of Harnack Inequalities

Now, we consider following Harnack-type inequality

$$\Phi(\bar{P}f(x)) \leq \bar{P}\Phi(f(y))e^{\Psi(x,y)}, \tag{3.14}$$

where Φ is a nonnegative convex function on $[0, \infty)$ and Ψ is a nonnegative function on $\mathbb{R}^d \times \mathbb{R}^d$.

Theorem 3.5. *Let E be a quasi-invariant linear expectation of $\bar{P}$, and $\Phi \in C^1([0,\infty))$ be an increasing function with $\Phi'(1) > 0$ and $\Phi(\infty) := \lim_{r\to\infty}\Phi(r) = \infty$ such that (3.14) holds.*

(1) *If $\lim_{y\to x}\{\psi(x,y)+\psi(y,x)\} = 0$ holds for all $x \in \mathbb{R}^d$, then $\bar{P}$ is strong Feller, i.e. $\bar{P}\mathcal{B}_b(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$.*

(2) *Let $\bar{P}f(x) = \sup_{\theta\in\Theta} P_\theta f(x)$. Then for all $\theta \in \Theta$, P_θ has a kernel p_θ with respect to E, $\bar{P}$ has a* sup*-kernel p with respect to E, and every invariant linear expectation of $\bar{P}$ is absolutely continuous with respect to E.*

(3) *If there exists $K > 0$ such that $\frac{1}{K}p_{\theta_1}(x,y) \leqslant p_{\theta_2}(x,y) \leqslant Kp_{\theta_1}(x,y)$, $\theta_1,\theta_2 \in \Theta$, $x, y \in \mathbb{R}^d$, where $p_\theta(x,y)$ is defined in (2), then $\bar{P}$ has at most one invariant linear expectation, and if it has one, a* sup*-kernel of $\bar{P}$ with respect to the invariant linear expectation is strictly positive.*

(4) *If $r\Phi^{-1}(r)$ is convex for $r \geqslant 0$, then a* sup*-kerner p of $\bar{P}$ with respect to E satisfies*

$$E\left[p(x,\cdot)p(y,\cdot)\right] \geq e^{-\Psi(x,y)}.$$

(5) *If E is a invariant linear expectation of $\bar{P}$, then*

$$\sup_{f\in\mathcal{B}_b^+(\mathbb{R}^d), E[\Phi(f)]\leq 1} \Phi(\bar{P}f(x)) \leq \frac{1}{E[e^{-\Psi(x,\cdot)}]}.$$

Proof.

(1) Let $0 < f \in \mathcal{B}_b(\mathbb{R}^d)$. Applying (3.14) to $f = 1 + \epsilon f$ for $\epsilon > 0$, we have

$$\Phi(1+\epsilon\bar{P}f(x)) \leq \{\bar{P}(\Phi(1+\epsilon f(y)))\}e^{\Psi(x,y)},\ x, y \in \mathbb{R}^d,\ \epsilon > 0.$$

By a Taylor expansion, we get

$$\Phi(1)+\epsilon\Phi'(1)\bar{P}f(x)+o(\epsilon) \leq \{\bar{P}(\Phi(1)+\epsilon\Phi'(1)f(y)+o(\epsilon))\}e^{\Psi(x,y)} \tag{3.15}$$

for small $\epsilon > 0$. Letting $y \to x$, we have

$$\epsilon\bar{P}f(x) \leq \epsilon \liminf_{y\to x} \bar{P}f(y) + o(\epsilon).$$

Then $\bar{P}f(x) \leq \liminf_{y\to x}\bar{P}f(y)$ for all $x \in \mathbb{R}^d$.

Moreover, letting $x \to y$ in (3.15), we have $\bar{P}f(y) \geq \limsup_{x\to y}\bar{P}f(x)$ for all $y \in \mathbb{R}^d$. Consequently, $\bar{P}f$ is continuous.

(2) To prove the existence of a sup-kernel, it suffices to prove $\bar{P}1_A(x) \leq E[p(x,\cdot)1_A]$ for some positive function p on $\mathbb{R}^d\times\mathbb{R}^d$.

We firstly prove that $E[1_A]=0$ implies $\bar{P}1_A\equiv 0$. Applying (3.14) to $f=1+n1_A$, we have

$$\Phi(1+n\bar{P}1_A(x))\leq \bar{P}\Phi(1+n1_A(y))e^{\Psi(x,y)},\ x,y\in\mathbb{R}^d.$$

Then

$$\Phi(1+n\bar{P}1_A(x))e^{-\Psi(x,y)}\leq \bar{P}\Phi(1+n1_A(y)),\ x,y\in\mathbb{R}^d.$$

It follows from E is a quasi-invariant expectation of $\bar{P}$ and $E[g]<\infty$ that

$$\begin{aligned}\Phi(1+n\bar{P}1_A(x)) &\leq \frac{E[\bar{P}\Phi(1+n1_A(\cdot))]}{E[e^{-\Psi(x,\cdot)}]}\\ &= \frac{E[\Phi(1+n1_A(\cdot))g]}{E[e^{-\Psi(x,\cdot)}]}\\ &\leq \frac{E[g\Phi(1)]}{E[e^{-\Psi(x,\cdot)}]}\\ &= \frac{\Phi(1)E[g]}{E[e^{-\Psi(x,\cdot)}]}\\ &< \infty.\end{aligned}$$

Since $\lim_{n\to\infty}\Phi(1+n)=\infty$, which implies that $\bar{P}1_A(x)=0$, $x\in\mathbb{R}^d$.

As $\bar{P}1_A(x)=\sup_{\theta\in\Theta}P_\theta 1_A(x)$, then $P_\theta 1_A(x)=0$, for all $\theta\in\Theta$. Therefore, there exists a p_θ on $\mathbb{R}^d\times\mathbb{R}^d$, such that $P_\theta f(x)=E[p_\theta(x,\cdot)f(\cdot)]$, $x\in\mathbb{R}^d$.

Moreover,

$$\bar{P}f(x)=\sup_{\theta\in\Theta}P_\theta f(x)=\sup_{\theta\in\Theta}E[p_\theta(x,\cdot)f(\cdot)]\leq E[p(x,\cdot)f(\cdot)],\ 0<f\in\mathcal{B}_b(\mathbb{R}^d),$$

where $p(x,\cdot)=\sup_{\theta\in\Theta}p_\theta(x,\cdot)$, we assume $E[p(x,\cdot)]<\infty$.

Furthermore, for any invariant expectation E_0 of $\bar{P}$, if $E[1_A]=0$, then $\bar{P}1_A=0$ implies that $E_0[1_A]=E_0[\bar{P}1_A]=0$. Therefore, E_0 is absolutely continuous with respect to E.

(3) Let p be a sup-kernel of $\bar{P}$ with respect to every invariant linear expectation E_0.

$$\bar{P}f(x)\leq E_0[p(x,\cdot)f(\cdot)]=:\widetilde{P}f(x),\ 0<f\in\mathcal{B}_b(\mathbb{R}^d).$$

We aim wo prove $p>0$. In fact, from the definition of $\widetilde{P}$, then $E_0[1_A]=0$ implies that $\widetilde{P}1_A=0$. To this end, it suffices to show that for any $x\in\mathbb{R}$, $\widetilde{P}1_A(x)=0$ implies that $E_0[1_A(x)]=0$. Since $\widetilde{P}1_A(x)\geq\bar{P}1_A(x)$, it suffices to show that $\bar{P}1_A(x)=0$ implies that $E_0[1_A(x)]=0$. Since $\bar{P}1_A(x)=0$, by applying (3.14) to $f=1+n1_A$, we obtain

$$\Phi(1+n\bar{P}1_A(y))\leq\bar{P}\Phi(1+n1_A)(x)e^{\Psi(y,x)}=\Phi(1)e^{\Psi(y,x)}.$$

Letting $n\to\infty$, we conclude that $\bar{P}1_A\equiv 0$, then $E_0[1_A]=E_0[\bar{P}1_A]=0$, which implies the sup-kernel $p(x,y)>0$.

Next we prove the uniqueness of invariant expectation. Let E_1 is a another invariant linear expectation. From (2), there exists a function f, for any g, such that $E_1[g] = E_0[fg]$, and $E_0[f] = 1$. We aim to prove that $f = 1, E_0 - a.e.$. Let $p(x,y) > 0$ be a sup-kernel of $\bar{P}$ with respect to E_0. Then, $\bar{P}f(x) \leq E_0[p(x,\cdot)f(\cdot)]$.

Moreover, since $\bar{P}f(x) = \sup_{\theta\in\Theta} P_\theta f(x)$, by (2), we know that $E_0[1_A] = 0$ implies $P_\theta 1_A = 0$.

Therefore, for any $\theta \in \Theta$, we have

$$\bar{P}f(x) \geq P_\theta f(x) = E_0[p_\theta(x,\cdot)f(\cdot)].$$

Let $P^*_\theta f(x) = \int_{\mathbb{R}^d} f(y)P^*_\theta(x,dy) = E_0[p_\theta(\cdot,x)f(\cdot)]$, $x \in \mathbb{R}^d$.

For any $0 < h \in L^1(E_0), 0 < g \in \mathcal{B}_b(\mathbb{R}^d)$, by Fubini theorem, we have

$$E_0[P^*_\theta hg] = E_0[E_0[p_\theta(\cdot,x)h(\cdot)]g(x)] = E_0[E_0[p_\theta(y,\cdot)g(\cdot)]h(y)] = E_0[hP_\theta g]. \tag{3.16}$$

Since E_0 is $\bar{P}$-invariant, and $P_\theta 1 = 1$, by (3.16), we have

$$E_0[(P^*_\theta 1)g] = E_0[P_\theta g] \leq E_0[\bar{P}g] = E_0[g],\ 0 < g \in \mathcal{B}_b(\mathbb{R}^d),$$

which implies $P^*_\theta 1 \leq 1$, E_0-a.e..

As $f \in L^1(E_0)$, by (3.16), we have

$$E_0[(P^*_\theta f)g] = E_0[fP_\theta g] = E_1[P_\theta g] \leq E_1[\bar{P}g] = E_1[g] = E_0[fg],\ 0 < g \in \mathcal{B}_b(\mathbb{R}^d).$$

which means $P^*_\theta f \leq f$, E_0-a.e..

By $P^*_\theta 1 \leq 1$, E_0-a.e., and Hölder inequality, we have

$$P^*_\theta\sqrt{f+1} \leq \sqrt{P^*_\theta(f+1)}\sqrt{P^*_\theta 1} \leq \sqrt{P^*_\theta f + P^*_\theta 1} \leq \sqrt{P^*_\theta f + 1},\ E_0-\text{a.e.}. \tag{3.17}$$

Furthermore, by (3.16) and $P^*_\theta f \leq f$, E_0-a.e., we obtain

$$E_0\left[P^*_\theta\sqrt{f+1}\right] = E_0\left[\sqrt{f+1}P_\theta 1\right] = E_0\left[\sqrt{f+1}\right] \geq E_0\left[\sqrt{P^*_\theta f+1}\right]. \tag{3.18}$$

From (3.17) and (3.18), we get

$$P^*_\theta\sqrt{f+1} = \sqrt{P^*_\theta f+1},\ E_0-\text{a.e.}. \tag{3.19}$$

By recalling (3.17) again, we have

$$P^*_\theta\sqrt{f+1} \leq \sqrt{P^*_\theta f+1}\sqrt{P^*_\theta 1},\ E_0-\text{a.e.}. \tag{3.20}$$

In contrast with (3.19), which implies $P^*_\theta 1 = 1$, E_0-a.e., then $P^*_\theta(x,\cdot)$ is a probability measure. Therefore, if and only if f is a constant under $P^*_\theta(x,\cdot)$ for E_0-a.e. x, the equation in (3.20) holds.

Moreover, since $p(x,y) > 0$ and for some $K > 0$, it holds that $\frac{1}{K}p_{\theta_1}(x,y) \leqslant p_{\theta_2}(x,y) \leqslant Kp_{\theta_1}(x,y)$, $\theta_1, \theta_2 \in \Theta$, then $p_\theta(x,y) > 0$. Therefore, $P^*_\theta 1_A = 0$ implies $E_0[1_A] = 0$, so f is a constant under E_0-a.e., which together with $E_0[f] = 1$ implies that $f = 1, E_0$-a.e..

(4) Since Φ is an increasing function, we have

$$\Phi(P_\theta f(x)) \leq \Phi(\bar{P}f(x)) \leq \bar{P}\Phi(f(y))e^{\Psi(x,y)} \leq E[p(y,\cdot)\Phi(f)(\cdot)]e^{\Psi(x,y)}. \tag{3.21}$$

From (2), there exists a p_θ on $\mathbb{R}^d \times \mathbb{R}^d$, sush that $P_\theta f(x) = E[p_\theta(x,\cdot)f(\cdot)]$.
Taking $f = n \wedge \Phi^{-1}(p(x,\cdot))$ in (3.21), we have

$$e^{-\Psi(x,y)}\Phi(E[p_\theta(x,\cdot)(n \wedge \Phi^{-1}(p(x,\cdot)))]) \leq E[p(y,\cdot)\Phi(n \wedge \Phi^{-1}(p(x,\cdot)))] < \infty.$$

Letting $n \to \infty$, by monotone convergence theorem, we get

$$E[p(y,\cdot)p(x,\cdot)] \geq e^{-\Psi(x,y)}\Phi(E[p_\theta(x,\cdot)(\Phi^{-1}(p_\theta(x,\cdot)))]). \tag{3.22}$$

Since $r\Phi^{-1}(r)$ is convex for $r \geqslant 0$, by Jensen's inequality, we have

$$E[p_\theta(x,\cdot)(\Phi^{-1}(p_\theta(x,\cdot)))] \geq \Phi^{-1}(1). \tag{3.23}$$

Combining (3.22) and (3.23), we obtain

$$E[p(y,\cdot)p(x,\cdot)] \geq e^{-\Psi(x,y)}.$$

(5) This result is obvious, we omit it here.

Next, we give an example which has at most one invariant linear expectation by means Harnack inequality.

Example 3.6. For $\theta \in \left\{\frac{1}{2}, 1\right\}$, let W_t be the stand 1-dimensional Brownian motion, consider the following SDE,

$$dX_t = -\theta X_t dt + \sqrt{2\theta}dW_t,\ X_0 = x.$$

Then $X_t = e^{-\theta t}x + \int_0^t \sqrt{2\theta}e^{-\theta(t-s)}dW_s$, $X_t \to N(0,1) = \mu_0$ in distribution as $t \to \infty$.

Let

$$P_\theta f = \int_R \frac{1}{\sqrt{2\pi(1-e^{-2\theta})}} \exp\left\{\frac{-(z-e^\theta x)^2}{2(1-e^{-2\theta})}\right\} f(z)dz.$$

Therefore, for every P_θ, there is a same invariant linear expectation E, where $E[f] = \int_{\mathbb{R}} f d\mu_0$. Thus,

$$E[\bar{P}f] = E\left[\sup_{\theta=\frac{1}{2},1} P_\theta f\right] \leq E\left[P_{\frac{1}{2}}f + P_1 f\right] = 2E[f],\ 0 \leq f \in \mathcal{B}_b(\mathbb{R}).$$

Then E is a quasi-invariant linear expectation of $\bar{P}$.

Moreover, for every P_θ, $\theta = \frac{1}{2}, 1$ and $\alpha > 1$, it holds that

$$(P_\theta f(x))^\alpha \leq P_\theta(f(y))^\alpha \exp\{C(\alpha,\theta)|x-y|^2\} \leq \bar{P}(f(y))^\alpha \exp\{C(\alpha,\theta)|x-y|^2\}.$$

Then,

$$(\bar{P}f(x))^\alpha \leq \bar{P}(f(y))^\alpha \exp\left\{\left(C\left(\alpha, \frac{1}{2}\right) + C(\alpha,1)\right)|x-y|^2\right\},\ 0 \leq f \in \mathcal{B}_b(\mathbb{R}).$$

By Theorem 3.5, $\bar{P}$ has at most one invariant linear expectation.

3.1.2 Application of Shift Harnack Inequalities

Now, we consider the following shift Harnack inequality

$$\Phi(\bar{P}f(x)) \leq \bar{P}\{\Phi \circ f(e+\cdot)\}(x)e^{C_\Phi(x,e)},\ 0 < f \in \mathcal{B}_b(\mathbb{R}^d), \tag{3.24}$$

for some $x, e \in \mathbb{R}^d$, and constant $e^{C_\Phi(x,e)} \geq 0$.

Theorem 3.7. *Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function such that (3.24) holds. If there exists a countable set Θ_0 and a family of positive function $\{g_\theta\}_{\theta \in \Theta_0}$ on $\mathbb{R}^d$, such that*

$$\bar{P}f(x) \leq \sup_{\theta \in \Theta_0} P_\theta(g_\theta f)(x),\ 0 < f \in \mathcal{B}_b(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

Then $\bar{P}$ has a sup *transition density $p(x, y)$ with respect to the Lebesgue measure.*

Proof. From (3.24), we have

$$\Phi(\bar{P}f(x))e^{-C_\Phi(x,e)} \leq \bar{P}\{\Phi \circ f(e+\cdot)\}(x). \tag{3.25}$$

Integrating both sides in (3.25) with respect to de, it holds that

$$\Phi(\bar{P}f(x)) \int_{\mathbb{R}^d} e^{-C_\Phi(x,e)}de \ \ \leq \ \ \int_{\mathbb{R}^d} \bar{P}\{\Phi \circ f(e+\cdot)\}(x)de. \tag{3.26}$$

Moreover, for any Lebesgue-null set A, it holds that

$$\int_{\mathbb{R}^d} P_\theta\{g_\theta \Phi \circ 1_A(e+\cdot)\}(x)de = 0.$$

Then

$$P_\theta\{g_\theta \Phi \circ 1_A(e+\cdot)\}(x) = 0, \text{ for a.e. e.}$$

Since Θ_0 is a countable set, thus $\bar{P}\{\Phi \circ 1_A(e+\cdot)\}(x) = 0$, for a.e. e.

Therefore,

$$\int_{\mathbb{R}^d} \bar{P}\{\Phi \circ 1_A(e+\cdot)\}(x)de = 0. \tag{3.27}$$

By the strictly increasing properties, we get $\Phi^{-1}(0) = 0$. Applying $f = 1_A$ in (3.26), combining with (3.27), we get

$$\bar{P}1_A(x) \leq \Phi^{-1}\left(\frac{\int_{\mathbb{R}^d} \bar{P}\{\Phi \circ 1_A(e+\cdot)\}(x)de}{\int_{\mathbb{R}^d} e^{-C_\Phi(x,e)}de}\right) = 0. \tag{3.28}$$

Then for any Lebesgue-null set A, we have $P_\theta 1_A(x) = 0$, which implies that there exists a density function p_θ on $\mathbb{R}^d \times \mathbb{R}^d$, such that

$$P_\theta f(x) = \int_{\mathbb{R}^d} p_\theta(x, y)f(y)dy.$$

Thus,

$$\bar{P}f(x) = \sup_{\theta \in \Theta} P_\theta f(x) \leq \int_{\mathbb{R}^d} p(x, y)f(y)dy,$$

where the sup density function $p(x, y) = \sup_{\theta \in \Theta} p_\theta(x, y)$.

Example 3.8. Consider the Example 3.6 again. For every P_θ, $\theta \in \{\frac{1}{2}, 1\}$ and $\alpha > 1$, it holds that

$$(P_\theta f(x))^\alpha \le P_\theta(f^\alpha(v+\cdot))(x)\exp\{C(\alpha,\theta)|x-y|^2\} \le \bar{P}(f^\alpha(v+\cdot))\exp\{C(\alpha,\theta)|x-y|^2\}.$$

Thus, there holds the shift Harnack inequality

$$(\bar{P}f(x))^\alpha \le \bar{P}(f^\alpha(v+\cdot))\exp\left\{\left(C\left(\alpha,\frac{1}{2}\right)+C(\alpha,1)\right)|x-y|^2\right\},\ f\in\mathcal{B}_b^+(\mathbb{R}).$$

Let $\bar{P}f = \sup_{\theta=\frac{1}{2},1} P_\theta f$, $f \in \mathcal{B}_b^+(\mathbb{R})$, where

$$P_\theta f(x) = \int_R \frac{1}{\sqrt{2\pi(1-e^{-2\theta})}}\exp\left\{\frac{-(z-e^\theta x)^2}{2(1-e^{-2\theta})}\right\} f(z)dz,\ \theta=\frac{1}{2},1.$$

Since

$$\begin{aligned}
&p_{\frac{1}{2}}(x,y)+p_1(x,y)\\
=\ & \frac{1}{\sqrt{2\pi(1-e^{-1})}}\exp\left\{\frac{-(y-e^{\frac{1}{2}}x)^2}{2(1-e^{-1})}\right\}+\frac{1}{\sqrt{2\pi(1-e^{-2})}}\exp\left\{\frac{-(y-ex)^2}{2(1-e^{-2})}\right\}\\
\le\ & \frac{1}{\sqrt{2\pi(1-e^{-1})}}\exp\left\{\frac{-(y-e^{\frac{1}{2}}x)^2}{2(1-e^{-1})}+\frac{-(y-ex)^2}{2(1-e^{-2})}\right\}.
\end{aligned}$$

Let

$$p(x,y) = \frac{1}{\sqrt{2\pi(1-e^{-1})}}\exp\left\{\frac{-(y-e^{\frac{1}{2}}x)^2}{2(1-e^{-1})}+\frac{-(y-ex)^2}{2(1-e^{-2})}\right\}.$$

From Theorem 3.7, we know that the $p(x,y)$ is a sup transition density of $\bar{P}$ with respect to the Lebesgue measure.

Acknowledgement. The author would like to thank Professor Feng-Yu Wang for guidance and helpful comments, as well as Xing Huang for corrections.

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