RANK DIFFERENCES FOR OVERPARTITIONS MODULO 6

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Abstract. In this article, we study the 3-dissection properties of ranks for overpartitions modulo 6. In this case, -1 appears as a unit root, so that double poles occur in the generating function. We prove two identities of generalized Lambert series by taking limits in Chan's identities, which are useful in generating various formulas with similar poles. We also relate these ranks to the third order mock theta functions $\omega(q)$ and $\rho(q)$.

Notation

Throughout this article we use the common q-series notation associated with infinite products:

$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad [a]_\infty := (a, q/a)_\infty,$$
$$(a_1, a_2, \dots, a_k)_\infty := (a_1)_\infty \cdots (a_k)_\infty, \qquad [a_1, a_2, \dots, a_k]_\infty := [a_1]_\infty \cdots [a_k]_\infty,$$
$$j(z;q) := (z;q)_\infty (q/z;q)_\infty (q;q)_\infty, \qquad J_{a,m} := j(q^a;q^m), \quad J_m := (q^m;q^m)_\infty.$$

For the sake of convergence, we always assume that $|q| < 1$. Also, we adopt a notation due to D. B. Sears [17]:

$$F(b_1, b_2, \dots, b_m) + \mathrm{idem}(b_1; b_2, \dots, b_m) := \sum_{i=1}^{m} F(b_i, b_2, \dots, b_{i-1}, b_1, b_{i+1}, \dots, b_m).$$

1. Introduction

An overpartition of a positive integer n is a partition of n, where the first occurrence of each different part may be overlined. The number of overpartitions of n is denoted by $\overline{p}(n)$. In particular, we set $\overline{p}(0) = 1$. Lovejoy [13] employed the classical definition of Dyson's rank, hereafter denoted the rank, as the largest part minus the number of parts in an overpartition. Let $\overline{N}(m, n)$ denote the number of overpartitions of n with

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the rank m. Lovejoy proved that the generating function of $\overline{N}(m,n)$ is given by

$$\overline{R}(z;q):=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)z^mq^n=\frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right). \tag{1.1}$$

Bringmann and Lovejoy [3] showed that $\overline{R}(z;q)$ is the holomorphic part of a harmonic weak Maass form of weight 3/2.

Let $\overline{N}(s,\ell,n)$ denote the number of overpartitions of n with rank congruent to s modulo ℓ. Lovejoy and Osburn [14] pointed out that the rank differences $\overline{N}(s,\ell,\ell n+d)-\overline{N}(t,\ell,\ell n+d)$ provide a measure of the extent to which the rank fails to produce a Ramanujan congruence of the type

$$\overline{p}(\ell n+d)\equiv 0 \pmod{\ell}.$$

They also expressed the generating functions of rank differences for $\ell=3,5$, in terms of infinite products and generalized Lambert series. The modulus $l=7$ has been determined by Jennings-Shaffer [11]. For even moduli, only special linear combinations of ranks were obtained previously. In [12], Ji, Zhang and Zhao studied 3-dissection properties of the form

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,n)+\overline{N}(1,6,n)-\overline{N}(2,6,n)-\overline{N}(3,6,n))q^n. \tag{1.2}$$

The difficulty of providing all rank differences $\overline{N}(s,6,n)-\overline{N}(t,6,n)$ $(0\leqslant s<t\leqslant 3)$ lies in the fact that -1 is a unit root of even moduli, so that $\overline{R}(-1;q)$ arises naturally in (1.1). Bringmann and Lovejoy [3] pointed out that $\overline{R}(-1;q)$ is more complicated since double poles occur. Similar situation happens in related problems associated with various types of ranks (such as crank, M_2-rank, etc.) for different types of partitions (see [1, 6, 7, 15, 16]).

In this article, we give 3-dissection properties for each residue in (1.2). Let

$$\overline{r}_s(d)=\sum_{n=0}^{\infty}\overline{N}(s,6,3n+d)q^n. \tag{1.3}$$

The main results are summarized in Theorems 1.1-1.3.

Theorem 1.1. *For $d=0$, we have*

$$\overline{r}_0(0)=\frac{J_6^{12}}{6J_{1,6}^8J_2^4J_{3,6}}+\frac{2J_6^3J_{3,6}}{3J_{1,6}^2J_2}-\frac{J_{1,6}^4J_2^2J_{3,6}^3}{3J_6^6}+\frac{2}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{(1+q^{3n})^2},$$

$$\overline{r}_1(0)=\frac{J_6^{12}}{6J_{1,6}^8J_2^4J_{3,6}}+\frac{J_{1,6}^4J_2^2J_{3,6}^3}{3J_6^6}-\frac{2}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{(1+q^{3n})^2},$$

$$\overline{r}_2(0) = \frac{J_6^{12}}{6J_{1,6}^8 J_2^4 J_{3,6}} - \frac{J_6^3 J_{3,6}}{3J_{1,6}^2 J_2} - \frac{J_{1,6}^4 J_2^2 J_{3,6}^3}{3J_6^6} + \frac{2}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{(1+q^{3n})^2},$$

$$\overline{r}_3(0) = \frac{J_6^{12}}{6J_{1,6}^8 J_2^4 J_{3,6}} + \frac{J_{1,6}^4 J_2^2 J_{3,6}^3}{3J_6^6} - \frac{2}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{(1+q^{3n})^2}.$$

Theorem 1.2. *For $d = 1$, we have*

$$\overline{r}_0(1) = \frac{J_6^{12}}{3J_{1,6}^7 J_2^4 J_{3,6}^2} + \frac{4J_6^3}{3J_{1,6} J_2} + \frac{J_{1,6}^5 J_2^2 J_{3,6}^2}{3J_6^6},$$

$$\overline{r}_1(1) = \frac{J_6^{12}}{3J_{1,6}^7 J_2^4 J_{3,6}^2} - \frac{J_{1,6}^5 J_2^2 J_{3,6}^2}{3J_6^6},$$

$$\overline{r}_2(1) = \frac{J_6^{12}}{3J_{1,6}^7 J_2^4 J_{3,6}^2} - \frac{2J_6^3}{3J_{1,6} J_2} + \frac{J_{1,6}^5 J_2^2 J_{3,6}^2}{3J_6^6},$$

$$\overline{r}_3(1) = \frac{J_6^{12}}{3J_{1,6}^7 J_2^4 J_{3,6}^2} - \frac{J_{1,6}^5 J_2^2 J_{3,6}^2}{3J_6^6}.$$

Theorem 1.3. *For $d = 2$, we have*

$$\overline{r}_0(2) = \frac{2J_6^{12}}{3J_{1,6}^6 J_2^4 J_{3,6}^3} - \frac{4J_6^3}{3J_2 J_{3,6}} + \frac{2J_{1,6}^6 J_2^2 J_{3,6}}{3J_6^6} - \frac{4}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{(1+q^{3n+1})^2} + \frac{4}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1+q^{3n+1}},$$

$$\overline{r}_1(2) = \frac{2J_6^{12}}{3J_{1,6}^6 J_2^4 J_{3,6}^3} + \frac{2J_6^3}{J_2 J_{3,6}} - \frac{2J_{1,6}^6 J_2^2 J_{3,6}}{3J_6^6} + \frac{4}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{(1+q^{3n+1})^2} - \frac{4}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1+q^{3n+1}},$$

$$\overline{r}_2(2) = \frac{2J_6^{12}}{3J_{1,6}^6 J_2^4 J_{3,6}^3} + \frac{2J_6^3}{3J_2 J_{3,6}} + \frac{2J_{1,6}^6 J_2^2 J_{3,6}}{3J_6^6} - \frac{4}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{(1+q^{3n+1})^2} + \frac{2}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1+q^{3n+1}},$$

$$\overline{r}_3(2) = \frac{2J_6^{12}}{3J_{1,6}^6 J_2^4 J_{3,6}^3} - \frac{4J_6^3}{J_2 J_{3,6}} - \frac{2J_{1,6}^6 J_2^2 J_{3,6}}{3J_6^6} + \frac{4}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{(1+q^{3n+1})^2}.$$

Theorems 1.1-1.3 carry information of rank sizes for different residues. For example, by Theorem 1.1, it is easy to derive that $\overline{N}(1,6,3n) = \overline{N}(3,6,3n)$ and $\overline{N}(0,6,3n) > \overline{N}(2,6,3n)$. Other comparisons take more efforts. For fixed d, numerical calculus suggests that all residues share a common main term. The growth rates of the second terms (some have the coefficient 0) overcome all the others left. This results in a total ordering relation.

Conjecture 1.4. *For $n \geqslant 11$, we have*

$$\overline{N}(0,6,3n) > \overline{N}(1,6,3n) = \overline{N}(3,6,3n) > \overline{N}(2,6,3n),$$

$$\overline{N}(0,6,3n+1) > \overline{N}(1,6,3n+1) = \overline{N}(3,6,3n+1) > \overline{N}(2,6,3n+1),$$

$$\overline{N}(1,6,3n+2) > \overline{N}(2,6,3n+2) > \overline{N}(0,6,3n+2) > \overline{N}(3,6,3n+2).$$

This should be verifiable by computing efficient asymptotic formulas of all terms, using standard analytic methods. However, this is far from the theme of this article and would take up a dozen pages. Therefore, we leave it as a conjecture [1].

Generalized Lambert series identities are widely used in the research of rank differences. In [5], Chan proved three generalized Lambert series expansions for infinite products. By taking limits in special variables, one can increase the order of poles in Chan's identities. In this article, we use this method to establish two identities aiming at decoupling parameters from the denominators. They are helpful in generating various identities with similar poles. We believe these formulas can also be used in handling functions similar to the overpartition rank.

In [9], Hickerson and Mortenson showed that a mock theta function can be expressed in terms of Appel-Lerch sums. Inspired by their work, we establish relations between the third order mock theta functions $\omega(q)$ and $\rho(q)$ and the ranks of overpartitions modulo 6, where $\omega(q)$ and $\rho(q)$ are defined by Watson [18]:

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)^2_{n+1}} \quad \text{and} \quad \rho(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(q;q^2)_{n+1}}{(q^3;q^6)_{n+1}}.$$

Theorem 1.5. *We have*

$$\begin{aligned}
\overline{r}_0(2)+\overline{r}_3(2)&=\frac{4}{9}\rho(q)-\frac{16}{9}\omega(q)+M(q),\\
\overline{r}_1(2)-\overline{r}_3(2)&=2\omega(q),\\
\overline{r}_2(2)+\overline{r}_3(2)&=-\frac{2}{9}\rho(q)-\frac{10}{9}\omega(q)+M(q),
\end{aligned}$$

where $M(q)$ is a weakly holomorphic modular form given by

$$M(q)=\frac{4J_6^{12}}{3J_{1,6}^6J_2^4J_{3,6}^3}.$$

This paper is organized as follows. In §2, we establish 3-dissection properties of ranks for overpartitions modulo 6. In §3 and §4, we introduce generalized Lambert series identities concerning double poles and single poles respectively. In §5, we introduce an algorithm which helps to transform $\mathcal{S}$-series raised in the new identities into sums of infinite products. Finally, we prove the relations between the ranks of overpartitions and mock theta functions in §6.

2. Ranks of Overpartitions modulo 6

In this section, we study 3-dissection properties of ranks of overpartitions modulo 6. Noting that $\overline{N}(s,\ell,n)=\overline{N}(\ell-s,\ell,n)$. Replacing z by $\xi_6=e^{\frac{\pi i}{3}}$ in (1.1), we have

$$\overline{R}(\xi_6;q)=\sum_{n=0}^{\infty}(\overline{N}(0,6,n)+\overline{N}(1,6,n)-\overline{N}(2,6,n)-\overline{N}(3,6,n))q^n \tag{2.1}$$

[1] At the time of submission, the conjecture has been proved by Ciolan [4].

$$= \frac{(-q)_\infty}{(q)_\infty}\left\{1+2\sum_{n=1}^{\infty}\frac{(2-\xi_6-\xi_6^{-1})(-1)^n q^{n^2+n}}{1-\xi_6 q^n-\xi_6^{-1}q^n+q^{2n}}\right\}$$

$$= \frac{2(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}.$$

Similarly, by replacing z by ξ_6^2, ξ_6^3 and 1 in (1.1) successively, we obtain

$$\begin{aligned}\overline{R}(\xi_6^2;q) &= \sum_{n=0}^{\infty}(\overline{N}(0,6,n)-\overline{N}(1,6,n)-\overline{N}(2,6,n)+\overline{N}(3,6,n))q^n \qquad (2.2)\\ &= \frac{6(-q)_\infty}{(q)_\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-q^{3n}}+\frac{(-q)_\infty}{(q)_\infty},\end{aligned}$$

$$\begin{aligned}\overline{R}(\xi_6^3;q) &= \sum_{n=0}^{\infty}(\overline{N}(0,6,n)-2\overline{N}(1,6,n)+2\overline{N}(2,6,n)-\overline{N}(3,6,n))q^n \qquad (2.3)\\ &= \frac{4(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{(1+q^n)^2},\end{aligned}$$

$$\begin{aligned}\overline{R}(1;q) &= \sum_{n=0}^{\infty}(\overline{N}(0,6,n)+2\overline{N}(1,6,n)+2\overline{N}(2,6,n)+\overline{N}(3,6,n))q^n \qquad (2.4)\\ &= \frac{(-q)_\infty}{(q)_\infty}.\end{aligned}$$

Now, we have a linear system of full rank which consists ranks for overpartitions of all residues modulo 6. Therefore, we are able to solve $\overline{N}(i,6,n)$ in terms of $\overline{R}(z;q)$.

$$\sum_{n=0}^{\infty}\overline{N}(0,6,n)q^n = \frac{1}{6}\left(\overline{R}(1;q)+2\overline{R}(\xi_6;q)+2\overline{R}(\xi_6^2;q)+\overline{R}(\xi_6^3;q)\right),$$

$$\sum_{n=0}^{\infty}\overline{N}(1,6,n)q^n = \frac{1}{6}\left(\overline{R}(1;q)+\overline{R}(\xi_6;q)-\overline{R}(\xi_6^2;q)-\overline{R}(\xi_6^3;q)\right),$$

$$\sum_{n=0}^{\infty}\overline{N}(2,6,n)q^n = \frac{1}{6}\left(\overline{R}(1;q)-\overline{R}(\xi_6;q)-\overline{R}(\xi_6^2;q)+\overline{R}(\xi_6^3;q)\right),$$

$$\sum_{n=0}^{\infty}\overline{N}(3,6,n)q^n = \frac{1}{6}\left(\overline{R}(1;q)-2\overline{R}(\xi_6;q)+2\overline{R}(\xi_6^2;q)-\overline{R}(\xi_6^3;q)\right).$$

Then, we can determine 3-dissection properties of $\overline{N}(i,6,n)$ by those of $\overline{R}(z;q)$. For $z=1$, it was proved by Hirschhorn and Sellers [10].

Lemma 2.1. We have

$$\overline{R}(1;q)=\frac{(-q)_\infty}{(q)_\infty}=\frac{J_{18}^{12}}{J_{3,18}^{8}J_6^4J_{9,18}}+q\frac{2J_{18}^{12}}{J_{3,18}^{7}J_6^4J_{9,18}^2}+q^2\frac{4J_{18}^{12}}{J_{3,18}^{6}J_6^4J_{9,18}^3}.$$

In §3 and §4, we prove the following lemmas concerning $\overline{R}(\xi_6^3;q)$ and $\overline{R}(\xi_6;q)$, $\overline{R}(\xi_6^2;q)$ successively.

Lemma 2.2. We have

$$\overline{R}(\xi_6^3;q)=\left(-\frac{2J_{3,18}^4J_6^2J_{9,18}^3}{J_{18}^6}+\frac{12}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n}}{(1+q^{9n})^2}\right)+q\frac{2J_{3,18}^5J_6^2J_{9,18}^2}{J_{18}^6}$$
$$+q^2\left(\frac{4J_{3,18}^6J_6^2J_{9,18}}{J_{18}^6}-\frac{24}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n}}{(1+q^{9n+3})^2}+\frac{16}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n}}{1+q^{9n+3}}\right).$$

Lemma 2.3. We have

$$\overline{R}(\xi_6;q)=\frac{J_{18}^3J_{9,18}}{J_{3,18}^2J_6}+q\frac{2J_{18}^3}{J_{3,18}J_6}+q^2\left(\frac{4J_{18}^3}{J_6J_{9,18}}-\frac{2}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n}}{1+q^{9n+3}}\right),$$
$$\overline{R}(\xi_6^2;q)=\frac{J_{18}^3J_{9,18}}{J_{3,18}^2J_6}+q\frac{2J_{18}^3}{J_{3,18}J_6}+q^2\left(\frac{4J_{18}^3}{J_6J_{9,18}}-\frac{6}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n}}{1-q^{9n+3}}\right).$$

It is worth noting that

$$\overline{R}(\xi_6^2;q)=\overline{R}(\xi_3;q)=\sum_{n=0}^{\infty}(\overline{N}(0,3,n)-\overline{N}(1,3,n))q^n,$$

so the identity for $\overline{R}(\xi_6^2;q)$ is exactly [14, Theorem 1] [2], where Lovejoy and Osburn determined rank differences of overpartitions modulo 3.

3. Generalized Lambert Series Identities: Double Poles

In this section, we consider $\overline{R}(\xi_6^3;q)$ first. In view of (2.3), a straightforward idea is to split the series into three sums according to the summation index, such as

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{(1+q^n)^2}=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+3n}}{(1+q^{3n})^2}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n+2}}{(1+q^{3n+1})^2}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+15n+6}}{(1+q^{3n+2})^2}.$$

Then Lambert series identities would help. In [5], Chan proved the following lemma.

Lemma 3.1. For non-negative integers $r<s$, we have

$$\frac{(a_0q,q/a_0,q,q)_\infty[a_1,a_2,\dots,a_r]_\infty}{[b_0,b_1,\dots,b_s]_\infty}$$

[2] Lemma 2.3 ommited -1 from [14, Theorem 1], since we assume the convention $\overline{p}(0)=1$.

$$= \frac{[a_0/b_0, a_1/b_0, \ldots, a_r/b_0]_\infty}{[b_1/b_0, b_2/b_0, \ldots, b_s/b_0]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{(s-r)n+1} q^{(s-r)n(n+1)/2} b_0 a_0^{-1}}{(1-b_0q^n)(1-b_0q^n/a_0)}$$

$$\times \left(\frac{a_0 a_1 \cdots a_r b_0^{s-r-1} q}{b_1 \cdots b_s} \right)^n + \mathrm{idem}(b_0; b_1, b_2, \ldots, b_s).$$

For $r = s$, this is true provided that $|q| < |\frac{a_0 \cdots a_r}{b_0 \cdots b_s}| < 1$.

For the sake of matching orders, Chan's identities will generate seven Lambert series, some of which are redundant. Therefore, we choose a roundabout method. We first make some adjustments,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^n)^2} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}(1-q^n+q^{2n})^2}{(1+q^{3n})^2} \tag{3.1}$$

$$= 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^{3n})^2} - 4\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(1+q^{3n})^2} + 3\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+3n}}{(1+q^{3n})^2}.$$

Then, consider the following 3-dissections of each Lambert series:

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^{3n})^2} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+3n}}{(1+q^{9n})^2} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n+2}}{(1+q^{9n+3})^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+15n+6}}{(1+q^{9n+6})^2},$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(1+q^{3n})^2} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+6n}}{(1+q^{9n})^2} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+12n+3}}{(1+q^{9n+3})^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+8}}{(1+q^{9n+6})^2},$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+3n}}{(1+q^{3n})^2} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n}}{(1+q^{9n})^2} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+15n+4}}{(1+q^{9n+3})^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+21n+10}}{(1+q^{9n+6})^2}.$$

The following lemma transforms each 3-dissection into one single Lambert series.

Lemma 3.2. We have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^{3n})^2} = \frac{J_{3,18}^6 J_6^3 J_{9,18}^2}{2J_{18}^9} \left(\frac{2}{3} - \frac{J_{9,18}^2 J_{18}^3}{4J_{3,18}^2 J_6} + \frac{J_{3,18}^6 J_6^3 J_{9,18}^2}{12 J_{18}^9} \right)$$

$$- \left(1 - \frac{2qJ_{3,18}}{J_{9,18}}\right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n+2}}{(1+q^{9n+3})^2},$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(1+q^{3n})^2} = \frac{J_{3,18}^6 J_6^3 J_{9,18}^2}{2J_{18}^9} \left(\frac{1}{3} + \frac{J_{9,18}^2 J_{18}^3}{4J_{3,18}^2 J_6} - \frac{J_{3,18}^6 J_6^3 J_{9,18}^2}{12 J_{18}^9} \right)$$

$$- \left(1 - \frac{2qJ_{3,18}}{J_{9,18}}\right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+5}}{(1+q^{9n+3})^2},$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+3n}}{(1+q^{3n})^2} = \frac{qJ_{3,18}^5 J_6^2 J_{9,18}^3}{2J_{18}^6} + \left(1 - \frac{2qJ_{3,18}}{J_{9,18}}\right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n}}{(1+q^{9n})^2}.$$

Then in view of (2.3), Lemma 2.2 follows by substituting Lemma 3.2 into (3.1) and using Lemma 2.1. To prove Lemma 3.2, we first take $a_0 = 1$, $b_0 = -1$, $b_1 = -q^3$, $b_2 = -q^6$ in Lemma 3.1, and get

$$\frac{J_{3,18}^5 J_6^2 J_{9,18}^3}{2J_{18}^6} = \frac{2J_{3,18}}{J_{9,18}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n}}{(1+q^{9n})^2} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+15n+3}}{(1+q^{9n+3})^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+21n+9}}{(1+q^{9n+6})^2}.$$

This proves the third formula. Though, for the other two formulas, it is hard to give a proper relationship directly by Lemma 3.1. Poles are twisted with the orders of q in numerators. Therefore, we introduce the following theorem.

First, for a sequence $\boldsymbol{a} = (a_1, \dots, a_r)$, we define series $\mathcal{S}(a_1, \dots, a_r)$ as

$$\mathcal{S}(a_1, \dots, a_r) := \mathcal{S}(a_1, \dots, a_r; q) = \sum_{u=1}^{r} \sum_{n=0}^{\infty} \left(\frac{1}{1 - a_u q^n} - \frac{1}{1 - a_u^{-1} q^{n+1}} \right). \tag{3.2}$$

We also write $\mathcal{S}(\boldsymbol{a}) = \mathcal{S}(a_1, \dots, a_r)$ for brevity.

Theorem 3.3. *Let $\boldsymbol{a} = (a_1, \dots, a_r)$ and $\boldsymbol{b} = (b_1, \dots, b_s)$. Then for non-negative integers $r < s$, we have*

$$\begin{aligned}
&\frac{(q)_\infty^2 [a_1, \dots, a_r]_\infty}{[b_1, \dots, b_s]_\infty} (1 - \mathcal{S}(\boldsymbol{a}) + \mathcal{S}(\boldsymbol{b})) \\
&\quad = \frac{[a_1/b_1, \dots, a_r/b_1]_\infty}{[b_2/b_1, \dots, b_s/b_1]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{(s-r)n} q^{(s-r)n(n+1)/2}}{(1 - b_1 q^n)^2} \left(\frac{a_1 \cdots a_r b_1^{s-r-1}}{b_2 \cdots b_s} \right)^n \\
&\qquad + \mathrm{idem}(b_1; b_2, \dots, b_s).
\end{aligned}$$

For $r = s$, this is true provided that $|q^2| < |\frac{a_1 \cdots a_r}{b_1 \cdots b_s}| < |q|$.

Proof. by setting $a_0 = 1$ and $b_0 = q$ in Lemma 3.1 and taking limits We need to compute the limits at $b_0 = q$. First, take $a_0 = 1$ in Lemma 3.1,

$$\begin{aligned}
\frac{(q)_\infty^4 [a_1, \dots, a_r]_\infty}{[b_0, b_1, \dots, b_s]_\infty} = &\frac{[b_0^{-1}, a_1/b_0, \dots, a_r/b_0]_\infty}{[b_1/b_0, \dots, b_s/b_0]_\infty} \\
&\times \sum_{n=-\infty}^{\infty} \frac{(-1)^{(s-r)n+1} q^{(s-r)n(n+1)/2} b_0}{(1 - b_0 q^n)^2} \left(\frac{a_1 \cdots a_r b_0^{s-r-1} q}{b_1 \cdots b_s} \right)^n \\
&+ \mathrm{idem}(b_0; b_1, \dots, b_s).
\end{aligned}$$

Denote the term on the left-hand side by L and those on the right-hand side by $R_0, \dots, R_s$ respectively. For R_1, we take $b_0 \to q$ directly and deduce that

$$\lim_{b_0 \to q} R_1 = \frac{[a_1/b_1, \dots, a_r/b_1]_\infty}{[b_2/b_1, \dots, b_s/b_1]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{(s-r)n} q^{(s-r)n(n+1)/2}}{(1 - b_1 q^n)^2} \left(\frac{a_1 \cdots a_r b_1^{s-r-1}}{b_2 \cdots b_s} \right)^n.$$

Thus it remains to show that

$$\lim_{b_0\to q}(L-R_0)=\frac{(q)_\infty^2[a_1,\ldots,a_r]_\infty}{[b_1,\ldots,b_s]_\infty}\left(1-\mathcal{S}(\boldsymbol{a})+\mathcal{S}(\boldsymbol{b})\right). \tag{3.3}$$

We separate terms containing poles from L and R_0 successively. First we rewrite L and R_0 as

$$L=\frac{(q)_\infty^4[a_1,\ldots,a_r]_\infty}{(b_0,b_0^{-1}q^2)_\infty[b_1,\ldots,b_s]_\infty}\cdot\frac{b_0}{b_0-q},$$

and

$$R_0=\frac{(1-b_0^{-1}q)(b_0,b_0^{-1}q^2)_\infty[a_1q/b_0,\ldots,a_rq/b_0]_\infty}{[b_1q/b_0,\ldots,b_sq/b_0]_\infty}$$
$$\times\sum_{n=-\infty}^{\infty}\frac{(-1)^{(s-r)(n+1)}q^{(s-r)n(n+1)/2}b_0}{(1-b_0q^n)^2}\left(\frac{a_1\cdots a_rb_0^{s-r}}{b_1\cdots b_s}\right)^{n+1}\left(\frac{q}{b_0}\right)^n.$$

In R_0, terms with $n\neq-1$ vanish when setting $b_0\to q$. Thus, in (3.3), we have

$$\begin{aligned}
&\lim_{b_0\to q}(L-R_0)\\
&=\lim_{b_0\to q}\frac{1}{b_0-q}\left(\frac{(q)_\infty^4[a_1,\ldots,a_r]_\infty}{(b_0,b_0^{-1}q^2)_\infty[b_1,\ldots,b_s]_\infty}\cdot b_0-\frac{(b_0,b_0^{-1}q^2)_\infty[a_1q/b_0,\ldots,a_rq/b_0]_\infty}{[b_1q/b_0,\ldots,b_sq/b_0]_\infty}\cdot q\right)\\
&=\lim_{b_0\to q}\frac{\mathrm{d}}{\mathrm{d}b_0}\left(\frac{(q)_\infty^4[a_1,\ldots,a_r]_\infty}{(b_0,b_0^{-1}q^2)_\infty[b_1,\ldots,b_s]_\infty}b_0-\frac{(b_0,b_0^{-1}q^2)_\infty[a_1q/b_0,\ldots,a_rq/b_0]_\infty}{[b_1q/b_0,\ldots,b_sq/b_0]_\infty}q\right)\\
&=:\lim_{b_0\to q}\frac{\mathrm{d}}{\mathrm{d}b_0}(L^*-R_0^*),
\end{aligned}$$

where the penultimate equation follows by L'Hôpital's rule.

For L^*, it is easy to obtain

$$\lim_{b_0\to q}\frac{\mathrm{d}L^*}{\mathrm{d}b_0}=\frac{(q)_\infty^2[a_1,\ldots,a_r]_\infty}{[b_1,\ldots,b_s]_\infty}.$$

For R_0^*, we have

$$\lim_{b_0\to q}R_0^*=\frac{q(q)_\infty^2[a_1,\ldots,a_r]_\infty}{[b_1,\ldots,b_s]_\infty}.$$

It follows by taking the logarithmic derivative that

$$\lim_{b_0\to q}\frac{\mathrm{d}\log R_0^*}{\mathrm{d}b_0}=\frac{\mathcal{S}(\boldsymbol{a})-\mathcal{S}(\boldsymbol{b})}{q},$$

where $\mathcal{S}$ is defined in (3.2). Therefore,

$$\lim_{b_0\to q}\frac{\mathrm{d}R_0^*}{\mathrm{d}b_0}=\lim_{b_0\to q}\left(R_0^*\frac{\mathrm{d}\log R_0^*}{\mathrm{d}b_0}\right)=\frac{(q)_\infty^2[a_1,\ldots,a_r]_\infty}{[b_1,\ldots,b_s]_\infty}\left(\mathcal{S}(\boldsymbol{a})-\mathcal{S}(\boldsymbol{b})\right).$$

Thus we complete the proof. ∎

Now, we replace q by q^9 and set $r = 1$, $s = 3$, $b_1 = -1$, $b_2 = -q^3$, $b_3 = -q^6$ in Theorem 3.3. Then by taking $a_1 = q^3$ and $a_1 = q^6$ successively, we obtain

$$\frac{J_6^3 J_{3,18}^6 J_{9,18}^2}{2J_{18}^9}\left(\frac{1}{2} - \mathcal{S}(q^3; q^9)\right) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+3n}}{(1+q^{9n})^2} \tag{3.4}$$

$$- \frac{2J_{3,18}}{J_{9,18}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n+3}}{(1+q^{9n+3})^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+15n+6}}{(1+q^{9n+6})^2},$$

$$\frac{J_6^3 J_{3,18}^6 J_{9,18}^2}{2J_{18}^9}\left(\frac{1}{2} - \mathcal{S}(q^6; q^9)\right) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+6n}}{(1+q^{9n})^2} \tag{3.5}$$

$$- \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+12n+3}}{(1+q^{9n+3})^2} + \frac{2J_{3,18}}{J_{9,18}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+9}}{(1+q^{9n+6})^2}.$$

Note that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+8}}{(1+q^{9n+6})^2} = - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+5}}{(1+q^{9n+3})^2}.$$

(3.4) and (3.5) prove the first two 3-dissections in Lemma 3.2 respectively provided that

$$\mathcal{S}(q^3; q^9) = -\mathcal{S}(q^6; q^9) = \frac{J_{9,18}^2 J_{18}^3}{4J_{3,18}^2 J_6} - \frac{J_{3,18}^6 J_6^3 J_{9,18}^2}{12J_{18}^9} - \frac{1}{6}. \tag{3.6}$$

We will prove this in §5.

4. Generalized Lambert Series Identities: Single Poles

In this section, we consider $\overline{R}(\xi_6; q)$ and $\overline{R}(\xi_6; q^2)$, which both contain generalized Lambert series with single poles. Consider their 3-dissections:

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1+q^{3n}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+3n}}{1+q^{9n}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n+2}}{1+q^{9n+3}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+15n+6}}{1+q^{9n+6}},$$

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n^2+n}}{1-q^{3n}} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{9n^2+3n}}{1-q^{9n}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n+2}}{1-q^{9n+3}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+15n+6}}{1-q^{9n+6}}.$$

Similarly to Lemma 3.2, the following lemma transforms these 3-dissections into one single series.

Lemma 4.1. We have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1+q^{3n}} = \left(2q\frac{J_{3,18}}{J_{9,18}} - 1\right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n+2}}{1+q^{9n+3}} + \frac{J_{3,18}^6 J_6^3 J_{9,18}^2}{2J_{18}^9},$$

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-q^{3n}}=\left(2q\frac{J_{3,18}}{J_{9,18}}-1\right)\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n+2}}{1-q^{9n+3}}+\frac{J_{3,18}^6J_6^3J_{9,18}^2}{6J_{18}^9}-\frac{1}{6}.$$

Then Lemma 2.3 follows by substituting Lemma 4.1 into (2.1), (2.2) and using Lemma 2.1. The first identity in Lemma 4.1 was proved by Ji, Zhang and Zhao [12] by using the following Chan's identity [5].

Lemma 4.2. For non-negative integers $r \leq s$, we have

$$\begin{aligned}\frac{[a_1,\ldots,a_r]_\infty (q)_\infty^2}{[b_0,b_1,\ldots,b_s]_\infty}&=\frac{[a_1/b_0,\ldots,a_r/b_0]_\infty}{[b_1/b_0,\ldots,b_s/b_0]_\infty}\\ &\quad\times\sum_{n=-\infty}^{\infty}\frac{(-1)^{(s-r+1)n}q^{(s-r+1)n(n+1)/2}}{1-b_0q^n}\left(\frac{a_1\cdots a_r b_0^{s-r}}{b_1\cdots b_s}\right)^n\\ &\quad+\mathrm{idem}(b_0;b_1,\ldots,b_s).\end{aligned}$$

For $r = s + 1$, this is true provided that $|q| < |\frac{a_1\cdots a_r}{b_0\cdots b_s}| < 1$.

Set $r = 1$ and $s = 2$ in Lemma 4.2. Then by replacing q by q^9 and taking $a_1 = q^3$, $b_0 = -1$, $b_1 = -q^3$, and $b_2 = -q^6$, we have

$$\frac{J_{3,18}^6J_6^3J_{9,18}^2}{2J_{18}^9}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+3n}}{1+q^{9n}}-\frac{2J_{3,18}}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n+3}}{1+q^{9n+3}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+15n+6}}{1+q^{9n+6}}.$$

This proves the first identity in Lemma 4.1. For the second one, one need to take efforts in the term $n = 0$. By taking $b_0 \to q$, we obtain the following theorem.

Theorem 4.3. *Let* $\boldsymbol{a} = (a_1, \ldots, a_r)$ *and* $\boldsymbol{b} = (b_1, \ldots, b_s)$. *Then for non-negative integers* $r \leqslant s$, *we have*

$$\begin{aligned}&\frac{[a_1,\ldots,a_r]_\infty}{[b_1,\ldots,b_s]_\infty}\left(1-\mathcal{S}(\boldsymbol{a})+\mathcal{S}(\boldsymbol{b})\right)\\ &\quad+\frac{[a_1,\ldots,a_r]_\infty}{[b_1,\ldots,b_s]_\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^{(s-r+1)n}q^{(s-r+1)n(n+1)/2-n}}{1-q^n}\left(\frac{a_1\cdots a_r}{b_1\cdots b_s}\right)^n\\ &=\frac{[a_1/b_1,\ldots,a_r/b_1]_\infty}{[b_1,b_2/b_1,\ldots,b_s/b_1]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{(s-r+1)n}q^{(s-r+1)n(n+1)/2-n}}{1-b_1q^n}\left(\frac{a_1\cdots a_r b_1^{s-r}}{b_2\cdots b_s}\right)^n\\ &\quad+\mathrm{idem}(b_1;b_2,\ldots,b_s).\end{aligned}$$

For $r = s + 1$, *this is true provided that* $|q^2| < |\frac{a_1\cdots a_r}{b_1\cdots b_s}| < |q|$.

Proof. The proof is similar to that of Theorem 3.3. Denote the term on the left-hand side of Lemma 4.2 by L', and those on the right-hand side by $R'_0, \ldots, R'_s$ respectively. Likewise by taking $b_0 \to q$ directly in terms other than R'_0, we get the right-hand side.

The difference arises in R_0'. The terms with $n \neq -1$ are no longer vanishing while taking $b_0 \to q$, which results in an extra Lambert series. In this case we have

$$\lim_{b_0\to q} R_0' = \lim_{b_0\to q} \frac{1}{b_0-q} \frac{[a_1/b_0,\dots,a_r/b_0]_\infty}{[b_1/b_0,\dots,b_s/b_0]_\infty} \cdot \frac{(-b_0)^{r-s} b_1\cdots b_s}{a_1\cdots a_r} \cdot q$$
$$+ \frac{[a_1/q,\dots,a_r/q]_\infty}{[b_1/q,\dots,b_s/q]_\infty} \sum_{\substack{n=-\infty\\ n\neq -1}}^{\infty} \frac{(-1)^{(s-r+1)n} q^{(s-r+1)n(n+1)/2+(s-r)n}}{1-q^{n+1}} \left(\frac{a_1\cdots a_r}{b_1\cdots b_s}\right)^n .$$

Thus, denoting the first term by R_0'', it suffices to show

$$\lim_{b_0\to q} (L' - R_0'') = \frac{[a_1,\dots,a_r]_\infty}{[b_1,\dots,b_s]_\infty} \left(1 - \mathcal{S}(\boldsymbol{a}) + \mathcal{S}(\boldsymbol{b})\right). \tag{4.1}$$

This can be proved following procedures similar to proving (3.3). □

Now we set $r = 1$ and $s = 2$ in Theorem 4.3. By replacing q by q^9 and taking $a_1 = -q^{12}$, $b_1 = q^3$ and $b_2 = q^6$, we obtain

$$\mathcal{S}(-q^3;q^9) = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n q^{9n^2+3n}}{1-q^{9n}} - \frac{2J_{3,18}}{J_{9,18}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+9n+3}}{1-q^{9n+3}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+15n+6}}{1-q^{9n+6}}.$$

This proves the second identity in Lemma 4.1 supplied that

$$\mathcal{S}(-q^3;q^9) = \frac{J_{3,18}^6 J_6^3 J_{9,18}^2}{6J_{18}^9} - \frac{1}{6}. \tag{4.2}$$

We will prove this in §5.

5. An Algorithm for $\mathcal{S}$-series

The generalized Lambert series $\mathcal{S}$ defined in (3.2) appears as an encumbrance in our expansions for infinite products. In this section, we show that $\mathcal{S}(\pm q^m; q^n)$ with m, n integers can be expanded as sums of infinite products. The following lemma shows that, for special $\boldsymbol{a}$, the function $\mathcal{S}(\boldsymbol{a})$ degenerates to concise forms.

Lemma 5.1. The function $\mathcal{S}$ has the following properties:

(1) $\mathcal{S}(-1) = -\frac{1}{2}$, $\mathcal{S}(-q) = \frac{1}{2}$;
(2) $\mathcal{S}(aq) = \mathcal{S}(a) + 1$, $\mathcal{S}(q/a) = -\mathcal{S}(a)$;
(3) Let $\boldsymbol{a} = (a_1,\dots,a_r)$. If $(q/a_1,\dots,q/a_r)$ is a permutation of $\boldsymbol{a}$, we have $\mathcal{S}(\boldsymbol{a}) = 0$;
(4) $\mathcal{S}(q^s;q^{-t}) = \mathcal{S}(q^{s+t};q^t)$.

Proof. Properties (1)-(3) are quite trivial. For (4), We have

$$\mathcal{S}(q^s;q^{-t}) = \sum_{n=0}^{\infty} \left(\frac{1}{1-q^s q^{-tn}} - \frac{1}{1-q^{-s}q^{-tn-t}}\right) = \sum_{n=0}^{\infty} \frac{q^{s-tn} - q^{-s-tn-t}}{(1-q^{s-tn})(1-q^{-s-tn-t})}.$$

We multiply both the denominator and numerator of each term by $q^{s+tn+t}q^{-s+tn}$, and deduce that

$$\mathcal{S}(q^s;q^{-t}) = \sum_{n=0}^{\infty}\left(\frac{1}{1-q^{s+tn+t}} - \frac{1}{1-q^{-s+tn}}\right) = \mathcal{S}(q^{s+t};q^t).$$

□

In view of (2) and (4), it suffices to consider $\mathcal{S}(\pm q^m;q^n)$ with m,n positive integers. The following lemma is due to Andrews, Lewis and Liu [2]. Chan [5] provided another proof using his generalized Lambert series identitites.

Lemma 5.2. For $|q|<1$, we have

$$\begin{aligned}\frac{[ab,bc,ca]_\infty(q)_\infty^2}{[a,b,c,abc]_\infty} = 1 &+ \sum_{n=0}^{\infty}\frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty}\frac{q^n/a}{1-q^n/a} + \sum_{n=0}^{\infty}\frac{bq^n}{1-bq^n} - \sum_{n=1}^{\infty}\frac{q^n/b}{1-q^n/b}\\ &+ \sum_{n=0}^{\infty}\frac{cq^n}{1-cq^n} - \sum_{n=1}^{\infty}\frac{q^n/c}{1-q^n/c} - \sum_{n=0}^{\infty}\frac{abcq^n}{1-abcq^n} + \sum_{n=1}^{\infty}\frac{q^n/abc}{1-q^n/abc}.\end{aligned} \tag{5.1}$$

We denote the infinite product on the left-hand side of Lemma 5.2 by $\mathcal{P}(a,b,c)$, which is

$$\mathcal{P}(a,b,c) = \mathcal{P}(a,b,c;q) = \frac{[ab,bc,ca]_\infty(q)_\infty^2}{[a,b,c,abc]_\infty}.$$

For the sake of brevity, we denote $\mathcal{P}(a,a,a)$ by $\mathcal{P}(a)$. Then, Lemma 5.2 shows that

$$\mathcal{P}(a,b,c) = 1 + \mathcal{S}(a,b,c) - \mathcal{S}(abc). \tag{5.2}$$

We are now equipped to propose an algorithm for $\mathcal{S}(\pm q^m;q^n)$ with arbitrary positive integers m and n. First in (5.2), by replacing q by q^n and setting $a=\pm q^m$, $b=\pm q^m$ and $c=-q^{n-2m}$, we have

$$\begin{aligned}\mathcal{P}(\pm q^m,\pm q^m,-q^{n-2m};q^n) &= 1+\mathcal{S}(\pm q^m,\pm q^m,-q^{n-2m};q^n) - \mathcal{S}(-q^n;q^n) \qquad (5.3)\\ &= \frac{1}{2} + 2\mathcal{S}(\pm q^m;q^n) - \mathcal{S}(-q^{2m};q^n).\end{aligned}$$

Therefore, in order to obtain expansions for $\mathcal{S}(\pm q^m;q^n)$ in terms of $\mathcal{P}$-functions, we need to calculate $\mathcal{S}(-q^{2m};q^n)$. Our strategy is to implement a recursive procedure using (5.2). Suppose that $n=3^s\cdot n'$ with $(3,n')=1$. We denote by k the order of 3 in the cyclic group $\mathbb{Z}_{n'}$. Thus, we have

$$3^k \equiv 1 \pmod{n'}$$

and accordingly

$$3^{s+k} \equiv 3^s \pmod{n}. \tag{5.4}$$

Then, by replacing all a, b, c with $-q^{3^{j-1}\cdot 2m}$ where $j = 1, \dots, s+k$ successively, we obtain a chain of identities as following:

$$
\begin{aligned}
j = 1: &\qquad \mathcal{P}(-q^{2m}; q^n) = 1 + 3\mathcal{S}(-q^{2m}; q^n) - \mathcal{S}(-q^{3\cdot 2m}; q^n);\\
&\qquad \vdots \\
j = s: &\qquad \mathcal{P}(-q^{3^{s-1}\cdot 2m}; q^n) = 1 + 3\mathcal{S}(-q^{3^{s-1}\cdot 2m}; q^n) - \mathcal{S}(-q^{3^s\cdot 2m}; q^n);\\
j = s+1: &\qquad \mathcal{P}(-q^{3^s\cdot 2m}; q^n) = 1 + 3\mathcal{S}(-q^{3^s\cdot 2m}; q^n) - \mathcal{S}(-q^{3^{s+1}\cdot 2m}; q^n);\\
&\qquad \vdots \\
j = s+k: &\qquad \mathcal{P}(-q^{3^{s+k-1}\cdot 2m}; q^n) = 1 + 3\mathcal{S}(-q^{3^{s+k-1}\cdot 2m}; q^n) - \mathcal{S}(-q^{3^{s+k}\cdot 2m}; q^n).
\end{aligned}
$$

In view of (5.4) and Lemma 5.1(2), we are now able to solve $\mathcal{S}(-q^{2m}; q^n)$. Concretely, the weighted summation of the first s identities with weights 3^{s-j} shows

$$\sum_{j=1}^{s} 3^{s-j}\mathcal{P}(-q^{3^{j-1}\cdot 2m}; q^n) = \frac{3^s - 1}{2} + 3^s\mathcal{S}(-q^{2m}; q^n) - \mathcal{S}(-q^{3^s\cdot 2m}; q^n).$$

The weighted summation of the last k identities with weights 3^{s+k-j} shows

$$\sum_{j=s+1}^{s+k} 3^{s+k-j}\mathcal{P}(-q^{3^{j-1}\cdot 2m}; q^n) = \frac{3^k - 1}{2} + 3^k\mathcal{S}(-q^{3^s\cdot 2m}; q^n) - \mathcal{S}(-q^{3^{s+k}\cdot 2m}; q^n).$$

Considering (5.4), we have

$$\mathcal{S}(-q^{3^{s+k}\cdot 2m}; q^n) = \mathcal{S}(-q^{3^s\cdot 2m}; q^n) + \frac{3^s(3^k - 1)\cdot 2m}{n}.$$

Then we are able to calculate $\mathcal{S}(-q^{2m}; q^n)$, and consequently $\mathcal{S}(\pm q^m; q^n)$ by (5.3). We summarize the algorithm as the following theorem.

Theorem 5.3. *Suppose that m and n are positive integers with $n = 3^s \cdot n'$ and $(3, n') = 1$. Denote by k the order of 3 in the cyclic group $\mathbb{Z}_{n'}$. Then, we have*

$$
\begin{aligned}
\mathcal{S}(\pm q^m; q^n) = \frac{2m - n}{2n} &+ \sum_{j=1}^{s+k} \frac{3^{k-j}}{2(3^k - 1)}\mathcal{P}(-q^{3^{j-1}\cdot 2m}; q^n)\\
&- \sum_{j=1}^{s} \frac{3^{-j}}{2(3^k - 1)}\mathcal{P}(-q^{3^{j-1}\cdot 2m}; q^n) + \frac{1}{2}\mathcal{P}(\pm q^m, \pm q^m, -q^{n-2m}; q^n).
\end{aligned}
$$

The length of the chain may be reduced for special m and n. We consider the first l identities in the chain. Their summation with weights 3^{l-j} gives

$$\sum_{j=1}^{l} 3^{l-j}\mathcal{P}(-q^{3^{j-1}\cdot 2m}; q^n) = \frac{3^l - 1}{2} + 3^l\mathcal{S}(-q^{2m}; q^n) - \mathcal{S}(-q^{3^l\cdot 2m}; q^n). \tag{5.5}$$

Lemma 5.1 provides values of $\mathcal{S}$ at special points, which would help to shorten the chain of identities. Suppose that

$$n = 3^{s_1} \cdot 2^{t_1} \cdot n' \quad \text{with} \quad (3, n') = 1 \text{ and } (2, n') = 1,$$
$$m = 3^{s_2} \cdot 2^{t_2} \cdot m' \quad \text{with} \quad (3, m') = 1 \text{ and } (2, m') = 1.$$

We consider two special cases.

Case I: $n' \mid m'$ and $t_1 \leqslant t_2 + 1$.

We take l by setting

$$l = \begin{cases} 0, & \text{when } s_2 \geqslant s_1, \\ s_1 - s_2, & \text{when } s_2 < s_1. \end{cases} \tag{5.6}$$

In this case, l is the least nonnegative integer such that

$$3^l \cdot 2m \equiv 0 \pmod{n}.$$

By Lemma 5.1, we have

$$\mathcal{S}(-q^{3^l \cdot 2m}; q^n) = \frac{3^l \cdot 2m}{n} + \mathcal{S}(-1; q^n) = \frac{3^l \cdot 2m}{n} - \frac{1}{2}. \tag{5.7}$$

Considering (5.5), we are able to obtain $\mathcal{S}(-q^{2m}; q^n)$, and consequently $\mathcal{S}(q^m; q^n)$.

Case II: $n' \mid m'$ and $t_1 = t_2 + 2$.

We take l as in (5.6). Now l is the least nonnegative integer such that

$$3^l \cdot 2m \equiv n/2 \pmod{n}.$$

The discussion is similar to that of Case I. A tiny difference lies in (5.7), where we now have

$$\mathcal{S}(-q^{3^l \cdot 2m}; q^n) = \frac{3^l \cdot 2m - n/2}{n} + \mathcal{S}(q^{\frac{n}{2}}; q^n) = \frac{3^l \cdot 2m}{n} - \frac{1}{2}.$$

We summarize these two cases as the following corollary.

Corollary 5.4. Let m and n be positive integers. Suppose that there exists a least nonnegative integer l such that $3^l \cdot 4m \equiv 0 \pmod{n}$. Then, we have

$$\mathcal{S}(\pm q^m; q^n) = \frac{2m-n}{2n} + \sum_{j=1}^{l} \frac{3^{-j}}{2} \mathcal{P}(-q^{3^{j-1} \cdot 2m}; q^n) + \frac{1}{2} \mathcal{P}(\pm q^m, \pm q^m, -q^{n-2m}; q^n).$$

For example, when $n = 3$, we have $l = 1$ in Corollary 5.4. We give the explicit expansion for $\mathcal{S}(\pm q; q^3)$ in terms of infinite products. This completes the proof of (3.6) and (4.2).

Corollary 5.5. We have

$$\mathcal{S}(q; q^3) = \frac{J_{3,6}^2 J_6^3}{4 J_{1,6}^2 J_2} - \frac{J_{1,6}^6 J_2^3 J_{3,6}^2}{12 J_6^9} - \frac{1}{6},$$

$$\mathcal{S}(-q;q^3)=\frac{J_{1,6}^6J_2^3J_{3,6}^2}{6J_6^9}-\frac{1}{6}.$$

6. Mock Theta Functions

Recall that the Appell-Lerch sum is defined as

$$m(x,q,z):=\frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^rq^{\binom{r}{2}}z^r}{1-q^{r-1}xz},$$

where $x,z\in\mathbb{C}^*$ with neither z nor xz an integral power of q. In [9], it is pointed out that the third order mock theta functions $\omega(q)$ and $\rho(q)$ can be expressed in term of $m(x,q,z)$ as follows,

$$\omega(q)=-2q^{-1}m(q,q^6,q^2)+\frac{J_6^3}{J_2J_{3,6}},$$
$$\rho(q)=q^{-1}m(q,q^6,-q). \tag{6.1}$$

A generalized Lambert series with simple poles is essentially an Appell-Lerch sums, so they play a the key role in relating rank differences with mock theta functions. This section is devoted to proving the relations between the rank differences of overpartitions and mock theta functions, as stated in Theorem 1.5.

First we recall the universal mock theta function $g_2(x,q)$ defined by Gordon and McIntosh [8]

$$g_2(x,q):=\frac{1}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(n+1)}}{1-xq^n}.$$

Hickerson and Mortenson [9] showed that $g_2(x,q)$ and $m(x,q,z)$ have the following relation,

$$g_2(x,q)=-x^{-1}m(x^{-2}q,q^2,x). \tag{6.2}$$

We are now in a position to give a proof of Theorem 1.5.

Proof of Theorem 1.5. From Theorem 1.3, we have

$$\overline{r}_0(2)+\overline{r}_3(2)=\frac{4}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}}+\frac{4J_6^{12}}{3J_{1,6}^6J_2^4J_{3,6}^3}-\frac{16J_6^3}{3J_2J_{3,6}}. \tag{6.3}$$

Replacing q by q^3 in (6.2) and setting $x=-q$, we have

$$g_2(-q,q^3)=q^{-1}m(q,q^6,-q),$$

and by (6.1), we deduce that

$$\rho(q)=g_2(-q,q^3).$$

Together with the identity in [18, p.63]

$$\omega(q)+2\rho(q)=\frac{3J_6^3}{J_2J_{3,6}},$$

we find that (6.3) can be transformed as follows:

$$\begin{aligned}\overline{r}_0(2)+\overline{r}_3(2)&=4\rho(q)-\frac{16}{9}(\omega(q)+2\rho(q))+\frac{4J_6^{12}}{3J_{1,6}^6J_2^4J_{3,6}^3}\\&=\frac{4}{9}\rho(q)-\frac{16}{9}\omega(q)+\frac{4J_6^{12}}{3J_{1,6}^6J_2^4J_{3,6}^3}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\overline{r}_1(2)-\overline{r}_3(2)&=-\frac{4}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}}+\frac{2J_6^3}{J_2J_{3,6}}\\&=2\omega(q),\end{aligned}$$

and

$$\begin{aligned}\overline{r}_2(2)+\overline{r}_3(2)&=\frac{2}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}}-\frac{10J_6^3}{3J_2J_{3,6}}+\frac{4J_6^{12}}{3J_{1,6}^6J_2^4J_{3,6}^3}\\&=2\rho(q)-\frac{10}{9}(\omega(q)+2\rho(q))+\frac{4J_6^{12}}{3J_{1,6}^6J_2^4J_{3,6}^3}\\&=-\frac{2}{9}\rho(q)-\frac{10}{9}\omega(q)+\frac{4J_6^{12}}{3J_{1,6}^6J_2^4J_{3,6}^3}.\end{aligned}$$

Thus we complete the proof of Theorem 1.5. □

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