Well-posedness and Regularity for Distribution Dependent SPDEs with Singular Drifts *

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Abstract

In this paper, the distribution dependent stochastic differential equation in a separable Hilbert space with a Dini continuous drift is investigated. The existence and uniqueness of weak and strong solutions are obtained. Moreover, some regularity results as well as gradient estimates and Wang's log-Harnack inequality are derived for the associated semigroup. In addition, Wang's Harnack inequality with power and shift Harnack inequality are also proved when the noise is additive. All of the results extend the ones in the distribution independent situation.

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1 Introduction

The distribution dependent stochastic differential equations (SDEs for short), also named McKean-Vlasov SDEs due to pioneering work [18, 25], can be described as the weak limit of N-particle interaction systems formed by N equations forced by independent Brownian motions. The subject has been extensively explored and it is still under investigation (see

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[2, 3, 9, 10, 12, 13, 14, 18, 25, 32] and references within). When the drifts are singular, there are a great number of results on the well-posedness , for instance, [4, 5, 6, 8, 15, 19, 23] and references therein. In [4, 5, 6], the existence of weak solutions in the additive noise case is shown by Girsanov's transform together with Schauder's fixed point theorem. However, this method does not work when the diffusion coefficients depend on distribution. The results in [8] are extended by the first author and his coauthor in [15], where the diffusion term is allowed to be distribution dependent. The pathwise uniqueness is proved by utilizing Zvonkin's transform [34] in [15, 19, 23], see references therein for distribution independent SDEs. The main idea of Zvonkin's transform is to remove the singular drifts, and it mainly depends on the regularity of a backward Kolmogrov equation with singular coefficients. In the infinite dimensional and distribution independent case, the author in [31] investigates the existence and uniqueness of solutions and log-Harnack inequality for semi-linear stochastic partial differential equations (SPDEs) with Dini continuous drifts by Zvonkin's transform. For distribution dependent semi-linear SPDEs, when the drift term satisfies the Lipschitz condition, the existence and uniqueness of the solution are obtained in [1]. Very recently, in [13] Heinemann studied the distribution dependent stochastic differential delay equations (DDSDDEs) in the variational framework. If the coefficients fulfill certain monotonicity assumptions, the DDSDDEs have unique strong solutions.

The present paper attempts to extend the results in [31] to the distribution dependent case. Meanwhile, Wang's Harnack inequality and shift Harnack inequality are also considered in special situations. In order to obtain the existence of weak solutions under a weak condition, the compactness method [11, chapter 8] as well as Skorohod representation and martingale representation theorem will be employed. It is crucial to construct a family of compact operators to deal with the stochastic convolution. Moreover, Zvonkin's transform combined with fixed point theorem can be used to investigate the strong well-posedness.

Using the method of coupling by change of measure, Wang's Harnack inequality, log-Harnack inequality and shift-Harnack inequality, introduced by F.-Y Wang in [26], [24] and [28] respectively, have been established and applied to various SDEs and SPDEs driven by Gaussian noises, see [17, 22, 24, 28, 29, 30, 33] and references therein. Different from the finite dimensional case [15, Theorem 2.5], due to the existence of a non-Lipschitzian term Au after Zvonkin's transform in Lemma 3.3 below, the coupling by change of measure, for instance in [29, Chapter 3], does not work even in the distribution independent case with multiplicative noise. To overcome this difficulty, [31] adopted the gradient-gradient estimate for Markovian semigroups to derive the log-Harnack inequality according to [29, Chapter 1]. However, this method is unavailable in the distribution dependent case since the solution is not a Markov process. Fortunately, we may employ the existed log-Harnack inequality in [31] and Girsanov's transform to obtain the desired log-Harnack inequality. The main idea is to derive the estimate of the relative entropy between two solutions with different initial distributions. To this end, we rewrite one of the two solutions by Girsanov's transform to be a new one with the same coefficients with another one, and then the log-Harnack inequality in [31] can be used. It seems that this method is an effective way to deal with the distribution dependent SDEs and SPDEs. As for the Harnack inequality and shift Harnack inequality, we adopt coupling by change of measure instead of Zvonkin's transform in the additive noise case.

Let $(\mathbb{H}, \langle,\rangle, |\cdot|)$ and $(\bar{\mathbb{H}}, \langle,\rangle_{\bar{\mathbb{H}}}, |\cdot|_{\bar{\mathbb{H}}})$ be two separable Hilbert spaces, and $W = (W_t)_{t\geq 0}$ be a cylindrical Brownian motion on $\bar{\mathbb{H}}$ with respect to a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. More precisely, $W_t = \sum_{n=1}^{\infty} B_t^n \bar{e}_n$ for a sequence of independent one dimensional Brownian motions $\{B_t^n\}_{n\geq 1}$ with respect to $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ and an orthonormal basis $\{\bar{e}_n\}_{n\geq 1}$ on $\bar{\mathbb{H}}$.

Let $\mathscr{P}$ be the collection of all probability measures on $\mathbb{H}$ equipped with the weak topology. For $\mu \in \mathscr{P}$, if $\mu(|\cdot|^p) := \int_{\mathbb{H}} |x|^p \mu(\mathrm{d}x) < \infty$ for some $p \geq 1$, we write $\mu \in \mathscr{P}_p$. For $p \geq 1$ and $\mu, \bar{\mu} \in \mathscr{P}_p$, the $\mathbb{W}_p$-Wasserstein distance between μ and $\bar{\mu}$ is defined by

$$\mathbb{W}_p(\mu, \bar{\mu}) = \inf_{\pi \in \mathcal{C}(\mu,\bar{\mu})} \Big(\int_{\mathbb{H}\times\mathbb{H}} |x-y|^p \pi(\mathrm{d}x, \mathrm{d}y) \Big)^{\frac{1}{p}},$$

where $\mathcal{C}(\mu, \bar{\mu})$ stands for the set of all couplings of μ and $\bar{\mu}$. For a random variable ξ, its law is written by $\mathscr{L}_\xi$, and write $\mathscr{L}_\xi|_{\mathbb{P}}$ as the distribution of ξ under $\mathbb{P}$.

Consider the following semi-linear distribution dependent SPDEs on $\mathbb{H}$:

E1 (1.1)
$$\mathrm{d}X_t = \{AX_t + b_t(X_t, \mathscr{L}_{X_t})\}\mathrm{d}t + Q_t(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t,$$

where $(A, \mathscr{D}(A))$ is a negative definite self-adjoint operator on $\mathbb{H}$, $b : [0,\infty) \times \mathbb{H} \times \mathscr{P} \to \mathbb{H}$ is measurable and locally bounded (i.e. bounded on bounded sets), and $Q : [0,\infty) \times \mathbb{H} \times \mathscr{P} \to \mathscr{L}(\bar{\mathbb{H}}; \mathbb{H})$ is measurable, where $\mathscr{L}(\bar{\mathbb{H}}; \mathbb{H})$ is the space of bounded linear operators from $\bar{\mathbb{H}}$ to $\mathbb{H}$. Let $\|\cdot\|$ and $\|\cdot\|_{\mathrm{HS}}$ denote the operator norm and the Hilbert-Schmidt norm respectively.

To characterize the singularity of b with respect to the second variable, set

$$\mathscr{D} = \Big\{ \phi : [0,+\infty) \to [0,+\infty) | \phi^2 \text{ is concave and } \phi \text{ is increasing with } \int_0^1 \frac{\phi(s)}{s}\mathrm{d}s < \infty \Big\}.$$

Throughout this paper, we assume that there exists an increasing function $K : (0,\infty) \to (0,\infty)$ such that A, b and Q satisfy the following conditions.

(a1) For some $\varepsilon \in (0,1)$, $(-A)^{\varepsilon-1}$ is of trace class. That is, $\sum_{n=1}^{\infty} \lambda_n^{\varepsilon-1} < \infty$ for $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ being all eigenvalues of $-A$ counting multiplicities with $-Ae_i = \lambda_i e_i, i \geq 1$ for an orthonormal basis $\{e_i\}_{i\geq 1}$ of $\mathbb{H}$.

(a2) The operator $Q : [0,\infty) \times \mathbb{H} \times \mathscr{P} \to \mathscr{L}(\bar{\mathbb{H}}; \mathbb{H}))$ is continuous and for each $t \geq 0$ and $\mu \in \mathscr{P}$, and $Q_t(\cdot, \mu)$ is in $C^2(\mathbb{H}; \mathscr{L}(\bar{\mathbb{H}}; \mathbb{H}))$ such that

$$\sup_{(t,x,\mu)\in[0,T]\times\mathbb{H}\times\mathscr{P}} \big(\|Q_t(x,\mu)\| + \|\nabla Q_t(x,\mu)\| + \|\nabla^2 Q_t(x,\mu)\| \big) \leq K(T), \quad T > 0,$$

here ∇ and ∇^2 stand for the first and second ordered gradient operator with respect to the space component respectively. Meanwhile, $(Q_t Q_t^*)(x,\mu)$ is invertible for each $(t,x,\mu) \in [0,\infty) \times \mathbb{H} \times \mathscr{P}$ with

$$\sup_{(t,x,\mu)\in[0,T]\times\mathbb{H}\times\mathscr{P}} \|(Q_t Q_t^*)(x,\mu)^{-1}\| \leq K(T), \quad T > 0.$$

Moreover, for any $x \in \mathbb{H}$, $t \geq 0$ and $\mu \in \mathscr{P}_2$, it holds

1.3 (1.2)
$$\lim_{n\to\infty} \|Q_t(x,\mu) - Q_t(\pi_n x, \mu)\|_{\mathrm{HS}}^2 = 0,$$

where π_n is the orthonormal projection from $\mathbb{H}$ to $\mathrm{span}\{e_1, e_2, \cdots, e_n\}$. In addition, for any $T > 0$, it holds

red (1.3)
$$\sup_{(t,x)\in[0,T]\times\mathbb{H}} \|Q_t(x,\mu) - Q_t(x,\nu)\|_{\mathrm{HS}}^2 \leq K(T)\mathbb{W}_2(\mu,\nu)^2, \quad \mu,\nu \in \mathscr{P}_2.$$

(a3) For any $t \in [0,T]$, b_t is continuous in $\mathbb{H} \times \mathscr{P}$. $\sup_{(x,\mu)\in\mathbb{H}\times\mathscr{P}} |b_t(x,\mu)|$ is locally bounded in t, and there exists $\phi \in \mathscr{D}$ such that

1.2 (1.4)
$$|b_t(x,\mu) - b_t(y,\nu)| \leq \phi(|x-y|) + K(t)\mathbb{W}_2(\mu,\nu), \quad t \geq 0, x, y \in \mathbb{H}, \mu, \nu \in \mathscr{P}_2.$$

Din **Remark 1.1.** *Recall that a non-negative increasing function ϕ on $[0,\infty)$ is called a Dini function if $\int_0^1 \frac{\phi(s)}{s}\mathrm{d}s < \infty$. So, (1.4) can be regarded as a Dini continuity condition in the space variable on the drift. Typical examples of $\phi \in \mathscr{D}$ include $\phi(s) = s^\alpha$ for $\alpha \in (0, \frac{1}{2})$ and $\phi(s) := \frac{K}{\log^{1+\delta}(c+s^{-1})}$ for constants $K, \delta > 0$ and c large enough such that ϕ^2 is concave.*

Definition 1.1. A continuous $\mathscr{F}_t$-adapted process $\{X_t\}_{t\geq 0}$ is called a mild solution to Equ. (1.1), if $\mathbb{P}$-a.s

Mil (1.5)
$$X_t = \mathrm{e}^{At}X_0 + \int_0^t \mathrm{e}^{A(t-s)} b_s(X_s, \mathscr{L}_{X_s})\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)} Q_s(X_s, \mathscr{L}_{X_s})\mathrm{d}W_s, \quad t \geq 0.$$

Moreover, if $\mathbb{E}|X_t|^2 < \infty$ for any $t \geq 0$, then the solution is said in $\mathscr{P}_2$. Equ. (1.1) is called strongly well-posed in $\mathscr{P}_2$, if for any $\mathscr{F}_0$-measurable random variable X_0 with $\mathscr{L}_{X_0} \in \mathscr{P}_2$, there exists a unique mild solution in $\mathscr{P}_2$.

(1) A couple $(\tilde{X}_t, \tilde{W}_t)_{t\geq 0}$ is called a weak solution to Equ. (1.1), if $\tilde{W}$ is a cylindrical Brownian motion with respect to a complete filtered probability space $(\tilde{\Omega}, \{\tilde{\mathscr{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$, and (1.5) holds for $(\tilde{X}_t, \tilde{W}_t)_{t\geq 0}$ in place of $(X_t, W_t)_{t\geq 0}$. Moreover, if $\mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}} \in \mathscr{P}_2$, the weak solution is called in $\mathscr{P}_2$.

(2) Equ.(1.1) is said to have weak uniqueness in $\mathscr{P}_2$, if any two weak solutions in $\mathscr{P}_2$ of (1.1) from common initial distribution are equal in law. Furthermore, we call weak well-posedness in $\mathscr{P}_2$ for Equ.(1.1) holds, if it has a weak solution from any initial distribution and has weak uniqueness in $\mathscr{P}_2$.

Some notations are listed which are necessary to state subsequent results and their proofs.

- Let $L^2(\Omega \to \mathbb{H}; \mathscr{F}_0)$ be the class of all random variables ξ which are $\mathscr{F}_0$-measurable and have finite second moment. Denote by $C([0,T];\mathbb{H})$ and $C([0,T];\mathscr{P}_2)$ the spaces consisted of all continuous functions from $[0,T]$ to $\mathbb{H}$ and $\mathscr{P}_2$ respectively. Let $\mathscr{B}_b(\mathbb{H})$ be the class of all bounded measurable functions on $\mathbb{H}$ and $L^p([0,T];\mathbb{H})$ be the space of the $\mathbb{H}$-valued functions defined on $[0,T]$ with finite p-th moment.

- For two Banach spaces E_1 and E_2 and $i = 1, 2$, $C^i(E_1; E_2)(C_b^i(E_1; E_2))$ denotes the collection of all functions from E_1 to E_2 with continuous (and bounded) Fréchet's derivatives up to order i.

- For a real-valued or $\mathbb{H}$-valued function f defined on $[0, T] \times \mathbb{H}$, let

$$\|f\|_{T,\infty} = \sup_{t\in[0,T],x\in\mathbb{H}} |f(t, x)|.$$

  Similarly, if f is an operator-valued map defined on $[0, T] \times \mathbb{H}$, let

$$\|f\|_{T,\infty} = \sup_{t\in[0,T],x\in\mathbb{H}} \|f(t, x)\|.$$

- The letter C with or without indices will denote an unimportant constant, whose values may change from one appearance to another.

This manuscript is organized as follows. In Section 2, we state the main results, including existence and uniqueness of solutions, Wang's Harnack inequality and shift Harnack inequality. Section 3 devotes to proving the existence and uniqueness of solutions through the compact method and Zvonkin's transform. Using the coupling by change of measure, the proofs of Wang's Harnack inequality and shift Harnack inequality will be given in Section 4.

2 Main results

The first result is concerning to the weak existence under a more general frame, where the coefficients are only assumed to be bounded and continuous. From now on, let T stand for any fixed time.

ws **Theorem 2.1.** *Assume* **(a1)**. *If* $\sup_{(x,\mu)\in\mathbb{H}\times\mathscr{P}}(|b_t(x, \mu)| + \|Q_t(x, \mu)\|)$ *is locally bounded with respect to* t *and* b_t, Q_t *are continuous in* $\mathbb{H} \times \mathscr{P}$ *for each* $t \geq 0$. *Then for any fixed* $T > 0$, *and* $\mu_0 \in \mathscr{P}$, *Equ.* (1.1) *has a weak solution up to time* T *with initial distribution* μ_0.

The next result ensures the well-posedness of (1.1) with the continuity of solutions in initial values.

T2.1 **Theorem 2.2.** *Assume* **(a1)**-**(a3)**. *Then the following assertions hold.*

(1) (1.1) *has weak well-posedness in* $\mathscr{P}_2$. *Let* $P_t^*\mu_0$ *be the unique distribution of the weak solution at time* $t \geq 0$ *with initial distribution* μ_0. *There exists a constant* $C(T) > 0$ *such that*

Pta (2.1)
$$\int_0^T \mathbb{W}_2(P_t^*\mu_0, P_t^*\nu_0)^2 \mathrm{d}t \leq C(T)\mathbb{W}_2(\mu_0, \nu_0)^2, \quad \mu_0, \nu_0 \in \mathscr{P}_2. \tag{2.1}$$

(2) *The strong well-posedness in $\mathscr{P}_2$ holds for* (1.1). *Moreover, there exists an increasing function $C : [0,\infty) \to [0,\infty)$ such that for any two solutions X_t and Y_t to* (1.1), *it holds*

X-Y
$$\int_0^T \mathbb{E}|X_s - Y_s|^2 \mathrm{d}s \le C(T)\mathbb{E}|X_0 - Y_0|^2, \quad T \ge 0. \tag{2.2}$$

For any $\mu \in \mathscr{P}_2$ and any $f \in \mathscr{B}_b(\mathbb{H})$, define

$$P_t f(\mu) = (P_t^*\mu)(f) := \int_{\mathbb{H}} f \mathrm{d}P_t^*\mu, \quad t \ge 0.$$

For a measurable space $(E, \mathcal{E})$, let $\mathscr{P}(E)$ denote the family of all probability measures on $(E, \mathcal{E})$. For $\mu, \nu \in \mathscr{P}(E)$, the relative entropy $\mathrm{Ent}(\nu|\mu)$ is defined by

$$\mathrm{Ent}(\nu|\mu) := \begin{cases} \int (\log \frac{\mathrm{d}\nu}{\mathrm{d}\mu})\,\mathrm{d}\nu, & \text{if } \nu \text{ is absolutely continuous with respect to } \mu, \\ \infty, & \text{otherwise;} \end{cases}$$

and the total variational distance $\|\mu - \nu\|_{\mathrm{TV}}$ is defined by

$$\|\mu - \nu\|_{\mathrm{TV}} := \sup_{A \in \mathcal{E}} |\mu(A) - \nu(A)|.$$

By Pinsker's inequality (see [21]),

ETX
$$\|\mu - \nu\|_{\mathrm{TV}}^2 \le \frac{1}{2}\mathrm{Ent}(\nu|\mu), \quad \mu, \nu \in \mathscr{P}(E). \tag{2.3}$$

Next, we consider Wang's log-Harnack inequality and Harnack inequality for the nonlinear semigroup P_t^*.

THar **Theorem 2.3.** *Assume* **(a1)-(a3)** *and that $Q_t(x,\mu)$ does not depend on μ. Then the following assertions hold.*

(1) *There exists an increasing function $C : [0,\infty) \to (0,\infty)$ such that for any $T > 0$, the log-Harnack inequality*

$$P_T \log f(\nu_0) \le \log P_T f(\mu_0) + \frac{C(T)}{T \wedge 1}\mathbb{W}_2(\mu_0, \nu_0)^2, \quad \mu_0, \nu_0 \in \mathscr{P}_2$$

holds for strictly positive function $f \in \mathscr{B}_b(\mathbb{H})$. Consequently, we have

pke
$$2\|P_T^*\mu_0 - P_T^*\nu_0\|_{\mathrm{TV}}^2 \le \mathrm{Ent}(P_T^*\mu_0|P_T^*\nu_0) \le \frac{C(T)}{T \wedge 1}\mathbb{W}_2(\mu_0, \nu_0)^2. \tag{2.4}$$

(2) *If $Q_t(x,\mu)$ does not depend on (x,μ), the Harnack inequality with power $p > 1$ holds for non-negative $f \in \mathscr{B}_b(\mathbb{H})$ and any $T > 0$, i.e.*

2.6
$$(P_T f(\mu_0))^p \le P_T f^p(\nu_0)\left(\mathbb{E}\exp\left\{\frac{p}{2(p-1)^2}\Phi(T)\right\}\right)^{p-1}, \quad \mu_0, \nu_0 \in \mathscr{P}_2, \tag{2.5}$$

where

$$\Phi(T)=K(T)\left(4T\phi^2\left(|X_0-Y_0|\right)+C(T)\mathbb{W}_2(\mu_0,\nu_0)^2+2\frac{|X_0-Y_0|^2}{T}\right), \tag{2.6}$$

with $\mathscr{L}_{X_0}=\mu_0$ *and* $\mathscr{L}_{Y_0}=\nu_0$. *Consequently,* $P_T^*\mu_0$ *is equivalent to* $P_T^*\nu_0$ *and it holds*

ap1
$$P_T\left\{\left(\frac{\mathrm{d}P_T^*\mu_0}{\mathrm{d}P_T^*\nu_0}\right)^{\frac{1}{p-1}}\right\}(\mu_0)\le\mathbb{E}\exp\left\{\frac{p}{2(p-1)^2}\Phi(T)\right\}. \tag{2.7}$$

The next assertion characterizes the shift Harnack inequality for P_t^*.

TsHar **Theorem 2.4.** *Assume* **(a1)-(a3)**. *If* $Q_t(x,\mu)$ *does not depend on* x, *then for any* $T>0$, $\mu_0\in\mathscr{P}_2$, $y\in\mathbb{H}$ *and non-negative* $f\in\mathscr{B}_b(\mathbb{H})$, *we have*

$$P_T\log f(\mu_0)\le\log(P_Tf(\mathrm{e}^{AT}y+\cdot))(\mu_0)+K(T)\left(T\phi^2(|y|)+\frac{|y|^2}{T}\right),\quad f\ge 1,$$

and

$$(P_Tf(\mu_0))^p\le P_T(f^p(\mathrm{e}^{AT}y+\cdot))(\mu_0)\exp\left[\frac{p}{(p-1)}K(T)\left(T\phi^2(|y|)+\frac{|y|^2}{T}\right)\right].$$

As an immediate result of Theorem 2.4 from [29, Theorem 1.4.4], we have

densitys **Corollary 2.5.** *Under the conditions of Theorem 2.4, for each* $y\in\mathbb{H}$ *and* $\mu_0\in\mathscr{P}_2$, $P_T^*\mu_0$ *is equivalent to* $(P_T^*\mu_0)(\cdot-\mathrm{e}^{AT}y)$. *Moreover, for any* $p>1$, *it holds*

$$P_T\left\{\left(\frac{\mathrm{d}P_T^*\mu_0}{\mathrm{d}[(P_T^*\mu_0)(\cdot-\mathrm{e}^{AT}y)]}\right)^{\frac{1}{p}}\right\}(\mu_0)\le\exp\left[\frac{1}{(p-1)}K(T)\left(T\phi^2(|y|)+\frac{|y|^2}{T}\right)\right].$$

3 Existence and Uniqueness

In this section, we investigate the existence and uniqueness of solutions to Equ.(1.1). Firstly, we will use the compactness method in the proof of [11, Theorem 8.1] to complete the proof of Theorem 2.1. Next, the strong uniqueness will be shown by the Zvonkin's transform which depends on distribution under **(a1)-(a3)**. Finally, the proof of Theorem 2.2 can be finished by the modified Yamada-Watanabe principle [16, Lemma 2.1].

3.1 Proof of Theorem 2.1

For the sake of reader's convenience, let us recall a result on the compact operators introduced in [11, Proposition 8.4].

CO **Lemma 3.1.** *Let $\{S(t)\}_{t>0}$ be a family of compact operators on $\mathbb{H}$. Then for any p, α satisfying $0 < \frac{1}{p} < \alpha \leq 1$, the operator G_α defined by*

$$G_\alpha f(t) = \int_0^t (t-s)^{\alpha-1} S(t-s) f(s) \mathrm{d}s, \quad t \in [0,T], \tag{3.1}$$

is compact from $L^p([0,T], \mathbb{H})$ into $C([0,T], \mathbb{H})$.

Now, we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1. The proof is divided into three steps.

Step 1. For each $n \geq 1$, let $\eta_n(s) = \lfloor \frac{s}{T/n} \rfloor \frac{T}{n}$, where $\lfloor \cdot \rfloor$ stands for the integer part. Let X_0 be an $\mathscr{F}_0$-measurable random variable with $\mathscr{L}_{X_0} = \mu_0$. For $t \in [0,T]$, define

fia'
$$X_t^n = \mathrm{e}^{At} X_0 + \int_0^t \mathrm{e}^{A(t-s)} b_s(X^n_{\eta_n(s)}, \mathscr{L}_{X^n_{\eta_n(s)}}) \mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)} Q_s(X^n_{\eta_n(s)}, \mathscr{L}_{X^n_{\eta_n(s)}}) \mathrm{d}W_s. \tag{3.2}$$

Due to **(a1)**, we have

RHS
$$\int_0^t r^{-\varepsilon} \|\mathrm{e}^{Ar}\|^2_{\mathrm{HS}} \mathrm{d}r = \sum_{i=1}^{+\infty} \int_0^t r^{-\varepsilon} \mathrm{e}^{-2\lambda_i r} \mathrm{d}r \leq 2^{\varepsilon-1} \Gamma(1-\varepsilon) \sum_{i=1}^{+\infty} \lambda_i^{\varepsilon-1} < \infty, \tag{3.3}$$

where Γ stands for Gamma-function. This, together with the condition that b and Q are bounded on $[0,T]$, implies that X_t^n in (3.2) is well-defined. Moreover, X^n has a continuous version (see [11, Theorem 5.9]).

Step 2. In this step, we aim to prove that $\{\mathscr{L}_{X^n}\}_{n \geq 1}$ is tight in the space of probability measures on $C([0,T]; \mathbb{H})$. Let G_α be as in (3.1) with e^{At} in place of $S(t)$. By (3.1), (3.3) and stochastic Fubini theorem, we have

$$\int_0^t \mathrm{e}^{A(t-s)} Q_s(X^n_{\eta_n(s)}, \mathscr{L}_{X^n_{\eta_n(s)}}) \mathrm{d}W_s = \frac{\sin \frac{\varepsilon\pi}{2}}{\pi} G_{\frac{\varepsilon}{2}} Y_n(t), \quad t \in [0,T],$$

where

$$Y_n(t) = \int_0^t (t-s)^{-\frac{\varepsilon}{2}} \mathrm{e}^{A(t-s)} Q_s(X^n_{\eta_n(s)}, \mathscr{L}_{X^n_{\eta_n(s)}}) \mathrm{d}W_s.$$

Define $\tilde{G} : \mathbb{H} \to C([0,T]; \mathbb{H})$ as

$$[\tilde{G}(x)](t) = \mathrm{e}^{At} x, \quad x \in \mathbb{H}, t \in [0,T].$$

It is not difficult to see that $\tilde{G}$ is a compact operator. Then X_t^n can be reformulated as

$$X_t^n = [\tilde{G}(X_0)](t) + G_1 \left(b_\cdot(X^n_{\eta_n(\cdot)}, \mathscr{L}_{X^n_{\eta_n(\cdot)}}) \right)(t) + \frac{\sin \frac{\varepsilon\pi}{2}}{\pi} G_{\frac{\varepsilon}{2}} Y_n(t), \quad t \in [0,T]. \tag{3.4}$$

Note that for $p > \frac{2}{\varepsilon}$ and each $n \geq 1$, it is clear that

$$\mathbb{E} \int_0^T |Y_n(t)|^p \mathrm{d}t \leq C_p \int_0^T \mathbb{E} \left(\int_0^t (t-s)^{-\varepsilon} \|\mathrm{e}^{A(t-s)} Q_s(X^n_{\eta_n(s)}, \mathscr{L}_{X^n_{\eta_n(s)}})\|^2_{\mathrm{HS}} \mathrm{d}s \right)^{\frac{p}{2}} \mathrm{d}t$$

$$\leq C_pT\sup_{t\in[0,T]}\sup_{(x,\mu)\in\mathbb{H}\times\mathscr{P}}\|Q_t(x,\mu)\|^p\left(\int_0^T r^{-\varepsilon}\|\mathrm{e}^{Ar}\|_{\mathrm{HS}}^2\mathrm{d}r\right)^{\frac{p}{2}} =: c_p<\infty,\quad \forall n\geq 1,$$

where C_p is a constant only depending on p, T and its value can change from line to line. Hence, we obtain

$$\mathbb{P}(|X_0|>r)\to 0,\quad r\to+\infty,$$

$$\mathbb{P}\left(\int_0^T|Y_n(s)|^p\mathrm{d}s>r^p\right)\leq\frac{1}{r^p}\mathbb{E}\int_0^T|Y_n(s)|^p\mathrm{d}s\leq\frac{c_p}{r^p}\to 0,\quad r\to+\infty,$$

and

$$\mathbb{P}\left(\int_0^T|b_s(X^n_{\eta_n(s)},\mathscr{L}_{X^n_{\eta_n(s)}})|^p\mathrm{d}s>r^p\right)\leq\frac{C_p\sup\limits_{t\in[0,T]}\sup\limits_{(x,\mu)\in\mathbb{H}\times\mathscr{P}}|b_t(x,\mu)|^pT}{r^p}\to 0,\quad r\to+\infty.$$

Therefore, for each $\delta>0$ small enough, there exists $r_\delta>0$ such that

$$\mathbb{P}\left(|X_0|\leq r_\delta,\ \left(\int_0^T|Y_n(s)|^p\mathrm{d}s\right)^{\frac{1}{p}}\leq r_\delta,\ \left(\int_0^T|b_s(X^n_{\eta_n(s)},\mathscr{L}_{X^n_{\eta_n(s)}})|^p\mathrm{d}s\right)^{\frac{1}{p}}\leq r_\delta\right)\geq 1-\delta.$$

This leads to $\mathscr{L}_{X^n}(K_\delta)\geq 1-\delta, n\geq 1$, where

$$K_\delta:=\left\{\tilde{G}x+G_1f+\frac{\sin\frac{\varepsilon\pi}{2}}{\pi}G_{\frac{\varepsilon}{2}}g:|x|\leq r_\delta,\left(\int_0^T|f(s)|^p\mathrm{d}s\right)^{\frac{1}{p}}\leq r_\delta,\left(\int_0^T|g(s)|^p\mathrm{d}s\right)^{\frac{1}{p}}\leq r_\delta\right\}$$

is compact in $C([0,T];\mathbb{H})$ by Lemma 3.1. So $\{\mathscr{L}_{X^n}\}_{n\geq 1}$ is tight.

Step 3. Due to the tightness of $\{\mathscr{L}_{X^n}\}_{n\geq 1}$, there exists a weakly convergent subsequence still denoted by $\{\mathscr{L}_{X^n}\}_{n\geq 1}$. By the Skorohod representation theorem [11, Theorem 2.4], there exists a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ and $C([0,T];\mathbb{H})$-valued stochastic processes $\tilde{X}^n$, $\tilde{X}$ such that $\mathscr{L}_{X^n}|_{\mathbb{P}}=\mathscr{L}_{\tilde{X}^n}|_{\tilde{\mathbb{P}}}$, and $\tilde{\mathbb{P}}$-a.s. $\tilde{X}^n$ converges in $C([0,T];\mathbb{H})$ to $\tilde{X}$ as $n\to\infty$, which implies that for any $t\in[0,T]$, $\mathscr{L}_{\tilde{X}^n_t}|_{\tilde{\mathbb{P}}}$ weakly converges to $\mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}}$. On the other hand, it follows from (3.2) that

fia'' (3.5)
$$\begin{aligned}(-A)^{-1}X^n_t=&\mathrm{e}^{At}(-A)^{-1}X_0+\int_0^t\mathrm{e}^{A(t-s)}(-A)^{-1}b_s(X^n_{\eta_n(s)},\mathscr{L}_{X^n_{\eta_n(s)}})\mathrm{d}s\\&+\int_0^t\mathrm{e}^{A(t-s)}(-A)^{-1}Q_s(X^n_{\eta_n(s)},\mathscr{L}_{X^n_{\eta_n(s)}})\mathrm{d}W_s,\quad t\in[0,T].\end{aligned}$$

In view of [7, Lemma 3.5], (3.5) implies

fi' (3.6)
$$\begin{aligned}(-A)^{-1}X^n_t=&(-A)^{-1}X_0+\int_0^t(-X^n_s)\mathrm{d}s+\int_0^t(-A)^{-1}b_s(X^n_{\eta_n(s)},\mathscr{L}_{X^n_{\eta_n(s)}})\mathrm{d}s\\&+\int_0^t(-A)^{-1}Q_s(X^n_{\eta_n(s)},\mathscr{L}_{X^n_{\eta_n(s)}})\mathrm{d}W_s.\end{aligned}$$

Let

$$N_t^n := (-A)^{-1}X_t^n - (-A)^{-1}X_0 + \int_0^t X_s^n \mathrm{d}s - \int_0^t (-A)^{-1} b_s(X^n_{\eta_n(s)}, \mathscr{L}_{X^n_{\eta_n(s)}})\mathrm{d}s, \quad t\in[0,T],$$

and $\tilde{N}^n$ be defined in the same way with X^n replaced by $\tilde{X}^n$. It is clear that $\{N_t^n\}_{t\in[0,T]}$ is a martingale with respect to the filtration $\mathscr{F}_t^n = \sigma\{X_s^n, s\le t\}$. Thanks to $\mathscr{L}_{X^n}|_{\mathbb{P}} = \mathscr{L}_{\tilde{X}^n}|_{\tilde{\mathbb{P}}}$ and the boundedness of Q and b, it is not difficult to prove that $\{\tilde{N}_t^n\}_{t\in[0,T]}$ is a martingale with respect to the filtration $\tilde{\mathscr{F}}_t^n = \sigma\{\tilde{X}_s^n, s\le t\}$ and the quadratic variation process is

$$\langle \tilde{N}^n\rangle_t = \int_0^t \Big((-A)^{-1}Q_s(\tilde{X}^n_{\eta_n(s)}, \mathscr{L}_{\tilde{X}^n_{\eta_n(s)}})\Big)\Big((-A)^{-1}Q_s(\tilde{X}^n_{\eta_n(s)}, \mathscr{L}_{\tilde{X}^n_{\eta_n(s)}})\Big)^* \mathrm{d}s, \quad t\in[0,T],$$

where $*$ stands for the adjoint operator. Noting that

$$|\tilde{X}^n_{\eta_n(s)} - \tilde{X}_s| \le |\tilde{X}^n_{\eta_n(s)} - \tilde{X}_{\eta_n(s)}| + |\tilde{X}_{\eta_n(s)} - \tilde{X}_s| \le \sup_{s\in[0,T]}|\tilde{X}^n_s - \tilde{X}_s| + |\tilde{X}_{\eta_n(s)} - \tilde{X}_s|,$$

we conclude that $\tilde{\mathbb{P}}$-a.s. $\tilde{X}^n_{\eta_n(s)}$ converges to $\tilde{X}_s$ as n goes to infinity. This combined with the continuity of b_t, Q_t implies that the process

$$\tilde{N}_t := (-A)^{-1}\tilde{X}_t - (-A)^{-1}\tilde{X}_0 + \int_0^t \tilde{X}_s\mathrm{d}s - \int_0^t (-A)^{-1}b_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s})\mathrm{d}s, \quad t\in[0,T]$$

is a martingale with respect to the filtration $\tilde{\mathscr{F}}_t = \sigma\{\tilde{X}_s, s\le t\}$ and the quadratic variation process is

$$\langle \tilde{N}\rangle_t = \int_0^t \Big((-A)^{-1}Q_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s})\Big)\Big((-A)^{-1}Q_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s})\Big)^*\mathrm{d}s, \quad t\in[0,T].$$

By the martingale representation theorem [11, Theorem 8.2], there exists a complete filtered probability space $(\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathscr{F}}}, \{\tilde{\tilde{\mathscr{F}}}_t\}, \tilde{\tilde{\mathbb{P}}})$, a cylindrical Brownian motion $\tilde{\tilde{W}}$ such that $\mathscr{L}_{\tilde{X}_s}|_{\tilde{\mathbb{P}}} = \mathscr{L}_{\tilde{X}_s}|_{\tilde{\tilde{\mathbb{P}}}}$ and

f

$$(3.7)\qquad \begin{aligned}(-A)^{-1}\tilde{X}_t =& (-A)^{-1}\tilde{X}_0 + \int_0^t(-\tilde{X}_s)\mathrm{d}s + \int_0^t (-A)^{-1}b_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s}|_{\tilde{\tilde{\mathbb{P}}}})\mathrm{d}s\\ &+ \int_0^t (-A)^{-1}Q_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s}|_{\tilde{\tilde{\mathbb{P}}}})\mathrm{d}\tilde{\tilde{W}}_s, \quad t\in[0,T].\end{aligned}$$

Again by [7, Lemma 3.5], (3.7) yields

$$\begin{aligned}(-A)^{-1}\tilde{X}_t =& \mathrm{e}^{At}(-A)^{-1}\tilde{X}_0 + \int_0^t (-A)^{-1}\mathrm{e}^{A(t-s)}b_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s}|_{\tilde{\tilde{\mathbb{P}}}})\mathrm{d}s\\ &+ \int_0^t (-A)^{-1}\mathrm{e}^{A(t-s)}Q_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s}|_{\tilde{\tilde{\mathbb{P}}}})\mathrm{d}\tilde{\tilde{W}}_s, \quad t\in[0,T],\end{aligned}$$

which derives

fig

$$(3.8)\quad \tilde{X}_t = \mathrm{e}^{At}\tilde{X}_0 + \int_0^t \mathrm{e}^{A(t-s)}b_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s}|_{\tilde{\tilde{\mathbb{P}}}})\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)}Q_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s}|_{\tilde{\tilde{\mathbb{P}}}})\mathrm{d}\tilde{\tilde{W}}_s, \quad t\in[0,T].$$

Thus, $(\tilde{X}_t, \tilde{\tilde{W}}_t)_{t\in[0,T]}$ is a weak solution of (1.1) with initial distribution μ_0. □

3.2 Proof of Theorem 2.2

In this part, the Zvonkin transform is used to obtain the strong uniqueness. Since Itô's formula for (1.1) is unavailable, we shall use finite dimensional approximation such that the formula can be applied. To this end, for $\lambda > 0$, $\mu \in C([0,T], \mathscr{P}_2)$ and $n \geq 1$, let $\mathbb{H}_n = \mathrm{span}\{e_1, \cdots, e_n\}$ and define

$$b_t^{\mu,n} = \pi_n b_t(\cdot, \mu_t) \circ \pi_n, \quad Q_t^{\mu,n} = \pi_n Q_t(\cdot, \mu_t) \circ \pi_n, \quad A_n = A \circ \pi_n.$$

Let $Z_{s,t}^n(z)$ solve

E-A-n (3.9)
$$\mathrm{d}Z_t^n = A_n Z_t^n \mathrm{d}t + Q_t^{\mu,n}(Z_t^n)\mathrm{d}W_t$$

with $Z_{s,s}^n(z) = z \in \mathbb{H}_n$ and $P_{s,t}^{\mu,n}$ be the associated semigroup. That is,

$$P_{s,t}^{\mu,n} f(x) = \mathbb{E} f(Z_{s,t}^n(x)), \quad x \in \mathbb{H}_n, f \in \mathscr{B}_b(\mathbb{H}_n), t \geq s \geq 0.$$

Consider

un (3.10)
$$u_s^n = \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{s,t}^{\mu,n}(\nabla_{b_t^{\mu,n}} u_t^n + b_t^{\mu,n})\mathrm{d}t, \quad s \in [0,T].$$

Due to [31, Lemma 2.3, Proposition 2.5], we have

L-PDE **Lemma 3.2.** *Assume* **(a1)-(a3)**. *Let $T > 0$ be fixed. Then there exists a constant $\lambda_0 > 0$ independent of n and $\mu \in C([0,T]; \mathscr{P}_2)$ such that for any $\lambda \geq \lambda_0$,* (3.10) *has a unique solution $u^{\lambda,\mu,n}$ which belongs to $C^1([0,T]; C_b^2(\mathbb{H}_n; \mathbb{H}_n))$ with*

g1' (3.11)
$$\|u^{\lambda,\mu,n}\|_{T,\infty} + \|\nabla u^{\lambda,\mu,n}\|_{T,\infty} + \left\|\nabla^2 u^{\lambda,\mu,n}\right\|_{T,\infty} \leq \frac{1}{5}, \quad n \geq 1.$$

Let $\Theta^{\lambda,\mu,n}(x) = x + u^{\lambda,\mu,n}(x), x \in \mathbb{H}_n$. Then we have the regularization of the finite dimensional approximation as follows.

L3.2 **Lemma 3.3.** *Assume* **(a1)-(a3)**. *For any $T > 0$, there exists a constant $\lambda(T) \geq \lambda_0$ such that for any $\zeta \in C([0,T]; \mathscr{P}_2)$ and adapted continuous process $(X_t)_{t \in [0,T]}$ on $\mathbb{H}$ with $\mathbb{P}$-a.s.*

3.3 (3.12)
$$X_t = \mathrm{e}^{At} X_0 + \int_0^t \mathrm{e}^{A(t-s)} b_s(X_s, \zeta_s)\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)} Q_s(X_s, \zeta_s)\mathrm{d}W_s, \quad t \in [0,T],$$

and any $\lambda \geq \lambda(T), n \geq 1$, $X_t^n := \pi_n X_t$ satisfies

3.4 (3.13)
$$\begin{aligned}
\Theta_t^{\lambda,\mu,n}(X_t^n) &= \mathrm{e}^{At}\Theta_0^{\lambda,\mu,n}(X_0^n) + \int_0^t \mathrm{e}^{A(t-s)} \nabla\Theta_s^{\lambda,\mu,n}(X_s^n)\pi_n Q_s(X_s, \zeta_s)\mathrm{d}W_s \\
&+ \int_0^t (\lambda - A)\mathrm{e}^{A(t-s)} u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{A(t-s)} \nabla\Theta_s^{\lambda,\mu,n}(X_s^n)\pi_n[b_s(X_s, \zeta_s) - b_s(X_s^n, \mu_s)]\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \mathrm{e}^{A(t-s)} \mathrm{tr}\{[(Q_s Q_s^*)(X_s, \zeta_s) - (Q_s Q_s^*)(X_s^n, \mu_s)]\nabla^2 u_s^{\lambda,\mu,n}(X_s^n)\}\mathrm{d}s, \quad t \in [0,T],
\end{aligned}$$

where

$$\mathrm{e}^{A(t-s)}\mathrm{tr}\{[(Q_sQ_s^*)(X_s,\zeta_s)-\langle Q_sQ_s^*)(X_s^n,\mu_s)]\nabla^2 u_s^{\lambda,\mu,n}(X_s^n)\}$$
$$:=\sum_{i=1}^n\left(\mathrm{e}^{-\lambda_i(t-s)}\mathrm{tr}\{[(Q_sQ_s^*)(X_s,\zeta_s)-(Q_sQ_s^*)(X_s^n,\mu_s)]\nabla^2\langle u_s^{\lambda,\mu,n}(X_s^n),e_i\rangle\}\right)e_i.$$

Proof. The proof mainly follows the idea of [31, Proposition 2.5]. However, due to the distribution dependence of b and Q, some additional terms will appear to the calculations.

For simplicity, let $b_t^\mu=b_t(\cdot,\mu_t)$ and $Q_t^\mu=Q_t(\cdot,\mu_t)$. For any second-order differential function F on $\mathbb{H}_n$, let $L_t^{\mu,n}$ be defined as

gen (3.14)
$$L_t^{\mu,n}F(z)=\langle Az,\nabla F(z)\rangle+\frac{1}{2}\sum_{i,j=1}^n\langle (Q_t^\mu(Q_t^\mu)^*)(z)e_i,e_j\rangle\nabla_{e_i}\nabla_{e_j}F(z),\quad z\in\mathbb{H}_n.$$

This together with [31, (2.6)], dominated convergence theorem and $u^{\lambda,\mu,n}=\pi_n u^{\lambda,\mu,n}\circ\pi_n$ implies

put (3.15)
$$\partial_s u_s^{\lambda,\mu,n}(z)=[(\lambda-L_s^{\mu,n})u_s^{\lambda,\mu,n}](z)-[\nabla_{b_s^{\mu,n}}u_s^{\lambda,\mu,n}+b_s^{\mu,n}](z),\quad z\in\mathbb{H}_n.$$

Since $X_s^n=\pi_nX_s$ solves the following equation

Xn (3.16)
$$\mathrm{d}X_s^n=AX_s^n\mathrm{d}s+\pi_nb_s(X_s,\zeta_s)\mathrm{d}s+\pi_nQ_s(X_s,\zeta_s)\mathrm{d}W_s,\quad s\in[0,T],$$

Itô's formula, (3.15), $u_s^{\lambda,\mu,n}=\pi_nu_s^{\lambda,\mu,n}\circ\pi_n$ and $[\nabla_{b_s^{\mu,n}}u_s^{\lambda,\mu,n}]\circ\pi_n=[\nabla_{b_s^\mu}u_s^{\lambda,\mu,n}]\circ\pi_n$ lead to

$$\begin{aligned}
\mathrm{d}u_s^{\lambda,\mu,n}(X_s^n)=&\langle\nabla u_s^{\lambda,\mu,n}(X_s^n),Q_s(X_s,\zeta_s)\mathrm{d}W_s\rangle+\partial_su_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s\\
&+\langle\nabla u_s^{\lambda,\mu,n}(X_s^n),b_s(X_s,\zeta_s)\rangle\mathrm{d}s+\langle AX_s^n,\nabla u_s^{\lambda,\mu,n}(X_s^n)\rangle\mathrm{d}s\\
&+\frac{1}{2}\sum_{i,j=1}^n\langle(Q_sQ_s^*)(X_s,\zeta_s)e_i,e_j\rangle\nabla_{e_i}\nabla_{e_j}u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s\\
=&\langle\nabla u_s^{\lambda,\mu,n}(X_s^n),Q_s(X_s,\zeta_s)\mathrm{d}W_s\rangle+\lambda u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s-[L_s^{\mu,n}u_s^{\lambda,\mu,n}](X_s^n)\mathrm{d}s\\
&-[\nabla_{b_s^{\mu,n}}u_s^{\lambda,\mu,n}+b_s^{\mu,n}](X_s^n)\mathrm{d}s+\langle\nabla u_s^{\lambda,\mu,n}(X_s^n),b_s(X_s,\zeta_s)\rangle\mathrm{d}s\\
&+\langle AX_s^n,\nabla u_s^{\lambda,\mu,n}(X_s^n)\rangle\mathrm{d}s+\frac{1}{2}\sum_{i,j=1}^n\langle(Q_sQ_s^*)(X_s,\zeta_s)e_i,e_j\rangle\nabla_{e_i}\nabla_{e_j}u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s\\
=&\langle\nabla u_s^{\lambda,\mu,n}(X_s^n),Q_s(X_s,\zeta_s)\mathrm{d}W_s\rangle+\lambda u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s\\
&+\langle\nabla u_s^{\lambda,\mu,n}(X_s^n),b_s(X_s,\zeta_s)-b_s^\mu(X_s^n)\rangle\mathrm{d}s-\pi_nb_s^\mu(X_s^n)\mathrm{d}s\\
&+\frac{1}{2}\sum_{i,j=1}^n\langle[(Q_sQ_s^*)(X_s,\zeta_s)-Q_s^\mu(Q_s^\mu)^*(X_s^n)]e_i,e_j\rangle\nabla_{e_i}\nabla_{e_j}u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s.
\end{aligned}$$

This together with (3.16) and $u_s^{\lambda,\mu,n}=\pi_nu_s^{\lambda,\mu,n}\circ\pi_n$ yields

$$\mathrm{d}[u_s^{\lambda,\mu,n}(X_s^n)+X_s^n]$$

$$
\begin{aligned}
=&A[X_s^n + u_s^{\lambda,\mu,n}(X_s^n)]\mathrm{d}s + (\lambda - A)u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s\\
&+ \langle \nabla u_s^{\lambda,\mu,n}(X_s^n), Q_s(X_s,\zeta_s)\mathrm{d}W_s\rangle + \pi_n Q_s(X_s,\zeta_s)\mathrm{d}W_s\\
&+ \langle \nabla u_s^{\lambda,\mu,n}(X_s^n), b_s(X_s,\zeta_s) - b_s^\mu(X_s^n)\rangle \mathrm{d}s + [\pi_n b_s(X_s,\mathscr{L}_{X_s}) - \pi_n b_s^\mu(X_s^n)]\mathrm{d}s\\
&+ \frac{1}{2}\sum_{i,j=1}^n \langle [(Q_sQ_s^*)(X_s,\zeta_s) - (Q_s^\mu(Q_s^\mu)^*)(X_s^n)]e_i, e_j\rangle \nabla_{e_i}\nabla_{e_j} u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s.
\end{aligned}
$$

Thus, we get

fi

$$
\begin{aligned}
&u_t^{\lambda,\mu,n}(X_t^n) + X_t^n\\
=&\mathrm{e}^{At}[u_0^{\lambda,\mu,n}(X_0^n) + X_0^n] + \int_0^t \mathrm{e}^{A(t-s)}(\lambda - A)u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s\\
&+ \int_0^t \mathrm{e}^{A(t-s)}\langle \nabla u_s^{\lambda,\mu,n}(X_s^n) + I, \pi_n Q_s(X_s,\zeta_s)\mathrm{d}W_s\rangle\\
&+ \int_0^t \mathrm{e}^{A(t-s)}\langle \nabla u_s^{\lambda,\mu,n}(X_s^n) + I, \pi_n b_s(X_s,\zeta_s) - \pi_n b_s^\mu(X_s^n)\rangle \mathrm{d}s\\
&+ \frac{1}{2}\int_0^t \mathrm{e}^{A(t-s)}\sum_{i,j=1}^n \langle (Q_sQ_s^*)(X_s,\zeta_s) - (Q_s^\mu(Q_s^\mu)^*)(X_s^n)e_i, e_j\rangle \nabla_{e_i}\nabla_{e_j} u_s^{\lambda,\mu,n}(X_s^n)\mathrm{d}s.
\end{aligned} \tag{3.17}
$$

The proof is finished. □

mor **Remark 3.4.** *The conditions in Lemma 3.3 are stronger than those required in [31, Proposition 2.5], where* **(a3")** *cannot ensure the existence of* $\nabla^2 u^{\lambda,\mu,n}$ *in* (3.13)*, which does not appear in the distribution independent setting. Moreover, different from the proof of [31, Proposition 2.5], we do not take limit in* (3.13) *with respect to* n *in order to avoid calculating* $\lim_{n\to\infty}\nabla^2 u_s^{\lambda,\mu,n}(X_s^n)$*. However, the present formula is enough for us to prove the strong uniqueness by* (3.13)*, see the proof of Theorem 2.2(2) below for more details.*

Proof of Theorem 2.2. Thanks to Theorem 2.1 and the modified Yamada-Watanabe principle [16, Lemma 2.1], it suffices to prove the uniqueness of (1.1) and (2.2). According to [31, Theorem 1.1], for any $\mu \in C([0,T];\mathscr{P}_2)$ and $X_0 \in L^2(\Omega \to \mathbb{H};\mathscr{F}_0)$, the following equation

DD

$$
\mathrm{d}X_t = \{AX_t + b_t(X_t,\mu_t)\}\mathrm{d}t + Q_t(X_t,\mu_t)\mathrm{d}W_t \tag{3.18}
$$

has a unique mild solution X_t. Let $\nu \in C([0,T];\mathscr{P}_2)$ and $Y_0 \in L^2(\Omega \to \mathbb{H};\mathscr{F}_0)$ and Y_t solve (3.18) with (μ, X_0) replaced by (ν, Y_0). Moreover, let $\Phi_t^{\mathscr{L}_{X_0}}(\mu)$ and $\Phi_t^{\mathscr{L}_{Y_0}}(\nu)$ be the distribution of X_t and Y_t respectively. Set $X_t^n = \pi_n X_t$ and $Y_t^n = \pi_n Y_t$.

Let λ be large enough such that the assertions in Lemma 3.3 and Lemma 3.2 hold. By

(3.13), we have $\mathbb{P}$-a.s.

3.6

$$\begin{aligned}
&\Theta_t^{\lambda,\mu,n}(X_t^n) - \Theta_t^{\lambda,\mu,n}(Y_t^n)\\
=&\mathrm{e}^{At}\left(\Theta_0^{\lambda,\mu,n}(X_0^n) - \Theta_0^{\lambda,\mu,n}(Y_0^n)\right)\\
&+\int_0^t \mathrm{e}^{A(t-s)}[\nabla\Theta_s^{\lambda,\mu,n}(X_s^n)\pi_n Q_s(X_s,\mu_s) - \nabla\Theta_s^{\lambda,\mu,n}(Y_s^n)\pi_n Q_s(Y_s,\nu_s)]\mathrm{d}W_s\\
&+\int_0^t (\lambda - A)\mathrm{e}^{A(t-s)}[u_s^{\lambda,\mu,n}(X_s^n) - u_s^{\lambda,\mu,n}(Y_s^n)]\mathrm{d}s\\
&+\int_0^t \mathrm{e}^{A(t-s)}\nabla\Theta_s^{\lambda,\mu,n}(X_s^n)\pi_n[b_s(X_s,\mu_s) - b_s(X_s^n,\mu_s)]\mathrm{d}s\\
&+\frac{1}{2}\int_0^t \mathrm{e}^{A(t-s)}\mathrm{tr}\{[(Q_sQ_s^*)(X_s,\mu_s) - (Q_sQ_s^*)(X_s^n,\mu_s)]\nabla^2 u_s^{\lambda,\mu,n}(X_s^n)\}\mathrm{d}s\\
&-\int_0^t \mathrm{e}^{A(t-s)}\nabla\Theta_s^{\lambda,\mu,n}(Y_s^n)\pi_n[b_s(Y_s,\nu_s) - b_s(Y_s^n,\mu_s)]\mathrm{d}s\\
&-\frac{1}{2}\int_0^t \mathrm{e}^{A(t-s)}\mathrm{tr}\{[(Q_sQ_s^*)(Y_s,\nu_s) - (Q_sQ_s^*)(Y_s^n,\mu_s)]\nabla^2 u_s^{\lambda,\mu,n}(Y_s^n)\}\mathrm{d}s,\quad t\in[0,T].
\end{aligned}\tag{3.19}$$

By the same argument as in [31, (3.7)] and Fatou's lemma, it is routine to obtain

$$\begin{aligned}
&\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\left|\int_0^t (\lambda - A)\mathrm{e}^{A(t-s)}(u_s^{\lambda,\mu,n}(X_s^n) - u_s^{\lambda,\mu,n}(Y_s^n))\mathrm{d}s\right|^2\mathrm{d}t\\
&\le \liminf_{n\to\infty}\mathbb{E}\int_0^l \mathrm{e}^{-2\lambda t}\left|\int_0^t (\lambda - A)\mathrm{e}^{A(t-s)}(u_s^{\lambda,\mu,n}(X_s^n) - u_s^{\lambda,\mu,n}(Y_s^n))\mathrm{d}s\right|^2\mathrm{d}t\\
&\le \frac{1}{4}\int_0^l \mathrm{e}^{-2\lambda t}\mathbb{E}|X_t - Y_t|^2\mathrm{d}t,\quad l\in[0,T].
\end{aligned}$$

Due to **(a2)**, Fatou's lemma and Lemma 3.2, there exists some function $\varepsilon(\lambda)\downarrow 0$ as $\lambda\uparrow\infty$ such that

$$\begin{aligned}
&\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\left|\int_0^t \mathrm{e}^{A(t-s)}[\nabla\Theta_s^{\lambda,\mu,n}(X_s^n)\pi_n Q_s(X_s,\mu_s) - \nabla\Theta_s^{\lambda,\mu,n}(Y_s^n)\pi_n Q_s(Y_s,\nu_s)]\mathrm{d}W_s\right|^2\mathrm{d}t\\
&\le \liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\mathbb{E}\left|\int_0^t \mathrm{e}^{A(t-s)}[\nabla\Theta_s^{\lambda,\mu,n}(X_s^n)Q_s(X_s,\mu_s) - \nabla\Theta_s^{\lambda,\mu,n}(Y_s^n)Q_s(Y_s,\nu_s)]\mathrm{d}W_s\right|^2\mathrm{d}t\\
&\le \varepsilon(\lambda)\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{E}|X_s - Y_s|^2\mathrm{d}s + \varepsilon(\lambda)\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s,\quad l\in[0,T].
\end{aligned}$$

Furthermore, it follows from **(a2)-(a3)**, Lemma 3.2 and dominated convergence theorem that

$$\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\left|\int_0^t \mathrm{e}^{A(t-s)}\nabla\Theta_s^{\lambda,\mu,n}(Y_s^n)\pi_n[b_s(Y_s,\nu_s) - b_s(Y_s^n,\mu_s)]\mathrm{d}s\right|^2\mathrm{d}t$$

$$
\begin{aligned}
&\le \tilde{\varepsilon}(\lambda)\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s + c\mathbb{E}\lim_{n\to\infty}\int_0^l |b_s(Y_s,\mu_s)-b_s(Y_s^n,\mu_s)|^2\,\mathrm{d}s\\
&= \tilde{\varepsilon}(\lambda)\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s,\quad l\in[0,T],
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\left|\int_0^t \mathrm{e}^{A(t-s)}\mathrm{tr}\{[(Q_sQ_s^*)(Y_s,\nu_s)-(Q_sQ_s^*)(Y_s^n,\mu_s)]\nabla^2 u_s^{\lambda,\mu,n}(Y_s^n)\}\mathrm{d}s\right|^2\mathrm{d}t\\
\le&\mathbb{E}\int_0^l \mathrm{e}^{-2\lambda t}\int_0^t \|\mathrm{e}^{A(t-s)}\|_{\mathrm{HS}}^2\,|\mathrm{tr}[(Q_sQ_s^*)(Y_s,\nu_s)-(Q_sQ_s^*)(Y_s,\mu_s)]|^2\,\mathrm{d}s\mathrm{d}t\\
&+\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\int_0^t \|\mathrm{e}^{A(t-s)}\|_{\mathrm{HS}}^2\,|\mathrm{tr}[(Q_sQ_s^*)(Y_s,\mu_s)-(Q_sQ_s^*)(Y_s^n,\mu_s)]|^2\,\mathrm{d}s\mathrm{d}t\\
\le&2K(T)^2\mathbb{E}\int_0^l \mathrm{e}^{-2\lambda t}\int_0^t \|\mathrm{e}^{A(t-s)}\|_{\mathrm{HS}}^2\,\|Q_s(Y_s,\nu_s)-Q_s(Y_s,\mu_s)\|_{\mathrm{HS}}^2\,\mathrm{d}s\mathrm{d}t\\
&+2K(T)^2\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\mathbb{E}\int_0^t \|\mathrm{e}^{A(t-s)}\|_{\mathrm{HS}}^2\,\|Q_s(Y_s,\mu_s)-Q_s(Y_s^n,\mu_s)\|_{\mathrm{HS}}^2\,\mathrm{d}s\mathrm{d}t\\
\le&2K(T)^3\int_0^l \mathrm{e}^{-2\lambda t}\int_0^t \|\mathrm{e}^{A(t-s)}\|_{\mathrm{HS}}^2\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s\mathrm{d}t\\
&+2K(T)^2\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\int_0^t \|\mathrm{e}^{A(t-s)}\|_{\mathrm{HS}}^2\,\|Q_s(Y_s,\mu_s)-Q_s(Y_s^n,\mu_s)\|_{\mathrm{HS}}^2\,\mathrm{d}s\mathrm{d}t\\
\le&2K(T)^3\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s\int_s^l \mathrm{e}^{-2\lambda(t-s)}\|\mathrm{e}^{A(t-s)}\|_{\mathrm{HS}}^2\mathrm{d}t\\
&+2K(T)^2\mathbb{E}\liminf_{n\to\infty}\int_0^l \|Q_s(Y_s,\mu_s)-Q_s(Y_s^n,\mu_s)\|_{\mathrm{HS}}^2\,\mathrm{d}s\int_s^l \mathrm{e}^{-2\lambda(t-s)}\|\mathrm{e}^{A(t-s)}\|_{\mathrm{HS}}^2\mathrm{d}t\\
\le&\tilde{\varepsilon}(\lambda)\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s + c\mathbb{E}\lim_{n\to\infty}\int_0^l \|Q_s(Y_s,\mu_s)-Q_s(Y_s^n,\mu_s)\|_{\mathrm{HS}}^2\,\mathrm{d}s\\
=&\tilde{\varepsilon}(\lambda)\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s,\quad l\in[0,T]
\end{aligned}
$$

for some $\tilde{\varepsilon}(\lambda)\downarrow 0$ as $\lambda\uparrow\infty$, where we use (1.2) in the last display. Similarly, dominated convergence theorem, **(a3)**, Lemma 3.2 and (1.2) lead to

$$
\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\left|\int_0^t \mathrm{e}^{A(t-s)}\nabla\Theta_s^{\lambda,\mu,n}(X_s^n)\pi_n[b_s(X_s,\mu_s)-b_s(X_s^n,\mu_s)]\mathrm{d}s\right|^2\mathrm{d}t=0,
$$

and

$$
\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\left|\int_0^t \mathrm{e}^{A(t-s)}\mathrm{tr}\{[(Q_sQ_s^*)(X_s,\mu_s)-(Q_sQ_s^*)(X_s^n,\mu_s)]\nabla^2 u_s^{\lambda,\mu,n}(X_s^n)\}\mathrm{d}s\right|^2\mathrm{d}t=0.
$$

Finally, by the monotone convergence theorem and Lemma 3.2, we arrive at

$$\begin{aligned}&\mathbb{E}\liminf_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\left|\Theta_t^{\lambda,\mu,n}(X_t^n)-\Theta_t^{\lambda,\mu,n}(Y_t^n)\right|^2\mathrm{d}t\\&\geq \frac{16}{25}\mathbb{E}\lim_{n\to\infty}\int_0^l \mathrm{e}^{-2\lambda t}\left|X_t^n-Y_t^n\right|^2\mathrm{d}t=\frac{16}{25}\mathbb{E}\int_0^l \mathrm{e}^{-2\lambda t}\left|X_t-Y_t\right|^2\mathrm{d}t.\end{aligned}$$

Combining all the estimates above, for λ large enough, we have

ga (3.20) $$\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{E}|X_s-Y_s|^2\mathrm{d}s\leq\frac{1}{2}\int_0^l \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s+c(T)\mathbb{E}|X_0-Y_0|^2,\quad l\in[0,T].$$

This combined with $\mathbb{W}_2(\Phi_s^{\mathscr{L}_{X_0}}(\mu),\Phi_s^{\mathscr{L}_{Y_0}}(\nu))^2\leq\mathbb{E}|X_s-Y_s|^2$ implies for λ large enough, it holds

gac (3.21) $$\int_0^T \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\Phi_s^{\mathscr{L}_{X_0}}(\mu),\Phi_s^{\mathscr{L}_{Y_0}}(\nu))^2\mathrm{d}s\leq\frac{1}{2}\int_0^T \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s+c(T)\mathbb{E}|X_0-Y_0|^2.$$

In particular, we have

ga' (3.22) $$\int_0^T \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\Phi_s^{\mathscr{L}_{X_0}}(\mu),\Phi_s^{\mathscr{L}_{X_0}}(\nu))^2\mathrm{d}s\leq\frac{1}{2}\int_0^T \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s.$$

The strong uniqueness of (1.1) follows immediately from (3.22). More precisely, for two solutions X_t and $\tilde{X}_t$ to (1.1) with the same initial $\xi\in L^2(\Omega\to\mathbb{H};\mathscr{F}_0)$, (3.22) yields

$$\int_0^T \mathrm{e}^{-2\lambda s}\mathbb{W}_2(\mathscr{L}_{X_s},\mathscr{L}_{\tilde{X}_s})^2\mathrm{d}s=0,$$

which together with $\mathscr{L}_{X_\cdot},\mathscr{L}_{\tilde{X}_\cdot}\in C([0,T];\mathscr{P}_2)$ implies $\mathscr{L}_{X_s}=\mathscr{L}_{\tilde{X}_s}, s\in[0,T]$. Thus, X_t and $\tilde{X}_t$ solve the same classical SPDE as in [31, Theorem 1.1], which gives $X_t=\tilde{X}_t, t\in[0,T]$. Finally, if X_t and Y_t are two solutions to (1.1), (3.20) holds for $\mu_s=\mathscr{L}_{X_s}$ and $\nu_s=\mathscr{L}_{Y_s}$. Again using $\mathbb{W}_2(\mu_s,\nu_s)^2\leq\mathbb{E}|X_s-Y_s|^2$, we deduce (2.2). □

wek **Remark 3.5.** *If $Q_t(x,\mu)$ does not depend on μ, the weak uniqueness can be ensured in the case that b is not weakly continuous in the distribution variable, see [16, Theorem 1.1(1)] and references therein for the condition that b is Lipschitz continuous in distribution variable under total variational distance. The crucial technique is Girsanov's transform, which is also available in infinite dimensional situation.*

4 Proof of Theorem 2.3 and Theorem 2.4

The main idea of the proof is to fix the distribution in the coefficients of Equ. (1.1), which goes back to the classical situation. Then the log-Harnack inequality from different initial distribution holds according to [31, (1.7)]. Next, we calculate the relative entropy for two solutions with different distributions in the coefficients of (1.1) but same initial distribution, which dominates the total variational distance of these two solutions by Pinsker's inequality. Combining the above two parts, the desired log-Harnack inequality follows. As for Wang's Harnack inequality and shift Harnack inequality, the coupling by change of measure is used.

4.1 Proof of Theorem 2.3

Proof. (1) According to [29, Theorem 1.4.2(2)], (2.4) follows from log-Harnack inequality and Pinsker's inequality. (2.7) is a direct conclusion of Wang's Harnack inequality, see [29, Theorem 1.4.2(1)]. So we only need to prove log-Harnack inequality and Wang's Harnack inequality.

Let $\mu_t = P_t^*\mu_0$ and $\nu_t = P_t^*\nu_0$. Let X_t be the solution to SPDEs

ECO
$$\mathrm{d}X_t = AX_t\mathrm{d}t + b_t(X_t, \mu_t)\mathrm{d}t + Q_t(X_t)\mathrm{d}W_t \tag{4.1}$$

with $\mathscr{L}_{X_0} = \mu_0$. Define

$$\gamma_s = Q_s^*(Q_sQ_s^*)^{-1}(X_s)[b_s(X_s, \mu_s) - b_s(X_s, \nu_s)], \qquad \bar{W}_t = W_t + \int_0^t \gamma_s \mathrm{d}s,$$

and

$$R_T = \exp\left\{-\int_0^T \langle \gamma_s, \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^T |\gamma_s|^2 \mathrm{d}s\right\}.$$

By **(a2)**-**(a3)** and (2.1), Girsanov's theorem yields that $\{\bar{W}_s\}_{s\in[0,T]}$ is a cylindrical Brownian motion under $\mathbb{Q}_T = R_T\mathbb{P}$. Moreover, from (1.4), **(a2)** and (2.1), it is clear that

RT
$$\begin{aligned}\log \mathbb{E}R_T^2 &= \log \mathbb{E}\exp\left\{-\int_0^T 2\langle \gamma_s, \mathrm{d}W_s\rangle - \int_0^T |\gamma_s|^2 \mathrm{d}s\right\}\\ &\leq C(T)\int_0^T \mathbb{W}_2(\mu_s, \nu_s)^2 \mathrm{d}s \leq C(T)\mathbb{W}_2(\mu_0, \nu_0)^2.\end{aligned} \tag{4.2}$$

for some constant $C(T) > 0$. Then we have

ECb
$$\mathrm{d}X_t = AX_t\mathrm{d}t + b_t(X_t, \nu_t)\mathrm{d}t + Q_t(X_t)\mathrm{d}\bar{W}_t. \tag{4.3}$$

Letting $\bar{\mu}_t$ be the distribution of X_t under $\mathbb{Q}_T$, we derive

mub
$$\bar{\mu}_T(f) = \mathbb{E}^{\mathbb{Q}_T} f(X_T) = \mathbb{E}(R_T f(X_T)) = \mathbb{E}(\mathbb{E}(R_T|X_T)f(X_T)), \quad f \in \mathscr{B}_b(\mathbb{H}). \tag{4.4}$$

This implies $\mathbb{P}$-a.s.

muc
$$\frac{\mathrm{d}\bar{\mu}_T}{\mathrm{d}\mu_T}(X_T) = \mathbb{E}(R_T|X_T). \tag{4.5}$$

On the other hand, according to the log-Harnack inequality in [31, (1.7)] and [29, Theorem 1.4.2(2)], there exists a constant $C > 0$ such that

$$\mathrm{Ent}(P_T^*\nu_0|\bar\mu_T) = \bar\mu_T\left(\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\right) \le \frac{C}{T\wedge 1}\mathbb{W}_2(\mu_0,\nu_0)^2.$$

Thus, by Young's inequality, Jensen's inequality, (4.2), (4.4) and (4.5), for any $f \in \mathscr{B}_b(\mathbb{H})$, one can arrive at

$$\begin{aligned}
&P_T\log f(\nu_0)\\
=&\mu_T\left(\frac{\mathrm{d}\bar\mu_T}{\mathrm{d}\mu_T}\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log f\right)\\
\le&\log P_Tf(\mu_0) + \mu_T\left(\frac{\mathrm{d}\bar\mu_T}{\mathrm{d}\mu_T}\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log\left(\frac{\mathrm{d}\bar\mu_T}{\mathrm{d}\mu_T}\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\right)\right)\\
=&\log P_Tf(\mu_0) + \mu_T\left(\frac{\mathrm{d}\bar\mu_T}{\mathrm{d}\mu_T}\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log\frac{\mathrm{d}\bar\mu_T}{\mathrm{d}\mu_T}\right) + \mu_T\left(\frac{\mathrm{d}\bar\mu_T}{\mathrm{d}\mu_T}\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\right)\\
=&\log P_Tf(\mu_0) + \bar\mu_T\left(\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log\frac{\mathrm{d}\bar\mu_T}{\mathrm{d}\mu_T}\right) + \bar\mu_T\left(\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\right)\\
\le&\log P_Tf(\mu_0) + \log\bar\mu_T\left(\frac{\mathrm{d}\bar\mu_T}{\mathrm{d}\mu_T}\right) + 2\bar\mu_T\left(\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\right)\\
\le&\log P_Tf(\mu_0) + \log\mathbb{E}R_T^2 + 2\bar\mu_T\left(\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\log\frac{\mathrm{d}P_T^*\nu_0}{\mathrm{d}\bar\mu_T}\right)\\
\le&\log P_Tf(\mu_0) + C(T)\mathbb{W}_2(\mu_0,\nu_0)^2 + \frac{C}{T\wedge 1}\mathbb{W}_2(\mu_0,\nu_0)^2\\
\le&\log P_Tf(\mu_0) + \frac{C(T)}{T\wedge 1}\mathbb{W}_2(\mu_0,\nu_0)^2
\end{aligned}$$

for some constant $C(T) > 0$.

(2) Recall $\mu_t = P_t^*\mu_0$ and $\nu_t = P_t^*\nu_0$. Let X_t, Y_t solve the equations respectively

EC1 (4.6)
$$\begin{aligned}
\mathrm{d}X_t &= AX_t\mathrm{d}t + b_t(X_t,\mu_t)\mathrm{d}t + Q_t\mathrm{d}W_t,\\
\mathrm{d}Y_t &= AY_t\mathrm{d}t + b_t(X_t,\mu_t)\mathrm{d}t + Q_t\mathrm{d}W_t + \mathrm{e}^{At}\frac{X_0-Y_0}{T}\mathrm{d}t
\end{aligned}$$

with $\mathscr{L}_{X_0} = \mu_0$ and $\mathscr{L}_{Y_0} = \nu_0$. Then we have $Y_t = X_t + \mathrm{e}^{At}\frac{(T-t)(Y_0-X_0)}{T}$. In particular, $Y_T = X_T$. Let

$$\tilde\Phi(t) = b_t(X_t,\mu_t) - b_t(Y_t,\nu_t) + \mathrm{e}^{At}\frac{X_0-Y_0}{T},\quad t\in[0,T],$$

and

$$M_s = \int_0^s \langle Q_u^*(Q_uQ_u^*)^{-1}\tilde\Phi(u),\mathrm{d}W_u\rangle,\quad s\in[0,T].$$

Set

$$\tilde{R}(s) = \exp\left(-M_s - \frac{1}{2}\langle M\rangle_s\right), \quad s \in [0, T],$$

and

$$\tilde{W}_s = W_s + \int_0^s Q_u^*(Q_u Q_u^*)^{-1}\tilde{\Phi}(u)\mathrm{d}u, \quad s \in [0, T].$$

In addition, combining **(a3)** with (2.1), there exists a constant $C > 0$ such that for any $t \in [0, T]$,

$$\begin{aligned}
\int_0^T |\tilde{\Phi}(t)|^2 \mathrm{d}t &\le \int_0^T \left\{ 2|b_t(X_t, \mu_t) - b_t(Y_t, \nu_t)|^2 + 2\left|\mathrm{e}^{At}\frac{X_0 - Y_0}{T}\right|^2 \right\} \mathrm{d}t \\
&\le \int_0^T 4\phi^2\left(\frac{T-t}{T}|X_0 - Y_0|\right)\mathrm{d}t + \int_0^T 4K(T)^2 \mathbb{W}_2(\mu_t, \nu_t)^2 \mathrm{d}t + 2\frac{|X_0 - Y_0|^2}{T} \\
&\le 4T\phi^2(|X_0 - Y_0|) + C(T)\mathbb{W}_2(\mu_0, \nu_0)^2 + 2\frac{|X_0 - Y_0|^2}{T}.
\end{aligned}$$

By Girsanov's theorem, $\{\tilde{W}_s\}_{s\in[0,T]}$ is a cylindrical Brownian motion under $\tilde{\mathbb{Q}} = \tilde{R}(T)\mathbb{P}$. Then the second equation in (4.6) can be rewritten as

E29 (4.7)
$$\mathrm{d}Y_t = AY_t\mathrm{d}t + b_t(Y_t, \nu_t)\mathrm{d}t + Q_t\mathrm{d}\tilde{W}_t.$$

Consider SPDEs

E2' (4.8)
$$\mathrm{d}\tilde{Y}_t = A\tilde{Y}_t\mathrm{d}t + b_t(\tilde{Y}_t, \mathscr{L}_{\tilde{Y}_t}|_{\tilde{\mathbb{Q}}})\mathrm{d}t + Q_t\mathrm{d}\tilde{W}_t$$

with $\tilde{Y}_0 = Y_0$, then $\mathscr{L}_{Y_0}|_{\mathbb{P}} = \mathscr{L}_{Y_0}|_{\tilde{\mathbb{Q}}} = \mathscr{L}_{\tilde{Y}_0}|_{\tilde{\mathbb{Q}}} = \nu_0$. Thus, by the weak uniqueness, $\mathscr{L}_{\tilde{Y}_t}|_{\tilde{\mathbb{Q}}} = \nu_t$, which implies $\tilde{Y}_t = Y_t$ and $\mathscr{L}_{Y_t}|_{\tilde{\mathbb{Q}}} = \nu_t$.

On the other hand, by Hölder's inequality, for any $p > 1$, it holds

$$P_T f(\nu_0) = \mathbb{E}^{\tilde{\mathbb{Q}}} f(Y_T) = \mathbb{E}^{\tilde{\mathbb{Q}}} f(X_T) \le (P_T f^p(\mu_0))^{\frac{1}{p}} \{\mathbb{E}\tilde{R}(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}}.$$

By the definition of $\tilde{R}(T)$ and **(a2)**, one can obtain

$$\begin{aligned}
&\mathbb{E}\tilde{R}(T)^{\frac{p}{p-1}} \\
&\le \mathbb{E}\left\{ \exp\left[-\frac{p}{p-1}M_T - \frac{1}{2}\frac{p^2}{(p-1)^2}\langle M\rangle_T\right] \times \exp\left[\frac{1}{2}\frac{p^2}{(p-1)^2} - \frac{1}{2}\frac{p}{p-1}\langle M\rangle_T\right]\right\} \\
&\le \mathbb{E}\left\{ \mathbb{E}\left\{ \exp\left[-\frac{p}{p-1}M_T - \frac{1}{2}\frac{p^2}{(p-1)^2}\langle M\rangle_T\right]\middle| \mathscr{F}_0\right\}\right. \\
&\quad \left. \times \exp\left\{\frac{p}{2(p-1)^2}K(T)\left(4T\phi^2(|X_0 - Y_0|) + C(T)\mathbb{W}_2(\mu_0, \nu_0)^2 + 2\frac{|X_0 - Y_0|^2}{T}\right)\right\}\right\} \\
&\le \mathbb{E}\exp\left\{\frac{p}{2(p-1)^2}K(T)\left(4T\phi^2(|X_0 - Y_0|) + C(T)\mathbb{W}_2(\mu_0, \nu_0)^2 + 2\frac{|X_0 - Y_0|^2}{T}\right)\right\}.
\end{aligned}$$

Thus, we derive the Harnack inequalities. □

4.2 Proof of Theorem 2.4

Proof. Recall $\mu_t = P_t^*\mu_0$. Let X_t, Y_t solve the equations

EC1s
$$\begin{aligned} \mathrm{d}X_t &= AX_t\mathrm{d}t + b_t(X_t,\mu_t)\mathrm{d}t + Q_t(\mu_t)\mathrm{d}W_t, \quad \mathscr{L}_{X_0} = \mu_0, \\ \mathrm{d}Y_t &= AY_t\mathrm{d}t + b_t(X_t,\mu_t)\mathrm{d}t + Q_t(\mu_t)\mathrm{d}W_t + \mathrm{e}^{At}\frac{y}{T}\mathrm{d}t, \quad Y_0 = X_0. \end{aligned} \tag{4.9}$$

Then we have $Y_t = X_t + \mathrm{e}^{At}\frac{ty}{T}$. In particular, $Y_T = X_T + \mathrm{e}^{AT}y$. Let

$$\bar{\Phi}(t) = b_t(X_t,\mu_t) - b_t(Y_t,\mu_t) + \mathrm{e}^{At}\frac{y}{T}, \quad t \in [0,T].$$

For any $t \in [0,T]$, set

$$\bar{R}(t) = \exp\left[-\int_0^t \langle (Q_u^*(Q_uQ_u^*)^{-1})(\mu_u)\bar{\Phi}(u), \mathrm{d}W_u\rangle - \frac{1}{2}\int_0^t |(Q_u^*(Q_uQ_u^*)^{-1})(\mu_u)\bar{\Phi}(u)|^2\mathrm{d}u \right],$$

and

$$\bar{W}_t = W_t + \int_0^t (Q_u^*(Q_uQ_u^*)^{-1})(\mu_u)\bar{\Phi}(u)\mathrm{d}u.$$

There exists a constant $C > 0$ such that for any $t \in [0,T]$,

NNOs
$$|\bar{\Phi}(t)| \le \phi\left(\left|\mathrm{e}^{At}\frac{ty}{T}\right|\right) + \left|\mathrm{e}^{At}\frac{y}{T}\right|. \tag{4.10}$$

Thus, we have

Phis
$$\int_0^T |\bar{\Phi}(s)|^2\mathrm{d}s \le 2T\phi^2(|y|) + 2\frac{|y|^2}{T}. \tag{4.11}$$

Girsanov's theorem implies that $\{\bar{W}_s\}_{s\in[0,T]}$ is a cylindrical Brownian motion under $\bar{\mathbb{Q}}_T = \bar{R}(T)\mathbb{P}$. Then the second equation in (4.9) can be reformulated as

E2s
$$\mathrm{d}Y_t = AY_t\mathrm{d}t + b_t(Y_t,\mu_t)\mathrm{d}t + Q_t(\mu_t)\mathrm{d}\bar{W}_t, \quad Y_0 = X_0. \tag{4.12}$$

Thus, the distribution of Y_T under the new probability $\bar{\mathbb{Q}}_T$ coincides with the one of X_T under $\mathbb{P}$.

On the other hand, by Young's inequality and Hölder's inequality respectively, we arrive at

$$\begin{aligned} P_T\log f(\mu_0) &= \mathbb{E}^{\bar{\mathbb{Q}}_T}\log f(Y_T) \\ &= \mathbb{E}^{\bar{\mathbb{Q}}_T}\log f(X_T + \mathrm{e}^{AT}y) \\ &\le \log P_Tf(\cdot + \mathrm{e}^{AT}y)(\mu_0) + \mathbb{E}\bar{R}(T)\log\bar{R}(T), \end{aligned}$$

and

$$P_Tf(\mu_0) = \mathbb{E}^{\bar{\mathbb{Q}}_T}f(Y_T)$$

$$= \mathbb{E}^{\bar{\mathbb{Q}}_T} f(X_T + \mathrm{e}^{AT}y) \le (P_T f^p(\cdot + \mathrm{e}^{AT}y))^{\frac{1}{p}}(\mu_0)\{\mathbb{E}\bar{R}(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}}.$$

It is standard to obtain

$$\mathbb{E}\bar{R}(T)\log\bar{R}(T) = \mathbb{E}^{\bar{\mathbb{Q}}_T}\log\bar{R}(T) = \frac{1}{2}\mathbb{E}^{\bar{\mathbb{Q}}_T}\int_0^T |(Q_u^*(Q_uQ_u^*)^{-1})(\mu_u)\bar{\Phi}(u)|^2\mathrm{d}u,$$

and by the same argument as in the estimate of $\mathbb{E}\tilde{R}(T)^{\frac{p}{p-1}}$ in Section 4.1, it holds

$$\mathbb{E}\bar{R}(T)^{\frac{p}{p-1}} \le \operatorname*{ess\,sup}_{\Omega}\exp\left\{\frac{p}{2(p-1)^2}\int_0^T |(Q_u^*(Q_uQ_u^*)^{-1})(\mu_u)\bar{\Phi}(u)|^2\mathrm{d}u\right\}.$$

Thus, the shift Harnack inequality follows from (4.11) and **(a2)**. □

References

AD [1] N.U. Ahmed, X. Ding, *A semilinear Mckean-Vlasov stochastic evolution equation in Hilbert space*, Stoch. Proc. Appl. 60(1995), 65-85.

BR1 [2] V. Barbu, M. Röckner, *Probabilistic representation for solutions to non-linear Fokker-Planck equations,* SIAM J. Math. Anal. 50(2018), 4246-4260.

BR2 [3] V. Barbu, M. Röckner, *From non-linear Fokker-Planck equations to solutions of distribution dependent SDE,* arXiv:1808.10706.

BBO [4] M. Bauer, T. M.-Brandis, *McKean-Vlasov equations on infinite-dimensional Hilbert spaces with irregular drift and additive fractional noise,* arXiv:1912.07427.

BB [5] M. Bauer, T. M-Brandis, *Existence and Regularity of Solutions to Multi-Dimensional Mean-Field Stochastic Differential Equations with Irregular Drift,* arXiv:1912.05932.

BBP [6] M. Bauer, T. M-Brandis, F. Proske,*Strong Solutions of Mean-Field Stochastic Differential Equations with irregular drift,* arXiv:1806.11451.

BG [7] Z. Brzeźniak, D. Gatarek, *Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces,* Stoch. Proc. Appl. 84(1999), 187-225.

CR [8] P. E. Chaudru de Raynal, *Strong well-posedness of McKean-Vlasov stochastic differential equation with Hölder drift,* DOI: 10.1016/j.spa.2019.01.006.

CMF [9] M. F. Chen, *From Markov chain to non-equilibrium particle systems*(second edition), World Scientific, 2004.

CF [10] L. Campi, M. Fischer, *N-player games and mean-field games with absorption,* Ann. Appl. Probab. 28(2016), 2188-2242.

DZ [11] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions,* Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1992.

FS [12] S. Feng, *Large deviation for empirical process of interacting particle system with unbounded jumps*, Ann. Prob. 22(1994), 2122–2151.

H [13] R. Heinemann, *Distribution-Dependent Stochastic Differential Delay Equations in finite and infinite dimensions,* arXiv:2005.07446.

HRW [14] X. Huang, M. Röckner, F.-Y. Wang, *Nonlinear Fokker–Planck equations for probability measures on path space and path-distribution dependent SDEs,* Discrete Contin. Dyn. Syst. 39(2019), 3017-3035.

HW [15] X. Huang, F.-Y. Wang, *Distribution dependent SDEs with singular coefficients,* Stoch. Proc. Appl. 129(2019), 4747-4770.

HW20 [16] X. Huang, F.-Y. Wang, *McKean-Vlasov SDEs with Drifts Discontinuous under Wasserstein Distance,* arXiv:2002.06877.

L [17] W. Liu, Harnack *inequality and applications for stochastic evolution equations with monotone drifts*, J. Evol. Equ. 9(2009), 747-770.

M [18] H. P. McKean, *A class of Markov processes associated with nonlinear parabolic equations,* Proc Natl Acad Sci U S A 56(1966), 1907-1911.

MV [19] Yu. S. Mishura, A. Yu. Veretennikov, *Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations,* arXiv:1603.02212.

O [20] M. Ondrejet, *Uniqueness for stochastic evolution equations in Banach spaces,* Dissertationes Math. (Rozprawy Mat.) 426(2004).

Pin [21] M. S. Pinsker, *Information and Information Stability of Random Variables and Processes,* Holden-Day, San Francisco, 1964.

RW10 [22] M. Röckner, F.-Y. Wang, *Harnack and functional inequalities for generalized Mehler semigroups,* J. Funct. Anal. 203(2007), 237-261.

RZ [23] M. Röckner, X. Zhang, *Well-posedness of distribution dependent SDEs with singular drifts,* arXiv:1809.02216.

RW [24] M. Röckner, F.-Y Wang, *Log-Harnack inequality for stochastic differential equations in Hilbert spaces and its consequences*, Infin. Dimenns. Anal. Quantum Probab. Relat. Top. 13(2010), 27-37.

V [25] A. A. Vlasov, *The vibrational properties of an electron gas,* Soviet Physics Uspekhi 10(1968), 721.

W97 [26] F.-Y. Wang, *On estimation of the logarithmic Sobolev constant and gradient estimates of heat semigroups*. Probab. Theory Relat. Fields. 108(1997), 87-101.

W11 [27] F.-Y. Wang, *Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds,* Ann. Probab. 39(2011), 1449-1467.

W14a [28] F.-Y. Wang, *Integration by parts formula and shift Harnack inequality for stochastic equations,* Ann. Probab. 42(2014), 994-1019.

Wbook [29] F.-Y. Wang, *Harnack Inequality and Applications for Stochastic Partial Differential Equations,* Springer, New York, 2013.

W07 [30] F.-Y. Wang, *Harnack inequality and applications for stochastic generalized porous media equations,* Ann. Probab. 35(2007), 1333-1350.

W [31] F.-Y. Wang, *Gradient estimate and applications for SDEs in Hilbert space with multiplicative noise and Dini continuous drift,* J. Differential Equations 260(2016), 2792-2829.

FYW1 [32] F.-Y. Wang, *Distribution-dependent SDEs for Landau type equations,* Stoch. Proc. Appl. 128(2018), 595-621.

WZ [33] F.-Y. Wang, T. Zhang, *Log-Harnack inequalities for semilinear SPDE with strongly multiplicative noise,* Stoch. Proc. Appl. 124(2014), 1261-1274.

AZ [34] A. K. Zvonkin, *A transformation of the phase space of a diffusion process that removes the drift,* Math. Sb. 93(1974), 129-149.