VANISHING COEFFICIENTS IN QUOTIENTS OF THETA FUNCTIONS OF MODULUS FIVE

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Abstract. Following the recent trends of investigating vanishing coefficients in infinite products, we show that such instances are very rare when the infinite product is among a family of theta-quotients of modulus five. We also prove that a general family of products of theta functions of modulus five is always able to be effectively 5-dissected.

1. Introduction

Throughout, some standard notation will be adopted:

$$(A;q)_\infty := \prod_{k=0}^{\infty}(1-Aq^k),$$
$$(A_1,A_2,\ldots,A_n;q)_\infty := (A_1;q)_\infty(A_2;q)_\infty\cdots(A_n;q)_\infty$$

and

$$\left(\begin{matrix}A_1,A_2,\ldots,A_n\\ B_1,B_2,\ldots,B_m\end{matrix};q\right)_\infty := \frac{(A_1;q)_\infty(A_2;q)_\infty\cdots(A_n;q)_\infty}{(B_1;q)_\infty(B_2;q)_\infty\cdots(B_m;q)_\infty}.$$

In a recent work [12] of the second author, it was proved that in the series expansions of

$$\frac{(q,q^4;q^5)_\infty^3}{(q^2,q^3;q^5)_\infty^2} \qquad \text{and} \qquad \frac{(q^2,q^3;q^5)_\infty^3}{(q,q^4;q^5)_\infty^2}, \tag{1.1}$$

there are no terms of the form q^{5n+4} and q^{5n+2}, respectively. This paper joins the recent trends of investigating vanishing coefficients in infinite products, almost all of which are products or quotients of theta functions. Such investigations were initiated by Richmond and Szekeres [10] in 1978, continued with the work of Andrews and Bressoud [2], Alladi and Gordon [1], and McLaughlin [8]. A more recent paper of Hirschhorn [7] further popularized this topic, with follow-ups by the second author [11] and Baruah and Kaur [3].

Vaguely speaking, a theta function is of the form $(\pm q^a, \pm q^{N-a}; q^N)_\infty$, where N is a positive integer and a is an integer between 1 and $N-1$. In some contexts, the power N is called the *modulus* of this theta function. Let us restrict our attention to quotients of theta functions of modulus 5. A nearly forgotten paper on their vanishing coefficients is due to

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S. H. Chan [4], in which it was proved as by-products of a general dissection formula that in the series expansions of

$$\frac{(q,q^4;q^5)_\infty}{(-q,-q^4;q^5)_\infty} \qquad \text{and} \qquad \frac{(q^2,q^3;q^5)_\infty}{(-q^2,-q^3;q^5)_\infty}, \tag{1.2}$$

there are no terms of the form q^{5n+4} and q^{5n+1}, respectively.

Let us set up the following definition.

Definition 1.1. A formal power series $\sum_{n=0}^\infty a_n q^n$ has the *coefficient-vanishing property of length* m if there exists an integer k among $0,1,\ldots,m-1$ such that $a_{mn+k}=0$ for all $n\geq 0$.

Hence, the theta-quotients of modulus 5 in (1.1) and (1.2) are instances of series satisfying the coefficient-vanishing property of length 5. Now a natural question is how rare are these series? Our first result indicates that such instances are, to some extent, *extremely* uncommon.

Theorem 1.2. *Let s and t be positive integers. Then*

(a). *The theta-quotient*

$$\sum_{n=0}^\infty \alpha_{s,t}(n)q^n = \frac{(q,q^4;q^5)_\infty^s}{(-q,-q^4;q^5)_\infty^t} \tag{1.3}$$

satisfies the coefficient-vanishing property of length 5 *if and only if* $(s,t)=(1,1)$. *Further,* $\alpha_{1,1}(5n+4)=0$.

(b). *The theta-quotient*

$$\sum_{n=0}^\infty \beta_{s,t}(n)q^n = \frac{(q^2,q^3;q^5)_\infty^s}{(-q^2,-q^3;q^5)_\infty^t} \tag{1.4}$$

satisfies the coefficient-vanishing property of length 5 *if and only if* $(s,t)=(1,1)$. *Further,* $\beta_{1,1}(5n+1)=0$.

(c). *The theta-quotient*

$$\sum_{n=0}^\infty \gamma_{s,t}(n)q^n = \frac{(q,q^4;q^5)_\infty^s}{(q^2,q^3;q^5)_\infty^t} \tag{1.5}$$

satisfies the coefficient-vanishing property of length 5 *if and only if* $(s,t)=(3,2)$ *or* $(4,1)$. *Further,* $\gamma_{3,2}(5n+4)=\gamma_{4,1}(5n+4)=0$.

(d). *The theta-quotient*

$$\sum_{n=0}^\infty \delta_{s,t}(n)q^n = \frac{(q^2,q^3;q^5)_\infty^s}{(q,q^4;q^5)_\infty^t} \tag{1.6}$$

satisfies the coefficient-vanishing property of length 5 *if and only if* $(s,t)=(3,2)$ *or* $(4,1)$. *Further,* $\delta_{3,2}(5n+2)=\delta_{4,1}(5n+4)=0$.

It is worth pointing out that both $\gamma_{4,1}(5n+4)=0$ and $\delta_{4,1}(5n+4)=0$ are by-products of 5-dissection formulas of their corresponding theta-quotient (see Theorem 2.1). In general, one may consider the following product of theta functions of modulus 5:

$$(q,q^4;q^5)_\infty^s(q^2,q^3;q^5)_\infty^t$$

where s and t are arbitrary integers. Interestingly, such a series is always able to be effectively 5-dissected into a nice form.

Let us first write theta functions (with a slight generalization so that the eta function is included) as

$$J_{a,N}:=\prod_{\substack{k\ge 1\\ k\equiv \pm a \bmod N}}(1-q^k)$$

for positive integer N and integer a between 0 and N. We denote by $\mathbf{\Pi}_\Theta(N)$ the collection of theta-products of the form $\prod_{0\le a\le N/2} J_{a,N}^{\delta_a}$ where δ_a is an integer for each a. An immediate observation is that if M divides N, then $\mathbf{\Pi}_\Theta(M)\subseteq\mathbf{\Pi}_\Theta(N)$.

Theorem 1.3. *Let $\mathbf{L}_\Theta(N)$ be the collection of finite linear summations $\sum aq^\ell\zeta$ where a is an integer, ℓ is a nonnegative integer and $\zeta\in\mathbf{\Pi}_\Theta(N)$. Then for any integers s and t, we are always able to dissect $(q,q^4;q^5)_\infty^s(q^2,q^3;q^5)_\infty^t$ as*

$$A_0(q^5)+qA_1(q^5)+q^2A_2(q^5)+q^3A_3(q^5)+q^4A_4(q^5)$$

such that $A_i(q)\in\mathbf{L}_\Theta(25)$ for each $0\le i\le 4$.

We will explain how the 5-dissection of $(q,q^4;q^5)_\infty^s(q^2,q^3;q^5)_\infty^t$ is explicitly obtained in Sect. 3. For more general products of theta functions, a computer-assisted algorithm for their dissections was raised by Chen, Du and Zhao [5].

2. Two 5-dissections

We are going to prove the following two 5-dissections, from which both $\gamma_{4,1}(5n+4)=0$ and $\delta_{4,1}(5n+4)=0$ follow as immediate consequences.

Theorem 2.1. *We have*

$$\frac{(q,q^4;q^5)_\infty^4}{(q^2,q^3;q^5)_\infty}=A_0(q^5)+qA_1(q^5)+q^2A_2(q^5)+q^3A_3(q^5)+q^4A_4(q^5), \tag{2.1}$$

where

$$\begin{aligned}
A_0(q)&=\frac{1}{(q,q^4;q^5)_\infty^8(q^2,q^3;q^5)_\infty}+\frac{7q}{(q,q^4;q^5)_\infty^3(q^2,q^3;q^5)_\infty^6},\\
A_1(q)&=\frac{-4}{(q,q^4;q^5)_\infty^7(q^2,q^3;q^5)_\infty^2}+\frac{-3q}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^7},\\
A_2(q)&=\frac{7}{(q,q^4;q^5)_\infty^6(q^2,q^3;q^5)_\infty^3}+\frac{-q}{(q,q^4;q^5)_\infty(q^2,q^3;q^5)_\infty^8},\\
A_3(q)&=\frac{-7}{(q,q^4;q^5)_\infty^5(q^2,q^3;q^5)_\infty^4}+\frac{q}{(q^2,q^3;q^5)_\infty^9},
\end{aligned}$$

$$A_4(q) = 0.$$

Also,

$$\frac{(q^2,q^3;q^5)_\infty^4}{(q,q^4;q^5)_\infty} = B_0(q^5) + qB_1(q^5) + q^2B_2(q^5) + q^3B_3(q^5) + q^4B_4(q^5), \tag{2.2}$$

where

$$\begin{aligned}
B_0(q) &= \frac{1}{(q,q^4;q^5)_\infty^9} + \frac{7q}{(q,q^4;q^5)_\infty^4(q^2,q^3;q^5)_\infty^5},\\
B_1(q) &= \frac{1}{(q,q^4;q^5)_\infty^8(q^2,q^3;q^5)_\infty} + \frac{7q}{(q,q^4;q^5)_\infty^3(q^2,q^3;q^5)_\infty^6},\\
B_2(q) &= \frac{-3}{(q,q^4;q^5)_\infty^7(q^2,q^3;q^5)_\infty^2} + \frac{4q}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^7},\\
B_3(q) &= \frac{-7}{(q,q^4;q^5)_\infty^6(q^2,q^3;q^5)_\infty^3} + \frac{q}{(q,q^4;q^5)_\infty(q^2,q^3;q^5)_\infty^8},\\
B_4(q) &= 0.
\end{aligned}$$

For their proofs, we first collect some necessary identities. Let us adopt the conventional notation in Hirschhorn's book [6, Eqs. (15.1.1) and (15.1.2)]:

$$G(q) = \frac{1}{(q,q^4;q^5)_\infty} \qquad \text{and} \qquad H(q) = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

Our first identity is a reformulation of the 5-dissection of Euler's product $(q;q)_\infty$ [9]; see also [6, Eq. (8.1.1)].

Lemma 2.2. *We have*

$$\frac{1}{G(q)H(q)} = G(q^5)H(q^5)\left(\frac{G(q^5)}{H(q^5)} - q - q^2\frac{H(q^5)}{G(q^5)}\right). \tag{2.3}$$

Our second identity comes from [6, Eq. (9.2.13)].

Lemma 2.3. *We have*

$$\frac{H(q)^5}{G(q)^5} = \frac{H(q^5)}{G(q^5)}\frac{\frac{G(q^5)^2}{H(q^5)^2} - 2q\frac{G(q^5)}{H(q^5)} + 4q^2 - 3q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2}}{\frac{G(q^5)^2}{H(q^5)^2} + 3q\frac{G(q^5)}{H(q^5)} + 4q^2 + 2q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2}}. \tag{2.4}$$

Our last identity follows from [6, Eq. (17.4.5)].

Lemma 2.4. *We have*

$$G(q^5)^{11}H(q^5) - q^{10}G(q^5)H(q^5)^{11} - 11q^5G(q^5)^6H(q^5)^6 = 1. \tag{2.5}$$

Now we turn to prove Theorem 2.1.

Proof of Theorem 2.1. Combining (2.3) and (2.4) and utilizing the fact that 1 can be rewritten by (2.5), we have

$$\begin{aligned}\frac{H(q)^2}{G(q)^8} &= \left(\frac{1}{G(q)H(q)}\right)^3 \cdot \frac{H(q)^5}{G(q)^5} \cdot 1\\ &= G(q^5)^2H(q^5)^4\left(\frac{G(q^5)}{H(q^5)} - q - q^2\frac{H(q^5)}{G(q^5)}\right)^3\\ &\quad\times\left(\frac{\frac{G(q^5)^2}{H(q^5)^2} - 2q\frac{G(q^5)}{H(q^5)} + 4q^2 - 3q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2}}{\frac{G(q^5)^2}{H(q^5)^2} + 3q\frac{G(q^5)}{H(q^5)} + 4q^2 + 2q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2}}\right)\\ &\quad\times\left(G(q^5)^{11}H(q^5) - q^{10}G(q^5)H(q^5)^{11} - 11q^5G(q^5)^6H(q^5)^6\right)\\ &= \Big(H(q^5)\left(G(q^5)^2 - qG(q^5)H(q^5) - q^2H(q^5)^2\right)^2\\ &\quad\times\big(G(q^5)^4 - 2qG(q^5)^3H(q^5) + 4q^2G(q^5)^2H(q^5)^2\\ &\qquad - 3q^3G(q^5)H(q^5)^3 + q^4H(q^5)^4\big)\Big)^2.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{H(q)}{G(q)^4} &= H(q^5)\left(G(q^5)^2 - qG(q^5)H(q^5) - q^2H(q^5)^2\right)^2\\ &\quad\times\big(G(q^5)^4 - 2qG(q^5)^3H(q^5) + 4q^2G(q^5)^2H(q^5)^2\\ &\qquad - 3q^3G(q^5)H(q^5)^3 + q^4H(q^5)^4\big).\end{aligned}$$

This is (2.1) after expansion. For (2.2), we write

$$\frac{G(q)^2}{H(q)^8} = \left(\frac{1}{G(q)H(q)}\right)^3 \cdot \frac{G(q)^5}{H(q)^5} \cdot 1.$$

The rest follows from analogous calculations. □

3. Proof of Theorem 1.3

Let us say that if a formal power series can be 5-dissected as in Theorem 1.3, it has a 5-dissection *of* $\mathbf{L}_\Theta(25)$*-type*. An important property of such series is that if f and g have 5-dissections of $\mathbf{L}_\Theta(25)$-type, so does fg. This is simply due to the fact that $\mathbf{\Pi}_\Theta(N)$ forms a multiplicative group.

Hence, to prove that for any integers s and t, $(q,q^4;q^5)_\infty^s(q^2,q^3;q^5)_\infty^t$ has a 5-dissection of $\mathbf{L}_\Theta(25)$-type, it suffices to show that all of

$$(q,q^4;q^5)_\infty,\ (q^2,q^3;q^5)_\infty,\ \frac{1}{(q,q^4;q^5)_\infty},\ \frac{1}{(q^2,q^3;q^5)_\infty}$$

have 5-dissections of $\mathbf{L}_\Theta(25)$-type. Moreover, an explicit 5-dissection can be derived once the aforementioned four series can be explicitly 5-dissected.

We first notice that the 5-dissection of $(q,q^2,q^3,q^4;q^5)_\infty$ is essentially (2.3).

$$(q,q^2,q^3,q^4;q^5)_\infty = \frac{(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty}\left(\begin{pmatrix} q^{10},q^{15} \\ q^5,q^{20} \end{pmatrix};q^{25}\right)_\infty - q - q^2\left(\begin{matrix} q^5,q^{20} \\ q^{10},q^{15} \end{matrix};q^{25}\right)_\infty\Bigg).$$

Also, the 5-dissection of $1/(q,q^2,q^3,q^4;q^5)_\infty$ comes directly from the 5-dissection of $1/(q;q)_\infty$ (see [6, Eq. (8.4.4)]).

$$\begin{aligned}\frac{1}{(q,q^2,q^3,q^4;q^5)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^5}\Bigg(&\left(\begin{matrix} q^{10},q^{15} \\ q^5,q^{20} \end{matrix};q^{25}\right)_\infty^4 + q\left(\begin{matrix} q^{10},q^{15} \\ q^5,q^{20} \end{matrix};q^{25}\right)_\infty^3 \\ &+ 2q^2\left(\begin{matrix} q^{10},q^{15} \\ q^5,q^{20} \end{matrix};q^{25}\right)_\infty^2 + 3q^3\left(\begin{matrix} q^{10},q^{15} \\ q^5,q^{20} \end{matrix};q^{25}\right)_\infty + 5q^4 \\ &- 3q^5\left(\begin{matrix} q^5,q^{20} \\ q^{10},q^{15} \end{matrix};q^{25}\right)_\infty + 2q^6\left(\begin{matrix} q^5,q^{20} \\ q^{10},q^{15} \end{matrix};q^{25}\right)_\infty^2 \\ &- q^7\left(\begin{matrix} q^5,q^{20} \\ q^{10},q^{15} \end{matrix};q^{25}\right)_\infty^3 + q^8\left(\begin{matrix} q^5,q^{20} \\ q^{10},q^{15} \end{matrix};q^{25}\right)_\infty^4\Bigg).\end{aligned}$$

Hence, both $(q,q^2,q^3,q^4;q^5)_\infty$ and $1/(q,q^2,q^3,q^4;q^5)_\infty$ have 5-dissections of $\mathbf{L}_\Theta(5)$-type (that is, they can be 5-dissected as in Theorem 1.3 but with $\mathbf{L}_\Theta(25)$ replaced by $\mathbf{L}_\Theta(5)$), and therefore of $\mathbf{L}_\Theta(25)$-type since $\mathbf{\Pi}_\Theta(5) \subseteq \mathbf{\Pi}_\Theta(25)$.

We next record two 5-dissections concerning the Rogers–Ramanujan continued fraction and its reciprocal due to Hirschhorn [6, Eqs. (16.3.4) and (16.3.10)].

$$\begin{aligned}\left(\begin{matrix} q,q^4 \\ q^2,q^3 \end{matrix};q^5\right)_\infty = \frac{1}{(q^5;q^5)_\infty}\Bigg(&\left(\begin{matrix} q^{30},q^{95},q^{125} \\ q^{15},q^{110} \end{matrix};q^{125}\right)_\infty - q\left(\begin{matrix} q^{20},q^{105},q^{125} \\ q^{10},q^{115} \end{matrix};q^{125}\right)_\infty \\ &+ q^2\left(\begin{matrix} q^{55},q^{70},q^{125} \\ q^{35},q^{90} \end{matrix};q^{125}\right)_\infty - q^{18}\left(\begin{matrix} q^{5},q^{120},q^{125} \\ q^{60},q^{65} \end{matrix};q^{125}\right)_\infty \\ &- q^4\left(\begin{matrix} q^{45},q^{80},q^{125} \\ q^{40},q^{85} \end{matrix};q^{125}\right)_\infty\Bigg)\end{aligned}$$

and

$$\begin{aligned}\left(\begin{matrix} q^2,q^3 \\ q,q^4 \end{matrix};q^5\right)_\infty = \frac{1}{(q^5;q^5)_\infty}\Bigg(&\left(\begin{matrix} q^{40},q^{85},q^{125} \\ q^{20},q^{105} \end{matrix};q^{125}\right)_\infty + q\left(\begin{matrix} q^{60},q^{65},q^{125} \\ q^{30},q^{95} \end{matrix};q^{125}\right)_\infty \\ &- q^7\left(\begin{matrix} q^{35},q^{90},q^{125} \\ q^{45},q^{80} \end{matrix};q^{125}\right)_\infty - q^3\left(\begin{matrix} q^{10},q^{115},q^{125} \\ q^{5},q^{120} \end{matrix};q^{125}\right)_\infty \\ &- q^{14}\left(\begin{matrix} q^{15},q^{110},q^{125} \\ q^{55},q^{70} \end{matrix};q^{125}\right)_\infty\Bigg).\end{aligned}$$

Hence, they also have 5-dissections of $\mathbf{L}_\Theta(25)$-type.

Finally, Theorem 2.1 tells us that both

$$\frac{(q,q^4;q^5)_\infty^4}{(q^2,q^3;q^5)_\infty} \qquad \text{and} \qquad \frac{(q^2,q^3;q^5)_\infty^4}{(q,q^4;q^5)_\infty}$$

have 5-dissections of $\mathbf{L}_\Theta(5)$-type, and hence of $\mathbf{L}_\Theta(25)$-type.

It turns out all of

$$(q,q^4;q^5)_\infty,\ (q^2,q^3;q^5)_\infty,\ \frac{1}{(q,q^4;q^5)_\infty},\ \frac{1}{(q^2,q^3;q^5)_\infty}$$

have 5-dissections of $\mathbf{L}_\Theta(25)$-type since for example,

$$(q,q^4;q^5)_\infty = \frac{(q^2,q^3;q^5)_\infty^4}{(q,q^4;q^5)_\infty} \cdot \frac{1}{(q,q^2,q^3,q^4;q^5)_\infty} \cdot \left(\begin{matrix} q,q^4 \\ q^2,q^3 \end{matrix};q^5\right)_\infty^3.$$

Further, their 5-dissections can be explicitly formulated by inserting the dissections of each multiplicand.

4. Proof of Theorem 1.2

Our last task is to demonstrate the rareness of coefficient-vanishing theta-quotients of modulus 5. Notice that the "if" parts of Theorem 1.2 have already been shown in the work of Chan (see (1.2)) and the second author (see (1.1)) together with Theorem 2.1. Therefore, it remains to prove the "only if" parts. To do so, a simple but effective approach is to examine the coefficients of small powers of q in the series expansion. Also, since the constant term 1 always exists, we only need to check the vanishing coefficients in arithmetic progressions $5n+\ell$ $(1\le \ell\le 4)$.

4.1. Part (a) of Theorem 1.2.

Recall that

$$\sum_{n=0}^{\infty}\alpha_{s,t}(n)q^n = \frac{(q,q^4;q^5)_\infty^s}{(-q,-q^4;q^5)_\infty^t}. \tag{4.1}$$

We obtain

$$\begin{aligned}
\alpha_{s,t}(1) &= -\big(s+t\big),\\
\alpha_{s,t}(2) &= \frac{1}{2}\big(s^2+2st+t^2-s+t\big),\\
\alpha_{s,t}(3) &= -\frac{1}{6}\big(s^3+3s^2t+3st^2+t^3-3s^2+3t^2+2s+2t\big)
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{s,t}(4) = \frac{1}{24}\big(&s^4+4s^3t+6s^2t^2+4st^3+t^4-6s^3-6s^2t+6st^2+6t^3\\
&+11s^2+10st+11t^2-30s-18t\big).
\end{aligned} \tag{4.2}$$

Here we only prove (4.2) while the evaluations of $\alpha_{s,t}(1)$, $\alpha_{s,t}(2)$ and $\alpha_{s,t}(3)$ follow from similar but simpler arguments. Also, it is easy to see that none of $\alpha_{s,t}(1)$, $\alpha_{s,t}(2)$ and $\alpha_{s,t}(3)$ are equal to 0 for positive integers s and t.

First, notice that the infinite product in (4.1) can be treated as the generating function of a weighted colored partition where the parts are from

$$\begin{aligned}&1_1, 1_2, \ldots, 1_s, \overline{1}_1, \overline{1}_2, \ldots, \overline{1}_t, 4_1, 4_2, \ldots, 4_s, \overline{4}_1, \overline{4}_2, \ldots, \overline{4}_t,\\ &6_1, 6_2, \ldots, 6_s, \overline{6}_1, \overline{6}_2, \ldots, \overline{6}_t, 9_1, 9_2, \ldots, 9_s, \overline{9}_1, \overline{9}_2, \ldots, \overline{9}_t, \ldots\end{aligned}$$

with each distinct overlined part appearing at most once. Further, the weight is 1 if the number of parts is even and -1 otherwise. To evaluate $\alpha_{s,t}(4)$, we need to consider two cases.

Case 1. The parts are exclusively from 4_* and $\overline{4}_*$. The number of such contributions is simply

$$-\binom{s}{1}\binom{t}{0}-\binom{s}{0}\binom{t}{1}.$$

Case 2. The parts are exclusively from 1_* and $\overline{1}_*$. If there are no overlined 1's, then the four non-overlined 1's are of one of the forms:

$$\begin{aligned}&1_* + 1_* + 1_* + 1_*,\\ &1_* + 1_* + 1_* + 1_{**},\\ &1_* + 1_* + 1_{**} + 1_{**},\\ &1_* + 1_* + 1_{**} + 1_{***},\\ &1_* + 1_{**} + 1_{***} + 1_{****},\end{aligned}$$

where we assume that two 1's have different subscripts if the number of stars are different. Hence, the number of such contributions is

$$\binom{s}{0}\left(\binom{t}{1}+\binom{t}{1}\binom{t-1}{1}+\binom{t}{2}+\binom{t}{1}\binom{t-1}{2}+\binom{t}{4}\right). \tag{4.3}$$

Similarly, if there is one overlined 1, the number of such contributions is

$$\binom{s}{1}\left(\binom{t}{1}+\binom{t}{1}\binom{t-1}{1}+\binom{t}{3}\right); \tag{4.4}$$

if there are two overlined 1's, the number of such contributions is

$$\binom{s}{2}\left(\binom{t}{1}+\binom{t}{2}\right); \tag{4.5}$$

if there are three overlined 1's, the number of such contributions is

$$\binom{s}{3}\binom{t}{1};$$

if there are four overlined 1's, the number of such contributions is

$$\binom{s}{4}\binom{t}{0}.$$

Joining all the contributions, we arrive at (4.2). Moreover, the only negative contribution comes from *Case 1*, which is $-s-t$. But there are two non-vanished terms in *Case 2*: $\binom{s}{0}\binom{t}{1} = t$ from (4.3) and $\binom{s}{1}\binom{t}{1} = st$ from (4.4). Also, when $s \geq 2$, we have another non-vanished term $\binom{s}{2}\binom{t}{1}$ from (4.5). Hence, $\alpha_{s,t}(4) = 0$ only if $s = t = 1$.

As consequence, the theta-quotient (4.1) satisfies the coefficient-vanishing property of length 5 only if $(s,t) = (1,1)$.

Remark 4.1. As suggested by the referee, the expressions of $\alpha_{s,t}(1), \ldots, \alpha_{s,t}(4)$ could also be achieved analytically. We simply need the fact that the series expansion of (4.1) up to the 4th power of q agrees with the series expansion of

$$\frac{\big((1-q)(1-q^4)\big)^s}{\big((1+q)(1+q^4)\big)^t}.$$

Further, this quotient can be readily expanded with a computer algebra system like *Mathematica*.

4.2. **Part (b) of Theorem 1.2.** Recall that

$$\sum_{n=0}^{\infty} \beta_{s,t}(n) q^n = \frac{(q^2, q^3; q^5)_\infty^s}{(-q^2, -q^3; q^5)_\infty^t}, \tag{4.6}$$

from which we have

$$\begin{aligned}
\beta_{s,t}(2) &= -\big(s+t\big),\\
\beta_{s,t}(3) &= -\big(s+t\big),\\
\beta_{s,t}(4) &= \frac{1}{2}\big(s^2+2st+t^2-s+t\big)
\end{aligned}$$

and

$$\beta_{s,t}(6) = -\frac{1}{6}\big(s^3+3s^2t+3st^2+t^3-6s^2-6st+5s-t\big). \tag{4.7}$$

It is trivial that none of $\beta_{s,t}(2)$, $\beta_{s,t}(3)$ and $\beta_{s,t}(4)$ are 0 for positive integers s and t. For $\beta_{s,t}(6) = 0$, we may factor from (4.7):

$$\beta_{s,t}(6) = -\frac{1}{6}\big(s+t-1\big)\big(s^2+2st+t^2-5s+t\big).$$

To find positive integer solutions (s,t) to $\beta_{s,t}(6) = 0$, we have

$$s^2+2st+t^2-5s+t = 0,$$

or

$$s + 2t + \frac{t^2}{s} + \frac{t}{s} = 5.$$

If $t\geq 2$, the left-hand side is definitely greater than 5 since $s\geq 1$. If $t=1$, it turns out that $s^2-3s+2=0$ so that $s=1$ or 2. But we further compute that $\beta_{2,1}(5\times 2+1)=\beta_{2,1}(11)=-3$.

Therefore, the theta-quotient (4.6) satisfies the coefficient-vanishing property of length 5 only if $(s,t)=(1,1)$.

4.3. **Part (c) of Theorem 1.2.** Recall that

$$\sum_{n=0}^{\infty}\gamma_{s,t}(n)q^n=\frac{(q,q^4;q^5)_\infty^s}{(q^2,q^3;q^5)_\infty^t}. \tag{4.8}$$

Likewise, we have

$$\begin{aligned}\gamma_{s,t}(1)&=-s,\\ \gamma_{s,t}(2)&=\frac{1}{2}\big(s^2-s+2t\big),\\ \gamma_{s,t}(3)&=-\frac{1}{6}\big(s^3-3s^2+6st+2s-6t\big)\end{aligned} \tag{4.9}$$

and

$$\gamma_{s,t}(4)=\frac{1}{24}\big(s^4-6s^3+12s^2t+11s^2-36st+12t^2-30s+12t\big).$$

We see that $\gamma_{s,t}(1)$ and $\gamma_{s,t}(2)$ are nonzero for positive integers s and t. We next treat the remaining two cases.

For $\gamma_{s,t}(3)=0$, we factor from (4.9):

$$\gamma_{s,t}(3)=-\frac{1}{6}\big(s-1\big)\big(s^2-2s+6t\big).$$

Hence, we have the solutions $(s,t)=(1,t)$ with $t\geq 1$. Further, when $s\geq 2$ and $t\geq 1$, $s^2-2s+6t>0$. Hence, for this case, there are no more solutions. We may further compute that, for $t\geq 1$,

$$\gamma_{1,t}(8)=\frac{1}{24}\big(t^4+6t^3-t^2-6t\big)=\frac{1}{24}t(t-1)(t+1)(t+6).$$

Hence, $\gamma_{1,t}(8)=0$ has only one positive integer solution $t=1$. However, $\gamma_{1,1}(5\times 3+3)=\gamma_{1,1}(18)=-1$. We conclude that for $\gamma_{s,t}(5n+3)$ is not always vanishing for any positive integers s and t.

For $\gamma_{s,t}(4)=0$, we have

$$s^4-6s^3+12s^2t+11s^2-36st+12t^2-30s+12t=0.$$

Notice that when $s\geq 6$ and $t\geq 1$, $s^4-6s^3\geq 0$, $12s^2t-36st>0$ and $11s^2-30s>0$. Hence $\gamma_{s,t}(4)\neq 0$. For $1\leq s\leq 5$, we simply analyze case by case and find three solutions $(1,2)$, $(3,2)$ and $(4,1)$. But one may compute that $\gamma_{1,2}(5\times 1+4)=\gamma_{1,2}(9)=1$.

We conclude that the theta-quotient (4.8) satisfies the coefficient-vanishing property of length 5 only if $(s,t)=(3,2)$ or $(4,1)$.

4.4. **Part (d) of Theorem 1.2.** Recall that

$$\sum_{n=0}^{\infty} \delta_{s,t}(n) q^n = \frac{(q^2, q^3; q^5)_\infty^s}{(q, q^4; q^5)_\infty^t}, \tag{4.10}$$

from which we have

$$\delta_{s,t}(1) = t,$$
$$\delta_{s,t}(2) = \frac{1}{2}\big(t^2 - 2s + t\big), \tag{4.11}$$
$$\delta_{s,t}(3) = \frac{1}{6}\big(t^3 - 6st + 3t^2 - 6s + 2t\big) \tag{4.12}$$

and

$$\delta_{s,t}(4) = \frac{1}{24}\big(t^4 - 12st^2 + 6t^3 + 12s^2 - 36st + 11t^2 - 12s + 30t\big). \tag{4.13}$$

Clearly, $\delta_{s,t}(1)$ is nonzero for positive integers s and t. We now treat the remaining cases.

For $\delta_{s,t}(2) = 0$, we deduce from (4.11) that $s = (t^2 + t)/2$. With this substitution, one may further compute that

$$\delta_{s,t}(7) = -\frac{1}{504} t\big(t - 2\big)\big(t + 3\big)\big(2t^4 - 23t^3 - 77t^2 + 2t - 30\big).$$

Setting to zero gives only one positive integer solution $t = 2$, which leads to the case $(s, t) = (3, 2)$.

For $\delta_{s,t}(3) = 0$, we factor from (4.12):

$$\delta_{s,t}(3) = \frac{1}{6}\big(t + 1\big)\big(t^2 - 6s + 2t\big).$$

Since t is a positive integer, we deduce from $\delta_{s,t}(3) = 0$ that $s = (t^2 + 2t)/6$. With this substitution, one may further compute that

$$\delta_{s,t}(8) = \frac{1}{217728} t\big(t + 6\big)\big(4t^6 + 200t^5 + 228t^4 - 5848t^3 - 689t^2 + 10938t + 7560\big).$$

Setting to zero gives no positive integer solutions. Hence, $\delta_{s,t}(5n+3)$ is not always vanishing for any positive integers s and t.

For the case $\delta_{s,t}(5n + 4)$, we also need to compute that

$$\begin{aligned}\delta_{s,t}(9) = \frac{1}{362880}\Big(&60480s^4 - 241920s^3 - 383040s^3t - 120960s^3t^2 - 10080s^3t^3\\ &+ 665280s^2 + 1049328s^2t + 695520s^2t^2 + 204120s^2t^3 + 30240s^2t^4\\ &+ 1512s^2t^5 - 120960s - 1247328st - 1520064st^2 - 696528st^3\\ &- 171360st^4 - 21672st^5 - 2016st^6 - 72st^7 + 403200t\\ &+ 15120s^4t + 484560t^2 + 632204t^3 + 233604t^4\\ &+ 52689t^5 + 7560t^6 + 546t^7 + 36t^8 + t^9\Big). \end{aligned} \tag{4.14}$$

One may treat $\delta_{s,t}(4)$ and $\delta_{s,t}(9)$ as polynomials in s. Then $\delta_{s,t}(4)$ and $\delta_{s,t}(9)$ are of degree 2 and 4, respectively. We further use the division algorithm to write

$$\delta_{s,t}(9) = \delta_{s,t}(4)u(s,t) + r(s,t)$$

where the remainder $r(s,t)$ has the form

$$r(s,t) = A(t)s + B(t)$$

with

$$A(t) = 1 + \frac{191}{84}t - \frac{11}{72}t^2 - \frac{451}{216}t^3 - \frac{79}{72}t^4 - \frac{7}{216}t^5 + \frac{1}{12}t^6 + \frac{5}{378}t^7$$

and

$$B(t) = -\frac{20}{9}t - \frac{131}{126}t^2 + \frac{7765}{4536}t^3 + \frac{497}{432}t^4 + \frac{79}{216}t^5 + \frac{29}{432}t^6 - \frac{1}{48}t^7 - \frac{17}{1512}t^8 - \frac{11}{9072}t^9.$$

If we have the system

$$\begin{cases} \delta_{s,t}(4) = 0, \\ \delta_{s,t}(9) = 0, \end{cases} \tag{4.15}$$

then $r(s,t) = 0$. If $A(t) \neq 0$, then $r(s,t) = 0$ implies that $s = -B(t)/A(t)$. With this substitution, the system (4.15) has no positive integer solution t. If $A(t) = 0$, we have one positive integer solution $t = 1$. Also, $B(1) = 0$ and hence $r(s,1) = 0$. Substituting $t = 1$ into (4.15), then there are two positive integer solutions $s = 1$ and 4. However, we find that $\delta_{1,1}(14) = -1$, which implies that $\delta_{1,1}(5n+4)$ is not always vanishing. Hence, we only have the case $(s,t) = (4,1)$.

It turns out that the theta-quotient (4.10) satisfies the coefficient-vanishing property of length 5 only if $(s,t) = (3,2)$ or $(4,1)$.

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