Bounded Solutions to an Energy Subcritical Non-linear Wave Equation on $\mathbb{R}^3$

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Abstract

In this work we consider an energy subcritical semi-linear wave equation ($3 < p < 5$)

$$\begin{cases} \partial_t^2 u - \Delta u = \phi(x)|u|^{p-1}u, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u|_{t=0} = u_0 \in \dot{H}^{s_p}(\mathbb{R}^3); \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s_p-1}(\mathbb{R}^3); \end{cases}$$

where $s_p = 3/2 - 2/(p-1)$ and the function $\phi : \mathbb{R}^3 \rightarrow [-1,1]$ is a radial continuous function with a limit at infinity. We prove that unless the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ has a nonzero radial solution $W \in C^2(\mathbb{R}^3) \cap \dot{H}^{s_p}(\mathbb{R}^3)$, any radial solution u with a finite uniform upper bound on the critical Sobolev norm $\|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p}(\mathbb{R}^3)}$ for all t in the maximal lifespan must be a global solution in time and scatter.

1 Introduction

1.1 Background: Pure Power-type Nonlinearity

The nonlinear wave equation ($s_p = \frac{3}{2} - \frac{2}{p-1}$)

$$\begin{cases} \partial_t^2 u - \Delta u = \zeta |u|^{p-1}u, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u|_{t=0} = u_0 \in \dot{H}^{s_p}(\mathbb{R}^3); \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s_p-1}(\mathbb{R}^3); \end{cases} \qquad (CP0)$$

has been extensively studied in a lot of previous works. There are two different cases: the defocusing one with $\zeta = -1$ and the focusing one with $\zeta = 1$. The latter case is usually more complicated and difficult to deal with.

Local theory Local theory usually follows a combination of suitable Strichartz estimates and a fixed-point argument. An almost complete version of Strichartz estimates can be found in [19]. Some of the endpoint cases are discussed in [25]. There are also Stichartz estimates that hold only in the radial case. Please see [33, 50], for instance. A fixed-point argument then gives a local theory for initial data in the critical Sobolev space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$. More details of local theory can be found in [17, 24, 39].

Small initial data If the initial data are small, the sign of ζ does not play an important role. For example, if $p > 1 + \sqrt{2}$, global existence, well-posedness and scattering of solutions with small initial data was discussed in the papers [4, 15, 16, 34, 43]. Some of these works deal with even worse nonlinear term $|u|^p$. By scattering we mean that the solution approaches a solution to the homogeneous linear wave equation as t tends to infinity.

Energy critical case The behaviour of solutions in the focusing case is much different from the one in the defocusing case if the initial data are large. For example, if the nonlinear term is energy critical ($p = 5$), any solution with a finite energy always exists for all time t and scatters in the defocusing case, see [20, 21, 46, 47]. On the other hand, the energy critical, focusing equation has a family of stationary radial solutions (called ground states)

$$\pm\lambda^{1/2}\left(1+\frac{\lambda^2|x|^2}{3}\right)^{-1/2}, \quad \lambda \in \mathbb{R}^+.$$

The solutions with a smaller energy than ground states either scatter in both two time directions or blow up in finite time, as shown in [27]. Solutions with an energy at most slightly more than the ground states and away from them are discussed in [35, 36]. Global behaviour of solutions with a very large energy is more difficult to understand. People are particularly interested in type II blow-up solutions, i.e. non-scattering solutions whose data remain bounded in $\dot{H}^1 \times L^2$ within its whole lifespan. Please see [10, 12, 13] and citation therein.

Super and subcritical case There are also lots of works on the energy subcritical case ($p < 5$) or the energy supercritical case ($p > 5$). For example, please see [9, 30, 40, 41] for blow-up behaviour of solutions to the focusing equation, [3] for self-similar solutions in the conformal case ($p = 3$), and [11, 28, 31] for conditional scattering theory in energy supercritical case. We are particularly interested in the following conditional scattering result[1] in the subcritical case, since this is most relevant to the topics of this work.

Theorem 1.1 (See [8] for $1+\sqrt{2} < p \leq 3$ and [48] for $3 < p < 5$)**.** *Assume* $1+\sqrt{2} < p < 5$. *Let u be a radial solution to the equation (CP0) with a maximal lifespan $(-T_-, T_+)$ satisfying a uniform boundedness condition*

$$\sup_{t\in[0,T_+)} \|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^{s_p}\times \dot{H}^{s_p-1}(\mathbb{R}^3)} < \infty.$$

Then $T_+ = +\infty$ and u scatters in the positive time direction.

Please note that recently Dodson-Lawrie-Mendelson-Murphy [7] proved that this result still holds for non-radial solutions if $3 < p < 5$. Dodson [5, 6] proved the global existence and scattering of solutions to the defocusing equation with $3 \leq p < 5$ whenever initial data are radial and in the critical Sobolev space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. Scattering theory based on conformal conservation law [18, 22] works in the non-radial case but only for initial data of higher regularity and stronger decay at infinity. We also have conditional scattering results similar to Theorem 1.1 in higher dimensions. See, for instance, [32, 45].

1.2 Topic and Result of this Work

We consider a semi-linear wave equation with an more general energy subcritical nonlinearity ($3 < p < 5$) and radial data.

$$\begin{cases} \partial_t^2 u - \Delta u = \phi(x)|u|^{p-1}u, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u|_{t=0} = u_0 \in \dot{H}^{s_p}(\mathbb{R}^3); \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s_p-1}(\mathbb{R}^3); \end{cases} \qquad (CP1)$$

Here $s_p = 3/2 - 2/(p-1)$ and the function $\phi : \mathbb{R}^3 \to [-1, 1]$ is a radial continuous function with a well-defined limit

$$\lim_{|x|\to\infty} \phi(x) = \phi(\infty).$$

[1]This is slightly different than the original result, as its uniform boundedness condition is only concerning the positive time direction. But a careful review on the proof reveals that this different version of theorem still holds.

The study of this equation may helps us understand the global behaviour of radial solutions to nonlinear wave equation in curved spaces. For example, if v is a suitable radial solution to the shifted wave equation in the hyperbolic space $\mathbb{H}^3$

$$\partial_t^2 v - (\Delta_{\mathbb{H}^3} + 1)v = \pm|v|^{p-1}v, \quad (x,t) \in \mathbb{H}^3 \times \mathbb{R},$$

which has been discussed in [1, 14, 51], then $u = \mathbf{T}v$ is a solution to

$$\partial_t^2 u - \Delta u = \pm\left(\frac{|x|}{\sinh|x|}\right)^{p-1} |u|^{p-1}u, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Here the transformation $\mathbf{T}$ is defined by (see also [44])

$$(\mathbf{T}v)(r,\Theta) = \frac{\sinh r}{r} v(r,\Theta), \quad (r,\Theta) \in \mathbb{R}^+ \times \mathbb{S}^2,$$

with (r,Θ) being the polar coordinates in both spaces $\mathbb{R}^3$ and $\mathbb{H}^3$.

Remark 1.2. *The assumption $\phi(x) \in [-1,1]$ is not essential. Because a continuous function $\phi(x)$ with a limit at infinity must be bounded. If u solves $\partial_t^2 u - \Delta u = \phi(x)|u|^{p-1}u$, then $v = cu$ satisfies the equation $\partial_t^2 v - \Delta v = c^{-(p-1)}\phi(x)|v|^{p-1}v$. We have $|c^{-(p-1)}\phi(x)| \leq 1$ as long as the constant c is sufficiently large.*

Main Result We introduce our main result of this work.

Theorem 1.3. *Any radial solution u to (CP1) with a maximal lifespan $(-T_-, T_+)$ satisfying*

$$\sup_{t\in[0,T_+)} \|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)} < \infty \tag{1}$$

must scatter in the positive time direction, unless the elliptic equation

$$-\Delta W = \phi(x)|W|^{p-1}W$$

has a nonzero radial solution $W_0 \in C^2(\mathbb{R}^3) \cap \dot{H}^{s_p}(\mathbb{R}^3)$.

Remark 1.4. *A similar result holds for the negative time direction as well, because the wave equation is time-reversible.*

Remark 1.5. *This is clear that if the elliptic equation does have a nonzero radial C^2 solution $W_0(x)$ in $\dot{H}^{s_p}(\mathbb{R}^3)$, then $u(x,t) = W_0(x)$ is a solution to the wave equation (CP1) independent of tine t. Its critical Sobolev norm remains the same for all time but it definitely does not scatter. Thus the assumption about the elliptic equation in Theorem 1.3 is not only a sufficient condition but also a necessary one. This kind of solutions are usually called solitons. According to Remark 7.4, a solution $W_0(x)$ as given above always comes with a nonzero limit*

$$\lim_{|x|\to\infty} |x|W_0(x).$$

Remark 1.6. *The existence of nontrivial radial $C^2(\mathbb{R}^3) \cap \dot{H}^{s_p}(\mathbb{R}^3)$ solutions to the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ can be determined if $\phi(x)$ satisfies some additional assumptions. For example, if $\phi(x) \leq 0$ for all $x \in \mathbb{R}^3$, then such solutions do not exist. In fact we may multiply both sides of the elliptic equation by W, integrate by parts in a ball $B(0,R)$, and obtain*

$$\int_{B(0,R)} |\nabla W|^2 dx - \frac{1}{2}\int_{|x|=R} \frac{d}{dr}(|W|^2) dS = \int_{B(0,R)} \phi(x)|W|^{p+1} dx \leq 0.$$

Since $W \in \dot{H}^{s_p}(\mathbb{R}^3)$ is radial, we have $|W(x)| \to 0$ as $|x| \to \infty$ by Lemma 2.15. Thus there exists a sequence $R_k \to \infty$ so that $\partial_r(|W|^2) \leq 0$ when $r = R_k$. Let $R = R_k$ in the equation above we obtain $\nabla W \equiv 0$ thus $W \equiv 0$. The case $\phi(x) > 0$ has also been considered in some previous works such as [42, 52]. In particular, T. Kusano and M. Naito [38] claim that if $\phi : [0,\infty) \to R^+$ satisfies

- *$\phi(r) \in C[0,\infty) \cap C^1(0,\infty)$;*
- *$\frac{d}{dr}\left[r^{(5-p)/2}\phi(r)\right]$ is nonnegative for all $t > 0$ but not identically zero;*

then every nonzero radial C^2 solution to the elliptic equation is oscillatory, i.e. it has a zero in any neighbourhood of infinity. Therefore none of these solutions are in the space $\dot{H}^{s_p}(\mathbb{R}^3)$ by Remark 1.5. This means the elliptic equation does not have a nontrivial radial $C^2(\mathbb{R}^3)\cap\dot{H}^{s_p}(\mathbb{R}^3)$ solutions under the assumptions above.

Remark 1.7. *In general, the global behaviour of a solution to a non-linear wave equation as $t \to T_+$ may be one of the following three cases*

(I) The solution scatters, i.e. it resembles the behaviour of a free wave[2]. More precisely, $T_+ = +\infty$ and there exists a pair $(u_0^+, u_1^+) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$, such that

$$\lim_{t\to+\infty} \left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - \mathbf{S}_L(t) \begin{pmatrix} u_0^+ \\ u_1^+ \end{pmatrix} \right\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)} = 0.$$

here $\mathbf{S}_L(t)$ is the linear wave propagation operator.

(II) The critical Sobolev norm of the solution is unbounded.

$$\limsup_{t\to T_+} \|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)} = +\infty$$

(III) The critical Sobolev norm of the solution is bounded but the solution does not scatter. One typical example is a soliton as mentioned above, if it exists.

Our main theorem claims that case (III) is possible only if there is a soliton $W \in C^2(\mathbb{R}^3) \cap \dot{H}^{s_p}(\mathbb{R}^3)$, i.e a classic solution to the elliptic equation that is also contained in the critical Sobolev space of (CP1).

1.3 Main Idea

As in the case with $\phi(x) \equiv \pm 1$, the idea is to apply the compactness-rigidity argument introduced by Kenig-Merle [26, 27]. We outline the proof and briefly describe the most important methods involved in the introduction here, before more details are given in later sections. The novelty of our argument lies in the compactness part. The lack of natural dilation makes this equation more difficult to deal with than the equation with a pure power-type nonlinearity. This problem can be solved by observing the fact that very high or low frequency solutions to (CP1) can be approximated by solutions to $\partial_t^2 u - \Delta u = c|u|^{p-1}u$, with $c \in \{\phi(0), \phi(\infty)\}$ is a constant. This is similar to the situation of profile decomposition in curved background, see [23, 37], for example.

1.3.1 Compactness

First of all, it suffices to verify that the statement $Sc(A)$ below is true for all $A > 0$, whenever a radial $C^2 \cap \dot{H}^{s_p}$ soliton does not exist, in order to prove the main theorem.

Statement 1.8 ($Sc(A)$)**.** *If $u(x,t)$ is a radial solution of the non-linear wave equation (CP1) with a maximal lifespan $(-T_-, T_+)$, so that*

$$\sup_{t\in[0,T_+)} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)} < A,$$

then $T_+ = \infty$ and the solution scatters in the positive time direction.

[2] A free wave is a solution to the homogenous linear wave equation $\partial_t^2 u - \Delta u = 0$

By the local theory given in section 2.2, we know that $Sc(A)$ holds for small $A > 0$. If the statement $Sc(A)$ failed for some $A > 0$, there would exist a positive number M, called the break-down point, so that $Sc(M)$ held but $Sc(A)$ failed for each $A > M$. Thus we can pick up a sequence $\{u_n\}$ of non-scattering solutions, so that

$$\sup_{t\in[0,T_n^+)} \|(u_n(\cdot,t), \partial_t u_n(\cdot,t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} \searrow M.$$

We are now seeking to take a limit and obtain a "critical element" u, which is a global solution of (CP1) and satisfies

- $\sup_{t\in\mathbb{R}} \|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)} = M.$
- The set $\{(u(\cdot,t), \partial_t u(\cdot,t)) | t \in \mathbb{R}\}$ is pre-compact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$.

Among the key gradients of the argument are the profile decomposition and non-linear profiles associated to it.

The profile decomposition Given a sequence of radial initial data $\{(u_{0,n}, u_{1,n})\}_{n\in\mathbb{Z}^+}$ which are uniformly bounded in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$, we can always find a subsequence of it, still denoted by $\{(u_{0,n}, u_{1,n})\}_{n\in\mathbb{Z}^+}$, a sequence of radial free waves, denoted by $\{V_j(x,t)\}_{j\in\mathbb{Z}^+}$, and a pair $(\lambda_{j,n}, t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}$ for each pair $(j,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, such that

- For each integer $J > 0$, we can write each pair of initial data in the subsequence into a sum of J major components plus an error term:

  $$(u_{0,n}, u_{1,n}) = \sum_{j=1}^{J} (V_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0)) + (w_{0,n}^J, w_{0,n}^J).$$

  Here $V_{j,n}$ is a modified version of V_j via the application of a dilation and a time translation:

  $$(V_{j,n}(x,t), \partial_t V_{j,n}(x,t)) = \left(\frac{1}{\lambda^{\frac{2}{p-1}}} V_j\left(\frac{x}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right), \frac{1}{\lambda^{\frac{2}{p-1}+1}} \partial_t V_j\left(\frac{x}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right); \quad (2)$$

  and $(w_{0,n}^J, w_{1,n}^J)$ represents an error term that gradually becomes negligible as J and n grow.

- The sequences $\{(\lambda_{j,n}, t_{j,n})\}_{n\in\mathbb{Z}^+}$ and $\{(\lambda_{j',n}, t_{j',n})\}_{n\in\mathbb{Z}^+}$ are "almost orthogonal" for $j \neq j'$. More precisely we have

  $$\lim_{n\to\infty} \left(\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}}\right) = +\infty.$$

- We can also assume $\lambda_{j,n} \to \lambda_j \in \{0, 1, \infty\}$ and $-t_{j,n}/\lambda_{j,n} \to t_j \in \mathbb{R} \cup \{\infty, -\infty\}$ as $n \to \infty$ for each fixed j, by possibly passing to a subsequence and/or adjusting the free waves $\{V_j\}_{j\in\mathbb{Z}^+}$.

The nonlinear profile In the case of a pure power-type nonlinearity, we can approximate the solution to (CP0) with initial data $(V_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0))$ by a nonlinear profile U_j, which is another solution to (CP0), up to a dilation and a time translation, and then add these approximations up to obtain an approximation of u_n, thanks to the almost orthogonality. The fact that the equation (CP0) is invariant under dilations and time translations plays a crucial role in this argument. The same argument no longer works for the equation (CP1), since the presence of $\phi(x)$ prevents the application of dilations in this purpose. However, we can overcome this difficulty if we allow the use of nonlinear profiles that are not necessarily solutions to (CP1) but possibly solutions to other related equations instead. In fact, the solution to (CP1) with initial data $(V_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0))$ can be approximated by a nonlinear profile U_j as described below, up to a dilation and a time translation.

I (Expanding Profile) If $\lambda_j = \infty$, then the profile spreads out in the space as $n \to \infty$. Eventually a given compact set won't contain any significant part of the profile. The combination of this fact and our assumption $\lim_{|x|\to\infty} \phi(x) = \phi(\infty)$ implies that the nonlinear term $\phi(x)|u|^{p-1}u$ works in a similar way as $\phi(\infty)|u|^{p-1}u$. As a result, the nonlinear profile U_j in this case is a solution to the nonlinear wave equation $\partial_t^2 u - \Delta u = \phi(\infty)|u|^{p-1}u$.

II (Stable Profile) If $\lambda_j = 1$, then the profile approaches a stationary scale as $n \to \infty$. Therefore the nonlinear profile U_j is still a solution to (CP1).

III (Concentrating Profile) If $\lambda_j = 0$, then the profile concentrates around the origin as $n \to \infty$. The nonlinear term $\phi(x)|u|^{p-1}u$ performs in almost the same way as $\phi(0)|u|^{p-1}u$. As a result, the nonlinear profile U_j is a solution to the wave equation $\partial_t^2 u - \Delta u = \phi(0)|u|^{p-1}u$.

1.3.2 Rigidity

In this part we need to prove the non-existence of a critical element as mentioned above unless the equation (CP1) has a nontrivial C^2 soliton in $\dot{H}^{s_p}$, i.e. a solution to the elliptic equation $-\Delta W = |W|^{p-1}W$ in $C^2 \cap \dot{H}^{s_p}(\mathbb{R}^3)$. This is because we can show that any critical element must coincides with such a soliton. The argument is similar to the one we used for the equation (CP0) and consists of three steps

(I) We first show that the critical element u must be more regular than we have assumed. More precisely, it is in the space $\dot{H}^1 \times L^2(\mathbb{R}^3 \setminus B(0,R))$ for all $R > 0$ and its behaviour near infinity is similar to that of $A/|x|$, where A is a constant independent of t.

(II) We then construct a solution W to the equation $-\Delta W = \phi(x)|W|^{p-1}W$, whose behaviour near infinity is similar to that of u. Please note that this can be done even if the elliptic equation does not have a nontrivial C^2 solution in $\dot{H}^{s_p}(\mathbb{R}^3)$. The function W is typically outside the space $\dot{H}^{s_p}(\mathbb{R}^3)$ when $A \neq 0$, sometimes defined only for large x, or identically zero if $A = 0$. However, if the elliptic equation does have a radial solution in $C^2 \cap \dot{H}^{s_p}(\mathbb{R}^3)$, this solution can always be constructed via our method.

(II) By applying "channel of energy" method, we show that u must be exactly the same as W. This gives a contradiction if the elliptic equation does not have a nonzero radial $C^2 \cap \dot{H}^{s_p}$ solution. Because under our assumption W is either outside the space $\dot{H}^{s_p}(\mathbb{R}^3)$ or identically zero, which is definitely different from u.

1.4 Structure of this Paper

In section 2 we introduce notations, local theory, and already-known results as a preparation for the proof of the main theorem. The first part of the proof comes with two sections: In section 3 we make a review on the profile decomposition, introduce non-linear profiles and prove some properties of the non-linear profiles. Next we carry on the compactness procedure and extract a critical element in section 4. The second part of proof consists of three sections: We show the additional regularity of the critical element in section 5, then consider the solutions to the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ in section 6, and finally finish the proof in section 7 via the "channel of energy" method. Please note that the methods used in section 5 and section 7 are similar to our previous argument for the equation (CP0). Therefore we skip most details and merely give most important statements and ideas in these two sections.

2 Preliminary Results

2.1 Notations

Definition 2.1. *The notation $A \lesssim B$ means that there exists a constant c so that the inequality $A \leq cB$ holds. We may also add a subscript to the symbol $\lesssim$ to indicate that the implicit constant c depends on the parameter(s) mentioned in the subscript but nothing else.*

Definition 2.2. *The function F is defined by $F(u) = |u|^{p-1}u$ throughout this paper unless otherwise specified.*

Definition 2.3. *We define $\mathbf{T}_\lambda$ to be the dilation operator*

$$\mathbf{T}_\lambda (u_0, u_1)(x) = \left(\frac{1}{\lambda^{3/2-s_p}} u_0 \left(\frac{x}{\lambda}\right), \frac{1}{\lambda^{5/2-s_p}} u_1 \left(\frac{x}{\lambda}\right) \right);$$

Here x is the spatial variable of functions.

Definition 2.4. *Let $\mathbf{S}_L(t)$ be the linear wave propagation operator. More precisely, if u is the solution to linear wave equation $\partial_t^2 u - \Delta u = 0$ with initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1)$, then we define*

$$\mathbf{S}_L(t_0)(u_0, u_1) = (u(\cdot, t_0), u_t(\cdot, t_0)), \qquad \mathbf{S}_L(t_0) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(\cdot, t_0) \\ u_t(\cdot, t_0) \end{pmatrix}.$$

In addition, we use the notation $\mathbf{S}_{L,0}(u_0, u_1)$ for the first component $u(\cdot, t_0)$ of the vector above.

2.2 Local Theory

We start by the Strichartz estimates, as they are the basis of our local theory.

Proposition 2.5 (Generalized Strichartz Inequalities)**.** *(Please see Proposition 3.1 of [19], here we use the Sobolev version in $\mathbb{R}^3$) Let $2 \leq q_1, q_2 \leq \infty$, $2 \leq r_1, r_2 < \infty$ and $\rho_1, \rho_2, s \in \mathbb{R}$ with*

$$\begin{aligned} &1/q_i + 1/r_i \leq 1/2; \quad i = 1, 2; \\ &1/q_1 + 3/r_1 = 3/2 - s + \rho_1; \\ &1/q_2 + 3/r_2 = 1/2 + s + \rho_2. \end{aligned}$$

Let u be the solution of the following linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = F_1(x, t), \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u|_{t=0} = u_0 \in \dot{H}^s(\mathbb{R}^3); \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s-1}(\mathbb{R}^3). \end{cases} \tag{3}$$

Then we have ($1/q_2 + 1/\bar{q}_2 = 1$, $1/r_2 + 1/\bar{r}_2 = 1$)

$$\begin{aligned} &\sup_{t \in [0,T]} \|(u(\cdot, t), \partial_t u(\cdot, t))\|_{\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)} + \|D_x^{\rho_1} u\|_{L^{q_1} L^{r_1}([0,T] \times \mathbb{R}^3)} \\ &\quad \leq C \left(\|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)} + \|D_x^{-\rho_2} F_1(x, t)\|_{L^{\bar{q}_2} L^{\bar{r}_2}([0,T] \times \mathbb{R}^3)} \right). \end{aligned}$$

The constant C does not depend on T.

Definition 2.6. *If I is a time interval, we define the following norms*

$$\begin{aligned} \|(u_0, u_1)\|_H &= \|(u_0, u_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}(\mathbb{R}^3)}; \\ \|u(x, t)\|_{Y(I)} &= \|u(x, t)\|_{L^{\frac{2p}{1+s_p}} L^{\frac{2p}{2-s_p}}(I \times \mathbb{R}^3)}; \\ \|v(x, t)\|_{Z(I)} &= \|u(x, t)\|_{L^{\frac{2}{1+s_p}} L^{\frac{2}{2-s_p}}(I \times \mathbb{R}^3)}. \end{aligned}$$

Here the space-time norm is defined in a standard way:

$$\|u(x,t)\|_{L^qL^r(I\times\mathbb{R}^3)} = \left(\int_I \left(\int_{\mathbb{R}^3} |u(x,t)|^r\,dx\right)^{q/r} dt\right)^{1/q}.$$

If V is a free wave, then the norm $\|(V(\cdot,t),\partial_t V(\cdot,t)\|_H$ is independent of t. Thus we may use the notation $\|V\|_H$ instead for simplicity.

The fixed-point argument If u is a solution to (3) on a time interval I containing 0, then we have the Strichartz estimates

$$\sup_{t\in I}\|(u(t),\partial_t u(t))\|_H + \|u\|_{Y(I)} \leq C\left[\|(u_0,u_1)\|_H + \|F_1\|_{Z(I)}\right].$$

Combining this with the inequalities

$$\|\phi F(u)\|_{Z(I)} \leq \|u\|_{Y(I)}^p;$$
$$\|\phi F(u_1)-\phi F(u_2)\|_{Z(I)} \leq C_p\|u_1-u_2\|_{Y(I)}\left[\|u_1\|_{Y(I)}^{p-1}+\|u_2\|_{Y(I)}^{p-1}\right];$$

and applying a fixed-point argument, we obtain a local theory as below. Since our argument is similar to those in a lot of earlier papers, we only give important statements but omit most of the proof here. Please see, for instance, [24, 39] for more details.

Definition 2.7 (Solutions). *We say that u is a solution of (CP1) in the time interval I, if $(u(\cdot,t),\partial_t u(\cdot,t)) \in C(I;\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3))$, with a finite norm $\|u\|_{Y(J)}$ for any bounded closed interval $J\subseteq I$ so that the integral equation*

$$u(\cdot,t) = \mathbf{S}_{L,0}(t)(u_0,u_1) + \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}[\phi F(u(\cdot,\tau))]d\tau$$

holds for all time $t\in I$.

Theorem 2.8 (Local solution). *For any initial data $(u_0,u_1)\in\dot{H}^{s_p}\times\dot{H}^{s_p-1}$, there is a maximal interval $(-T_-(u_0,u_1),T_+(u_0,u_1))$ in which the equation has a unique solution.*

Theorem 2.9 (Scattering with small data). *There exists $\delta=\delta(p)>0$ such that if the norm of the initial data $\|(u_0,u_1)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}}<\delta$, then the Cauchy problem (CP1) has a global-in-time solution u with $\|u\|_{Y(-\infty,+\infty)}\leq C_p\|(u_0,u_1)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}}$. Here both the constants $\delta(p)$ and C_p can be chosen independent of the coefficient function $|\phi(x)|\leq 1$.*

Corollary 2.10. *There exists a function $\eta:\mathbb{R}^+\to\mathbb{R}^+$, such that if $\|(u_0,u_1)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}}\geq C_1>0$, then the solution u to (CP1) with the initial data (u_0,u_1) satisfies*

$$\inf_{t\in(-T_-,T_+)}\|(u(\cdot,t),\partial_t u(\cdot,t)\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)}\geq\eta(C_1),$$

Lemma 2.11 (Standard finite-time blow-up criterion). *If $T_+<\infty$, then $\|u\|_{Y([0,T_+))}=\infty$.*

Theorem 2.12 (Perturbation theory). *Fix $3<p<5$. Let M be a positive constant. There exists a constant $\varepsilon_0=\varepsilon_0(M,p)>0$, such that if an approximation solution $\tilde{u}$ defined on $\mathbb{R}^3\times I$ ($0\in I$) and a pair of initial data $(u_0,u_1)\in\dot{H}^{s_p}\times\dot{H}^{s_p-1}$ satisfy*

$$(\partial_t^2-\Delta)(\tilde{u})-\phi F(\tilde{u})=e(x,t), \qquad (x,t)\in\mathbb{R}^3\times I;$$
$$\|\tilde{u}\|_{Y(I)}<M; \qquad \|(\tilde{u}(\cdot,0),\partial_t\tilde{u}(\cdot,0))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}}<\infty;$$
$$\varepsilon \doteq \|e(x,t)\|_{Z(I)}+\|\mathbf{S}_{L,0}(t)(u_0-\tilde{u}(\cdot,0),u_1-\partial_t\tilde{u}(\cdot,0))\|_{Y(I)}<\varepsilon_0;$$

there exists a solution $u(x,t)$ of (CP1) defined in the interval I with the initial data (u_0,u_1) and satisfying

$$\|u(x,t)-\tilde{u}(x,t)\|_{Y(I)} < C(M,p)\varepsilon.$$

$$\sup_{t\in I}\left\|\begin{pmatrix} u(t)\\ \partial_t u(t)\end{pmatrix} - \begin{pmatrix} \tilde{u}(t)\\ \partial_t \tilde{u}(t)\end{pmatrix} - \mathbf{S}_L(t)\begin{pmatrix} u_0-\tilde{u}(0)\\ u_1-\partial_t\tilde{u}(0)\end{pmatrix}\right\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} < C(M,p)\varepsilon.$$

Proof. Let us first prove the perturbation theory when M is sufficiently small. Let I_1 be the maximal lifespan of the solution $u(x,t)$ to the Cauchy problem (CP1) with the given initial data (u_0,u_1) and assume $[0,T]\subseteq I\cap I_1$. By the Strichartz estimates, we have

$$\begin{aligned}
\|\tilde{u}-u\|_{Y([0,T])} &\leq \|\mathbf{S}_{L,0}(t)(u_0-\tilde{u}(0),u_1-\tilde{u}(0))\|_{Y([0,T])} + C_p\|e+\phi F(\tilde{u})-\phi F(u)\|_{Z([0,T])}\\
&\leq \varepsilon + C_p\|e\|_{Z([0,T])} + C_p\|F(\tilde{u})-F(u)\|_{Z([0,T])}\\
&\leq \varepsilon + C_p\varepsilon + C_p\|\tilde{u}-u\|_{Y([0,T])}\left(\|\tilde{u}\|^{p-1}_{Y([0,T])} + \|\tilde{u}-u\|^{p-1}_{Y([0,T])}\right)\\
&\leq C_p\varepsilon + C_p\|\tilde{u}-u\|_{Y([0,T])}\left(M^{p-1} + \|\tilde{u}-u\|^{p-1}_{Y([0,T])}\right).
\end{aligned}$$

Here the notation C_p may represent different constants at different places. By a continuity argument in T, there exist $M_0 = M_0(p) > 0$ and $\varepsilon_0 = \varepsilon_0(p) > 0$, such that if $M \leq M_0$ and $\varepsilon < \varepsilon_0$, we have

$$\|\tilde{u}-u\|_{Y([0,T])} \leq C_p\varepsilon.$$

Observing that the estimate above is independent of time T and works as well for an interval $[T,0]$ if $T<0$, we are actually able to conclude $I\subseteq I_1$ by the finite-time blow-up criterion and obtain

$$\|\tilde{u}-u\|_{Y(I)} \leq C_p\varepsilon.$$

In addition, by the Strichartz estimate we have

$$\begin{aligned}
&\sup_{t\in I}\left\|\begin{pmatrix} u(t)\\ \partial_t u(t)\end{pmatrix} - \begin{pmatrix} \tilde{u}(t)\\ \partial_t \tilde{u}(t)\end{pmatrix} - \mathbf{S}_L(t)\begin{pmatrix} u_0-\tilde{u}(0)\\ u_1-\partial_t\tilde{u}(0)\end{pmatrix}\right\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}}\\
&\quad\leq C_p\|\phi F(u)-\phi F(\tilde{u})-e\|_{Z(I)}\\
&\quad\leq C_p\left(\|e\|_{Z(I)} + \|F(u)-F(\tilde{u})\|_{Z(I)}\right)\\
&\quad\leq C_p\left[\varepsilon + \|u-\tilde{u}\|_{Y(I)}\left(\|\tilde{u}\|^{p-1}_{Y(I)} + \|u-\tilde{u}\|^{p-1}_{Y(I)}\right)\right]\\
&\quad\leq C_p\varepsilon.
\end{aligned}$$

This finishes the proof as M is sufficiently small. To deal with the general case, we can separate the time interval I into finite number of subintervals $\{I_j\}$, so that $\|\tilde{u}\|_{Y(I_j)} < M_0$, and then iterate our argument above. □

Remark 2.13. *If K is a compact subset of the space $\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)$, then there exists $T = T(K) > 0$ such that for any $(u_0,u_1)\in K$, $T_+(u_0,u_1) > T(K)$. This is a direct corollary from the perturbation theory.*

2.3 Known Results for a Constant ϕ

In this subsection we make a review on the already-known results concerning radial solutions to the equation

$$\begin{cases} \partial_t^2 u - \Delta u = c|u|^{p-1}u;\\ u|_{t=0} = u_0 \in \dot{H}^{s_p}(\mathbb{R}^3);\\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s_p-1}(\mathbb{R}^3); \end{cases} \tag{4}$$

Here c is a constant. The case with $c=\pm 1$, namely the equation (CP0), has been discussed in the author's previous work [48], whose main result has been mentioned in the introduction as

Theorem 1.1. The case $c = 0$ is trivial, since u becomes a solution to linear wave equation. If u is a solution to (4) with $c \notin \{-1, 0, 1\}$, then $|c|^{1/(p-1)}u$ is a solution to (CP0); and vice versa. This transformation immediately gives

Proposition 2.14. *Let u be a radial solution to the equation* (4) *with a maximal lifespan $(-T_-, T_+)$ and a uniform boundedness condition*

$$\sup_{t\in[0,T_+)} \|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)} < \infty.$$

Then $T_+ = \infty$ and u scatters in the positive time direction.

2.4 Properties of Radial $\dot{H}^s$ Functions

Lemma 2.15. *(Please see lemma 3.2 of [28]) Let $1/2 < s < 3/2$. Any radial $\dot{H}^s(\mathbb{R}^3)$ function u satisfies the inequality*

$$|u(x)| \lesssim_s \frac{\|u\|_{\dot{H}^s(\mathbb{R}^3)}}{|x|^{\frac{3}{2}-s}}.$$

Remark 2.16. *This actually means that a radial $\dot{H}^s$ function is uniformly continuous in $\mathbb{R}^3 \setminus B(0,R)$ if $R > 0$.*

Lemma 2.17. *Let K be a compact subset of $\dot{H}^s(\mathbb{R}^3)$, $1/2 < s < 3/2$. Then we have*

$$\sup_{|x|>R, u\in K} |x|^{\frac{3}{2}-s}|u(x)| \to 0, \quad \text{as } R \to +\infty;$$

$$\sup_{|x|<r, u\in K} |x|^{\frac{3}{2}-s}|u(x)| \to 0, \quad \text{as } r \to 0^+$$

Proof. A Combination of the compactness with Lemma 2.15 shows that it suffices to prove this lemma when K contains a single element. Please see Appendix of [48] for a proof of this special case. ☐

3 Profile Decomposition

3.1 Linear Profile Decomposition

Theorem 3.1 (Profile Decomposition)**.** *Let A be a constant. Given a sequence of radial initial data $\{(u_{0,n}, u_{1,n})\}_{n\in\mathbb{Z}^+}$ so that $\|(u_{0,n}, u_{1,n})\|_H \leq A$, there exist a subsequence of it, still denoted by $(u_{0,n}, u_{1,n})$; a sequence of radial free waves $V_j(x,t) = \mathbf{S}_L(t)(v_{j,0}, v_{j,1})$, $j \in \mathbb{Z}^+$; a pair $(\lambda_{j,n}, t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}$ for each pair $(j,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$; such that*

(i) Given a positive integer J, each pair of initial data in the subsequence can be expressed as a sum of the first J major components plus an error term

$$\begin{aligned}(u_{0,n}, u_{1,n}) &= \sum_{j=1}^{J} \left(\frac{1}{\lambda_{j,n}^{3/2-s_p}} V_j\left(\frac{\cdot}{\lambda_{j,n}}, \frac{-t_{j,n}}{\lambda_{j,n}}\right), \frac{1}{\lambda_{j,n}^{5/2-s_p}} \partial_t V_j\left(\frac{\cdot}{\lambda_{j,n}}, \frac{-t_{j,n}}{\lambda_{j,n}}\right)\right) + (w_{0,n}^J, w_{1,n}^J) \\ &= \sum_{j=1}^{J} \mathbf{S}_L(-t_{j,n})\mathbf{T}_{\lambda_{j,n}}(v_{0,j}, v_{1,j}) + (w_{0,n}^J, w_{1,n}^J);\end{aligned}$$

(ii) If $j \neq j'$, then the sequences $\{(\lambda_{j,n}, t_{j,n})\}_{n\in\mathbb{Z}^+}$ and $\{(\lambda_{j',n}, t_{j',n})\}_{n\in\mathbb{Z}^+}$ are "almost orthogonal", i.e. we have the limit

$$\lim_{n\to\infty} \left(\frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}} \right) = +\infty.$$

(iii) $\limsup_{n\to\infty} \left\|\mathbf{S}_L(t)(w_{0,n}^J, w_{1,n}^J)\right\|_{Y(\mathbb{R})} \to 0$ *as* $J \to \infty$.

(iv) For each given $J \geq 1$*, we have*

$$\|(u_{0,n}, u_{1,n})\|_H^2 = \sum_{j=1}^{J} \|V_j\|_H^2 + \left\|(w_{0,n}^J, w_{1,n}^J)\right\|_H^2 + o_{J,n}(1).$$

Here $o_{J,n}(1) \to 0$ *as* $n \to \infty$.

(v) We have the limits $\lambda_{j,n} \to \lambda_j \in \{0, 1, \infty\}$ *and* $-t_{j,n}/\lambda_{j,n} \to t_j \in [-\infty, \infty]$ *as* $n \to \infty$ *for each* j.

Please see [2] for the proof. There are a few remarks.

- This original paper deals with the energy critical case $s_p = 1$. But the same argument works for all $1/2 < s_p < 1$ as well.
- The original paper works for non-radial initial data as well. In this work we only consider the radial case.
- The original theorem is proved under an additional assumption labelled (1.6) there. But this condition can be eliminated according to Remark 5 on Page 159 of that paper. The elimination of this condition also implies that λ_j, the limit of the sequence $\lambda_{j,n}$ as $n \to \infty$, may converge to 1 or $+\infty$, besides 0, as given in part (v) above.

We need to prove a few lemmata before the introduction of the non-linear profiles.

Lemma 3.2. *If* $j \neq j'$*, then we have the almost orthogonality*

$$\lim_{n\to\infty} \left\langle \mathbf{S}_L(-t_{j,n})\mathbf{T}_{\lambda_{j,n}}(v_{0,j}, v_{1,j}), \mathbf{S}_L(-t_{j',n})\mathbf{T}_{\lambda_{j',n}}(v_{0,j'}, v_{1,j'}) \right\rangle_H = 0.$$

Proof. We first rewrite the dot product into

$$\begin{aligned} &\left\langle \mathbf{S}_L(-t_{j,n})\mathbf{T}_{\lambda_{j,n}}(v_{0,j}, v_{1,j}), \mathbf{S}_L(-t_{j',n})\mathbf{T}_{\lambda_{j',n}}(v_{0,j'}, v_{1,j'}) \right\rangle_H \\ &= \left\langle \mathbf{T}_{\lambda_{j,n}/\lambda_{j',n}}(v_{0,j}, v_{1,j}), \mathbf{S}_L\left(\frac{t_{j,n} - t_{j',n}}{\lambda_{j',n}}\right)(v_{0,j'}, v_{1,j'}) \right\rangle_H . \end{aligned}$$

We can immediately finish the proof by the almost orthogonal condition (ii) and Fourier analysis. □

Lemma 3.3. *Let* $\{(w_{0,n}, w_{1,n})\}_{n\in\mathbb{Z}^+}$ *be a bounded sequence in* H*, i.e.* $\|(w_{0,n}, w_{1,n})\|_H \leq A$ *so that* $\|\mathbf{S}_{L,0}(t)(w_{0,n}, w_{1,n})\|_{Y(\mathbb{R})} \to 0$*. Then we have the weak limit* $(w_{0,n}, w_{1,n}) \rightharpoonup 0$ *in* H*.*

Proof. If the weak limit $(w_{0,n}, w_{1,n}) \rightharpoonup 0$ were false, we could assume $(w_{0,n}, w_{1,n}) \rightharpoonup (w_0, w_1) \neq 0$ in H by possibly passing to a subsequence. Because the map $(u_0, u_1) \to \mathbf{S}_{L,0}(t)(u_0, u_1)$ is a bounded linear operator from the space H to $Y(\mathbb{R})$ by the Strichartz estimates, we also have a weak limit $\mathbf{S}_{L,0}(t)(w_{0,n}, w_{1,n}) \rightharpoonup \mathbf{S}_{L,0}(t)(w_0, w_1)$ in the space $Y(\mathbb{R})$. On the other hand, the same sequence $\mathbf{S}_{L,0}(t)(w_{0,n}, w_{1,n})$ has a strong limit zero in the space $Y(\mathbb{R})$ by the given conditions. As a result, we have $\mathbf{S}_{L,0}(t)(w_0, w_1) = 0 \Longrightarrow (w_0, w_1) = 0$. This is a contradiction. □

Lemma 3.4. *Assume* $\|(w_{0,n}, w_{1,n})\|_H \leq A$ *and* $\|\mathbf{S}_{L,0}(t)(w_{0,n}, w_{1,n})\|_{Y(\mathbb{R})} \to 0$*. Let* I *be a closed time interval and* $(U_0(x,t), U_1(x,t)) \in C(I; H)$*. If* I *contains a neighbourhood of* ∞ *or* $-\infty$*, we also assume*

$$\lim_{t\to\pm\infty} \left\| \begin{pmatrix} U_0(\cdot,t) \\ U_1(\cdot,t) \end{pmatrix} - \mathbf{S}_L(t) \begin{pmatrix} u_0^\pm \\ u_1^\pm \end{pmatrix} \right\|_H = 0$$

for some pair(s) $(u_0^\pm, u_1^\pm) \in H$*. Then for any two sequences* $\{\lambda_n : \lambda_n > 0\}_{n\in\mathbb{Z}^+}$ *and* $\{t_n : t_n \in I\}_{n\in\mathbb{Z}^+}$*, we have the limit*

$$\langle \mathbf{T}_{\lambda_n}(U_0(\cdot, t_n), U_1(\cdot, t_n)), (w_{0,n}, w_{1,n}) \rangle_H \to 0.$$

Proof. First of all, we can rewrite the pairing into

$$\begin{aligned}
&\langle \mathbf{T}_{\lambda_n}(U_0(\cdot,t_n),U_1(\cdot,t_n)),(w_{0,n},w_{1,n})\rangle_H\\
=&\langle (U_0(\cdot,t_n),U_1(\cdot,t_n)),\mathbf{T}_{1/\lambda_n}(w_{0,n},w_{1,n})\rangle_H\\
=&\langle \mathbf{S}_L(-t_n)(U_0(\cdot,t_n),U_1(\cdot,t_n)),\mathbf{S}_L(-t_n)\mathbf{T}_{1/\lambda_n}(w_{0,n},w_{1,n})\rangle_H.
\end{aligned}$$

According to the conditions given, we have

- The set $\{\mathbf{S}_L(-t)(U_0(\cdot,t),U_1(\cdot,t))|t\in I\}$ is pre-compact in H.
- The sequence $\mathbf{S}_L(-t_n)\mathbf{T}_{1/\lambda_n}(w_{0,n},w_{1,n})$ is bounded and converges weakly to 0 in the space H, by Lemma 3.3 and

$$\begin{aligned}
\left\|\mathbf{S}_L(-t_n)\mathbf{T}_{1/\lambda_n}(w_{0,n},w_{1,n})\right\|_H &= \|(w_{0,n},w_{1,n})\|_H \leq A;\\
\left\|\mathbf{S}_{L,0}(t)\mathbf{S}_L(-t_n)\mathbf{T}_{1/\lambda_n}(w_{0,n},w_{1,n})\right\|_{Y(\mathbb{R})} &= \|\mathbf{S}_{L,0}(t)(w_{0,n},w_{1,n})\|_{Y(\mathbb{R})} \to 0.
\end{aligned}$$

Therefore the pairing converges to zero. □

3.2 Nonlinear Profiles

In this subsection we introduce the nonlinear profiles and prove some properties of them. Recall the notation $F(u)=|u|^{p-1}u$.

Definition 3.5 (A nonlinear profile). *Fix $\tilde{\phi}$ to be either the function ϕ or a constant function c. Let $V(x,t)=\mathbf{S}_{L,0}(t)(v_0,v_1)$ be a free wave and $\tilde{t}\in[-\infty,\infty]$ be a time. We say that $U(x,t)$ is a nonlinear profile associated to $(V,\tilde{\phi},\tilde{t})$ if $U(x,t)$ is a solution to the nonlinear wave equation*

$$\partial_t^2 u - \Delta u = \tilde{\phi}F(u) \tag{5}$$

with a maximal timespan I so that I contains a neighbourhood[3] *of $\tilde{t}$ and*

$$\lim_{t\to\tilde{t}} \|(U(\cdot,t),\partial_t U(\cdot,t))-(V(\cdot,t),\partial_t V(\cdot,t))\|_H = 0.$$

Remark 3.6. *Given a triple $(V,\tilde{\phi},\tilde{t})$ as above, one can show there is always a unique nonlinear profile. Please see Remark 2.13 in [26] for the idea of proof. In particular, if $\tilde{t}$ is finite, then the nonlinear profile U is simply the solution to the equation* (5) *with the initial data $(U(\cdot,\tilde{t}),\partial_t U(\cdot,\tilde{t}))=(V(\cdot,\tilde{t}),\partial_t V(\cdot,\tilde{t}))$. We will also use the fact that the nonlinear profile automatically scatters in the positive time direction if $\tilde{t}=+\infty$.*

Definition 3.7 (Nonlinear Profiles). *For each linear profile V_j in a profile decomposition as given in Theorem 3.1, we assign a nonlinear profile U_j to it in the following way*

- *If $\lambda_j=0$, then U_j is chosen as the nonlinear profile associated to $(V_j,\phi(0),t_j)$;*
- *If $\lambda_j=1$, then U_j is chosen as the nonlinear profile associated to (V_j,ϕ,t_j);*
- *If $\lambda_j=\infty$, then U_j is chosen as the nonlinear profile associated to $(V_j,\phi(\infty),t_j)$.*

In either case, we use the notation I_j for the maximal lifespan of U_j and define

$$U_{j,n}(x,t) \doteq \frac{1}{\lambda_{j,n}^{3/2-s_p}} U_j\left(\frac{x}{\lambda_{j,n}},\frac{t-t_{j,n}}{\lambda_{j,n}}\right).$$

[3] A neighbourhood of infinity is $(M,+\infty)$, if $\tilde{t}=+\infty$; or $(-\infty,M)$, if $\tilde{t}=-\infty$.

Remark 3.8. *By the definition of nonlinear profile, for each j we have the limit*

$$\lim_{n\to\infty} \|(U_{j,n}(\cdot,0), \partial_t U_{j,n}(\cdot,0)) - \mathbf{S}_L(-t_{j,n})\mathbf{T}_{\lambda_{j,n}}(v_{0,j}, v_{1,j})\|_H = 0.$$

Lemma 3.9. *If $j \neq j'$, then we have the following almost orthogonality.*

$$\lim_{n\to\infty} \langle (U_{j,n}(\cdot,0), \partial_t U_{j,n}(\cdot,0)), (U_{j',n}(\cdot,0), \partial_t U_{j',n}(\cdot,0))\rangle_H = 0.$$

Proof. This is a direct corollary of Remark 3.8 and Lemma 3.2. □

Lemma 3.10 (Almost Orthogonality of $U_{j,n}$)**.** *Assume $\|\tilde{U}_j\|_{Y(I'_j)} < \infty$ for $j = 1, 2$. Let $\{(\lambda_{1,n}, t_{1,n})\}_{n\in\mathbb{Z}^+}$ and $\{(\lambda_{2,n}, t_{2,n})\}_{n\in\mathbb{Z}^+}$ be two "almost orthogonal" sequences of pairs, i.e.*

$$\lim_{n\to+\infty} \left(\frac{\lambda_{2,n}}{\lambda_{1,n}} + \frac{\lambda_{1,n}}{\lambda_{2,n}} + \frac{|t_{1,n} - t_{2,n}|}{\lambda_{1,n}}\right) = +\infty.$$

If $\{J_n\}$ is a sequence of time intervals, such that $J_n \subseteq (t_{1,n} + \lambda_{1,n} I'_1) \cap (t_{2,n} + \lambda_{2,n} I'_2)$ holds for all sufficiently large positive integers n, then we have

$$N(n) = \left\|\tilde{U}_{1,n}\tilde{U}_{2,n}\right\|_{L_t^{\frac{p}{1+s_p}} L_x^{\frac{p}{2-s_p}}(J_n\times\mathbb{R}^3)} \to 0, \qquad \text{as } n \to \infty.$$

Here $\tilde{U}_{j,n}$ is defined as usual

$$\tilde{U}_{j,n}(x,t) = \frac{1}{\lambda_{j,n}^{3/2-s_p}} \tilde{U}_j\left(\frac{x}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}}\right).$$

Proof. (See also Lemma 2.7 in [29]) First of all, by defining $\tilde{U}_j(x,t) = 0$ for $t \notin I'_j$, we can always assume $I'_j = \mathbb{R}$ and $J_n = \mathbb{R}$. Observing the continuity of the map

$$\Phi : Y(\mathbb{R}) \times Y(\mathbb{R}) \to l^\infty, \quad \Phi(\tilde{U}_1, \tilde{U}_2) = \left\{\left\|\tilde{U}_{1,n}\tilde{U}_{2,n}\right\|_{L_t^{\frac{p}{1+s_p}} L_x^{\frac{p}{2-s_p}}(\mathbb{R}\times\mathbb{R}^3)}\right\}_{n\in\mathbb{Z}^+};$$

we can also assume, without loss of generality, that

$$\left|\tilde{U}_j(x,t)\right| \leq M_j, \text{ for any } (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \qquad \operatorname{Supp}(\tilde{U}_j) \subseteq \{(x,t) : |x|, |t| < R_j\}$$

for each $j = 1, 2$ and some constants M_j and R_j, since the functions satisfying these conditions are dense in the space $Y(\mathbb{R})$. If the conclusion were false, we would find a sequence $n_1 < n_2 < n_3 < \cdots$ and a positive constant ε_0 such that $N(n_k) \geq \varepsilon_0$. There are three cases

(I) $\limsup_{k\to\infty} \lambda_{1,n_k}/\lambda_{2,n_k} = \infty$. In this case the product $\tilde{U}_{1,n_k}\tilde{U}_{2,n_k}$ is supported in the $(3+1)$-dimensional circular cylinder centred at $(0, t_{2,n_k})$ with radius $\lambda_{2,n_k} R_2$ and height $2\lambda_{2,n_k} R_2$ since $\tilde{U}_{2,n_k}$ is supported in this cylinder. On the other hand, we also have

$$\left|\tilde{U}_{1,n_k}\tilde{U}_{2,n_k}\right| \leq \lambda_{1,n_k}^{-3/2+s_p}\lambda_{2,n_k}^{-3/2+s_p} M_1 M_2.$$

A basic computation shows

$$N(n_k) = \left\|\tilde{U}_{1,n_k}\tilde{U}_{2,n_k}\right\|_{L_t^{\frac{p}{1+s_p}} L_x^{\frac{p}{2-s_p}}(\mathbb{R}\times\mathbb{R}^3)} \leq C(p) M_1 M_2 R_2^{3-2s_p}\left(\frac{\lambda_{2,n_k}}{\lambda_{1,n_k}}\right)^{3/2-s_p}.$$

This upper bound tends to zero as $\lambda_{1,n_k}/\lambda_{2,n_k} \to \infty$. Thus we have a contradiction.

(II) $\limsup_{k\to\infty} \lambda_{2,n_k}/\lambda_{1,n_k} = \infty$. This can be handled in the same way as case (I).

(III) $\lambda_{1,n_k} \simeq \lambda_{2,n_k}$. By the "almost orthogonality" of the sequences of pairs, we also have

$$\frac{|t_{1,n_k} - t_{2,n_k}|}{\lambda_{1,n_k}} \to \infty.$$

This implies $\mathrm{Supp}(\tilde{U}_{1,n_k}) \cap \mathrm{Supp}(\tilde{U}_{2,n_k}) = \varnothing$ when k is sufficiently large thus gives a contradiction.

□

Lemma 3.11. *Let $I'_j \subseteq I_j$ with $\|U_j(x,t)\|_{Y(I'_j)} < \infty$. Suppose that $\{J_n\}$ is a sequence of time intervals, so that given $J \in \mathbb{Z}^+$ we have $J_n \subseteq \cap_{j=1}^{J}(t_{j,n} + \lambda_{j,n} I'_j)$ for sufficiently large n. Then for each $J \in \mathbb{Z}^+$ the following limits hold.*

$$\lim_{n\to\infty} \left\| F\left(\sum_{j=1}^{J} U_{j,n}\right) - \sum_{j=1}^{J} F(U_{j,n}) \right\|_{Z(J_n)} = 0.$$

$$\limsup_{n\to\infty} \left\| \sum_{j=1}^{J} U_{j,n} \right\|_{Y(J_n)} \leq \left(\sum_{j=1}^{J} \|U_j\|_{Y(I'_j)}^p \right)^{1/p}.$$

Proof We use an induction

$$\begin{aligned}
& \limsup_{n\to\infty} \left\| F\left(\sum_{j=1}^{J} U_{j,n}\right) - F\left(\sum_{j=1}^{J-1} U_{j,n}\right) - F(U_{J,n}) \right\|_{Z(J_n)} \\
= \; & \limsup_{n\to\infty} \left\| \left[U_{J,n} \int_0^1 F'\left(\tau U_{J,n} + \sum_{j=1}^{J-1} U_{j,n}\right) d\tau \right] - \left[U_{J,n} \int_0^1 F'(\tau U_{J,n}) d\tau \right] \right\|_{Z(J_n)} \\
= \; & \limsup_{n\to\infty} \left\| \left(U_{J,n} \sum_{j=1}^{J-1} U_{j,n} \right) \left(\int_0^1 \int_0^1 F''\left(\tau U_{J,n} + \tilde{\tau} \sum_{j=1}^{J-1} U_{j,n}\right) d\tilde{\tau} d\tau \right) \right\|_{Z(J_n)} \\
\leq \; & \limsup_{n\to\infty} C_p \left(\sum_{j=1}^{J-1} \|U_{J,n} U_{j,n}\|_{L^{\frac{p}{1+s_p}} L^{\frac{p}{2-s_p}} (J_n \times \mathbb{R}^3)} \right) \left(\sum_{j=1}^{J} \|U_{j,n}\|_{Y(J_n)} \right)^{p-2} \\
\leq \; & \limsup_{n\to\infty} C_p \left(\sum_{j=1}^{J-1} \|U_{J,n} U_{j,n}\|_{L^{\frac{p}{1+s_p}} L^{\frac{p}{2-s_p}} (J_n \times \mathbb{R}^3)} \right) \left(\sum_{j=1}^{J} \|U_j\|_{Y(I'_j)} \right)^{p-2} \\
= \; & 0.
\end{aligned}$$

In the last step we use Lemma 3.10. This finishes the proof of the first limit. The second limit is a corollary:

$$\begin{aligned}
\limsup_{n\to\infty}\left\|\sum_{j=1}^{J} U_{j,n}\right\|_{Y(J_n)}^{p} &= \limsup_{n\to\infty}\left\|F\left(\sum_{j=1}^{J} U_{j,n}\right)\right\|_{Z(J_n)}\\
&\leq \limsup_{n\to\infty}\left(\sum_{j=1}^{J}\|F(U_{j,n})\|_{Z(J_n)}\right)\\
&\leq \limsup_{n\to\infty}\left(\sum_{j=1}^{J}\|U_{j,n}\|_{Y(J_n)}^{p}\right)\\
&\leq \sum_{j=1}^{J}\|U_j\|_{Y(I'_j)}^{p}.
\end{aligned}$$

Remark 3.12. *The same result still holds if we arbitrarily select a few profiles from U_j's. More precisely, if the inequality $\|U_{j_k}\|_{Y(I'_{j_k})} < \infty$ holds for each positive integers $j_1 < j_2 < \cdots < j_m$, then we have*

$$\begin{aligned}
\lim_{n\to\infty}\left\|F\left(\sum_{k=1}^{m} U_{j_k,n}\right) - \sum_{k=1}^{m} F(U_{j_k,n})\right\|_{Z(J_n)} &= 0;\\
\limsup_{n\to\infty}\left\|\sum_{k=1}^{m} U_{j_k,n}\right\|_{Y(J_n)} &\leq \left(\sum_{k=1}^{m}\|U_{j_k}\|_{Y(I'_{j_k})}^{p}\right)^{1/p};
\end{aligned}$$

as long as $J_n \subseteq \cap_{k=1}^{m}(t_{j_k,n} + \lambda_{j_k,n} I'_{j_k})$ holds for all sufficiently large n.

Lemma 3.13 (Commutator Estimate)**.** *Assume $I'_j \subseteq I_j$ so that $\|U_j\|_{Y(I'_j)} < \infty$. If we define the error term*

$$e_{j,n} = (\partial_t^2 - \Delta)U_{j,n} - \phi F(U_{j,n}),$$

then we have $\lim_{n\to\infty}\|e_{j,n}\|_{Z(\lambda_{j,n}I'_j + t_{j,n})} = 0$.

Proof. First of all, applying a space-time dilation we have

$$\partial_t^2 U_j - \Delta U_j = \tilde{\phi}(x)F(U_j) \implies (\partial_t^2 - \Delta)U_{j,n} = \tilde{\phi}\left(\frac{x}{\lambda_{j,n}}\right)F(U_{j,n}).$$

Here $\tilde{\phi}(x)$ is chosen as in Definition 3.7. Thus we have

$$\begin{aligned}
\|e_{j,n}\|_{Z(\lambda_{j,n}I'_j + t_{j,n})} &= \left\|\left(\tilde{\phi}\left(\frac{x}{\lambda_{j,n}}\right) - \phi(x)\right)F(U_{n,j})\right\|_{Z(\lambda_{j,n}I'_j + t_{j,n})}\\
&= \left\|\left(\tilde{\phi}(x) - \phi(\lambda_{j,n}x)\right)F(U_j)\right\|_{Z(I'_j)} \to 0
\end{aligned}$$

by the dominated convergence theorem and the (almost everywhere) point-wise limit $\phi(\lambda_{j,n}x) \to \tilde{\phi}(x)$. ☐

4 Compactness Procedure

In this section we prove the existence of a critical element and its compactness properties.

Theorem 4.1. *If $Sc(A)$ breaks down at $A = M$, then there exists a radial critical element u, also called a minimal blow-up*[4] *solution, such that it satisfies*

(i) Its maximal lifespan is $\mathbb{R}$;

(ii) It fails to scatter in both time directions with $\|u\|_{Y([0,\infty))} = \|u\|_{Y((-\infty,0])} = +\infty$.

(iii) The upper bound of its critical Sobolev norm is equal to M.

$$\sup_{t\in\mathbb{R}} \|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}(\mathbb{R}^3)} = M.$$

(iv) The set $\{(u(\cdot,t), \partial_t u(\cdot,t))|t \in \mathbb{R}\}$ is pre-compact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$.

Remark 4.2. *Theorem 4.1 also holds in non-radial case. In the non-radial case we need to consider spatial translations in the profile decomposition as well, i.e. we introduce additional parameters $x_{j,n} \in \mathbb{R}^3$ and substitute x by $x - x_{j,n}$ in (2). The way to deal with these addition parameters $x_{j,n}$ can be found in [49], which discusses energy critical wave equation with a similar nonlinearity. We also need to apply the non-radial version of Theorem 1.1 in this argument. Our rigidity argument in this work, however, only works for radial solutions.*

4.1 Setup

Now let us assume that the statement $Sc(A)$ breaks down at $A = M$. We spend the rest of this section to show the existence of a critical element. First of all, we can take a sequence of non-scattering radial solutions $v_n(x,t)$ with maximal lifespans $(-\tilde{T}_n^-, \tilde{T}_n^+)$ so that

$$\|v_n\|_{Y([0,\tilde{T}_n^+))} = +\infty; \qquad \sup_{t\in[0,\tilde{T}_n^+)} \|(v_n(\cdot,t), \partial_t v_n(\cdot,t))\|_H < M + 2^{-n}.$$

Using the first condition above, we can find a time $\tilde{t}_n \in [0, \tilde{T}_n^+)$ for each n, such that $\|v_n\|_{Y([0,\tilde{t}_n])} = 2^n$. Time translations then give a sequence of new solutions to (CP1) by the formula $u_n(x,t) \doteq v_n(x, t + \tilde{t}_n)$. These solutions $\{u_n\}$ satisfy:

(i) Each solution u_n blows up in the positive time direction, i.e $\|u_n\|_{Y([0,T_n^+))} = +\infty$.

(ii) $\|u_n\|_{Y((-T_n^-,0])} > 2^n$.

(iii) The inequality $\|(u_n(\cdot,t), \partial_t u_n(\cdot,t)\|_H < M + 2^{-n}$ holds for each $t \in [0, T_n^+)$ and for each $t < 0$ that satisfies $\|u_n\|_{Y([t,0])} \leq 2^n$.

Here the notation $(-T_n^-, T_n^+)$ represents the maximal lifespan of u_n. We apply the profile decomposition (Theorem 3.1) on the sequence of initial data $\{(u_{0,n}, u_{1,n})\} = \{(u_n(\cdot,0), \partial_t u_n(\cdot,0))\}$, introduce the nonlinear profiles U_j and then define the approximation solutions $U_{j,n}$ as described in Section 3. The conclusion (iv) of the profile decomposition gives

$$\sum_{j=1}^{\infty} \|V_j\|_H^2 = \sum_{j=1}^{\infty} \|(v_{j,0}, v_{j,1})\|_H^2 \leq M^2. \tag{6}$$

This implies that $\|(U_j(\cdot,t_j), \partial_t U_j(\cdot,t_j))\|_H \to 0$ as $j \to \infty$ since the definition of the nonlinear profiles implies (if $t_j = \pm\infty$, the left hand below is in the sense of limit as $t \to t_j$)

$$\|(U_j(\cdot,t_j), \partial_t U_j(\cdot,t_j))\|_H = \|V_j\|_H.$$

According to Theorem 2.9, it follows that U_j scatters in both time directions when $j > J_0$ for some sufficiently large J_0. In addition, we have

$$\|U_j\|_{Y(\mathbb{R})} \lesssim_p \|(U_j(\cdot,t_j), \partial_t U_j(\cdot,t_j))\|_H = \|V_j\|_H, \text{ if } j > J_0 \implies \sum_{j=J_0+1}^{\infty} \|U_j\|_{Y(\mathbb{R})}^p < \infty. \tag{7}$$

[4] Blow-up here simply means non-scattering. The solution does not necessarily blow up in finite time.

4.2 A Single Profile May Survive

In this subsection we show all but one profile must be zero. Let us suppose that there were at least two nonzero profiles, say U_1 and U_2. This implies

$$\varepsilon_0 = \min\{\|V_1\|_H, \|V_2\|_H\} = \min\{\|(v_{1,0}, v_{1,1})\|_H, \|(v_{2,0}, v_{2,1})\|_H\} > 0. \tag{8}$$

According to (6), we can always assume

$$\sum_{j=J_0+1}^{\infty} \|V_j\|_H^2 = \sum_{j=J_0+1}^{\infty} \|(v_{j,0}, v_{j,1})\|_H^2 < \frac{\varepsilon_0^2}{9}, \tag{9}$$

by possibly raising the value of J_0.

Lemma 4.3. *Given any $J > J_0$, we have*

$$\limsup_{n\to\infty} \left\| \sum_{j=J_0+1}^{J} (U_{j,n}(\cdot, 0), \partial_t U_{j,n}(\cdot, 0)) \right\|_H < \frac{\varepsilon_0}{3}.$$

Proof. By Remark 3.8 and Lemma 3.2, we have

$$\begin{aligned}\limsup_{n\to\infty} \left\| \sum_{j=J_0+1}^{J} (U_{j,n}(\cdot, 0), \partial_t U_{j,n}(\cdot, 0)) \right\|_H^2 &= \limsup_{n\to\infty} \left\| \sum_{j=J_0+1}^{J} \mathbf{S}_L(-t_{j,n}) \mathbf{T}_{\lambda_{j,n}}(v_{0,j}, v_{1,j}) \right\|_H^2 \\ &= \limsup_{n\to\infty} \sum_{j=J_0+1}^{J} \left\| \mathbf{S}_L(-t_{j,n}) \mathbf{T}_{\lambda_{j,n}}(v_{0,j}, v_{1,j}) \right\|_H^2 \\ &= \limsup_{n\to\infty} \sum_{j=J_0+1}^{J} \|V_j\|_H^2 < \frac{\varepsilon_0^2}{9}.\end{aligned}$$

□

Remark 4.4. *A similar argument as above shows that if $j_1 < j_2 < \cdots < j_m$ are positive integers, then we have*

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{m} (U_{j_k,n}(\cdot, 0), \partial_t U_{j_k,n}(\cdot, 0)) \right\|_H^2 = \sum_{k=1}^{m} \|V_{j_k}\|_H^2.$$

Asymptotic behaviour If $j > J_0$, then we have already shown that the nonlinear profile U_j scatters. We always choose $I'_j = \mathbb{R}$ in this case. Otherwise, if $j \leq J_0$, let us consider the behaviour of $U_j(x,t)$ as t goes to $+\infty$. There are two cases:

(I) U_j scatters in the positive time direction. Let us choose a time interval

$$I'_j = [t_j^-, +\infty) = \begin{cases} [t_j^-, +\infty), & \text{if } -t_{j,n}/\lambda_{j,n} \to t_j \in \mathbb{R}, \text{ here we fix } t_j^- \in (-\infty, t_j) \cap I_j; \\ (-\infty, +\infty), & \text{if } -t_{j,n}/\lambda_{j,n} \to -\infty; \\ [t_j^-, +\infty), & \text{if } -t_{j,n}/\lambda_{j,n} \to +\infty, \text{ here we fix } t_j^- \in I_j. \end{cases}$$

(II) U_j does not scatter with a maximal lifespan $(-T_j^-, T_j^+)$, thus $t_j < +\infty$. By our assumption on M (if $\lambda_j = 1$) or Proposition 2.14 (if $\lambda_j \in \{0, +\infty\}$), we always have

$$\sup_{t\in(t_j, T_j^+)} \|(U_j(\cdot, t), \partial_t U_j(\cdot, t))\|_H \geq M.$$

As a result, we can find a time $T_j \in (t_j, T_j^+)$ so that

$$\|(U_j(\cdot, T_j), \partial_t U_j(\cdot, T_j))\|_H > \sqrt{M^2 - \frac{1}{2}[\eta(\varepsilon_0/2)]^2}, \tag{10}$$

where the function η is the one given in Corollary 2.10, and choose

$$I'_j = [t_j^-, T_j] = \begin{cases} [t_j^-, T_j], & \text{if } -t_{j,n}/\lambda_{j,n} \to t_j \in \mathbb{R}, \text{ here we fix } t_j^- \in (-\infty, t_j) \cap I_j; \\ (-\infty, T_j], & \text{if } -t_{j,n}/\lambda_{j,n} \to -\infty. \end{cases}$$

In summary, we always have $\|U_j\|_{Y(I'_j)} < \infty$. If n is sufficiently large, we have $-t_{j,n}/\lambda_{j,n}$ is contained in the interior of I'_j for all j. Without loss of generality, we can assume that this happens for all j, n.

Approximation Solutions Now let us define

$$\bar{t}_n = \sup\left\{t > 0 \,\middle|\, t \in \cap_{j=1}^{J_0} \left(t_{j,n} + \lambda_{j,n} I'_j\right)\right\}.$$

This is either a positive number or undefined. The second case may happen only if all profiles U_j scatter in the positive time direction. In this case we interpret $\bar{t}_n \equiv \infty$. The definition actually implies

$$[0, \bar{t}_n] \subseteq \cap_{j=1}^{\infty} \left(t_{j,n} + \lambda_{j,n} I'_j\right).$$

According to the profile decomposition and Remark 3.8, we can write

$$(u_{0,n}, u_{1,n}) = \sum_{j=1}^{J} (U_{j,n}(\cdot, 0), \partial_t U_{j,n}(\cdot, 0)) + (w_{0,n}^J, w_{1,n}^J). \tag{11}$$

with

$$\limsup_{n\to\infty} \left\|(w_{0,n}^J, w_{1,n}^J)\right\|_H \leq M; \qquad \limsup_{n\to\infty} \left\|\mathbf{S}_{L,0}(t)(w_{0,n}^J, w_{1,n}^J)\right\|_{Y(\mathbb{R})} \to 0 \text{ as } J \to 0. \tag{12}$$

Please note that this new error term $(w_{0,n}^J, w_{1,n}^J)$ is different from the one given in the linear profile decomposition. It also covers the error created by the substitution of the linear profiles by their nonlinear counterparts. In addition, we have that the sum $S_{J,n} \doteq \sum_{j=1}^{J} U_{j,n}$ is a solution of the equation

$$\partial_t^2 u - \Delta u = \phi F(u) + Err_{J,n} \tag{13}$$

in the time interval $[0, \bar{t}_n]$. Here the error term $Err_{J,n}$ is defined by

$$Err_{J,n} = -\phi F\left(\sum_{j=1}^{J} U_{j,n}\right) + \sum_{j=1}^{J} \phi F(U_{j,n}) + \sum_{j=1}^{J} \left[\partial_t^2 U_{j,n} - \Delta U_{j,n} - \phi F(U_{j,n})\right].$$

By Lemma 3.11, Lemma 3.13 and the inequality (7), we have

$$\lim_{n\to\infty} \|Err_{J,n}\|_{Z([0,\bar{t}_n])} = 0; \tag{14}$$

$$\limsup_{n\to\infty} \|S_{J,n}\|_{Y([0,\bar{t}_n])}^p \leq \sum_{j=1}^{J} \|U_j\|_{Y(I'_j)}^p \leq \sum_{j=1}^{J_0} \|U_j\|_{Y(I'_j)}^p + \sum_{j=J_0+1}^{\infty} \|U_j\|_{Y(\mathbb{R})}^p < \infty. \tag{15}$$

The upper bound in the second line above is independent of J.

Proposition 4.5. *Let U_j be non-linear profiles introduced in Subsection 4.1. The exists at least one profile U_j so that it does not scatter at the positive direction.*

Proof. If it were false, then we would have $\bar{t}_n = \infty$ for all $n \in \mathbb{Z}^+$. We can choose a sequence $\{J_k, n_k\}_{k\in\mathbb{Z}^+}$, such that

$$\lim_{k\to\infty} \left\| \mathbf{S}_{L,0}(t)(w_{0,n_k}^{J_k}, w_{1,n_k}^{J_k}) \right\|_{Y(\mathbb{R})} = 0;$$

$$\lim_{k\to\infty} \|Err_{J_k,n_k}\|_{Z([0,\infty))} = 0;$$

$$\|S_{J_k,n_k}\|_{Y([0,\infty))}^p \leq \sum_{j=1}^{J_0} \|U_j\|_{Y(I_j')}^p + \sum_{j=J_0+1}^{\infty} \|U_j\|_{Y(\mathbb{R})}^p + 1 < \infty.$$

Observing equation (11) and (13), we can apply the long-time perturbation theory on the approximation solutions S_{J_k,n_k}, the initial data (u_{0,n_k}, u_{1,n_k}) as well as the time interval $[0,\infty)$ and finally obtain that u_{n_k} scatters in the positive time direction if k is sufficiently large. This is a contradiction. □

Now we know $\bar{t}_n \in (0,\infty)$. In addition, for each large n, there is a $j \leq J_0$ such that U_j does not scatter in the positive time direction with $\bar{t}_n = \lambda_{j,n} T_j + t_{j,n}$. By passing to a subsequence, we can assume that the same $j = j_0$ works for all sufficiently large n.

Proposition 4.6. *The pairs $(U_{j_0,n}(\cdot,\bar{t}_n), \partial_t U_{j_0,n}(\cdot,\bar{t}_n))$ and $(U_{j,n}(\cdot,\bar{t}_n), \partial_t U_{j,n}(\cdot,\bar{t}_n))$ are almost orthogonal in the space $H = \dot{H}^{s_p} \times \dot{H}^{s_p-1}$ if $j \neq j_0$. Namely, we have*

$$\lim_{n\to\infty} \langle (U_{j_0,n}(\cdot,\bar{t}_n), \partial_t U_{j_0,n}(\cdot,\bar{t}_n)), (U_{j,n}(\cdot,\bar{t}_n), \partial_t U_{j,n}(\cdot,\bar{t}_n)) \rangle_H = 0.$$

Proof. We have

$$(U_{j_0,n}(\bar{t}_n), \partial_t U_{j_0,n}(\bar{t}_n)) = \left(\frac{1}{\lambda_{j_0,n}^{3/2-s_p}} U_{j_0}\left(\frac{x}{\lambda_{j_0,n}}, T_{j_0}\right), \frac{1}{\lambda_{j,n}^{5/2-s_p}} \partial_t U_{j_0}\left(\frac{x}{\lambda_{j_0,n}}, T_{j_0}\right) \right);$$

$$(U_{j,n}(\bar{t}_n), \partial_t U_{j,n}(\bar{t}_n)) = \left(\frac{1}{\lambda_{j,n}^{3/2-s_p}} U_j\left(\frac{x}{\lambda_{j,n}}, \frac{\bar{t}_n - t_{j,n}}{\lambda_{j,n}}\right), \frac{1}{\lambda_{j,n}^{5/2-s_p}} \partial_t U_j\left(\frac{x}{\lambda_{j,n}}, \frac{\bar{t}_n - t_{j,n}}{\lambda_{j,n}}\right) \right).$$

Since the dot product is dilation-invariant, we can rewrite the dot product in question into

$$\left\langle (U_{j_0}(x,T_{j_0}), \partial_t U_{j_0}(x,T_{j_0})), \left(\left(\frac{\lambda_{j_0,n}}{\lambda_{j,n}}\right)^{\frac{3}{2}-s_p} U_j\left(\frac{\lambda_{j_0,n} x}{\lambda_{j,n}}, t_n'\right), \left(\frac{\lambda_{j_0,n}}{\lambda_{j,n}}\right)^{\frac{5}{2}-s_p} \partial_t U_j\left(\frac{\lambda_{j_0,n} x}{\lambda_{j,n}}, t_n'\right) \right) \right\rangle.$$

Here $t_n' = \dfrac{\bar{t}_n - t_{j,n}}{\lambda_{j,n}} = \dfrac{T_{j_0}\lambda_{j_0,n} + t_{j_0,n} - t_{j,n}}{\lambda_{j,n}} \in I_j'$. By the inequality $|t_n'| \geq -\dfrac{\lambda_{j_0,n}}{\lambda_{j,n}}|T_{j_0}| + \dfrac{|t_{j_0,n} - t_{j,n}|}{\lambda_{j,n}}$ and the almost orthogonal condition, we have

$$\lim_{n\to\infty} \frac{\lambda_{j_0,n}}{\lambda_{j,n}} + \frac{\lambda_{j,n}}{\lambda_{j_0,n}} + |t_n'| = +\infty.$$

By the fact that $U_j(t)$ always scatters in the corresponding time direction whenever I_j' contains a neighbourhood of ∞ or $-\infty$, the limit above implies that the second factor in the dot product above converges weakly to zero in H, this finishes the proof. □

Approximation Solutions without $U_{j_0,n}$ Now let us define $(J \geq J_0)$

$$S_{J,n}' = \sum_{1\leq j\leq J, j\neq j_0} U_{j,n}.$$

The function $S'_{J,n}$ is the solution to

$$\partial_t^2 u - \Delta u = \phi F(u) + Err'_{J,n} \tag{16}$$

The error term is given by

$$Err'_{J,n} = -\phi F\left(S'_{J,n}\right) + \sum_{1\leq j\leq J, j\neq j_0} \phi F(U_{j,n}) + \sum_{1\leq j\leq J, j\neq j_0} \left[\partial_t^2 U_{j,n} - \Delta U_{j,n} - \phi F(U_{j,n})\right].$$

By Remark 3.12, and Lemma 3.13, we have

$$\lim_{n\to\infty} \|Err'_{J,n}\|_{Z([0,\bar{t}_n])} = 0; \tag{17}$$

$$\limsup_{n\to\infty} \left\|S'_{J,n}\right\|^p_{Y([0,\bar{t}_n])} \leq \sum_{1\leq j\leq J, j\neq j_0} \|U_j\|^p_{Y(I'_j)} \leq \sum_{j=1}^{J_0} \|U_j\|^p_{Y(I'_j)} + \sum_{j=J_0+1}^{\infty} \|U_j\|^p_{Y(\mathbb{R})} < \infty. \tag{18}$$

Choice of $n(J)$ For each $J > J_0$, we can choose a large positive integer $n(J)$ so that (See (12), (14), (15), (17), (18), Lemma 4.3 and Lemma 4.6)

$$n(J) > J; \tag{19}$$

$$\left\|S_{J,n(J)}\right\|_{Y([0,\bar{t}_{n(J)}])} \leq \left(\sum_{j=1}^{J_0} \|U_j\|^p_{Y(I'_j)} + \sum_{j=J_0+1}^{\infty} \|U_j\|^p_{Y(\mathbb{R})} + 1\right)^{1/p}; \tag{20}$$

$$\|S'_{J,n(J)}\|_{Y([0,\bar{t}_{n(J)}])} \leq \left(\sum_{j=1}^{J_0} \|U_j\|^p_{Y(I'_j)} + \sum_{j=J_0+1}^{\infty} \|U_j\|^p_{Y(\mathbb{R})} + 1\right)^{1/p}; \tag{21}$$

$$\left\|\sum_{j=J_0+1}^{J} \left(U_{j,n(J)}(\cdot,0), \partial_t U_{j,n(J)}(\cdot,0)\right)\right\|_H \leq \frac{\varepsilon_0}{3}; \tag{22}$$

$$\|Err_{J,n(J)}\|_{Z([0,\bar{t}_{n(J)}])} \leq 2^{-J}; \tag{23}$$

$$\|Err'_{J,n(J)}\|_{Z([0,\bar{t}_{n(J)}])} \leq 2^{-J}; \tag{24}$$

$$\left|\left\langle \begin{pmatrix} U_{j_0,n(J)}(\bar{t}_{n(J)}) \\ \partial_t U_{j_0,n(J)}(\bar{t}_{n(J)}) \end{pmatrix}, \begin{pmatrix} U_{j,n(J)}(\bar{t}_{n(J)}) \\ \partial_t U_{j,n(J)}(\bar{t}_{n(J)}) \end{pmatrix} \right\rangle_H\right| \leq \frac{2^{-J}}{J}, \text{ if } 1\leq j\leq J,\ j\neq j_0; \tag{25}$$

$$\lim_{J\to\infty} \left\|\mathbf{S}_{L,0}(t)(w^J_{0,n(J)}, w^J_{1,n(J)})\right\|_{Y(\mathbb{R})} = 0; \tag{26}$$

$$\left\|(w^J_{0,n(J)}, w^J_{1,n(J)})\right\|_H \leq M + 1. \tag{27}$$

Combining the equation (11), (13) and the inequalities (20), (23), (26), (27), we can apply long-time perturbation theory on the approximation solution $S_{J,n(J)}$, the initial data $(u_{0,n(J)}, u_{1,n(J)})$ as well as the time interval $[0, \bar{t}_{n(J)}]$, conclude $\bar{t}_{n(J)}$ is in the maximal lifespan of $u_{n(J)}$ and obtain

$$\lim_{J\to\infty} \left\|u_{n(J)} - S_{J,n(J)}\right\|_{Y([0,\bar{t}_{n(J)}])} = 0.$$

$$\lim_{J\to\infty} \left\|\begin{pmatrix} u_{n(J)}(\cdot,\bar{t}_{n(J)}) \\ \partial_t u_{n(J)}(\cdot,\bar{t}_{n(J)}) \end{pmatrix} - \begin{pmatrix} S_{J,n(J)}(\cdot,\bar{t}_{n(J)}) \\ \partial_t S_{J,n(J)}(\cdot,\bar{t}_{n(J)}) \end{pmatrix} - \mathbf{S}_L(\bar{t}_{n(J)}) \begin{pmatrix} w^J_{0,n(J)} \\ w^J_{1,n(J)} \end{pmatrix}\right\|_H = 0,$$

if J is sufficiently large. Therefore we have

$$\begin{aligned} &\limsup_{J\to\infty} \left\|\left(u_{n(J)}(\cdot,\bar{t}_{n(J)}), \partial_t u_{n(J)}(\cdot,\bar{t}_{n(J)})\right)\right\|_H \\ &\quad = \limsup_{J\to\infty} \left\|\begin{pmatrix} S_{J,n(J)}(\cdot,\bar{t}_{n(J)}) \\ \partial_t S_{J,n(J)}(\cdot,\bar{t}_{n(J)}) \end{pmatrix} + \mathbf{S}_L(\bar{t}_{n(J)}) \begin{pmatrix} w^J_{0,n(J)} \\ w^J_{1,n(J)} \end{pmatrix}\right\|_H. \end{aligned} \tag{28}$$

By (26), (27), Lemma 3.4 and the identity

$$\begin{pmatrix} U_{j_0,n(J)}(\cdot,\bar{t}_{n(J)}) \\ \partial_t U_{j_0,n(J)}(\cdot,\bar{t}_{n(J)}) \end{pmatrix} = \mathbf{T}_{\lambda_{j_0,n(J)}} \begin{pmatrix} U_{j_0}(\cdot,T_{j_0}) \\ \partial_t U_{j_0}(\cdot,T_{j_0}) \end{pmatrix},$$

We have

$$\lim_{J\to\infty} \left\langle \begin{pmatrix} U_{j_0,n(J)}(\cdot,\bar{t}_{n(J)}) \\ \partial_t U_{j_0,n(J)}(\cdot,\bar{t}_{n(J)}) \end{pmatrix}, \mathbf{S}_L(\bar{t}_{n(J)}) \begin{pmatrix} w_{0,n(J)}^J \\ w_{1,n(J)}^J \end{pmatrix} \right\rangle_H = 0.$$

Combining this with (25) and (28), we obtain

$$\begin{aligned}
&\limsup_{J\to\infty} \left\| \left(u_{n(J)}(\cdot,\bar{t}_{n(J)}), \partial_t u_{n(J)}(\cdot,\bar{t}_{n(J)})\right) \right\|_H^2 \\
&= \limsup_{J\to\infty} \left(\left\| \begin{pmatrix} U_{j_0,n(J)}(\cdot,\bar{t}_{n(J)}) \\ \partial_t U_{j_0,n(J)}(\cdot,\bar{t}_{n(J)}) \end{pmatrix} \right\|_H^2 + \left\| \begin{pmatrix} S'_{J,n(J)}(\cdot,\bar{t}_{n(J)}) \\ \partial_t S'_{J,n(J)}(\cdot,\bar{t}_{n(J)}) \end{pmatrix} + \mathbf{S}_L(\bar{t}_{n(J)}) \begin{pmatrix} w_{0,n(J)}^J \\ w_{1,n(J)}^J \end{pmatrix} \right\|_H^2 \right) \\
&= \left\| \begin{pmatrix} U_{j_0}(\cdot,T_{j_0}) \\ \partial_t U_{j_0}(\cdot,T_{j_0}) \end{pmatrix} \right\|_H^2 + \limsup_{J\to\infty} \left\| \begin{pmatrix} S'_{J,n(J)}(\cdot,\bar{t}_{n(J)}) \\ \partial_t S'_{J,n(J)}(\cdot,\bar{t}_{n(J)}) \end{pmatrix} + \mathbf{S}_L(\bar{t}_{n(J)}) \begin{pmatrix} w_{0,n(J)}^J \\ w_{1,n(J)}^J \end{pmatrix} \right\|_H^2 . \qquad (29)
\end{aligned}$$

Now let us find a lower bound on the second term above. First of all, by (22) we have

$$\left\| \begin{pmatrix} S'_{J,n(J)}(\cdot,0) \\ \partial_t S'_{J,n(J)}(\cdot,0) \end{pmatrix} + \begin{pmatrix} w_{0,n(J)}^J \\ w_{1,n(J)}^J \end{pmatrix} \right\|_H \geq \left\| \begin{pmatrix} S'_{J_0,n(J)}(\cdot,0) \\ \partial_t S'_{J_0,n(J)}(\cdot,0) \end{pmatrix} + \begin{pmatrix} w_{0,n(J)}^J \\ w_{1,n(J)}^J \end{pmatrix} \right\|_H - \frac{\varepsilon_0}{3}. \qquad (30)$$

By (26), (27) and the identity $\left(\frac{-t_{j,n(J)}}{\lambda_{j,n(J)}} \in I'_j\right)$

$$\begin{pmatrix} S'_{J_0,n(J)}(\cdot,0) \\ \partial_t S'_{J_0,n(J)}(\cdot,0) \end{pmatrix} = \sum_{1\leq j\leq J_0, j\neq j_0} \mathbf{T}_{\lambda_{j,n(J)}} \begin{pmatrix} U_j\left(\cdot, \frac{-t_{j,n(J)}}{\lambda_{j,n(J)}}\right) \\ \partial_t U_j\left(\cdot, \frac{-t_{j,n(J)}}{\lambda_{j,n(J)}}\right) \end{pmatrix},$$

we can apply Lemma 3.4 and obtain

$$\lim_{J\to\infty} \left\langle \begin{pmatrix} S'_{J_0,n(J)}(\cdot,0) \\ \partial_t S'_{J_0,n(J)}(\cdot,0) \end{pmatrix}, \begin{pmatrix} w_{0,n(J)}^J \\ w_{1,n(J)}^J \end{pmatrix} \right\rangle_H = 0.$$

Using this limit, Remark 4.4 and the lower bound (8), we have

$$\begin{aligned}
\liminf_{J\to\infty} \left\| \begin{pmatrix} S'_{J_0,n(J)}(\cdot,0) \\ \partial_t S'_{J_0,n(J)}(\cdot,0) \end{pmatrix} + \begin{pmatrix} w_{0,n(J)}^J \\ w_{1,n(J)}^J \end{pmatrix} \right\|_H^2 &\geq \liminf_{J\to\infty} \left\| \begin{pmatrix} S'_{J_0,n(J)}(\cdot,0) \\ \partial_t S'_{J_0,n(J)}(\cdot,0) \end{pmatrix} \right\|_H^2 \\
&= \sum_{1\leq j\leq J_0, j\neq j_0} \|V_j\|_H^2 \geq \varepsilon_0^2.
\end{aligned}$$

Combining this estimate with (30), we have an estimate on the norm of

$$\begin{pmatrix} u'_{0,J} \\ u'_{1,J} \end{pmatrix} \doteq \begin{pmatrix} S'_{J,n(J)}(\cdot,0) \\ \partial_t S'_{J,n(J)}(\cdot,0) \end{pmatrix} + \begin{pmatrix} w_{0,n(J)}^J \\ w_{1,n(J)}^J \end{pmatrix} \qquad (31)$$

given by

$$\liminf_{J\to\infty} \|(u'_{0,J}, u'_{1,J})\|_H \geq \frac{2\varepsilon_0}{3}. \qquad (32)$$

Let u'_J be the solution to (CP1) with the initial data $(u'_{0,J}, u'_{1,J})$. By equation (16), identity (31), estimates (21), (24), (26), we can apply the long-time perturbation theory on the approximation solution $S'_{J,n(J)}$, the initial data $(u'_{0,J}, u'_{1,J})$ as well as the time interval $[0, \bar{t}_{n(J)}]$, thus conclude that $\bar{t}_{n(J)}$ is contained in the maximal lifespan of u'_J for large J with

$$\lim_{J\to\infty} \left\| \begin{pmatrix} u'_J(\cdot, \bar{t}_{n(J)}) \\ \partial_t u'_J(\cdot, \bar{t}_{n(J)}) \end{pmatrix} - \begin{pmatrix} S'_{J,n(J)}(\cdot, \bar{t}_{n(J)}) \\ \partial_t S'_{J,n(J)}(\cdot, \bar{t}_{n(J)}) \end{pmatrix} - \mathbf{S}_L(\bar{t}_{n(J)}) \begin{pmatrix} w^J_{0,n(J)} \\ w^J_{1,n(J)} \end{pmatrix} \right\|_H = 0, \tag{33}$$

This implies

$$\begin{aligned} \limsup_{J\to\infty} \left\| \begin{pmatrix} S'_{J,n(J)}(\cdot, \bar{t}_{n(J)}) \\ \partial_t S'_{J,n(J)}(\cdot, \bar{t}_{n(J)}) \end{pmatrix} + \mathbf{S}_L(\bar{t}_{n(J)}) \begin{pmatrix} w^J_{0,n(J)} \\ w^J_{1,n(J)} \end{pmatrix} \right\|_H^2 &= \limsup_{J\to\infty} \left\| \begin{pmatrix} u'_J(\cdot, \bar{t}_{n(J)}) \\ \partial_t u'_J(\cdot, \bar{t}_{n(J)}) \end{pmatrix} \right\|_H^2 \\ &\geq [\eta(\varepsilon_0/2)]^2. \end{aligned}$$

In the last step above, we use the lower bound on the initial data (32) and Corollary 2.10. Combining this with (10) and (29), we obtain

$$\limsup_{J\to\infty} \left\| \left(u_{n(J)}(\cdot, \bar{t}_{n(J)}), \partial_t u_{n(J)}(\cdot, \bar{t}_{n(J)})\right) \right\|_H^2 \geq M^2 - \frac{1}{2}[\eta(\varepsilon_0/2)]^2 + [\eta(\varepsilon_0/2)]^2 > M^2.$$

This contradicts with our assumption (iii) on u_n.

4.3 Extraction of a Critical Element

Now there is only one nonzero profile U_1 with a maximal lifespan I_1. The profile decomposition can be rewritten into

$$\begin{aligned} (u_{0,n}, u_{1,n}) &= \left(U_{1,n}(\cdot, 0), \partial_t U_{1,n}(\cdot, 0)\right) + (w_{0,n}, w_{1,n}) \\ &= \left(\frac{1}{\lambda_{1,n}^{3/2-s_p}} U_1\left(\frac{\cdot}{\lambda_{1,n}}, \frac{-t_{1,n}}{\lambda_{1,n}}\right), \frac{1}{\lambda_{1,n}^{5/2-s_p}} \partial_t U_1\left(\frac{\cdot}{\lambda_{1,n}}, \frac{-t_{1,n}}{\lambda_{1,n}}\right)\right) + (w_{0,n}, w_{1,n}) \end{aligned} \tag{34}$$

Here $U_{1,n}$ is defined for all $t \in t_{1,n} + \lambda_{1,n} I_1$ and satisfies the equation

$$\partial_t^2 u - \Delta u = \phi F(u) + Err_{1,n}, \tag{35}$$

so that if $I'_1 \subseteq I_1$ is any time interval such that $\|U_1\|_{Y(I'_1)} < \infty$, then we have the identity

$$\|U_{1,n}\|_{Y(t_{1,n}+\lambda_{1,n} I'_1)} = \|U_1\|_{Y(I'_1)} < \infty, \tag{36}$$

and limits

$$\limsup_{n\to\infty} \|(w_{0,n}, w_{1,n})\|_H \leq M; \tag{37}$$

$$\lim_{n\to\infty} \|\mathbf{S}_{L,0}(t)(w_{0,n}, w_{1,n})\|_{Y(\mathbb{R})} = 0; \tag{38}$$

$$\lim_{n\to\infty} \|Err_{1,n}\|_{Z(t_{1,n}+\lambda_{1,n} I'_1)} = 0; \tag{39}$$

$$\lim_{t\to t_1} \|(U_1(\cdot, t), \partial_t U_1(\cdot, t))\|_H = \|V_1\|_H \leq M. \tag{40}$$

We have already shown that U_1 does not scatter in the positive time direction by Proposition 4.5, thus here $-t_{1,n}/\lambda_{1,n} \to t_1 < +\infty$.

U_1 fails to scatter in the negative direction Let us do the negative direction by a contradiction. If U_1 scatters in the negative time direction, we can choose an interval $I'_1 = (-\infty, t_1^+]$, where $t_1^+ > t_1$ is a fixed time in I_1, so that $\|U_1\|_{Y(I'_1)} < \infty$. By the profile decomposition (34), the fact that $U_{1,n}$ satisfies equation (35), inequality (36), limits (38), (39) and the fact that

$(-\infty, 0] \subseteq t_{1,n} + \lambda_{1,n} I'_1$ holds for large n, we are able to apply the long-time perturbation theory on the approximation solution $U_{1,n}$, the initial data $(u_{0,n}, u_{1,n})$ as well as the time interval $(-\infty, 0]$, thus to conclude that $(-\infty, 0]$ is contained in the maximal lifespan of u_n if n is large with

$$\lim_{n\to\infty} \|u_n - U_{1,n}\|_{Y((-\infty,0])} = 0.$$

This means that $\limsup_{n\to\infty} \|u_n\|_{Y((-\infty,0])} \le \|U_1\|_{Y(I'_1)} < \infty$. This contradicts with our assumption (ii) on $\{u_n\}$. One direct corollary is that t_1 is finite.

Upper bound on the norm of U_1 Since t_1 is finite, we have $\|(U_1(\cdot, t_1), \partial_t U_1(\cdot, t_1))\|_H \le M$ by (40). Now we claim

$$\sup_{t\in I_1} \|(U_1(\cdot, t), \partial_t U_1(\cdot, t))\|_H \le M. \tag{41}$$

If this were false, we would have $T_1 \in I_1 \setminus \{t_1\}$ such that $\|(U_1(\cdot, T_1), \partial_t U_1(\cdot, T_1))\|_H > M$. A Contradiction can be found immediately by the following lemma.

Lemma 4.7. *Assume $T \in I_1 \setminus \{t_1\}$. If we define $\bar{t}_n = t_{1,n} + \lambda_{1,n} T$ and*

$$J_n = \begin{cases} [\bar{t}_n, 0] & \text{if } T < t_1; \\ [0, \bar{t}_n] & \text{if } T > t_1; \end{cases}$$

then for sufficiently large n we have

(i) $\bar{t}_n$ is contained in the maximal lifespan of u_n;

(ii) $\limsup_{n\to\infty} \|u_n\|_{Y(J_n)} < \infty$;

(iii) $\limsup_{n\to\infty} \left\| \begin{pmatrix} u_n(\cdot, \bar{t}_n) \\ \partial_t u_n(\cdot, \bar{t}_n) \end{pmatrix} \right\|_H^2 \ge \left\| \begin{pmatrix} U_1(\cdot, T) \\ \partial_t U_1(\cdot, T) \end{pmatrix} \right\|_H^2 + \limsup_{n\to\infty} \left\| \begin{pmatrix} w_{0,n} \\ w_{1,n} \end{pmatrix} \right\|_H^2.$

Proof. Without loss of generality we can assume $T < t_1$. Let us pick up a real number $t_1^+ \in (t_1, \infty) \cap I_1$ and choose $I'_1 = [T, t_1^+] \subset I_1$. Thus we have $\|U_1\|_{Y(I'_1)} < \infty$. One can also check that $\bar{t}_n < 0$ and $[\bar{t}_n, 0] \subseteq t_{1,n} + \lambda_{1,n} I'_1$ hold if n is sufficiently large. As a result, we have

$$\|U_{1,n}\|_{Y([\bar{t}_n,0])} \le \|U_1\|_{Y(I'_1)} < \infty. \tag{42}$$

Now we are able to apply the long-time perturbation theory on approximation solution $U_{1,n}$, initial data $(u_{0,n}, u_{1,n})$ and the interval $[\bar{t}_n, 0]$ if n is large by using the profile decomposition (34), the approximation equation (35), limits (37), (38), (39) and the uniform upper bound (42). We conclude that $[\bar{t}_n, 0]$ is contained in the lifespan of u_n if n is large and

$$\lim_{n\to\infty} \|u_n - U_{1,n}\|_{Y([\bar{t}_n,0])} = 0; \tag{43}$$

$$\lim_{n\to\infty} \left\| \begin{pmatrix} u_n(\cdot, \bar{t}_n) \\ \partial_t u_n(\cdot, \bar{t}_n) \end{pmatrix} - \begin{pmatrix} U_{1,n}(\cdot, \bar{t}_n) \\ \partial_t U_{1,n}(\cdot, \bar{t}_n) \end{pmatrix} - \mathbf{S}_L(\bar{t}_n) \begin{pmatrix} w_{0,n} \\ w_{1,n} \end{pmatrix} \right\|_H = 0. \tag{44}$$

Thus the conclusion (ii) can be proved by using inequality (42) and limit (43):

$$\limsup_{n\to\infty} \|u_n\|_{Y([\bar{t}_n,0])} = \limsup_{n\to\infty} \|U_{1,n}\|_{Y([\bar{t}_n,0])} \le \|U_1\|_{Y(I'_1)} < \infty.$$

Furthermore, limit (44) gives us

$$\begin{aligned}\limsup_{n\to\infty}\left\|\begin{pmatrix}u_n(\cdot,\bar{t}_n)\\ \partial_t u_n(\cdot,\bar{t}_n)\end{pmatrix}\right\|_H^2 &= \limsup_{n\to\infty}\left\|\begin{pmatrix}U_{1,n}(\cdot,\bar{t}_n)\\ \partial_t U_{1,n}(\cdot,\bar{t}_n)\end{pmatrix}+\mathbf{S}_L(\bar{t}_n)\begin{pmatrix}w_{0,n}\\ w_{1,n}\end{pmatrix}\right\|_H^2\\ &= \limsup_{n\to\infty}\left\|\mathbf{T}_{\lambda_{1,n}}\begin{pmatrix}U_1(\cdot,T)\\ \partial_t U_1(\cdot,T)\end{pmatrix}+\mathbf{S}_L(\bar{t}_n)\begin{pmatrix}w_{0,n}\\ w_{1,n}\end{pmatrix}\right\|_H^2\\ &\geq \lim_{n\to\infty}\left\|\mathbf{T}_{\lambda_{1,n}}\begin{pmatrix}U_1(\cdot,T)\\ \partial_t U_1(\cdot,T)\end{pmatrix}\right\|_H^2+\limsup_{n\to\infty}\left\|\mathbf{S}_L(\bar{t}_n)\begin{pmatrix}w_{0,n}\\ w_{1,n}\end{pmatrix}\right\|_H^2\\ &= \left\|\begin{pmatrix}U_1(\cdot,T)\\ \partial_t U_1(\cdot,T)\end{pmatrix}\right\|_H^2+\limsup_{n\to\infty}\left\|\begin{pmatrix}w_{0,n}\\ w_{1,n}\end{pmatrix}\right\|_H^2.\end{aligned}$$

Here we use (37), (38) and apply Lemma 3.4. This finishes the proof of conclusion (iii). □

The nonlinear profile is a critical element Now we can conclude that U_1 is a solution to $\partial_t^2 u - \Delta u = \phi F(u)$. This is equivalent to saying $\lambda_{1,n} \to \lambda_1 = 1$. If this were false, we would have that U_1 is a solution to $\partial_t^2 u - \Delta u = cF(u)$, where c is a constant. Using (41) and applying Proposition 2.14, we conclude that U_1 scatters in both two time directions. This is a contradiction.

The least upper bound of H norm Finally we can conclude

$$\sup_{t\in I_1}\|(U_1(\cdot,t),\partial_t U_1(\cdot,t))\|_H = M. \tag{45}$$

According to (41), we only to show the upper bounds above can not be smaller than M. This is trivial since we have assumed that Sc(M) holds and we have shown that U_1 fails to scatter in both two time directions.

Summary Now we have extracted a critical element U_1. It blows up in both time directions and satisfies (45). In order to finish the proof of Theorem 4.1, we still need to show that the maximal lifespan $I_1 = \mathbb{R}$ and the pre-compactness of the set $\{(U_1(\cdot,t),\partial_t U_1(\cdot,t))|t\in\mathbb{R}\}$. This will be done in Subsection 4.4.

4.4 Almost Periodicity

We start by proving some further properties concerning the single-profile representation (34).

Proposition 4.8. *The sequence of error terms $(w_{0,n}, w_{1,n})$ in (34) converges to zero strongly in the space H. Namely, we have*

$$\lim_{n\to\infty}\|(w_{0,n},w_{1,n})\|_H = 0.$$

Proof. By (45), we can pick up a sequence $T_k \in I_1 \setminus \{t_1\}$, such that $\|(U_1(\cdot,T_k),\partial_t U_1(\cdot,T_k))\|_H > M - 2^{-k}$. Applying Lemma 4.7 with $T = T_k$, we obtain

$$\limsup_{n\to\infty}\|(w_{0,n},w_{1,n})\|_H^2 \leq M^2 - (M-2^{-k})^2$$

by its conclusion (ii), (iii) and our assumption (iii) on $\{u_n\}$. We can finish the proof by making $k\to\infty$. □

Compactness of initial data Now we can give a compactness result.

Proposition 4.9. *Let $\{(u_{0,n}, u_{1,n})\}_{n\in\mathbb{Z}^+}$ be a sequence of radial initial data and $\{u_n\}$ be their corresponding solutions to (CP1), so that $\{u_n\}$ satisfies the conditions (i), (ii) and (iii) listed at the beginning of Subsection 4.1. Then there exists a subsequence of the initial data, so that it converges strongly in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$.*

Proof. We have already found a subsequence, still denoted $\{(u_{0,n}, u_{1,n})\}$, so that the single-profile representation (34) holds. Combining the facts $\lambda_{1,n} \to \lambda_1 = 1$, $-t_{1,n}/\lambda_{1,n} \to t_1 \in I_1$, $(U_1(\cdot,t), \partial_t U_1(\cdot,t)) \in C(I_1; \dot{H}^{s_p} \times \dot{H}^{s_p-1})$ and Proposition 4.8, we have the strong limit

$$(u_{0,n}, u_{1,n}) \to (U_1(\cdot,t_1), \partial_t U_1(\cdot,t_1)) \quad \text{in } \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3).$$

□

Almost periodicity of the critical element Now we are able to conclude the set

$$\{(u(\cdot,t), \partial_t u(\cdot,t)| t \in I_1\}$$

is pre-compact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. In fact, if $\{t_n\}_{n\in\mathbb{Z}^+}$ is a sequence of time in I_1, then the time-translated solutions $U_1(t+t_n)$ solve (CP1) and satisfy the conditions (i), (ii), (iii) listed at the beginning of Subsection 4.1, with initial data $(U_1(\cdot,t_n), \partial_t U_1(\cdot,t_n))$. Now we can apply Proposition 4.9, conclude that the sequence $\{(U_1(\cdot,t_n), \partial_t U_1(\cdot,t_n))\}$ has a convergent subsequence and thus finish the proof.

4.5 Global Existence and Completion of the Proof

According to Remark 2.13, the compactness result above implies that there exists a positive constant ε, such that

$$t_0 \in I_1 \Longrightarrow [t_0 - \varepsilon, t_0 + \varepsilon] \subseteq I_1.$$

This means that $I_1 = \mathbb{R}$. Collecting all information about U_1, finally we are able to finish the proof of Theorem 4.1.

5 Further Properties of the Critical Element

In this section we show that the critical element has to satisfy further regularity conditions. The argument is similar to the one we used for the special case $\phi(x) \equiv \pm 1$. The radial assumption plays an important role in this argument. If $u(x,t)$ is a radial function, then we use the notation $u(r,t)$ for the value $u(x,t)$ when $|x| = r$. The main idea is that if u is a radial solution to

$$\partial_t^2 u - \Delta u = F_1(|x|, t),$$

then the function $w(r,t) \doteq r u(r,t)$ is a solution to the one-dimensional wave equation

$$\partial_t^2 w - \partial_r^2 w = r F_1(r,t).$$

A direct calculation shows

Lemma 5.1. *Let $(u(x,t_0), \partial_t u(x,t_0))$ be radial and in the energy space $\dot{H}^1 \times L^2$ locally (possibly away from the origin), then for any $0 < a < b < \infty$, we have the identity*

$$\frac{1}{4\pi} \int_{a<|x|<b} (|\nabla u|^2 + |\partial_t u|^2) dx = \left(\int_a^b [(\partial_r w)^2 + (\partial_t w)^2] dr \right) + \left(a u^2(a) - b u^2(b) \right)$$

holds. Here we take the value of the functions at time t_0.

Higher regularity First of all, we claim that u is always more regular away from the origin.

Proposition 5.2. *Assume* $3 < p < 5$. *Let* u *be a radial solution to the wave equation*

$$\partial_t^2 u - \Delta u = F_1(|x|, t),$$

defined for all $t \in \mathbb{R}$ *so that*

- *The set* $\{(u(\cdot, t), \partial_t u(\cdot, t)) | t \in \mathbb{R}\}$ *is pre-compact in the space* $\dot{H}^{s_p} \times \dot{H}^{s_p - 1}(\mathbb{R}^3)$.
- *The function* $F_1(|x|, t)$ *is in the space* $Z(I)$ *for any bounded time interval* I *and satisfies the inequality* $|F_1(|x|, t)| \leq C_0 |x|^{-\frac{2p}{p-1}}$ *for all* $(x, t) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$.

Then $(u(\cdot, t), \partial_t u(\cdot, t)) \in C(\mathbb{R}; \dot{H}^1 \times L^2(\mathbb{R}^3 \setminus B(0, R)))$ *with*

$$\int_{R<|x|<4R} \left(|\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2\right) dx \leq C_1 R^{-2(1-s_p)}. \tag{46}$$

Here the constant C_1 *is independent of* t *and* R.

Proof. A similar result has been proved in the author's previous work [48], where we considered a special case $F_1(|x|, t) = |u|^{p-1} u$. This general case can be proved in exactly the same way. In general, we define $w(r, t) = r u(r, t)$, then prove $\partial_r w(\cdot, t), \partial_t w(\cdot, t) \in C(\mathbb{R}_t; L^2([R, \infty)))$ for $R > 0$ with

$$\begin{aligned} &\int_R^{4R} \left[(\partial_r w(r, t))^2 + (\partial_t w(r, t))^2\right] dr \\ &\leq \frac{1}{2} \int_R^{4R} \left[\left(\int_0^\infty (r + t') F_1(r + t', t - t') dt'\right)^2 + \left(\int_0^\infty (r + t') F_1(r + t', t + t') dt'\right)^2\right] dr \\ &\leq C_2 R^{-2(1-s_p)}. \end{aligned} \tag{47}$$

More details can be found in Section 4 of the work mentioned above. The main ingredients of the proof include the transformation $u \to w$ as given above, the characteristic line method of one-dimensional wave equation, Duhamel's formula, strong Huygens' principle and smooth approximation techniques. □

Behaviour near infinity Next we consider the behaviour of $u(x, t)$ as $|x| \to +\infty$

Proposition 5.3. *Assume* $3 < p < 5$. *Let* u *be a radial solution to the wave equation*

$$\partial_t^2 u - \Delta u = F(|x|, u, t),$$

defined on all $t \in \mathbb{R}$ *so that*

- *The set* $\{(u(\cdot, t), \partial_t u(\cdot, t)) | t \in \mathbb{R}\}$ *is pre-compact in the space* $\dot{H}^{s_p} \times \dot{H}^{s_p - 1}(\mathbb{R}^3)$.
- *The function* $F(r, u, t)$ *satisfies* $|F(r, u, t)| \leq |u|^p$.

Then we have constants $A \in \mathbb{R}$, $C_3, C_4, C_5 > 0$ *independent of* t, r *and* x, *such that*

- *The solution satisfies*

$$|u(x, t)| \leq \frac{C_3}{|x|}, \qquad \left|u(x, t) - \frac{A}{|x|}\right| \leq \frac{C_4}{|x|^{p-2}};$$

- *We have an estimate on the local energy*

$$\int_{r<|x|<4r} \left(|\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2\right) dx \leq C_5 r^{-1}.$$

Proof. Let us briefly outline the proof. More details of this argument can be found in Section 7 of [48]. Since $w = ru$ solves one-dimensional wave equation $\partial_t^2 w - \Delta w = rF(r,u,t)$, we recall d'Alembert formula and write

$$
\begin{aligned}
w(r,t_0) =& \frac{1}{2}\left[w\left(\frac{r}{2}, t_0 - \frac{r}{2}\right) + w\left(\frac{3r}{2}, t_0 - \frac{r}{2}\right)\right] + \frac{1}{2}\int_{r/2}^{3r/2} \partial_t w\left(s, t_0 - \frac{r}{2}\right) ds \\
&+ \frac{1}{2}\int_0^{r/2}\int_{\frac{r}{2}+t}^{\frac{3r}{2}-t} sF\left(s, u\left(s, t_0 - \frac{r}{2} + t\right), t_0 - \frac{r}{2} + t\right) ds\, dt. \qquad (48)
\end{aligned}
$$

Let us fix $\beta_0 = \frac{3}{2} - s_p$. For each $\beta \in [\beta_0, 1)$ we define a function

$$f_\beta(r) = \sup_{t\in\mathbb{R}, |x|\geq r} |x|^\beta |u(x,t)|,$$

which helps us compare the decay rate of u with that of $|x|^{-\beta}$ as $|x| \to \infty$. Let us assume $f_\beta(r) \to 0$ as $r \to \infty$, which is true at least for $\beta = \beta_0$, thanks to Lemma 2.17. Now we use the assumption $|F(r,u,t)| \leq |u|^p$, the upper bound $|u(r,t)| \leq r^{-\beta} f(r) \leq r^{-\beta} f(r/2)$ and the inequality (47) with $F_1(r,t) = F(r,u,t)$ on the right hand of (48), divide both sides by $r^{1-\beta}$ and finally obtain

$$r^\beta |u(r,t_0)| \leq \left[g(\beta) + C_p f_\beta^{p-1}\left(\frac{r}{2}\right) r^{2-(p-1)\beta}\right] f_\beta\left(\frac{r}{2}\right). \qquad (49)$$

Here

$$g(\beta) = \frac{1}{2}\left[\left(\frac{3}{2}\right)^{1-\beta} + \left(\frac{1}{2}\right)^{1-\beta}\right] < 1,$$

We observe that the right hand side of (49) is a non-increasing function of r, take the least upper bound on both sides for all $r > r_0$ and obtain

$$f_\beta(r_0) \leq \left[g(\beta) + C_p f_\beta^{p-1}\left(\frac{r_0}{2}\right) r_0^{2-(p-1)\beta}\right] f_\beta\left(\frac{r_0}{2}\right).$$

Since $2 - (p-1)\beta \leq 0$ and $\lim_{r_0\to\infty} f_\beta(r_0/2) = 0$, we have

$$f_\beta(r) \leq \frac{g(\beta)+1}{2} f_\beta(r/2)$$

when r is sufficiently large. This implies that f_β decays at a rate at least comparable to that of a small negative power of r. As a result, we can increase the value of β, iterate the argument above and conclude that given any $\beta \in [\beta_0, 1)$, we have $|u(r,t)| \lesssim_\beta r^{-\beta}$ for sufficiently large r. Next we fix $\beta = \beta(p)$ so that $p\beta - 3 > 0$, then plug $|F_1(r,t)| = |F(r,u,t)| \leq |u(r,t)|^p \lesssim r^{-p\beta}$ in inequality (47) and obtain the following estimates for large R:

$$\int_R^{4R} \left[(\partial_r w(r,t))^2 + (\partial_t w(r,t))^2\right] dr \lesssim R^{5-2p\beta} \qquad \Rightarrow \qquad \int_R^\infty |\partial_r w(r,t)| dr \lesssim R^{3-p\beta}.$$

This implies that the limit of $w(r,t)$ as $r \to \infty$ always exists for any given t since $3 - p\beta < 0$. Furthermore, the limit does not depend on t by the $L^2([R,4R])$ estimate of $\partial_t w(r,t)$ above. This limit is the constant A. Next we may combine the $L^1([R,\infty))$ estimate of $|\partial_r w|$ above with the compactness assumption of $(u, \partial_t u)$ to conclude $|w(r,t)|$ is uniformly bounded for all r,t. This proves the inequality $|u(x,t)| \leq C_3/|x|$. Finally we plug $|F_1(r,t)| = |F(r,u,t)| \leq |u(r,t)|^p \leq C_3^p r^{-p}$ in inequality (47) and conclude

$$\int_R^{4R} \left[(\partial_r w(r,t))^2 + (\partial_t w(r,t))^2\right] dr \lesssim R^{5-2p} \qquad \Rightarrow \qquad \int_R^\infty |\partial_r w(r,t)| dr \lesssim R^{3-p}.$$

This immediately gives $|w(r,t) - A| \leq C_4 r^{3-p} \Rightarrow |u(x,t) - A/|x|| \leq C_4 |x|^{2-p}$. Finally we apply Lemma 5.1 and obtain

$$\int_{R<|x|<4R} \left(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\right) dx \lesssim R^{-1}. \quad R > 1.$$

We combine this estimate (for $R > 1$) with Proposition 5.2 (for $R \leq 1$) to finish the proof. □

6 Stationary Solutions

In this section we construct a stationary solution to (CP1), i.e. a solution to the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ with a similar asymptotic behaviour to a critical element u when $|x|$ is large. More precisely, we prove

Proposition 6.1. *Given any constant A, there exists a unique radial C^2 solution $W(x) = W_{A,\phi}(x)$ to the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ so that*

$$\left|W(x) - \frac{A}{|x|}\right| \lesssim \frac{1}{|x|^{p-2}}, \qquad |\nabla W(x)| \lesssim \frac{1}{|x|^2}, \qquad \text{if } |x| \gg 1.$$

In addition, the solution W and its maximal domain [5] Ω satisfy either of the following

(I) $\Omega = \{x \in \mathbb{R}^3 : |x| > R_W\}$ and $\limsup_{|x|\to R_W^+} |W(x)| = +\infty$. Here $R_W > 0$ is a constant.

(II) $\Omega = \mathbb{R}^3 \setminus \{0\}$ and $W \notin \dot{H}^{s_p}(\mathbb{R}^3)$. In this case $R_W = 0$.

(III) $\Omega = \mathbb{R}^3$ and $W \in \dot{H}^{s_p}(\mathbb{R}^3)$. In this case we also define $R_W = 0$.

Remark 6.2. *If $A = 0$, then we always have $W(x) \equiv 0$.*

6.1 Existence and Uniqueness

Proposition 6.3. *Given any constant A, there exists a unique radial C^2 solution $W(x)$ to the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ so that*

$$\left|W(x) - \frac{A}{|x|}\right| \lesssim \frac{1}{|x|^{p-2}}, \qquad |\nabla W(x)| \lesssim \frac{1}{|x|^2}, \qquad \text{if } |x| \gg 1.$$

In addition, if W has a maximal domain $\{x \in \mathbb{R}^3 : |x| > R_W\}$ with a radius $R_W > 0$, then $\limsup_{|x|\to R_W^+} |W(x)| = +\infty$.

Proof. Let us rewrite $W(x)$ into the form $W(x) = \dfrac{A + \rho(|x|)}{|x|}$. Here ρ is a C^2 function defined for large positive real numbers r. The asymptotic behaviour of W near infinity implies $\rho(r), \rho'(r) \to 0$ as $r \to \infty$. The elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ can be rewritten in term of ρ:

$$\rho''(r) = -\frac{\phi(r)F(\rho(r) + A)}{r^{p-1}}, \qquad F(u) = |u|^{p-1}u. \tag{50}$$

The first step is to show this equation has a unique solution defined on the interval $[R_1, \infty)$ via a fixed-point argument, where R_1 is a large radius to be determined later. We define a complete metric space

$$X = \left\{\rho : \rho \in C([R_1, \infty); [-1, 1]),\ \lim_{r\to+\infty} \rho(r) = 0\right\}$$

[5]The maximal domain Ω means that $W \in C^2(\Omega)$ solves the elliptic equation but we can not extend f to a C^2 solution in a larger radially symmetric and connected region in $\mathbb{R}^3$. Here Ω is assumed to contain a neighbourhood of infinity.

with the distance $d(\rho_1, \rho_2) = \sup_{r\in[R_1,\infty)} |\rho_1(r) - \rho_2(r)|$ and a map

$$(\mathbf{L}\rho)(r) = \int_r^\infty \int_s^\infty \left(-\frac{\phi(t)F(\rho(t)+A)}{t^{p-1}}\right) dt\, ds.$$

Since the absolute value of the integrand never exceeds $(1+|A|)^p/t^{p-1}$, this integral defines a continuous function on $[R_1,\infty)$. In addition, we have

$$|(\mathbf{L}\rho)(r)| \leq \int_r^\infty \int_s^\infty \frac{(1+|A|)^p}{t^{p-1}} dt\, ds \leq \frac{C_p(1+|A|)^p}{r^{p-3}}; \tag{51}$$

$$|(\mathbf{L}\rho_1)(r) - (\mathbf{L}\rho_2)(r)| \leq \int_r^\infty \int_s^\infty \frac{p(1+|A|)^{p-1} d(\rho_1,\rho_2)}{t^{p-1}} dt\, ds \leq \frac{C_p(1+|A|)^{p-1}}{r^{p-3}} d(\rho_1,\rho_2).$$

As a result, if we choose a sufficiently large $R_1 = R_1(A,p)$, then the map $\mathbf{L}$ is a contraction map on the space X. This enables us to apply a fixed-point argument and find a solution ρ to the equation (50) defined on $[R_1,\infty)$. Furthermore, we have an estimate on $\rho'(r)$ when $r \geq R_1$:

$$|\rho'(r)| = \left|\int_r^\infty \left(\frac{\phi(t)F(\rho(t)+A)}{t^{p-1}}\right) dt\right| \leq \frac{C_p(1+|A|)^p}{r^{p-2}}.$$

Combing this upper bound on $\rho'(r)$ with the upper bound (51) on $\rho(r)$, we obtain the behaviour of $W(x)$ as $|x|$ is large:

$$\left|W(x) - \frac{A}{|x|}\right| = \frac{|\rho(|x|)|}{|x|} \lesssim_{p,A} \frac{1}{|x|^{p-2}};$$

$$|\nabla W(x)| = \left|\frac{\rho'(|x|)}{|x|} - \frac{A+\rho(|x|)}{|x|^2}\right| \lesssim_{p,A} \frac{1}{|x|^2}.$$

The second step is to extend the solution $\rho(r)$ to its maximal interval of existence (R_W,∞). Basic theory of ordinary differential equations guarantees that the solution ρ (thus W in its maximal domain) is unique in its maximal interval of existence. We still need to prove

$$\limsup_{|x|\to R_W^+} |W(x)| = +\infty$$

if $R_W > 0$. This is equivalent to saying the upper limit of $|\rho(r)|$ as $r \to R_W^+$ is infinity. If this were false, then we could assume $|\rho(r)| \leq M$ for all $r > R_W$. But this implies that both $\rho''(r)$ and $\rho'(r)$ are also bounded when r is close to the blow-up point R_W, according to equation (50). This contradicts with basic theory of ordinary differential equations. □

6.2 Classification of Solutions

In the subsection we consider the classification of solutions W obtained via Proposition 6.3 thus finish the proof of Proposition 6.1. There are three kinds of solutions:

(I) The solution W is only defined for points away from the origin, i.e. $R_W > 0$. Proposition 6.3 guarantees that $W(x)$ is unbounded when $|x| \to R_W^+$. Therefore W is not in the space $\dot{H}^{s_p}(\mathbb{R}^3)$, thanks to Lemma 2.15. More precisely, it is impossible to find a radial function $u \in \dot{H}^{s_p}(\mathbb{R}^3)$, such that $u(x) = W(x)$ for all x with $|x| > R_W$.

(II) The solution W is well-defined everywhere except for at the origin . But we have

$$\limsup_{|x|\to 0^+} |x|^{\frac{3}{2}-s_p}|W(x)| > 0.$$

Thus it is impossible to define W at the origin so that $W \in C^2(\mathbb{R}^3)$. According to Lemma 2.17, we know $W \notin \dot{H}^{s_p}(\mathbb{R}^3)$. Examples of this type of solution are given by the case $\phi(x) \equiv 1$. Please see Section 9 of [48] for more details.

(III) The solution W is well-defined for all $x \in \mathbb{R}^3 \setminus \{0\}$, and satisfies

$$\lim_{|x|\to 0^+} |x|^{\frac{3}{2}-s_p}|W(x)| = 0.$$

It turns out that this solution satisfies $W \in C^2(\mathbb{R}^3)$, as shown in the Proposition 6.5 below. A Combination of this C^2 smoothness with the decay rate of the gradient ∇W near infinity guarantees that $W \in \dot{W}^{1,q}$ for all $q > 3/2$. By Sobolev embedding we have $W \in \dot{H}^s(\mathbb{R}^3)$ for all $s \in (1/2, 1]$, in particular for $s = s_p$. One example in this case can be given explicitly by the function

$$W(x) = \frac{3}{\sqrt{3|x|^2+1}},$$

which solves the elliptic equation $-\Delta W = (1/9)W^5$. As a result, it also solves the elliptic equation $-\Delta W = \phi_p(x)|W|^{p-1}W$ if we choose

$$\phi_p(x) = (1/9)|W(x)|^{5-p} = 3^{3-p}(3|x|^2+1)^{(p-5)/2}.$$

Next we prove a solution W in category (III) above must satisfy $W \in C^2(\mathbb{R}^3)$. We start by a technical lemma.

Lemma 6.4. *Let $W \in C^2(B(0,r_0) \setminus \{0\})$ be a radial solution to the elliptic equation $-\Delta W = q(|x|)W$, where $q(r)$ is a continuous function defined on $(0, r_0)$ satisfying $\lim_{r\to 0^+} r^2 q(r) = 0$. In addition, there is a constant $\varepsilon \in (0, 1/2)$, such that $\lim_{|x|\to 0^+} |x|^{1-\varepsilon}W(x) = 0$. Then there exist two constants $r_1 \in (0, r_0)$ and $C_1 > 0$, such that the inequality $|W(x)| \leq C_1|x|^{-\varepsilon}$ holds for all $0 < |x| < r_1$.*

Proof. This is trivial if $W \equiv 0$, thus we assume that W is not identically zero in any neighbourhood of the origin. First of all, we define a new function $v : (0, r_0) \to \mathbb{R}$ by $v(|x|) = |x|^{1-\varepsilon}W(x)$. According to the assumption on W we have $v(r) \to 0$ as $r \to 0^+$. A basic calculation shows that v satisfies the equation

$$v''(r) + \frac{2\varepsilon}{r}v'(r) = \eta(r)v(r) \tag{52}$$

Here $\eta(r) = \dfrac{\varepsilon(1-\varepsilon)}{r^2} - q(r)$. By the assumption on $q(r)$, there exists a small positive number $r_1 \in (0, r_0)$, such that $\eta(r) > 0$ for all $r \in (0, r_1]$.

Step 1 We claim that $v(r)$ has neither a positive local maximum nor a negative local minimum on $(0, r_1)$. If $v(r)$ had a positive local maximum at $r = r_2 \in (0, r_1)$, then we would have $v''(r_2) \leq 0$, $v'(r_2) = 0$ and $\eta(r_2)v(r_2) > 0$. This violates the equation (52). The same argument rules out the existence of any negative local minimum.

Step 2 The function v is never zero in the interval $(0, r_1]$. If this were false, then we would find a number $r_2 \in (0, r_1]$, so that $v(r_2) = 0$. Since $v(r)$ is a nontrivial C^2 function defined on $(0, r_2]$ with $\lim_{r\to 0^+} v(r) = 0$ and $v(r_2) = 0$, it must have either a local positive maximum or a local negative minimum in the interval $(0, r_2)$. This is a contradiction. A direct corollary follows that $v(r)$ is either always positive or always negative in the interval $(0, r_1]$. Without loss of generality, we assume $v(r) > 0$ for all $r \in (0, r_1]$.

Step 3 The derivative $v'(r) > 0$ for all $r \in (0, r_1]$. In fact, a negative derivative $v'(r_2) < 0$ at a point $r_2 \in (0, r_1]$ would imply the existence of a positive local maximum in the interval $(0, r_2)$, since we have $v(r) > 0$ for $r \in (0, r_1]$ and $v(r) \to 0$ as $r \to 0$. Therefore we have $v'(r) \geq 0$ for all $r \in (0, r_1]$. Furthermore, if $v'(r_2) = 0$ at a point $r_2 \in (0, r_1]$, then $v''(r_2) \leq 0$ because we always have $v'(r) \geq 0$ for $r \in (0, r_2)$. Again this contradicts with equation (52), since we have already shown $v(r_2) > 0$.

Step 4 Now in the interval $(0, r_1)$ we can rewrite (52) into an inequality

$$v''(r) + \frac{2\varepsilon}{r} v'(r) > 0 \quad \Longrightarrow \quad \frac{d}{dr}\{\ln[v'(r)]\} > \frac{-2\varepsilon}{r}$$

Integrating this from r to r_1, we obtain

$$\ln[v'(r_1)] - \ln[v'(r)] > -2\varepsilon \ln r_1 + 2\varepsilon \ln r \quad \Longrightarrow \quad v'(r) < Cr^{-2\varepsilon}, \quad \text{if } 0 < r < r_1.$$

Here $C = r_1^{2\varepsilon} v'(r_1) > 0$. Combining this inequality with $\lim_{r\to 0^+} v(r) = 0$, we obtain that if $0 < r = |x| < r_1$, then

$$0 < v(r) < C_1 r^{1-2\varepsilon} \quad \Longrightarrow \quad |W(x)| < C_1 |x|^{-\varepsilon}.$$

Here $C_1 = C/(1-2\varepsilon)$. ☐

Proposition 6.5. *Let $W \in C^2(\mathbb{R}^3 \setminus \{0\})$ be a radial solution to the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ so that*

$$\lim_{|x|\to 0^+} |x|^{\frac{3}{2}-s_p}|W(x)| = 0.$$

Then we can extend the domain of W to the whole space $\mathbb{R}^3$ by continuity so that $W \in C^2(\mathbb{R}^3)$ gives a classic solution to the elliptic equation above.

Proof. Let us define $y : \mathbb{R}^+ \to \mathbb{R}$ by $y(|x|) = W(x)$ and choose a small constant $0 < \varepsilon < \min\{1/p, 1 - \frac{2}{p-1}\}$. Applying Lemma 6.4 with $q(r) = \phi(r)|y(r)|^{p-1}$, we obtain an estimate $|y(r)| \leq C_1 r^{-\varepsilon}$ for small $r \in (0, r_1)$. In addition, a basic calculation shows that y satisfies the equation

$$(ry)'' + r\phi(r)|y|^{p-1}y = 0.$$

By the upper bound $|y(r)| \leq C_1 r^{-\varepsilon}$ and our assumption $p\varepsilon < 1$, we have that $|(ry)''| = r|\phi(r)||y|^p < C_1^p$ is bounded for all $0 < r < \min\{r_1, 1\}$. Therefore the limit

$$\lim_{r\to 0^+} (ry)' = C_2$$

exists with $|(ry)' - C_2| \leq C_1^p r$. Basic integration immediately shows (recall $ry \to 0$ as $r \to 0^+$)

$$|ry(r) - C_2 r| \leq \frac{C_1^p}{2} r^2 \quad \Longrightarrow \quad |y(r) - C_2| \leq \frac{C_1^p}{2} r.$$

Therefore the function W extends to a continuous function on $\mathbb{R}^3$. Since the right hand of $-\Delta W = \phi(x)|W|^{p-1}W$ is continuous, we can gain two derivatives and conclude $W \in C^2(\mathbb{R}^3)$ by basic knowledge in Laplace's equation. ☐

7 Non-existence of Critical Element

The following theorem shows that critical element may not exist unless there exists a soliton $W \in C^2(\mathbb{R}^3) \cap \dot{H}^{s_p}(\mathbb{R}^3)$.

Theorem 7.1. *If $u(x,t)$ is a radial solution to (CP1) defined for all $t \in \mathbb{R}$ so that its trajectory $\{(u(\cdot,t), \partial_t u(\cdot,t)) : t \in \mathbb{R}\}$ is pre-compact in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, then there exists a radial C^2 solution W to the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$, along with a radius $R_W \geq 0$, both of which are given in Proposition 6.1, so that $u(x,t) = W(x)$ holds for all (x,t) with $|x| > R_W$.*

Proof of main theorem We first temporarily assume Theorem 7.1 and prove the main theorem, then go back to the proof of this theorem. Let us assume that the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$ does not have a nonzero radial solution $W \in C^2(\mathbb{R}^3) \cap \dot{H}^{s_p}(\mathbb{R}^3)$. If the main theorem failed, then $Sc(A)$ would break down at $A = M$ for a positive number M. We then obtain a critical element u by Theorem 4.1. Next we apply theorem 7.1 to conclude that u coincides with a solution W to the elliptic equation for all (x,t) with $|x| > R_W$. The solution W and radius R_W here are given by Proposition 6.1 and satisfy either of the three conditions (I), (II) or (III) there. The case (I) and (II) can not happen because these solutions can not be extended to an $\dot{H}^{s_p}(\mathbb{R}^3)$ function u. Thus W satisfies condition (III), i.e. $W \in C^2(\mathbb{R}^3) \cap \dot{H}^{s_p}(\mathbb{R}^3)$ and $R_W = 0$. We have assumed that such a solution must be zero, thus the critical element u is also zero. This is a contradiction.

Idea to prove Theorem 7.1 The rest of this section is devoted to the proof of Theorem 7.1. We first give a brief idea. According to Proposition 5.3, there exists a constant A so that we have $u(x,t) \sim A/|x|$ when $|x|$ is large. Proposition 6.1 then gives a solution $W(x) \in C^2(\{x \in \mathbb{R}^3 : |x| > R_W\})$ to the elliptic equation $-\Delta W = \phi(x)|W|^{p-1}W$, so that its behaviour near infinity is close to that of the critical element u. More precisely we have

$$|u(x,t) - W(x)| \lesssim \frac{1}{|x|^{p-2}}, \quad |x| \gg 1.$$

We need to show that $u(x,t) = W(x)$ as long as $|x| > R_W$. The argument consists of two steps. In the first step we show the identity holds for very large x. Then in the second step we prove that the identity has to hold for all (x,t) with $|x| > R_W$, otherwise we can obtain a contradiction. Each step is summarized into a proposition, which works for a more general nonlinear term as well. Both propositions are proved via the "channel of energy" method in exactly the same way as in the special case $\phi(x) \equiv 1$. Thus in Subsection 7.1 we only give the statements of these two propositions and omit the details of proof. More details can be found in [13, 48]. Finally we combine all the ingredients mentioned above and give a proof of Theorem 7.1 in the final subsection.

7.1 Abstract Theory

Assumptions Assume $3 < p < 5$. Let $W \in C^2(\{x \in \mathbb{R}^3 : |x| > R_W\})$ be a radial solution to the elliptic equation

$$-\Delta W = F(|x|, W),$$

where $F : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$|F(r,u)| \leq |u|^p;$$
$$|F(r,u_1) - F(r,u_2)| \leq C_6 |u_1 - u_2| \left(|u_1|^{p-1} + |u_2|^{p-1}\right);$$

so that the inequalities $|W(x)| \lesssim \frac{1}{|x|}$, $|\nabla W(x)| \lesssim \frac{1}{|x|^2}$ hold when $|x|$ is large. We say $u(x,t)$ is a solution to the equation $\partial_t^2 u - \Delta u = F(|x|, u)$ in the time interval I, if $(u(\cdot,t), \partial_t u(\cdot,t)) \in C(I; \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3))$, with finite norm $\|u\|_{Y(J)} < \infty$ for any closed bounded interval $J \subseteq I$, so that the integral equation

$$u(\cdot,t) = \mathbf{S}_{L,0}(t)(u(\cdot,t_0), \partial_t u(\cdot,t_0)) + \int_{t_0}^{t} \frac{\sin[(t-\tau)\sqrt{-\Delta}]}{\sqrt{-\Delta}} F(\cdot, u(\cdot,\tau)) d\tau$$

holds for all $t, t_0 \in I$.

Proposition 7.2. *Let $W(x)$ and $F(r,u)$ be as above . Suppose that $u(x,t)$ is a radial solution of the equation*

$$\partial_t^2 u - \Delta u = F(|x|, u)$$

defined for all $t \in \mathbb{R}$ so that (the implicit constant in the inequalities does not depend on t)

(I) We have $u(x,t)$ and $W(x)$ are very close to each other as $|x|$ is large

$$|u(x,t) - W(x)| \lesssim \frac{1}{|x|^{p-2}}, \quad t \in \mathbb{R}, |x| > R.$$

(II) The following inequality holds for each $t \in \mathbb{R}$ and $r > 0$.

$$\int_{r<|x|<4r} \left(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\right) dx \lesssim r^{-1}.$$

Then there exists a constant $R_0 > R_W$ independent of t such that

$$u(x,t) - W(x) = 0, \qquad \partial_t u(x,t) = 0$$

hold for all $t \in \mathbb{R}$ and $|x| > R_0$.

Essential Radius of Support If the pair $(u(x,t), u_1(x,t))$ coincide with $(W(x), 0)$ for large x, we can define the essential radius of support for their difference by

$$R(t) = \min\{R \geq R_W : (u(x,t) - W(x), \partial_t u(x,t)) = (0,0) \text{ holds for } |x| > R\}.$$

Theorem 7.3 (Behavior of "compactly supported" solutions)**.** *Let $W(x)$, $F(r,u)$ be as above and I be a time interval containing a neighbourhood of t_0. Suppose $u(x,t)$ is a radial solution of the equation*

$$\partial_t^2 u - \Delta u = F(|x|, u)$$

on the time interval I satisfying

(I) $(u(x,t), \partial_t u(x,t)) \in C(I; \dot{H}^1 \times L^2(\mathbb{R}^3 \setminus B(0,R)))$ for each $R > 0$.

(II) The pair $(u(x,t_0) - W(x), \partial_t u(x,t_0))$ is compactly supported with an essential radius of support $R(t_0) > R_1 > R_W$.

Then there exists a constant $\tau = \tau(p, R_1, C_6, W) > 0$, such that the essential radius of support defined above satisfies the identity

$$R(t) = R(t_0) + |t - t_0|$$

for each $t \in [t_0, t_0 + \tau] \cap I$ or for each $t \in [t_0 - \tau, t_0] \cap I$.

7.2 Proof of Proposition 7.1

Preparation Let $u(x,t)$ be a radial solution to (CP1) defined for all $t \in \mathbb{R}$ so that the trajectory $\{(u(\cdot,t), \partial_t u(\cdot,t)) : t \in \mathbb{R}\}$ is pre-compact in $\dot{H}^{s_p} \times \dot{H}^{s_p - 1}$. First of all, we may apply Proposition 5.3 and obtain three constants A, C_4, C_5 so that

$$\left|u(x,t) - \frac{A}{|x|}\right| \leq \frac{C_4}{|x|^{p-2}}; \qquad \int_{r<|x|<4r} \left(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2\right) dx \leq C_5 r^{-1}. \tag{53}$$

Proposition 6.1 then gives a radial solution W to the elliptic equation $-\Delta W = -|W|^{p-1} W$ along with a radius R_W so that $W \in C^2(\{x \in \mathbb{R}^3 : |x| > R_W\})$ and

$$\left|W(x) - \frac{A}{|x|}\right| \lesssim \frac{1}{|x|^{p-2}}, \qquad |\nabla W(x)| \lesssim \frac{1}{|x|^2}, \qquad \text{if } |x| \gg 1.$$

Step 1 Now we can apply Proposition 7.2 on the solution u and $W(x)$ given above to conclude that there exists a radius $R_0 > R_W$ independent of t so that

$$(u(x,t), \partial_t u(x,t)) = (W(x), 0), \quad \text{for } |x| > R_0.$$

As a result, we know the essential radius of support $R(t)$ for the difference $(u(x,t) - W(x), \partial_t u(x,t))$ is well-defined and satisfies the inequality$R(t) \leq R_0$ for all t.

Step 2 Now we can prove $u(x,t) = W(x)$ for all time t and $|x| > R_W$. If this were false, we could find a time, say $t = 0$, so that $R(t) > R_W$ and deduce a contradiction. We start by choosing $R_1 = [R(0) + R_W]/2$ and applying Proposition 7.3 with $I = \mathbb{R}$ and $t_0 = 0$. Without loss of generality, we assume the radius of support $R(t)$ increases in the positive time direction. More precisely we have

$$R(t) = R(0) + t, \quad \text{for } t \in [0, \tau].$$

Since $R(\tau) > R(0) > R_1$, we are able to apply Proposition 7.3 at time $t_0 = \tau$ again with the same constant R_1. Our conclusion is that $R(t)$ has to increase in a linear manner in at least one time direction for the same time period τ. This must be the positive time direction since we have assumed that $R(t)$ decreases in the negative time direction at time $t_0 = \tau$. Therefore we obtain

$$R(t) = R(0) + t, \quad \text{for } t \in [0, 2\tau].$$

Repeating this argument, we have $R(t) = R(0) + t$ for all $t > 0$. This contradicts with the uniform upper bound $R(t) \leq R_0$.

Remark 7.4. *Given any nonzero radial solution to (CP1) defined for all $t \in \mathbb{R}$ with a pre-compact trajectory, we claim that the well-defined limit (see* (53) *above)*

$$A = \lim_{|x| \to \infty} |x| u(x,t)$$

is alway nonzero. Otherwise we would choose $W(x) \equiv 0$ and $R_W = 0$ in the proof above and finally conclude that $u(x,t) = W(x) = 0$.

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