



A Berry-Esseen bound of order $\frac{1}{\sqrt{n}}$ for martingales

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Résumé. Renz [13] has established a rate of convergence $1/\sqrt{n}$ in the central limit theorem for martingales with some restrictive conditions. In the present paper a modification of the methods, developed by Bolthausen [2] and Grama and Haeusler [6], is applied for obtaining the same convergence rate for a class of more general martingales. An application to linear processes is discussed.

Keywords. Martingales, Berry-Esseen bound, Rate of convergence.

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Renz [13] a établi un taux de convergence $1/\sqrt{n}$ dans le théorème de la limite centrale pour les martingales avec certaines conditions restrictives. Dans le présent article, une modification des méthodes, développées par Bolthausen [2] et Grama et Haeusler [6], est appliquée pour obtenir le même taux de convergence pour une classe de martingales plus générales. Une application aux processus linéaires est discutée.

1. Introduction and main result

For $n \in \mathbf{N}$, let $(\xi_i, \mathcal{F}_i)_{i=0, \dots, n}$ be a finite sequence of martingale differences defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\xi_0 = 0$ and $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_n \subseteq \mathcal{F}$ are increasing σ -fields. Denote

$$X_0 = 0, \quad X_k = \sum_{i=1}^k \xi_i, \quad k = 1, \dots, n.$$

Then $X = (X_k, \mathcal{F}_k)_{k=0, \dots, n}$ is a martingale. Denote by $\langle X \rangle$ the conditional variance of X :

$$\langle X \rangle_0 = 0, \quad \langle X \rangle_k = \sum_{i=1}^k \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}], \quad k = 1, \dots, n.$$

Define

$$D(X_n) = \sup_{x \in \mathbf{R}} \left| \mathbf{P}(X_n \leq x) - \Phi(x) \right|,$$

where $\Phi(x)$ is the distribution function of the standard normal random variable. Denote by $\xrightarrow{\mathbf{P}}$ the convergence in probability as $n \rightarrow \infty$. According to the martingale central limit theorem, the ‘‘conditional Lindeberg condition’’

$$\sum_{i=1}^n \mathbf{E}[\xi_i^2 \mathbf{1}_{\{|\xi_i| \geq \epsilon\}} | \mathcal{F}_{i-1}] \xrightarrow{\mathbf{P}} 0, \quad \text{for each } \epsilon > 0,$$

and the ‘‘conditional normalizing condition’’ $\langle X \rangle_n \xrightarrow{\mathbf{P}} 1$ together implies asymptotic normality of X_n , that is, $D(X_n) \rightarrow 0$ as $n \rightarrow \infty$.

The convergence rate of $D(X_n)$ has attracted a lot of attentions. For instance, Bolthausen [2] proved that if $|\xi_i| \leq \epsilon_n$ for a number ϵ_n and $\langle X \rangle_n = 1$ a.s., then $D(X_n) \leq c\epsilon_n^3 n \log n$, where, here and after, c is an absolute constant not depending on ϵ_n and n . El Machkouri and Ouchti [3] improved the factor $\epsilon_n^3 n \log n$ in Bolthausen’s bound to $\epsilon_n \log n$ under the following more general condition

$$\mathbf{E}[|\xi_i|^3 | \mathcal{F}_{i-1}] \leq \epsilon_n \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}] \quad \text{a.s. for all } i = 1, 2, \dots, n.$$

For more related results, we refer to Ouchti [12] and Mourrat [11]. Recently, Fan [4] proved that if there exist a positive constant ρ and a number ϵ_n , such that

$$\mathbf{E}[|\xi_i|^{2+\rho} | \mathcal{F}_{i-1}] \leq \epsilon_n^\rho \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}] \quad \text{a.s. for all } i = 1, 2, \dots, n,$$

and $\langle X \rangle_n = 1$ a.s., then $D(X_n) \leq c_\rho \hat{\epsilon}_n$, where

$$\hat{\epsilon}_n = \begin{cases} \epsilon_n^\rho, & \text{if } \rho \in (0, 1), \\ \epsilon_n |\log \epsilon_n|, & \text{if } \rho \geq 1, \end{cases}$$

and c_ρ is a constant depending only on ρ . Fan [4] also showed that this Berry-Esseen bound is optimal. In particular, if $\epsilon_n \asymp 1/\sqrt{n}$, then we have $\epsilon_n |\log \epsilon_n| \asymp (\log n)/\sqrt{n}$. Thus, we cannot obtain the classical convergence rate $1/\sqrt{n}$ for general martingales.

However, the convergence rate $1/\sqrt{n}$ for martingales is possible to be attained with some additional restrictive conditions. For instance, Renz [13] proved that if there exists a constant $\rho > 0$ such that

$$\mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}] = 1/n, \quad \mathbf{E}[\xi_i^3 | \mathcal{F}_{i-1}] = 0 \quad \text{and} \quad \mathbf{E}[|\xi_i|^{3+\rho} | \mathcal{F}_{i-1}] \leq cn^{-(3+\rho)/2}, \quad \text{a.s.}, \quad (1)$$

then it holds

$$D(X_n) = O\left(\frac{1}{\sqrt{n}}\right). \quad (2)$$

He also showed that this result is not true for $\rho = 0$. More martingale Berry-Esseen bounds of convergence rate $1/\sqrt{n}$ can be found in Bolthausen [2] and Kir’yanova and Rotar [10].

In this paper we are interested in extending (2) to a class of more general martingales. The following theorem is our main result.

Theorem 1. *Assume that there exist some numbers $\rho \in (0, +\infty)$, $\epsilon_n \in (0, \frac{1}{2}]$ and $\delta_n \in [0, \frac{1}{2}]$ such that for all $1 \leq i \leq n$,*

$$|\langle X \rangle_n - 1| \leq \delta_n^2, \quad (3)$$

$$\mathbf{E}[\xi_i^3 | \mathcal{F}_{i-1}] = 0 \quad (4)$$

and

$$\mathbf{E}[|\xi_i|^{3+\rho} | \mathcal{F}_{i-1}] \leq \epsilon_n^{1+\rho} \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}] \quad a.s. \quad (5)$$

Then

$$D(X_n) \leq c_\rho(\epsilon_n + \delta_n),$$

where c_ρ depends only on ρ . In addition, it holds $c_\rho = O(\rho^{-1})$, $\rho \rightarrow 0$.

Notice that under the conditions of Renz [13], the conditions of Theorem 1 are satisfied with $\delta_n = 0$ and $\epsilon_n \asymp 1/\sqrt{n}$. Thus Theorem 1 extends Renz's result to a class of more general martingales.

Thanks to the additional condition (4), the Berry-Esseen bound (6) improves the bound of Fan [4] by replacing $\epsilon_n |\log \epsilon_n|$ with ϵ_n .

Relaxing the condition (3), we have the following analogue estimation of Fan (cf. (26) of [4]).

Theorem 2. Assume that there exist some numbers $\rho \in (0, +\infty)$ and $\epsilon_n \in (0, \frac{1}{2}]$ such that for all $1 \leq i \leq n$,

$$\mathbf{E}[\xi_i^3 | \mathcal{F}_{i-1}] = 0$$

and

$$\mathbf{E}[|\xi_i|^{3+\rho} | \mathcal{F}_{i-1}] \leq \epsilon_n^{1+\rho} \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}] \quad a.s.$$

Then, for all $p \geq 1$,

$$D(X_n) \leq c_\rho \epsilon_n + c_p \left(\mathbf{E}[|\langle X \rangle_n - 1|^p] + \mathbf{E} \left[\max_{1 \leq i \leq n} |\xi_i|^{2p} \right] \right)^{1/(2p+1)}, \quad (6)$$

where c_ρ and c_p depend only on ρ and p , respectively.

It is easy to see that when $p \rightarrow \infty$,

$$\left(\mathbf{E}[|\langle X \rangle_n - 1|^p] \right)^{1/(2p+1)} \rightarrow \|\langle X \rangle_n - 1\|_\infty^{1/2},$$

which coincides with δ_n of Theorem 1.

2. Application

We first extend Theorem 1 to triangular arrays with infinity many terms in each line. For $n \in \mathbf{N}$, let $(\xi_{n,i}, \mathcal{F}_{n,i})_{i=-\infty}^n$ be a sequence of martingale differences defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where the adapted filtration is $\{\emptyset, \Omega\} = \mathcal{F}_{-\infty} \subset \dots \subset \mathcal{F}_{n,n-1} \subset \mathcal{F}_{n,n} \subset \mathcal{F}$. Denote $X_{n,k} = \sum_{i=-\infty}^k \xi_{n,i}$, $k \leq n$. Then $(X_{n,k}, \mathcal{F}_{n,k})_{k=-\infty}^n$ is a martingale. Let $\langle X \rangle_{n,k} = \sum_{i=-\infty}^k \mathbf{E}[\xi_{n,i}^2 | \mathcal{F}_{n,i-1}]$, $k \leq n$. In particular, denote $X_n := X_{n,n}$ and $\langle X \rangle_n := \langle X \rangle_{n,n}$.

With some slight modification on the proof, Theorem 1 still holds in this new setting. Now we apply Theorem 1 with this new setting to the partial sum of linear processes. Let $(\varepsilon_i)_{i \in \mathbf{Z}}$ be a sequence of identically distributed martingale differences adapted to the filtration $(\mathcal{F}_i)_{i \in \mathbf{Z}}$. We consider the causal linear process in the form

$$Y_k = \sum_{j=-\infty}^k a_{k-j} \varepsilon_j, \quad (7)$$

where the martingale differences have finite variance and the sequence of real coefficients satisfies $\sum_{i=0}^{\infty} a_i^2 < \infty$. Without loss of generality, let the variance of the martingale

difference to be 1. We say the linear process has long memory if $\sum_{i=0}^{\infty} |a_i| = \infty$. In this case, we assume that $a_0 = 1$ and

$$a_i = \ell(i)i^{-\alpha}, \quad i > 0, \quad \text{with } 1/2 < \alpha < 1. \quad (8)$$

Here $\ell(\cdot)$ is a slowly varying function. On the other hand, we say the linear process has short memory if $\sum_{i=0}^{\infty} |a_i| < \infty$ and $\sum_{i=0}^{\infty} a_i \neq 0$. The third case is $\sum_{i=0}^{\infty} |a_i| < \infty$ and $\sum_{i=0}^{\infty} a_i = 0$.

The long memory linear processes covers the well-known fractional ARIMA processes (cf. Granger and Joyeux [7]; Hosking [9]), which play an important role in financial time series modeling and application. As a special case, let $0 < d < 1/2$ and B be the backward shift operator with $B\varepsilon_k = \varepsilon_{k-1}$ and consider

$$Y_k = (1 - B)^{-d}\varepsilon_k = \sum_{i=0}^{\infty} a_i \varepsilon_{k-i}, \quad \text{where } a_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}.$$

For this example we have $\lim_{n \rightarrow \infty} a_n/n^{d-1} = 1/\Gamma(d)$. Note that these processes have long memory because $\sum_{j=0}^{\infty} |a_j| = \infty$.

The partial sum $S_n = \sum_{k=1}^n Y_k$ of causal linear process (7) can be written as $S_n = \sum_{i=-\infty}^n b_{n,i} \varepsilon_i$, where $b_{n,i} = \sum_{j=0}^{n-i} a_j$ for $0 < i \leq n$, and $b_{n,i} = \sum_{j=1-i}^{n-i} a_j$ for $i \leq 0$. The variance of S_n is $B_n^2 = \text{var}(S_n) = \sum_{i=-\infty}^n b_{n,i}^2$. Now let $X_{n,k} = \sum_{i=-\infty}^k b_{n,i} \varepsilon_i / B_n$. Then $X_n = X_{n,n} = S_n / B_n$ and $\langle X \rangle_n = \sum_{i=-\infty}^n b_{n,i}^2 \mathbf{E}[\varepsilon_i^2 | \mathcal{F}_{i-1}] / B_n^2$. If we assume $|\langle X \rangle_n - 1| \leq \delta_n^2$ for some $\delta_n \in [0, \frac{1}{2}]$, $\mathbf{E}[\varepsilon_i^3 | \mathcal{F}_{i-1}] = 0$ and $\mathbf{E}[|\varepsilon_i|^{3+\rho} | \mathcal{F}_{i-1}] \leq d^{1+\rho} \mathbf{E}[\varepsilon_i^2 | \mathcal{F}_{i-1}]$ a.s. for all $i \in \mathbf{Z}$ and some constant d_ρ , then, by Theorem 1,

$$\sup_{x \in \mathbf{R}} |\mathbf{P}(S_n/B_n \leq x) - \Phi(x)| \leq c_\rho(\epsilon_n + \delta_n),$$

where $\epsilon_n = d_\rho \sup_{i \leq n} |b_{n,i}| / B_n$.

In the case that $\sum_{i=0}^{\infty} |a_i| < \infty$, $\sup_{i \leq n} |b_{n,i}| \leq \sum_{i=0}^{\infty} |a_i| < \infty$ and it is well known that B_n^2 has order n . Hence ϵ_n has order $1/\sqrt{n}$ in this case. In the long memory case $\sum_{i=0}^{\infty} |a_i| = \infty$, if we assume (8), B_n^2 has order $n^{3-2\alpha} \ell^2(n)$ (e.g., Wu and Min [14]) and $\sup_{i \leq n} |b_{n,i}|$ has order $n^{1-\alpha} \ell(n)$ (see Beknazaryan et al. [1] for upper bound and Fortune et al. [5] for lower bound in the case $d = 1$). Hence in this case ϵ_n also has order $1/\sqrt{n}$. In either case the Berry-Esseen bound has order $1/\sqrt{n}$ if $\delta_n = O(n^{-1/2})$. In particular, if we in addition assume that the innovations $(\varepsilon_i)_{i \in \mathbf{Z}}$ are independent, then $\delta_n = 0$ and the Berry-Esseen bound $\sup_{x \in \mathbf{R}} |\mathbf{P}(S_n/B_n \leq x) - \Phi(x)|$ has order $1/\sqrt{n}$. Here the condition $\mathbf{E}[\varepsilon_i^3 | \mathcal{F}_{i-1}] = 0$ is needed to have the Berry-Esseen bound of order $1/\sqrt{n}$. We cannot have this order from the result of Fan [4].

3. Proofs of theorems

3.1. Preliminary lemmas

In the proofs of theorems, we need the following technical lemmas. The first two lemmas can be found in Fan [4] (cf. Lemmas 3.1 and 3.2 therein).

Lemma 3. *If there exists an $s > 3$ such that*

$$\mathbf{E}[|\xi_i|^s | \mathcal{F}_{i-1}] \leq \epsilon_n^{s-2} \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}],$$

then, for any $t \in [3, s)$,

$$\mathbf{E}[|\xi_i|^t | \mathcal{F}_{i-1}] \leq \epsilon_n^{t-2} \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}].$$

Lemma 4. *If there exists an $s > 3$ such that*

$$\mathbf{E}[|\xi_i|^s | \mathcal{F}_{i-1}] \leq \epsilon_n^{s-2} \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}],$$

then

$$\mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}] \leq \epsilon_n^2.$$

The next two technical lemmas are due to Bolthausen (cf. Lemmas 1 and 2 of [2]).

Lemma 5. *Let X and Y be random variables. Then*

$$\sup_u \left| \mathbf{P}(X \leq u) - \Phi(u) \right| \leq c_1 \sup_u \left| \mathbf{P}(X + Y \leq u) - \Phi(u) \right| + c_2 \left\| \mathbf{E}[Y^2 | X] \right\|_{\infty}^{1/2},$$

where c_1 and c_2 are two positive constants.

Lemma 6. *Let $G(x)$ be an integrable function on \mathbf{R} of bounded variation $\|G\|_V$, X be a random variable and $a, b \neq 0$ are real numbers. Then*

$$\mathbf{E} \left[G \left(\frac{X + a}{b} \right) \right] \leq \|G\|_V \sup_u \left| \mathbf{P}(X \leq u) - \Phi(u) \right| + \|G\|_1 |b|,$$

where $\|G\|_1$ is the $L_1(\mathbf{R})$ norm of $G(x)$.

In the proof of Theorem 2, we also need the following lemma of El Machkouri and Ouchti [3].

Lemma 7. *Let X and Y be two random variables. Then, for $p \geq 1$,*

$$D(X + Y) \leq 2D(X) + 3 \left\| \mathbf{E}[Y^{2p} | X] \right\|_1^{1/(2p+1)}. \quad (9)$$

3.2. Proof of Theorem 1

By Lemma 3, we only need to consider the case of $\rho \in (0, 1]$. We follow the method of Grama and Haeusler [6]. Let $T = 1 + \delta_n^2$. We introduce a modification of the conditional variance $\langle X \rangle_n$ as follows:

$$V_k = \langle X \rangle_k \mathbf{1}_{\{k < n\}} + T \mathbf{1}_{\{k = n\}}. \quad (10)$$

It is easy to see that $V_0 = 0, V_n = T$, and that $(V_k, \mathcal{F}_k)_{k=0, \dots, n}$ is a predictable process. Set

$$\gamma = \epsilon_n + \delta_n.$$

Let c_* be some positive and sufficient large constant. Define the following non-increasing discrete time predictable process

$$A_k = c_*^2 \gamma^2 + T - V_k, \quad k = 1, \dots, n. \quad (11)$$

Obviously, we have $A_0 = c_*^2 \gamma^2 + T$ and $A_n = c_*^2 \gamma^2$. In addition, for $u, x \in \mathbf{R}$, and $y > 0$, denote

$$\Phi_u(x, y) = \Phi \left(\frac{u - x}{\sqrt{y}} \right). \quad (12)$$

Let $\mathcal{N} = \mathcal{N}(0, 1)$ be a standard normal random variable, which is independent of X_n . Using a smoothing procedure, by Lemma 5, we deduce that

$$\begin{aligned}
\sup_u \left| \mathbf{P}(X_n \leq u) - \Phi(u) \right| &\leq c_1 \sup_u \left| \mathbf{P}(X_n + c_* \gamma \mathcal{N} \leq u) - \Phi(u) \right| + c_2 \gamma \\
&= c_1 \sup_u \left| \mathbf{E}[\Phi_u(X_n, A_n)] - \Phi(u) \right| + c_2 \gamma \\
&\leq c_1 \sup_u \left| \mathbf{E}[\Phi_u(X_n, A_n)] - \mathbf{E}[\Phi_u(X_0, A_0)] \right| \\
&\quad + c_1 \sup_u \left| \mathbf{E}[\Phi_u(X_0, A_0)] - \Phi(u) \right| + c_2 \gamma \\
&= c_1 \sup_u \left| \mathbf{E}[\Phi_u(X_n, A_n)] - \mathbf{E}[\Phi_u(X_0, A_0)] \right| \\
&\quad + c_1 \sup_u \left| \Phi\left(\frac{u}{\sqrt{c_*^2 \gamma^2 + T}}\right) - \Phi(u) \right| + c_2 \gamma. \tag{13}
\end{aligned}$$

It is obvious that

$$\left| \Phi\left(\frac{u}{\sqrt{c_*^2 \gamma^2 + T}}\right) - \Phi(u) \right| \leq c_3 \left| \frac{1}{\sqrt{c_*^2 \gamma^2 + T}} - 1 \right| \leq c_4 \gamma. \tag{14}$$

Returning to (13), we get

$$\sup_u \left| \mathbf{P}(X_n \leq u) - \Phi(u) \right| \leq c_1 \sup_u \left| \mathbf{E}[\Phi_u(X_n, A_n)] - \mathbf{E}[\Phi_u(X_0, A_0)] \right| + c_5 \gamma. \tag{15}$$

By a simple telescoping, we know that

$$\mathbf{E}[\Phi_u(X_n, A_n)] - \mathbf{E}[\Phi_u(X_0, A_0)] = \mathbf{E}\left[\sum_{k=1}^n \left(\Phi_u(X_k, A_k) - \Phi_u(X_{k-1}, A_{k-1})\right)\right]. \tag{16}$$

Taking into account the fact that

$$\frac{\partial^2}{\partial x^2} \Phi_u(x, y) = 2 \frac{\partial}{\partial y} \Phi_u(x, y),$$

we get

$$\mathbf{E}[\Phi_u(X_n, A_n)] - \mathbf{E}[\Phi_u(X_0, A_0)] = J_1 + J_2 - J_3, \tag{17}$$

where

$$\begin{aligned}
J_1 = \mathbf{E} \left[\sum_{k=1}^n \left(\Phi_u(X_k, A_k) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial x} \Phi_u(X_{k-1}, A_k) \xi_k \right. \right. \\
\left. \left. - \frac{1}{2} \frac{\partial^2}{\partial x^2} \Phi_u(X_{k-1}, A_k) \xi_k^2 - \frac{1}{6} \frac{\partial^3}{\partial x^3} \Phi_u(X_{k-1}, A_k) \xi_k^3 \right) \right], \tag{18}
\end{aligned}$$

$$J_2 = \frac{1}{2} \mathbf{E} \left[\sum_{k=1}^n \frac{\partial^2}{\partial x^2} \Phi_u(X_{k-1}, A_k) \left(\Delta \langle X \rangle_k - \Delta V_k \right) \right], \tag{19}$$

$$J_3 = \mathbf{E} \left[\sum_{k=1}^n \left(\Phi_u(X_{k-1}, A_{k-1}) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial y} \Phi_u(X_{k-1}, A_k) \Delta V_k \right) \right], \tag{20}$$

where $\Delta \langle X \rangle_k = \langle X \rangle_k - \langle X \rangle_{k-1}$.

Now, we need to give some estimates of J_1, J_2 and J_3 . To this end, we introduce some notations. Denote by ϑ_i some random variables satisfying $0 \leq \vartheta_i \leq 1$, which may represent different values at different places. For the rest of the paper, φ stands for the density function of the standard normal random variable.

Control of J_1 : For convenience's sake, let $T_{k-1} = (u - X_{k-1})/\sqrt{A_k}$, $k = 1, 2, \dots, n$. It is easy to see that

$$\begin{aligned} B_k &=: \Phi_u(X_k, A_k) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial x} \Phi_u(X_{k-1}, A_k) \xi_k \\ &\quad - \frac{1}{2} \frac{\partial^2}{\partial x^2} \Phi_u(X_{k-1}, A_k) \xi_k^2 - \frac{1}{6} \frac{\partial^3}{\partial x^3} \Phi_u(X_{k-1}, A_k) \xi_k^3 \\ &= \Phi\left(T_{k-1} - \frac{\xi_k}{\sqrt{A_k}}\right) - \Phi(T_{k-1}) + \Phi'(T_{k-1}) \frac{\xi_k}{\sqrt{A_k}} \\ &\quad - \frac{1}{2} \Phi''(T_{k-1}) \left(\frac{\xi_k}{\sqrt{A_k}}\right)^2 + \frac{1}{6} \Phi'''(T_{k-1}) \left(\frac{\xi_k}{\sqrt{A_k}}\right)^3. \end{aligned}$$

To estimate the right hand side of the last equality, we distinguish two cases.

Case 1: $|\xi_k/\sqrt{A_k}| \leq 2 + |T_{k-1}|/2$. By a four-term Taylor expansion, it is obvious that if $|\xi_k/\sqrt{A_k}| \leq 1$, then

$$\begin{aligned} |B_k| &= \left| \frac{1}{24} \Phi^{(4)}\left(T_{k-1} - \vartheta \frac{\xi_k}{\sqrt{A_k}}\right) \left| \frac{\xi_k}{\sqrt{A_k}} \right|^4 \right| \\ &\leq \left| \Phi^{(4)}\left(T_{k-1} - \vartheta \frac{\xi_k}{\sqrt{A_k}}\right) \right| \left| \frac{\xi_k}{\sqrt{A_k}} \right|^{3+\rho}. \end{aligned}$$

If $|\xi_k/\sqrt{A_k}| > 1$, by a three-term Taylor expansion, then

$$\begin{aligned} |B_k| &\leq \frac{1}{2} \left(\left| \Phi''' \left(T_{k-1} - \vartheta \frac{\xi_k}{\sqrt{A_k}} \right) \right| + \left| \Phi'''(T_{k-1}) \right| \right) \left| \frac{\xi_k}{\sqrt{A_k}} \right|^3 \\ &\leq \left| \Phi''' \left(T_{k-1} - \vartheta' \frac{\xi_k}{\sqrt{A_k}} \right) \right| \left| \frac{\xi_k}{\sqrt{A_k}} \right|^3 \\ &\leq \left| \Phi''' \left(T_{k-1} - \vartheta' \frac{\xi_k}{\sqrt{A_k}} \right) \right| \left| \frac{\xi_k}{\sqrt{A_k}} \right|^{3+\rho}, \end{aligned}$$

where

$$\vartheta' = \begin{cases} \vartheta, & \text{if } \left| \Phi''' \left(T_{k-1} - \vartheta \frac{\xi_k}{\sqrt{A_k}} \right) \right| \geq \left| \Phi'''(T_{k-1}) \right|, \\ 0, & \text{if } \left| \Phi''' \left(T_{k-1} - \vartheta \frac{\xi_k}{\sqrt{A_k}} \right) \right| < \left| \Phi'''(T_{k-1}) \right|. \end{cases}$$

Using the inequality $\max\{|\Phi'''(t)|, |\Phi''''(t)|\} \leq \varphi(t)(2 + t^4)$, we find that

$$\begin{aligned} \left| B_k \mathbf{1}_{\{|\xi_k/\sqrt{A_k}| \leq 2 + |T_{k-1}|/2\}} \right| &\leq \varphi\left(T_{k-1} - \vartheta_1 \frac{\xi_k}{\sqrt{A_k}}\right) \left(2 + \left(T_{k-1} - \vartheta_1 \frac{\xi_k}{\sqrt{A_k}}\right)^4 \right) \left| \frac{\xi_k}{\sqrt{A_k}} \right|^{3+\rho} \\ &\leq g_1(T_{k-1}) \left| \frac{\xi_k}{\sqrt{A_k}} \right|^{3+\rho}, \end{aligned} \tag{21}$$

where

$$g_1(z) = \sup_{|t-z| \leq 2+|z|/2} \varphi(t)(2 + t^4).$$

Case 2: $|\xi_k/\sqrt{A_k}| > 2 + |T_{k-1}|/2$. It is obvious that, for $|\Delta x| > 1 + |x|/2$,

$$\begin{aligned} & \left| \Phi(x - \Delta x) - \Phi(x) + \Phi'(x) \Delta x - \frac{1}{2} \Phi''(x) (\Delta x)^2 + \frac{1}{6} \Phi'''(x) (\Delta x)^3 \right| \\ & \leq \left(\left| \frac{\Phi(x - \Delta x) - \Phi(x)}{|\Delta x|^3} \right| + |\Phi'(x)| + |\Phi''(x)| + |\Phi'''(x)| \right) |\Delta x|^3 \\ & \leq \left(8 \left| \frac{\Phi(x - \Delta x) - \Phi(x)}{(2 + |x|)^3} \right| + |\Phi'(x)| + |\Phi''(x)| + |\Phi'''(x)| \right) |\Delta x|^3 \\ & \leq \left(\frac{\tilde{c}}{(2 + |x|)^3} + |\Phi'(x)| + |\Phi''(x)| + |\Phi'''(x)| \right) |\Delta x|^3 \\ & \leq \frac{\hat{c}}{(2 + |x|)^3} |\Delta x|^3 \\ & \leq \frac{\hat{c}}{(2 + |x|)^3} |\Delta x|^{3+\rho}. \end{aligned}$$

Hence, we have

$$\left| B_k \mathbf{1}_{\{|\xi_k/\sqrt{A_k}| > 2 + |T_{k-1}|/2\}} \right| \leq g_2(T_{k-1}) \left| \frac{\xi_k}{\sqrt{A_k}} \right|^{3+\rho}, \quad (22)$$

where

$$g_2(z) = \frac{\hat{c}}{(2 + |z|)^3}.$$

Denote

$$G(z) = g_1(z) + g_2(z).$$

Combining (21) and (22) together, we get

$$|B_k| \leq G(T_{k-1}) \left| \frac{\xi_k}{\sqrt{A_k}} \right|^{3+\rho}. \quad (23)$$

Therefore,

$$\left| J_1 \right| = \left| \mathbf{E} \left[\sum_{k=1}^n B_k \right] \right| \leq \mathbf{E} \left[\sum_{k=1}^n G(T_{k-1}) \left| \frac{\xi_k}{\sqrt{A_k}} \right|^{3+\rho} \right]. \quad (24)$$

Next, we consider conditional expectation of $|\xi_k|^{3+\rho}$. By condition (5), we get

$$\mathbf{E}[|\xi_k|^{3+\rho} | \mathcal{F}_{k-1}] \leq \epsilon_n^{1+\rho} \Delta \langle X \rangle_k, \quad (25)$$

where $\Delta \langle X \rangle_k = \langle X \rangle_k - \langle X \rangle_{k-1}$ and we know that

$$\Delta \langle X \rangle_k = \Delta V_k = V_k - V_{k-1}, \quad 1 \leq k < n, \quad \Delta \langle X \rangle_n \leq \Delta V_n, \quad (26)$$

then

$$\mathbf{E}[|\xi_k|^{3+\rho} | \mathcal{F}_{k-1}] \leq \epsilon_n^{1+\rho} \Delta V_k. \quad (27)$$

By (24) and (27), we obtain

$$|J_1| \leq R_1 := \epsilon_n^{1+\rho} \left[\sum_{k=1}^n \frac{G(T_{k-1})}{A_k^{(3+\rho)/2}} \Delta V_k \right]. \quad (28)$$

To estimate R_1 , we introduce the time change τ_t as follow: for any real $t \in [0, T]$,

$$\tau_t = \min\{k \leq n : V_k \geq t\}, \quad \text{where } \min \emptyset = n. \quad (29)$$

Obviously, for any $t \in [0, T]$, the stopping time τ_t is predictable. In addition, $(\sigma_k)_{k=1, \dots, n+1}$ (with $\sigma_1 = 0$) stands for the increasing sequence of moments when the increasing and stepwise function $\tau_t, t \in [0, T]$, has jumps. It is easy to see that $\Delta V_k = \int_{[\sigma_k, \sigma_{k+1})} dt$, and that $k = \tau_t$ for $t \in [\sigma_k, \sigma_{k+1})$. Since $\tau_T = n$, we have

$$\sum_{k=1}^n \frac{G(T_{k-1})}{A_k^{(3+\rho)/2}} \Delta V_k = \sum_{k=1}^n \int_{[\sigma_k, \sigma_{k+1})} \frac{G(T_{\tau_t-1})}{A_{\tau_t}^{(3+\rho)/2}} dt = \int_0^T \frac{G(T_{\tau_t-1})}{A_{\tau_t}^{(3+\rho)/2}} dt. \quad (30)$$

Let $a_t = c_*^2 \gamma^2 + T - t$. Because of $\Delta V_{\tau_t} \leq 2\epsilon_n^2 + 2\delta_n^2$ (cf. Lemma 4), we know that

$$t \leq V_{\tau_t} = V_{\tau_t-1} + \Delta V_{\tau_t} \leq t + 2\epsilon_n^2 + 2\delta_n^2, \quad t \in [0, T]. \quad (31)$$

Assume $c_* \geq 2$, then we have

$$\frac{1}{2} a_t \leq A_{\tau_t} = c_*^2 \gamma^2 + T - V_{\tau_t} \leq a_t, \quad t \in [0, T]. \quad (32)$$

Note that $G(z)$ is symmetric and is non-increasing in $z \geq 0$. The last bound implies that

$$R_1 \leq 2^{(3+\rho)/2} \epsilon_n^{1+\rho} \int_0^T \frac{1}{a_t^{(3+\rho)/2}} \mathbf{E} \left[G \left(\frac{u - X_{\tau_t-1}}{a_t^{1/2}} \right) \right] dt. \quad (33)$$

Note also that $G(z)$ is a symmetric integrable function of bounded variation. By Lemma 6, it is obvious that

$$\mathbf{E} \left[G \left(\frac{u - X_{\tau_t-1}}{a_t^{1/2}} \right) \right] \leq c_6 \sup_z \left| \mathbf{P}(X_{\tau_t-1} \leq z) - \Phi(z) \right| + c_7 \sqrt{a_t}. \quad (34)$$

Because of $c_* \geq 2, V_{\tau_t-1} = V_{\tau_t} - \Delta V_{\tau_t}, V_{\tau_t} \geq t$ and $\Delta V_{\tau_t} \leq 2\epsilon_n^2 + 2\delta_n^2$, we obtain

$$V_n - V_{\tau_t-1} = V_n - V_{\tau_t} + \Delta V_{\tau_t} \leq 2\epsilon_n^2 + 2\delta_n^2 + T - t \leq a_t. \quad (35)$$

Therefore

$$\begin{aligned} \mathbf{E} \left[(X_n - X_{\tau_t-1})^2 \middle| \mathcal{F}_{\tau_t-1} \right] &= \mathbf{E} \left[\sum_{k=\tau_t}^n \mathbf{E} [\xi_k^2 | \mathcal{F}_{k-1}] \middle| \mathcal{F}_{\tau_t-1} \right] \\ &= \mathbf{E} [\langle X \rangle_n - \langle X \rangle_{\tau_t-1} | \mathcal{F}_{\tau_t-1}] \\ &\leq \mathbf{E} [V_n - V_{\tau_t-1} | \mathcal{F}_{\tau_t-1}] \\ &\leq a_t. \end{aligned}$$

Then, by Lemma 5, we deduce that for any $t \in [0, T]$,

$$\sup_z \left| \mathbf{P}(X_{\tau_t-1} \leq z) - \Phi(z) \right| \leq c_8 \sup_z \left| \mathbf{P}(X_n \leq z) - \Phi(z) \right| + c_9 \sqrt{a_t}. \quad (36)$$

Combining (28), (33), (34) and (36) together, we get

$$|J_1| \leq c_{10} \epsilon_n^{1+\rho} \int_0^T \frac{1}{a_t^{(3+\rho)/2}} dt \sup_z \left| \mathbf{P}(X_n \leq z) - \Phi(z) \right| + c_{11} \epsilon_n^{1+\rho} \int_0^T \frac{1}{a_t^{1+\rho/2}} dt. \quad (37)$$

Taking some elementary computations, it follows that

$$\int_0^T \frac{1}{a_t^{(3+\rho)/2}} dt = \int_0^T \frac{1}{(c_*^2 \gamma^2 + T - t)^{(3+\rho)/2}} dt \leq \frac{2}{c_*^{1+\rho} (1 + \rho) \gamma^{1+\rho}} \quad (38)$$

and

$$\int_0^T \frac{1}{a_t^{1+\rho/2}} dt = \int_0^T \frac{1}{(c_*^2 \gamma^2 + T - t)^{1+\rho/2}} dt \leq \frac{2}{c_*^\rho \rho \gamma^\rho}. \quad (39)$$

This yields

$$|J_1| \leq \frac{c_{12}}{c_*^{1+\rho}} \sup_z \left| \mathbf{P}(X_n \leq z) - \Phi(z) \right| + \frac{c_{\rho,1} \epsilon_n}{\rho}. \quad (40)$$

Control of J_2 : Since $0 \leq \Delta V_k - \Delta \langle X \rangle_k \leq 2\delta^2 \mathbf{1}_{\{k=n\}}$, we have

$$|J_2| \leq \mathbf{E} \left[\frac{1}{2A_n} |\varphi'(T_{n-1})(\Delta V_n - \Delta \langle X \rangle_n)| \right].$$

Denote $\tilde{G}(z) = \sup_{|z-t| \leq 1} |\varphi'(t)|$, and then $|\varphi'(z)| \leq \tilde{G}(z)$ for any real z . Since $A_n = c_*^2 \gamma^2$, then we get the following estimation:

$$|J_2| \leq \frac{1}{c_*^2} \mathbf{E} [\tilde{G}(T_{n-1})].$$

Note that \tilde{G} is non-increasing in $z \geq 0$, and thus it has bounded variation on \mathbf{R} . By Lemma 6, we get

$$|J_2| \leq \frac{c_{13}}{c_*^2} \sup_z |\mathbf{P}(X_{n-1} \leq z) - \Phi(z)| + c_{*,2}(\epsilon_n + \delta_n). \quad (41)$$

Then, by Lemma 5, we deduce that

$$\sup_z |\mathbf{P}(X_{n-1} \leq z) - \Phi(z)| \leq c_{14} \sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| + c_{15} \epsilon_n. \quad (42)$$

This yields

$$|J_2| \leq \frac{c_{16}}{c_*^2} \sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| + c_{\rho,2}(\epsilon_n + \delta_n). \quad (43)$$

Control of J_3 . By a two-term Taylor expansion, it follows that

$$|J_3| = \frac{1}{8} \mathbf{E} \left[\sum_{k=1}^n \frac{1}{(A_k - \vartheta_k \Delta A_k)^2} \varphi''' \left(\frac{u - X_{k-1}}{\sqrt{A_k - \vartheta_k \Delta A_k}} \right) (\Delta A_k)^2 \right].$$

Note that $c_* \geq 2$, $\Delta A_k \leq 0$ and, by Lemma 4, $|\Delta A_k| = \Delta V_k \leq 2\epsilon_n^2 + 2\delta_n^2$. We obtain

$$A_k \leq A_k - \vartheta_k \Delta A_k \leq c_*^2 \gamma^2 + T - V_k + 2\epsilon_n^2 + 2\delta_n^2 \leq 2A_k. \quad (44)$$

Denote $\hat{G}(z) = \sup_{|t-z| \leq 2} |\varphi'''(t)|$. Then $\hat{G}(z)$ is symmetric, and is non-increasing in $z \geq 0$. Using (44), we get

$$|J_3| \leq (2\epsilon_n^2 + 2\delta_n^2) \mathbf{E} \left[\sum_{k=1}^n \frac{1}{A_k^2} \hat{G} \left(\frac{T_{k-1}}{\sqrt{2}} \right) \Delta V_k \right]. \quad (45)$$

By an argument similar to that of (40), we get

$$\begin{aligned} |J_3| &\leq \frac{c_{17}(2\epsilon_n^2 + 2\delta_n^2)}{c_*^2 \gamma^2} \sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| + \frac{2c_{18}(2\epsilon_n^2 + 2\delta_n^2)}{c_* \gamma} \\ &\leq \frac{c_{19}}{c_*^2} \sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| + \frac{4c_{18}(\epsilon_n + \delta_n)^2}{c_* \gamma} \\ &\leq \frac{c_{19}}{c_*^2} \sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| + c_{\rho,3}(\epsilon_n + \delta_n). \end{aligned} \quad (46)$$

Combining (17), (40), (43) and (46) together, we get

$$\left| \mathbf{E}[\Phi_u(X_n, A_n)] - \mathbf{E}[\Phi_u(X_0, A_0)] \right| \leq \frac{c_{20}}{c_*^{1+\rho}} \sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| + \frac{\hat{c}_\rho}{\rho}(\epsilon_n + \delta_n),$$

By (15), we know that

$$\sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| \leq \frac{c_{21}}{c_*^{1+\rho}} \sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| + \frac{\tilde{c}_\rho}{\rho}(\epsilon_n + \delta_n),$$

from which, choosing $c_*^{1+\rho} = \max\{2c_{21}, 2^{1+\rho}\}$, we get

$$\sup_z |\mathbf{P}(X_n \leq z) - \Phi(z)| \leq \frac{2\tilde{c}_\rho(\epsilon_n + \delta_n)}{\rho}. \quad (47)$$

3.3. Proof of Theorem 2

Following the method of Bolthausen [2], we enlarge the sequence $(\xi_i, \mathcal{F}_i)_{1 \leq i \leq n}$ to $(\hat{\xi}_i, \hat{\mathcal{F}}_i)_{1 \leq i \leq N}$ such that $\langle \hat{X} \rangle_N := \sum_{i=1}^N \mathbf{E}[\hat{\xi}_i^2 | \hat{\mathcal{F}}_{i-1}] = 1$ a.s., and then apply Theorem 1 to the enlarged sequence. Consider the stopping time

$$\tau = \sup\{k \leq n : \langle X \rangle_k \leq 1\}. \quad (48)$$

Assume that $0 \leq \varepsilon \leq \varepsilon_n$. Let $r = \lfloor \frac{1 - \langle X \rangle_\tau}{\varepsilon^2} \rfloor$, where $\lfloor x \rfloor$ denotes the ‘‘integer part’’ of x . It is easy to see that $r \leq \lfloor \frac{1}{\varepsilon^2} \rfloor$. Set $N = n + r + 1$. Let $(\zeta_i)_{i \geq 1}$ be a sequence of independent Rademacher random variables, which is independent of the martingale differences $(\xi_i)_{1 \leq i \leq n}$. Consider the random variables $(\hat{\xi}_i, \hat{\mathcal{F}}_i)_{1 \leq i \leq N}$ defined as follows:

$$\hat{\xi}_i = \begin{cases} \xi_i & \text{a.s.,} & \text{if } i \leq \tau, \\ \varepsilon \zeta_i & \text{a.s.,} & \text{if } \tau + 1 \leq i \leq \tau + r, \\ (1 - \langle X \rangle_\tau - r\varepsilon^2)^{1/2} \zeta_i & \text{a.s.,} & \text{if } i = \tau + r + 1, \\ 0 & \text{a.s.,} & \text{if } \tau + r + 1 \leq i \leq N, \end{cases}$$

and $\hat{\mathcal{F}}_i = \sigma(\hat{\xi}_1, \hat{\xi}_2, \dots, \hat{\xi}_i)$.

Clearly, $(\hat{\xi}_i, \hat{\mathcal{F}}_i)_{1 \leq i \leq N}$ still forms a martingale difference sequence with respect to the enlarged filtration. Then $\hat{X}_k = \sum_{i=1}^k \hat{\xi}_i$, $k = 0, \dots, N$, with $\hat{X}_0 = 0$, is also a martingale. Moreover, it holds that $\langle \hat{X} \rangle_N = 1$, $\mathbf{E}[\hat{\xi}_i^3 | \hat{\mathcal{F}}_{i-1}] = 0$ and

$$\mathbf{E}[|\hat{\xi}_i|^{3+\rho} | \hat{\mathcal{F}}_{i-1}] \leq \varepsilon_n^{1+\rho} \mathbf{E}[\zeta_i^2 | \hat{\mathcal{F}}_{i-1}], \quad \text{a.s.}$$

By Theorem 1, we have

$$D(\hat{X}_N) \leq \frac{c_\rho \varepsilon_n}{\rho}. \quad (49)$$

Using Lemma 7, we obtain that

$$\begin{aligned} D(X_n) &\leq 2D(\hat{X}_N) + 3 \left\| \mathbf{E}[|X_n - \hat{X}_N|^{2p} | \hat{X}_N] \right\|_1^{1/(2p+1)} \\ &\leq \frac{2c_\rho \varepsilon_n}{\rho} + 3 \left(\mathbf{E}[|\hat{X}_N - X_n|^{2p}] \right)^{1/(2p+1)}. \end{aligned} \quad (50)$$

Since τ is a stopping time and

$$\hat{X}_N - X_n = \sum_{i=\tau+1}^N (\hat{\xi}_i - \xi_i), \quad \text{where put } \xi_i = 0 \text{ for } i > n, \quad (51)$$

$(\hat{\xi}_i - \xi_i, \hat{\mathcal{F}}_i)_{i \geq \tau+1}$ still forms a martingale difference sequence. Applying Theorem 2.11 of Hall and Heyde [8], we get

$$\begin{aligned} \mathbf{E}[|\hat{X}_N - X_n|^{2p}] &\leq \mathbf{E} \left[\max_{\tau+1 \leq i \leq N} |\hat{X}_i - X_i|^{2p} \right] \\ &\leq c_p \left(\mathbf{E} \left[\left| \sum_{i=\tau+1}^N \mathbf{E}[(\hat{\xi}_i - \xi_i)^2 | \hat{\mathcal{F}}_{i-1}] \right|^p \right] + \mathbf{E} \left[\max_{\tau+1 \leq i \leq N} |\hat{\xi}_i - \xi_i|^{2p} \right] \right). \end{aligned} \quad (52)$$

As $\mathbf{E}[\xi_i \hat{\xi}_i | \hat{\mathcal{F}}_{i-1}] = 0$ for all $i \geq \tau + 1$, we have

$$\sum_{i=\tau+1}^N \mathbf{E}[(\hat{\xi}_i - \xi_i)^2 | \hat{\mathcal{F}}_{i-1}] = \sum_{i=\tau+1}^N \mathbf{E}[\hat{\xi}_i^2 | \hat{\mathcal{F}}_{i-1}] + \sum_{i=\tau+1}^n \mathbf{E}[\xi_i^2 | \hat{\mathcal{F}}_{i-1}] = 1 - 2\langle X \rangle_\tau + \langle X \rangle_n.$$

Noting that $1 - \mathbf{E}[\xi_{\tau+1}^2 | \mathcal{F}_\tau] \leq \langle X \rangle_\tau$. Consequently, using the inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, $p \geq 1$, and Jensen's inequality, we derive that

$$\begin{aligned} \left| \sum_{i=\tau+1}^N \mathbf{E}[(\hat{\xi}_i - \xi_i)^2 | \hat{\mathcal{F}}_{i-1}] \right|^p &\leq \left| \langle X \rangle_n - 1 + 2\mathbf{E}[\xi_{\tau+1}^2 | \mathcal{F}_\tau] \right|^p \\ &\leq 2^{2p-1} \left(|\langle X \rangle_n - 1|^p + \left| \mathbf{E}[\xi_{\tau+1}^2 | \mathcal{F}_\tau] \right|^p \right) \\ &\leq 2^{2p-1} \left(|\langle X \rangle_n - 1|^p + \mathbf{E}[\xi_{\tau+1}^{2p} | \mathcal{F}_\tau] \right). \end{aligned} \quad (53)$$

Taking expectations on both sides of the last inequality, we deduce that

$$\begin{aligned} \mathbf{E} \left[\left| \sum_{i=\tau+1}^N \mathbf{E}[(\hat{\xi}_i - \xi_i)^2 | \hat{\mathcal{F}}_{i-1}] \right|^p \right] &\leq 2^{2p-1} \left(\mathbf{E} \left[|\langle X \rangle_n - 1|^p \right] + \mathbf{E}[\xi_{\tau+1}^{2p}] \right) \\ &\leq 2^{2p-1} \left(\mathbf{E} \left[|\langle X \rangle_n - 1|^p \right] + \mathbf{E} \left[\max_{1 \leq i \leq n} |\xi_i|^{2p} \right] \right). \end{aligned} \quad (54)$$

Similarly, using the inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, $p \geq 1$,

$$\begin{aligned} \mathbf{E} \left[\max_{\tau+1 \leq i \leq N} |\hat{\xi}_i - \xi_i|^{2p} \right] &\leq 2^{2p-1} \mathbf{E} \left[\max_{\tau+1 \leq i \leq N} \left(|\xi_i|^{2p} + |\hat{\xi}_i|^{2p} \right) \right] \\ &\leq 2^{2p-1} \left(\mathbf{E} \left[\max_{1 \leq i \leq n} |\xi_i|^{2p} \right] + \varepsilon^{2p} \right). \end{aligned} \quad (55)$$

Combining (52), (54) and (55) together, we obtain

$$\mathbf{E} \left[|\hat{X}_N - X_n|^{2p} \right] \leq \hat{c}_p \left(\mathbf{E} \left[|\langle X \rangle_n - 1|^p \right] + \mathbf{E} \left[\max_{1 \leq i \leq n} |\xi_i|^{2p} \right] + \varepsilon^{2p} \right). \quad (56)$$

Finally, applying the last inequality to (50) and let $\varepsilon \rightarrow 0$, then we have

$$D(X_n) \leq \tilde{c}_\rho \frac{\epsilon_n}{\rho} + \tilde{c}_p \left(\mathbf{E} \left[|\langle X \rangle_n - 1|^p \right] + \mathbf{E} \left[\max_{1 \leq i \leq n} |\xi_i|^{2p} \right] \right)^{1/(2p+1)}.$$

This completes the proof of Theorem 2.

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