INSTABILITY OF THE SOLITARY WAVES FOR THE GENERALIZED BOUSSINESQ EQUATIONS

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Abstract. In this work, we consider the following generalized Boussinesq equation

$$\partial_t^2 u - \partial_x^2 u + \partial_x^2(\partial_x^2 u + |u|^p u) = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R},$$

with $0 < p < \infty$. This equation has the traveling wave solutions $\phi_\omega(x - \omega t)$, with the frequency $\omega \in (-1,1)$ and ϕ_ω satisfying

$$-\partial_{xx}\phi_\omega + (1-\omega^2)\phi_\omega - \phi_\omega^{p+1} = 0.$$

Bona and Sachs (1988) proved that the traveling wave $\phi_\omega(x - \omega t)$ is orbitally stable when $0 < p < 4$, $\frac{p}{4} < \omega^2 < 1$. Liu (1993) proved the orbital instability under the conditions $0 < p < 4$, $\omega^2 < \frac{p}{4}$ or $p \geq 4$, $\omega^2 < 1$. In this paper, we prove the orbital instability in the degenerate case $0 < p < 4, \omega^2 = \frac{p}{4}$.

1. Introduction

In this paper, we consider the stability theory of the following generalized Boussinesq equation

$$\partial_t^2 u - \partial_x^2 u + \partial_x^2(\partial_x^2 u + |u|^p u) = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

with the initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{1.2}$$

Here $0 < p < \infty$.

The Boussinesq equation is a model describing the phenomenon that the propagating water wave occurs transformation affected by water flow, barrier and so on. The water wave will generate complex phenomenon of scattering, reflecting, dissipation of energy and other physical changes.

The equation (1.1) has the solitary wave solution $u(x,t) = \phi_\omega(x - \omega t)$, where ϕ_ω is the ground state solution of the following elliptic equation

$$-\partial_{xx}\phi_\omega + (1-\omega^2)\phi_\omega - \phi_\omega^{p+1} = 0, \qquad |\omega| < 1. \tag{1.3}$$

The ground state solution ϕ_ω is an even function and it has the property of exponential decay, that is, $|\phi_\omega| \leq C_1 e^{-C_2|x|}$ for some $C_1, C_2 > 0$ and $|\partial_x \phi_\omega| \leq C_3 e^{-C_4|x|}$ for some $C_3, C_4 > 0$.

The equation (1.1) has the equivalent system form

$$\begin{cases} u_t = v_x, \\ v_t = (-u_{xx} + u - |u|^p u)_x. \end{cases} \tag{1.4}$$

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Then the system (1.4) has the following solitary wave solution

$$\begin{pmatrix} u \\ v \end{pmatrix}(t,x) = \begin{pmatrix} \phi_\omega(x-\omega t) \\ -\omega\phi_\omega(x-\omega t) \end{pmatrix}.$$

For the $H^1 \times L^2$-solution $(u,v)^T$ of (1.1)–(1.2), the momentum Q and the energy E are conserved under the flow, where

$$Q\begin{pmatrix} u \\ v \end{pmatrix} = \int_{\mathbb{R}} uv \, \mathrm{d}x; \tag{1.5}$$

$$E\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}} (|u_x|^2 + |u|^2 + |v|^2)\, dx - \frac{1}{p+2}\int_{\mathbb{R}} |u|^{p+2} \, \mathrm{d}x. \tag{1.6}$$

There are several related results for the generalized Boussinesq equation. For the local existence result, Liu [9] proved the system (1.4) is locally well-posed in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. For the stability theories, Bona and Sachs [2] proved when $0 < p < 4$, $\frac{p}{4} < \omega^2 < 1$, the solitary wave solution is orbitally stable. Liu [9] proved the orbital instability if $0 < p < 4$ and $\omega^2 < \frac{p}{4}$, or $p \geq 4$ and $\omega^2 < 1$. Liu [10] proved that when the wave speed $\omega = 0$, the solitary wave solution is strongly unstable by blowup. Later, Liu, Ohta and Todorova [11] further showed that when $0 < p < \infty$ and $0 < 2(p+2)\omega^2 < p$, the solitary wave solution is strongly unstable by blowup. For the abstract Hamiltonian systems, we refer to Grillakis, Shatah and Strauss [5], [6] for the general stability/instability theories, in which the Vakhitov-Kolokolov's stability criterions of the solitary waves were confirmed except the degenerate cases. In the degenerate cases, it was also proved by Comech and Pelinovsky [4] (see also [14]) that the solitary wave solution is orbitally instable under some regularity restrictions in the nonlinearity (for example, p should be suitable large in our cases). In this paper, we consider the stability theory on the solitary wave solutions of the generalized Boussinesq equation and aim to show the instability in the degenerate cases without any regularity restriction. It is worth noting that none of the frameworks of Grillakis, Shatah and Strauss [5, 6] and Comech and Pelinovsky [4] are available in our cases, either because of the degeneration or because of insufficient regularity of the nonlinearity.

Before starting our theorem, we give some definitions. Let $v_0 = \int_{-\infty}^x u_1(y)\, \mathrm{d}y$, $\vec{u} = (u,v)^T$, $\vec{u}_0 = (u_0, v_0)^T$, and $\overrightarrow{\Phi_\omega} = (\phi_\omega, -\omega\phi_\omega)^T$. For $\varepsilon > 0$, we denote the set $U_\varepsilon(\overrightarrow{\Phi_\omega})$ as

$$U_\varepsilon(\overrightarrow{\Phi_\omega}) = \{\vec{u} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) : \inf_{y\in\mathbb{R}} \|\vec{u} - \overrightarrow{\Phi_\omega}(\cdot - y)\|_{H^1 \times L^2} < \varepsilon\}. \tag{1.7}$$

Definition 1.1. *We say that the solitary wave solution $\phi_\omega(x-\omega t)$ of (1.1) is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1\times L^2} < \delta$, then the solution $\vec{u}(t)$ of (1.1) with $\vec{u}(0) = \vec{u}_0$ exists for all $t \in \mathbb{R}$, and $\vec{u}(t) \in U_\varepsilon(\overrightarrow{\Phi_\omega})$ for all $t \in \mathbb{R}$. Otherwise, $\phi_\omega(x-\omega t)$ is said to be unstable.*

Then the main result in the present paper is

Theorem 1.2. *Let $0 < p < 4$, $\omega \in (-1,1)$ and ϕ_ω be the solution of (1.3). If $|\omega| = \sqrt{\frac{p}{4}}$, then the solitary waves solution $\phi_\omega(x-\omega t)$ is orbitally unstable.*

The main method that we use in the present paper is from [19], in which the instability of the standing wave solutions of the Klein-Gordon equation in the degenerate cases was proved. Instead of construction of the Lyapunov functional, the argument in [19] is to use the monotonicity of the virial quantity to control the modulations. However, this argument is much problem dependent, the key ingredients in our proofs are the following.

(1) The non-standard modulation and coercivity properties are given. More precisely, define the functional S_ω as

$$S_\omega(\vec{u}) = E(\vec{u}) + \omega Q(\vec{u}).$$

Inspired by [12, 13, 18], we establish the following non-standard coercivity properties. We show that for some suitable directions $\overrightarrow{\Gamma_\omega}, \overrightarrow{\Psi_\omega} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that the following coercivity properties hold. Suppose that $\vec{\eta} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies

$$\left\langle \vec{\eta}, \overrightarrow{\Gamma_\omega} \right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle = 0,$$

then

$$\left\langle S_\omega''(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle \gtrsim \|\vec{\eta}\|^2_{H^1 \times L^2}.$$

The choices of $\overrightarrow{\Gamma_\omega}, \overrightarrow{\Psi_\omega}$ play important roles in our estimation. $\overrightarrow{\Psi_\omega}$ can be regarded as the negative direction. However, we remark that $\overrightarrow{\Gamma_\omega} \notin Ker(S_\omega''(\overrightarrow{\Phi_\omega}))$, which is much different from the standard. Moreover, by suitably setting the translation and scaling parameters y, λ, we can establish the modulation by writing

$$\vec{u} = \left(\vec{\eta} + \overrightarrow{\Phi_{\lambda(t)}}\right)(\cdot - y(t))$$

such that $\vec{\eta}$ verifies similar orthogonal conditions above (by replacing $\overrightarrow{\Gamma_\omega}, \overrightarrow{\Psi_\omega}$ with $\overrightarrow{\Gamma_\lambda}, \overrightarrow{\Psi_\lambda}$ respectively).

(2) A subtle control on the modulated translation parameter is obtained. Instead of the rough control of the modulation parameter y as $\dot{y} - \lambda = O(\|\vec{\eta}\|_{H^1 \times L^2})$, we obtain the following finer estimate,

$$\dot{y} - \lambda = \|\phi_\lambda\|^{-2}_{L^2}\Big[Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] - \|\phi_\lambda\|^{-2}_{L^2}\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] + O(\|\vec{\eta}\|^2_{H^1 \times L^2}).$$

The subtle estimate is benefited from the choices of $\overrightarrow{\Gamma_\omega}, \overrightarrow{\Psi_\omega}$ in the first step and the dynamic of the solution. This estimate has great effects when we set up the structure of virial identity $I'(t)$ in the following.

(3) The monotonicity of the virial quantity is constructed. The key ingredient here is to suitably define a quantity $I(t)$ and obtain its monotonicity. To this end, the crucial issue is to prove the following structure of $I'(t)$ as

$$I'(t) = \rho(\vec{u}_0) + h(\lambda) + R(\vec{u}),$$

where for some positive constant C_1, C_2,

$$\rho(\vec{u}_0) \geq C_1 a,$$
$$h(\lambda) \geq C_2(\lambda - \omega)^2 + O\big(a(\lambda - \omega)^2\big) + o(\lambda - \omega)^2,$$

and $R(\vec{u})$ is a remainder term which can be dominated by ρ and h. Here a is the difference between the initial data and the soliton. The obstacles in the proof come from non-conservation terms among $I'(t)$, and the cancelation of one-order terms with respect to $\vec{\eta}$ and λ, these make much technical complexity. By a delicate analysis and the utilization of the estimates above, we overcome all difficulties and finally obtain the monotonicity of $I(t)$.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we show the coercivity property of the Hessian $S''_\omega(\overrightarrow{\Phi_\omega})$. In Section 4, we show the existence of modulation parameters. In Section 5, we control the modulation parameters obtained in Section 4. In section 6, we show the localized virial identities. Finally, we prove the main theorem in section 7.

2. Preliminary

2.1. **Notations.** For $f, g \in L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{R})$, we define

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) \,\mathrm{d}x$$

and regard $L^2(\mathbb{R})$ as a real Hilbert space. Similarly, for $\vec{f}, \vec{g} \in \big(L^2(\mathbb{R})\big)^2 = \big(L^2(\mathbb{R}, \mathbb{R})\big)^2$, we define

$$\langle \vec{f}, \vec{g} \rangle = \int_{\mathbb{R}} \vec{f}(x)^T \cdot \vec{g}(x) \,\mathrm{d}x.$$

For a function $f(x)$, its L^q-norm $\|f\|_{L^q} = \Big(\int_{\mathbb{R}} |f(x)|^q \,\mathrm{d}x\Big)^{\frac{1}{q}}$ and its H^1-norm $\|f\|_{H^1} = (\|f\|_{L^2}^2 + \|\partial_x f\|_{L^2}^2)^{\frac{1}{2}}$. For $\vec{f} = (f, g)^T$, its $H^1 \times L^2$-norm $\|\vec{f}\|_{H^1 \times L^2} = (\|f\|_{H^1}^2 + \|g\|_{L^2}^2)^{\frac{1}{2}}$.

Further, we write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \le CY$ for some constant $C > 0$. We use the notation $X \sim Y$ to denote $X \lesssim Y \lesssim X$. We also use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$; and use $o(Y)$ to denote any quantity X such that $X/Y \to 0$ if $Y \to 0$. Throughout the whole paper, the letter C will denote various positive constants which are of no importance in our analysis.

2.2. **Some basic definitions and properties.** In the rest of this paper, we consider the case of $0 < p < 4$, and $\omega_c = \sqrt{\frac{p}{4}}$, $\omega = \pm\omega_c$. Let $\vec{u} = (u, v)^T$, $\overrightarrow{\Phi_\omega} = (\phi_\omega, -\omega\phi_\omega)^T$. Recall the conserved equalities,

$$Q(\vec{u}) = \int_{\mathbb{R}} uv \,\mathrm{d}x,$$
$$E(\vec{u}) = \frac{1}{2}(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|v\|_{L^2}^2) - \frac{1}{p+2}\|u\|_{L^{p+2}}^{p+2}.$$

First, we give some basic properties on the momentum and energy.

Lemma 2.1. *Let $|\omega| = \sqrt{\frac{p}{4}}$, then the following equality holds,*

$$\partial_\lambda Q(\overrightarrow{\Phi_\lambda})\Big|_{\lambda=\omega} = 0.$$

Proof. Note that for $\lambda \in (-1, 1)$

$$Q(\overrightarrow{\Phi_\lambda}) = -\lambda\|\phi_\lambda\|_{L^2}^2. \tag{2.1}$$

By rescaling in (1.3) we find

$$\phi_\lambda(x) = (1-\lambda^2)^{\frac{1}{p}}\phi_0\left(\sqrt{1-\lambda^2}x\right). \tag{2.2}$$

This implies that

$$Q(\overrightarrow{\Phi_\lambda}) = -\lambda(1-\lambda^2)^{\frac{2}{p}-\frac{1}{2}}\|\phi_0\|_{L^2}^2.$$

By a straightforward computation, we have

$$\partial_\lambda Q(\overrightarrow{\Phi_\lambda}) = -(1-\lambda^2)^{\frac{2}{p}-\frac{3}{2}}\Big(1-\frac{4}{p}\lambda^2\Big)\|\phi_0\|_{L^2}^2.$$

Finally, we substitute $\lambda^2 = \frac{p}{4}$ into the equality above and thus complete the proof. □

Now we define the functional S_ω as

$$S_\omega(\vec{u}) = E(\vec{u}) + \omega Q(\vec{u}). \tag{2.3}$$

Then we have

$$Q'(\vec{u}) = \begin{pmatrix} v \\ u \end{pmatrix}, \tag{2.4}$$

$$E'(\vec{u}) = \begin{pmatrix} -\partial_{xx}u + u - |u|^p u \\ v \end{pmatrix}, \tag{2.5}$$

$$S'_\omega(\vec{u}) = \begin{pmatrix} -u_{xx} + u - |u|^p u + \omega v \\ v + \omega u \end{pmatrix}.$$

Note that $S'_\omega(\overrightarrow{\Phi_\omega}) = \vec{0}$. Moreover, for the vector $\vec{f} = (f,g)^T$, a direct computation shows

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{f} = \begin{pmatrix} -\partial_{xx}f + f - (p+1)\phi_\omega^p f + \omega g \\ g + \omega f \end{pmatrix}, \tag{2.6}$$

and for any vector $\vec{\xi}$, $\vec{\eta}$,

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi}, \vec{\eta}\Big\rangle = \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\xi}\Big\rangle.$$

Moreover, taking the derivative of $S'_\omega(\overrightarrow{\Phi_\omega}) = \vec{0}$ with respect to ω gives

$$S''_\omega(\overrightarrow{\Phi_\omega})\partial_\omega\overrightarrow{\Phi_\omega} = -Q'(\overrightarrow{\Phi_\omega}). \tag{2.7}$$

Then a consequence of Lemma 2.1 is

Corollary 2.2. *Let* $\lambda \in (-1,1)$, $|\omega| = \omega_c$, *then*

$$S_\lambda(\overrightarrow{\Phi_\lambda}) - S_\lambda(\overrightarrow{\Phi_\omega}) = o\big((\lambda-\omega)^2\big).$$

Proof. From the definition of $S_\omega(\vec{u})$ in (2.3), we have

$$S_\lambda(\overrightarrow{\Phi_\lambda}) - S_\lambda(\overrightarrow{\Phi_\omega}) = S_\omega(\overrightarrow{\Phi_\lambda}) - S_\omega(\overrightarrow{\Phi_\omega}) + (\lambda-\omega)\Big(Q(\overrightarrow{\Phi_\lambda}) - Q(\overrightarrow{\Phi_\omega})\Big).$$

Recall that $S'_\omega(\overrightarrow{\Phi_\omega}) = \vec{0}$, then we use Taylor's expansion to calculate

$$\begin{aligned} &S_\lambda(\overrightarrow{\Phi_\lambda}) - S_\lambda(\overrightarrow{\Phi_\omega}) \\ =&\frac{1}{2}\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\overrightarrow{\Phi_\lambda} - \overrightarrow{\Phi_\omega}\Big), \Big(\overrightarrow{\Phi_\lambda} - \overrightarrow{\Phi_\omega}\Big)\Big\rangle \end{aligned}$$

$$+(\lambda-\omega)\Big(Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big)+o\big((\lambda-\omega)^2\big). \tag{2.8}$$

Note that

$$\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}=(\lambda-\omega)\partial_\omega\overrightarrow{\Phi_\omega}+o(\lambda-\omega),$$

then we find

$$\begin{aligned}
&\Big\langle S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big),\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big)\Big\rangle\\
&\quad=(\lambda-\omega)^2\Big\langle S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\partial_\omega\overrightarrow{\Phi_\omega},\partial_\omega\overrightarrow{\Phi_\omega}\Big\rangle+o\big((\lambda-\omega)^2\big)\\
&\quad=-(\lambda-\omega)^2\Big\langle Q'\big(\overrightarrow{\Phi_\omega}\big),\partial_\omega\overrightarrow{\Phi_\omega}\Big\rangle+o\big((\lambda-\omega)^2\big)\\
&\quad=-(\lambda-\omega)^2\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big)\Big|_{\lambda=\omega}+o\big((\lambda-\omega)^2\big),
\end{aligned}$$

here we have used equality (2.7) in the second step. Using Lemma 2.1, we have

$$\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big)\Big|_{\lambda=\omega}=0.$$

Hence,

$$\Big\langle S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big),\Big(\overrightarrow{\Phi_\lambda}-\overrightarrow{\Phi_\omega}\Big)\Big\rangle=o\big((\lambda-\omega)^2\big),$$

and

$$Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)=o\big(\lambda-\omega\big).$$

Taking these two results into (2.8), we obtain the desired estimate. □

3. Coercivity

In this section, we have a coercivity property on the Hessian of the action $S''_\omega\big(\overrightarrow{\Phi_\omega}\big)$. First, we study the kernel of $S''_\omega\big(\overrightarrow{\Phi_\omega}\big)$ in the following lemma. The proof is standard, and it is a consequence of the result from [17].

Lemma 3.1. *The kernel of $S''_\omega\big(\overrightarrow{\Phi_\omega}\big)$ satisfies that*

$$Ker\Big(S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\Big)=\big\{C\partial_x\overrightarrow{\Phi_\omega}:C\in\mathbb{R}\big\}.$$

Proof. Firstly, we need to show the relationship "⊃". For any $\vec{f}\in\big\{C\partial_x\overrightarrow{\Phi_\omega}:C\in\mathbb{R}\big\}$, using the equation (1.3), we have

$$S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\vec{f}=S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\big(C\partial_x\overrightarrow{\Phi_\omega}\big)=C\begin{pmatrix}\partial_x\big(-\partial_{xx}\phi_\omega+(1-\omega^2)\phi_\omega-\phi_\omega^{p+1}\big)\\ -\omega\phi'_\omega+\omega\phi'_\omega\end{pmatrix}=\vec{0}.$$

Then it implies that $\vec{f}\in Ker\Big(S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\Big)$, and we have the conclusion

$$Ker\Big(S''_\omega\big(\overrightarrow{\Phi_\omega}\big)\Big)\supset\big\{C\partial_x\overrightarrow{\Phi_\omega}:C\in\mathbb{R}\big\}.$$

Secondly, we prove the reverse relationship "$\subset$". For any $\vec{f} \in Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big)$, by the expression of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.6), we have

$$\begin{cases} -\partial_{xx} f + (1-\omega^2) f - (p+1)\phi_\omega^p f = 0, \\ g + \omega f = 0. \end{cases} \tag{3.1}$$

By the work of Weinstein [17], the only solutions to (3.1) are

$$\begin{cases} f = C\partial_x \phi_\omega, \\ g = -C\omega\partial_x\phi_\omega, \end{cases} \quad C \in \mathbb{R}.$$

This implies that $\vec{f} \in \{C\partial_x \overrightarrow{\Phi_\omega} : C \in \mathbb{R}\}$ and we have

$$Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big) \subset \{C\partial_x \overrightarrow{\Phi_\omega} : C \in \mathbb{R}\}.$$

Finally, combining the two relationship gives us

$$Ker\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big) = \{C\partial_x \overrightarrow{\Phi_\omega} : C \in \mathbb{R}\}.$$

This gives the proof of the lemma. □

The second lemma is the uniqueness of the negative eigenvalue of $S''_\omega(\overrightarrow{\Phi_\omega})$.

Lemma 3.2. $S''_\omega(\overrightarrow{\Phi_\omega})$ *exists only one negative eigenvalue.*

Proof. It is known that the operator $-\partial_{xx} + (1-\omega^2) - (p+1)\phi_\omega^p$ has only one negative eigenvalue (see [17]), and we denote it by λ_{-1}. Then there exists a unique associated eigenvector $\zeta \in H^1(\mathbb{R})$, such that

$$-\partial_{xx}\zeta + (1-\omega^2)\zeta - (p+1)\phi_\omega^p \zeta = \lambda_{-1}\zeta. \tag{3.2}$$

Using the expression of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.6), we have

$$\begin{aligned} &\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\overrightarrow{\Phi_\omega}, \overrightarrow{\Phi_\omega}\Big\rangle \\ &\quad = \int_{\mathbb{R}} (-\partial_{xx}\phi_\omega + \phi_\omega - (p+1){\phi_\omega}^{p+1} - \omega^2\phi_\omega, -\omega\phi_\omega + \omega\phi_\omega) \cdot \begin{pmatrix} \phi_\omega \\ -\omega\phi_\omega \end{pmatrix} dx \\ &\quad = -\, p\|\phi_\omega\|_{L^{p+2}}^{p+2} < 0. \end{aligned}$$

This implies that $S''_\omega(\overrightarrow{\Phi_\omega})$ has at least one negative eigenvalue, says μ_0. Assume its associated eigenvector $\vec{\eta_0} = (\xi_0, \eta_0)^T$, that is,

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta_0} = \mu_0\vec{\eta_0}.$$

Using (2.6) again, the last equality yields

$$\begin{cases} -\partial_{xx}\xi_0 + \xi_0 - (p+1)\phi_\omega^p \xi + \omega\eta_0 = \mu_0\xi_0, \\ \eta_0 + \omega\xi_0 = \mu_0\eta_0. \end{cases}$$

From the second equality we have $\eta_0 = -\frac{\omega}{1-\mu_0}\xi_0$. Then we substitute it into the first equality to get

$$-\partial_{xx}\xi_0 + (1-\omega^2)\xi_0 - (p+1)\phi_\omega^p\xi_0 = \mu_0\Big(\frac{\omega^2}{1-\mu_0} + 1\Big)\xi_0.$$

Hence, by (3.2), the equation above has only one solution–pair (μ_0, ξ_0) with $\mu_0\Big(\frac{\omega^2}{1-\mu_0}+1\Big) = \lambda_{-1}$, $\xi_0 = \zeta$, then $(\mu_0, \vec{\eta}_0)$ is exact the pair satisfying

$$\mu_0 = \frac{1}{2}\left(\lambda_{-1} + \omega^2 + 1 - \sqrt{\lambda_{-1}^2 + 2(\omega^2-1)\lambda_{-1} + (\omega^2+1)^2}\right), \qquad \vec{\eta}_0 = \begin{pmatrix} \zeta \\ \dfrac{\omega\zeta}{\mu_0 - 1} \end{pmatrix}. \tag{3.3}$$

This implies that $S''_\omega(\overrightarrow{\Phi_\omega})$ has exactly one simple negative eigenvalue. This completes the proof of the Lemma. □

The next lemma gives one of the negative direction of $S''_\omega(\overrightarrow{\Phi_\omega})$.

Lemma 3.3. *Let*

$$\vec{\psi}_\omega = \frac{1}{2\omega}\begin{pmatrix} \partial_\omega \phi_\omega \\ -\omega \partial_\omega \phi_\omega \end{pmatrix}, \qquad \overrightarrow{\Psi_\omega} = \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix}.$$

Then

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega = \overrightarrow{\Psi_\omega}. \tag{3.4}$$

Moreover, if $|\omega| = \omega_c$*, then*

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \Big\rangle < 0.$$

Proof. Taking the derivative of equation (1.3) with respect to ω, we have

$$-\partial_{xx}(\partial_\omega \phi_\omega) + (1-\omega^2)\partial_\omega \phi_\omega - (p+1)\phi_\omega^p \partial_\omega \phi_\omega = 2\omega \phi_\omega. \tag{3.5}$$

Using (2.6), we have

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega = \frac{1}{2\omega}\begin{pmatrix} -\partial_{xx}(\partial_\omega \phi_\omega) + (1-\omega^2)\partial_\omega \phi_\omega - (p+1)\phi_\omega^p \partial_\omega \phi_\omega \\ 0 \end{pmatrix}.$$

This combining with (3.5) gives

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega = \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix} = \overrightarrow{\Psi_\omega}. \tag{3.6}$$

Now we show $\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \Big\rangle < 0$. From (3.6), we have

$$\begin{aligned}\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \Big\rangle &= \Big\langle \overrightarrow{\Psi_\omega}, \vec{\psi}_\omega \Big\rangle = \int_{\mathbb{R}} (\phi_\omega, 0) \cdot \frac{1}{2\omega}\begin{pmatrix} \partial_\omega \phi_\omega \\ -\omega \partial_\omega \phi_\omega \end{pmatrix} \mathrm{d}x \\ &= \frac{1}{2\omega}\int_{\mathbb{R}} \phi_\omega \, \partial_\omega \phi_\omega \, \mathrm{d}x = \frac{1}{4\omega}\partial_\omega \|\phi_\omega\|_{L^2}^2. \end{aligned} \tag{3.7}$$

Note that $\partial_\omega\big(\omega\|\phi_\omega\|_{L^2}^2\big) = \|\phi_\omega\|_{L^2}^2 + \omega\partial_\omega\|\phi_\omega\|_{L^2}^2$, then we substitute this into (3.7) to obtain

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \Big\rangle = -\frac{1}{4\omega^2}\partial_\omega(-\omega\|\phi_\omega\|_{L^2}^2) - \frac{1}{4\omega^2}\|\phi_\omega\|_{L^2}^2.$$

Using (2.1), we further get

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \Big\rangle = -\frac{1}{4\omega^2}\partial_\omega\Big(Q(\overrightarrow{\Phi_\omega})\Big) - \frac{1}{4\omega^2}\|\phi_\omega\|_{L^2}^2.$$

Finally, by Lemma 2.1, we have

$$\left\langle S_\omega''(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle = -\frac{1}{4\omega^2}\|\phi_\omega\|_{L^2}^2 < 0.$$

This completes the proof. □

Now we prove the following coercivity property.

Proposition 3.4. *Let* $|\omega| = \omega_c$. *Suppose that* $\vec{\eta} = (\xi, \eta)^T \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *satisfies*

$$\left\langle \vec{\eta}, \partial_x \overrightarrow{\Phi_\omega} \right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle = 0, \tag{3.8}$$

where $\overrightarrow{\Psi_\omega} = (\phi_\omega, 0)^T$. *Then*

$$\left\langle S_\omega''(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle \gtrsim \|\vec{\eta}\|_{H^1\times L^2}^2.$$

Proof. From the expression of $S_\omega''(\overrightarrow{\Phi_\omega})$ in (2.6), we can write $S_\omega''(\overrightarrow{\Phi_\omega})$ as

$$S_\omega''(\overrightarrow{\Phi_\omega}) = L + V,$$

where $L = \begin{pmatrix} -\partial_{xx} + 1 & \omega \\ \omega & 1 \end{pmatrix}$, and $V = \begin{pmatrix} -(p+1)\phi_\omega^p & 0 \\ 0 & 0 \end{pmatrix}$. Hence V is a compact perturbation of the self-adjoint operator L.

Step 1. Analyse the spectrum of $S_\omega''(\overrightarrow{\Phi_\omega})$.

We firstly compute the essential spectrum of L. Note that for any $\vec{f} = (f, g)^T \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$,

$$\begin{aligned} \langle L\vec{f}, \vec{f} \rangle &= \left\langle \begin{pmatrix} -\partial_{xx} + 1 & \omega \\ \omega & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle \\ &= \int_{\mathbb{R}} (-\partial_{xx} f + f + \omega g, \omega f + g) \cdot \begin{pmatrix} f \\ g \end{pmatrix} \mathrm{d}x \\ &= \|\partial_x f\|_{L^2}^2 + \|f\|_{L^2}^2 + 2\omega\langle f, g\rangle + \|g\|_{L^2}^2 \\ &= \|\vec{f}\|_{H^1\times L^2}^2 + 2\omega\langle f, g\rangle. \end{aligned} \tag{3.9}$$

For the term $2\omega\langle f, g\rangle$, applying Hölder's and Young's inequalities, we have

$$|2\omega\langle f, g\rangle| \le |\omega| \|\vec{f}\|_{H^1\times L^2}^2.$$

Taking this estimate into (3.9), we have

$$\langle L\vec{f}, \vec{f} \rangle \ge (1 - |\omega|)\|\vec{f}\|_{H^1\times L^2}^2.$$

Since $|\omega| < 1$, we get

$$\langle L\vec{f}, \vec{f} \rangle \gtrsim \|\vec{f}\|_{H^1\times L^2}^2.$$

This means that there exists $\delta > 0$ such that the essential spectrum of L is $[\delta, +\infty)$. By Weyl's Theorem, $S_\omega''(\overrightarrow{\Phi_\omega})$ and L share the same essential spectrum. So we obtain the essential spectrum of $S_\omega''(\overrightarrow{\Phi_\omega})$. Recall that we have obtained the only one negative eigenvalue μ_0 of $S_\omega''(\overrightarrow{\Phi_\omega})$ in Lemma 3.2, and the kernel of $S_\omega''(\overrightarrow{\Phi_\omega})$ in Lemma 3.1. So the discrete spectrum of $S_\omega''(\overrightarrow{\Phi_\omega})$ is μ_0, 0, and the essential spectrum is $[\delta, +\infty)$.

Step 2. Positivity.

The argument here is inspired by [1, 8]. By Lemma 3.2, we have the unique negative eigenvalue μ_0 and eigenvector $\vec{\eta}_0$ of $S''_\omega(\overrightarrow{\Phi_\omega})$. For convenience, we normalize the eigenvector $\vec{\eta}_0$ such that $\|\vec{\eta}_0\|_{L^2\times L^2} = 1$. Hence for vector $\vec{\eta} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, by spectral decomposition theorem we can write the decomposition of $\vec{\eta}$ along the spectrum of $S''_\omega(\overrightarrow{\Phi_\omega})$,

$$\vec{\eta} = a_\eta \vec{\eta}_0 + b_\eta \partial_x \overrightarrow{\Phi_\omega} + \vec{g}_\eta,$$

where a_η, $b_\eta \in \mathbb{R}$ and $\vec{g}_\eta$ in the positive eigenspace of $S''_\omega(\overrightarrow{\Phi_\omega})$ satisfies

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle \geq \sigma \|\vec{g}_\eta\|^2_{L^2\times L^2}, \qquad \sigma > 0. \tag{3.10}$$

Since $\vec{\eta}$ satisfies the orthogonality condition $\left\langle \vec{\eta}, \partial_x \overrightarrow{\Phi_\omega} \right\rangle = 0$ in (3.8), we have $b_\eta = 0$, and thus

$$\vec{\eta} = a_\eta \vec{\eta}_0 + \vec{g}_\eta. \tag{3.11}$$

Substituting (3.11) into $\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle$, we get

$$\begin{aligned}\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})(a_\eta\vec{\eta}_0 + \vec{g}_\eta), a_\eta\vec{\eta}_0 + \vec{g}_\eta \right\rangle \\ &= a_\eta^2 \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}_0, \vec{\eta}_0 \right\rangle + 2\mu_0 a_\eta \left\langle \vec{g}_\eta, \vec{\eta}_0 \right\rangle + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle.\end{aligned}$$

Due to the orthogonality property of eigenvector $\langle \vec{g}_\eta, \vec{\eta}_0 \rangle = 0$, we have

$$\begin{aligned}\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle &= a_\eta^2 \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}_0, \vec{\eta}_0 \right\rangle + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle \\ &= \mu_0 a_\eta^2 \langle \vec{\eta}_0, \vec{\eta}_0 \rangle + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle \\ &= \mu_0 a_\eta^2 + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle.\end{aligned} \tag{3.12}$$

To $\vec{\psi}_\omega$, by spectral decomposition theorem again and noting that $\left\langle \vec{\psi}_\omega, \partial_x \overrightarrow{\Phi_\omega} \right\rangle = 0$, we write

$$\vec{\psi}_\omega = a\vec{\eta}_0 + \vec{g},$$

where $a \in \mathbb{R}$, $\vec{g}$ in the positive eigenspace of $S''_\omega(\overrightarrow{\Phi_\omega})$. A similar computation as above shows that

$$\begin{aligned}\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})(a\vec{\eta}_0 + \vec{g}), a\vec{\eta}_0 + \vec{g} \right\rangle \\ &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})(a\vec{\eta}_0), a\vec{\eta}_0 \right\rangle + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \\ &= \mu_0 a^2 + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle.\end{aligned}$$

For convenience, let $-\delta_0 = \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle$. Then by Lemma 3.3, we know $\delta_0 > 0$. Moreover, we have

$$-\delta_0 = \mu_0 a^2 + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle. \tag{3.13}$$

Using the orthogonality assumption $\left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle = 0$ in (3.8) and (3.4), we have

$$\begin{aligned}
0 &= \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle = \left\langle a_\eta \vec{\eta}_0 + \vec{g}_\eta, S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega \right\rangle \\
&= \left\langle a_\eta \vec{\eta}_0 + \vec{g}_\eta, S''_\omega(\overrightarrow{\Phi_\omega})(a\vec{\eta}_0 + \vec{g}) \right\rangle \\
&= \left\langle a_\eta \vec{\eta}_0, S''_\omega(\overrightarrow{\Phi_\omega})(a\vec{\eta}_0) \right\rangle + \left\langle \vec{g}_\eta, S''_\omega(\overrightarrow{\Phi_\omega})\vec{g} \right\rangle \\
&= \mu_0 a a_\eta \langle \vec{\eta}_0, \vec{\eta}_0 \rangle + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}_\eta \right\rangle \\
&= \mu_0 a a_\eta + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}_\eta \right\rangle.
\end{aligned}$$

So we get the equality

$$0 = \mu_0 a a_\eta + \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}_\eta \right\rangle.$$

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
(\mu_0 a a_\eta)^2 &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}_\eta \right\rangle^2 \\
&\leq \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle.
\end{aligned}$$

This gives

$$(-\mu_0 a^2)(-\mu_0 a_\eta^2) \leq \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle. \tag{3.14}$$

The last equality combining with (3.13) implies

$$-\mu_0 a_\eta^2 \leq \frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle}{-\mu_0 a^2} = \frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0},$$

that is,

$$\mu_0 a_\eta^2 \geq -\frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0}. \tag{3.15}$$

Inserting (3.15) into (3.12), we obtain

$$\begin{aligned}
\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle &\geq \left(1 - \frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0}\right) \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle \\
&= \frac{\delta_0}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0} \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta \right\rangle.
\end{aligned}$$

Recalling that $\vec{g}_\eta$ satisfies (3.10), we have

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle \geq \frac{\delta_0 \sigma}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g} \right\rangle + \delta_0} \|\vec{g}_\eta\|^2_{L^2 \times L^2}. \tag{3.16}$$

From the expression of $\vec{\eta}$ in (3.11) and the inequality (3.15), we have

$$\|\vec{\eta}\|^2_{L^2 \times L^2} = \|a_\eta \vec{\eta}_0 + \vec{g}_\eta\|^2_{L^2 \times L^2} = a_\eta^2 + \|\vec{g}_\eta\|^2_{L^2 \times L^2}$$

$$\leq \frac{1}{-\mu_0} \cdot \frac{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}\right\rangle \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}_\eta, \vec{g}_\eta\right\rangle}{\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}, \vec{g}\right\rangle + \delta_0} + \|\vec{g}_\eta\|^2_{L^2\times L^2}$$
$$\lesssim \|\vec{g}_\eta\|^2_{L^2\times L^2}.$$

Therefore, this together with (3.16) gives

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle \gtrsim \|\vec{g}_\eta\|^2_{L^2\times L^2} \gtrsim \left\|\vec{\eta}\right\|^2_{L^2\times L^2}. \tag{3.17}$$

To obtain the final conclusion, we still need to estimate

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle \gtrsim \|\vec{\eta}\|^2_{H^1\times L^2}.$$

Using (2.6), we have

$$\begin{aligned}\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\rangle &= \int_{\mathbb{R}} (-\partial_{xx}\xi + \xi - (p+1)\phi_\omega^p\xi + \omega\eta, \eta + \omega\xi) \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mathrm{d}x \\ &= \|\partial_x\xi\|^2_{L^2} + \|\vec{\eta}\|^2_{L^2\times L^2} + 2\omega \int_{\mathbb{R}} \xi\eta \,\mathrm{d}x - (p+1)\int_{\mathbb{R}} |\phi_\omega|^p \xi^2 \,\mathrm{d}x.\end{aligned}$$

Thus by Hölder's and Young's inequalities and (3.17), we get

$$\begin{aligned}\|\partial_x\xi\|^2_{L^2} &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle - 2\omega\int_{\mathbb{R}} \xi\eta\,\mathrm{d}x + (p+1)\int_{\mathbb{R}} |\phi_\omega|^p\xi^2\,\mathrm{d}x - \|\vec{\eta}\|^2_{L^2\times L^2} \\ &\leq \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle + 2|\omega|\|\xi\|_{L^2}\|\eta\|_{L^2} + (p+1)\|\phi_\omega\|^p_{L^\infty}\|\xi\|^2_{L^2} \\ &\leq \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle + \Big(|\omega| + (p+1)\|\phi_\omega\|^p_{L^\infty}\Big)\|\vec{\eta}\|^2_{L^2\times L^2} \\ &\lesssim \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle + \|\vec{\eta}\|^2_{L^2\times L^2} \lesssim \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle.\end{aligned} \tag{3.18}$$

Therefore, together (3.17) and (3.18), we obtain

$$\|\vec{\eta}\|^2_{H^1\times L^2} = \|\partial_x\xi\|^2_{L^2} + \|\vec{\eta}\|^2_{L^2\times L^2} \lesssim \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle.$$

Thus we obtain the desired result. □

Applying Proposition 3.4, we have the following corollary, which is the non-standard coercivity property we need in this paper and is one of key ingredients in our proof. The corollary shows that we can replace the orthogonal condition from the kernel by a suitable defined vector which essentially effect on the estimates of the translation parameter in Section 5.

Corollary 3.5. *Let* $|\omega| = \omega_c$. *Suppose that* $\vec{\eta} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *satisfies*

$$\left\langle \vec{\eta}, \overrightarrow{\Gamma_\omega}\right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega}\right\rangle = 0, \tag{3.19}$$

where $\overrightarrow{\Gamma_\omega} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, *and* $\partial_x\overrightarrow{\Gamma_\omega} = \overrightarrow{\Psi_\omega} = (\phi_\omega, 0)^T$. *Then*

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta}\right\rangle \gtrsim \left\|\vec{\eta}\right\|^2_{H^1\times L^2}. \tag{3.20}$$

Proof. We define

$$\vec{\xi} = \vec{\eta} + b\partial_x \overrightarrow{\Phi_\omega}, \qquad \vec{\xi} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}).$$

Choosing

$$b = -\frac{\left\langle \vec{\eta}, \partial_x \overrightarrow{\Phi_\omega} \right\rangle}{\|\partial_x \overrightarrow{\Phi_\omega}\|^2_{L^2 \times L^2}},$$

then

$$\left\langle \vec{\xi}, \partial_x \overrightarrow{\Phi_\omega} \right\rangle = 0.$$

Moreover, by (3.19), we have

$$\left\langle \vec{\xi}, \overrightarrow{\Psi_\omega} \right\rangle = \left\langle \vec{\eta} + b\partial_x \overrightarrow{\Phi_\omega}, \overrightarrow{\Psi_\omega} \right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\omega} \right\rangle + b\left\langle \partial_x \overrightarrow{\Phi_\omega}, \overrightarrow{\Psi_\omega} \right\rangle. \tag{3.21}$$

Recalling that ϕ_ω is an even function, we have

$$b\left\langle \partial_x \overrightarrow{\Phi_\omega}, \overrightarrow{\Psi_\omega} \right\rangle = b \int_{\mathbb{R}} \Big(\partial_x \phi_\omega, (-\omega)\partial_x \phi_\omega \Big) \cdot \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix} \mathrm{d}x = b \int_{\mathbb{R}} \partial_x \phi_\omega \phi_\omega \, \mathrm{d}x = 0.$$

Hence, $\left\langle \vec{\xi}, \overrightarrow{\Psi_\omega} \right\rangle = 0$. Therefore, $\vec{\xi}$ satisfies the orthogonality conditions (3.8) in Proposition 3.4. Hence, using Proposition 3.4 and $S''_\omega(\overrightarrow{\Phi_\omega})\partial_x \overrightarrow{\Phi_\omega} = \vec{0}$, we get

$$\begin{aligned}
\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta}, \vec{\eta} \right\rangle &= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\vec{\xi} - b\partial_x \overrightarrow{\Phi_\omega}\Big), \Big(\vec{\xi} - b\partial_x \overrightarrow{\Phi_\omega}\Big) \right\rangle \\
&= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi}, \vec{\xi} \right\rangle - 2b\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\partial_x \overrightarrow{\Phi_\omega}, \vec{\xi} \right\rangle + b^2\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\partial_x \overrightarrow{\Phi_\omega}, \partial_x \overrightarrow{\Phi_\omega} \right\rangle \\
&= \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi}, \vec{\xi} \right\rangle \gtrsim \|\vec{\xi}\|^2_{H^1 \times L^2},
\end{aligned}$$

where we have used the self-adjoint property of the operator $S''_\omega(\overrightarrow{\Phi_\omega})$ in the second step.

Now we claim that $\|\vec{\xi}\|^2_{H^1 \times L^2} \gtrsim \|\vec{\eta}\|^2_{H^1 \times L^2}$. Indeed, using the orthogonality assumption (3.19), we have

$$\left\langle \vec{\xi}, \overrightarrow{\Gamma_\omega} \right\rangle = \left\langle \vec{\eta} + b\partial_x \overrightarrow{\Phi_\omega}, \overrightarrow{\Gamma_\omega} \right\rangle = -b \int_{\mathbb{R}} (\phi_\omega, -\omega\phi_\omega) \cdot \begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix} = -b\|\phi_\omega\|^2_{L^2}.$$

Thus, by Hölder's inequality, we have

$$|b| = \frac{\left| \left\langle \vec{\xi}, \overrightarrow{\Gamma_\omega} \right\rangle \right|}{\|\phi_\omega\|^2_{L^2}} \lesssim \|\vec{\xi}\|_{H^1 \times L^2}. \tag{3.22}$$

Now from (3.22),

$$\|\vec{\eta}\|_{H^1 \times L^2} = \left\| \vec{\xi} - b\partial_x \overrightarrow{\Phi_\omega} \right\|_{H^1 \times L^2} \le \|\vec{\xi}\|_{H^1 \times L^2} + |b| \left\| \partial_x \overrightarrow{\Phi_\omega} \right\|_{H^1 \times L^2} \lesssim \|\vec{\xi}\|_{H^1 \times L^2}.$$

This completes the proof. □

4. Modulation

We now suppose for contradiction that the solitary wave solution is stable, that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that when

$$\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1\times L^2} < \delta,$$

we have

$$\vec{u} \in U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big). \tag{4.1}$$

Then the modulation theory as presented in the following shows that by choosing suitable parameters, the orthogonality conditions in Corollary 3.5 can be verified. The modulation is obtained via the standard Implicit Function Theorem.

Proposition 4.1. *(Modulation). Let $|\omega| = \omega_c$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\vec{u} \in U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big)$, the following properties are verified. There exist C^1-functions*

$$y : \mathbb{R} \to \mathbb{R}, \quad \lambda : \mathbb{R} \to \mathbb{R}^+$$

such that if we define $\vec{\eta}$ by

$$\vec{\eta}(t) = \vec{u}\big(t, \cdot + y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}}, \tag{4.2}$$

then $\vec{\eta}$ satisfies the following orthogonality conditions for any $t \in \mathbb{R}$,

$$\Big\langle \vec{\eta}, \overrightarrow{\Gamma_{\lambda(t)}} \Big\rangle = \Big\langle \vec{\eta}, \overrightarrow{\Psi_{\lambda(t)}} \Big\rangle = 0. \tag{4.3}$$

where $\overrightarrow{\Gamma_\lambda} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, and $\partial_x \overrightarrow{\Gamma_\lambda} = \overrightarrow{\Psi_\lambda} = \begin{pmatrix} \phi_\lambda \\ 0 \end{pmatrix}$. Moreover,

$$\|\vec{\eta}\|_{H^1\times L^2} + |\lambda - \omega| \lesssim \varepsilon. \tag{4.4}$$

Proof. Define

$$p = (\vec{u}; \lambda, y), \qquad p_0 = (\overrightarrow{\Phi_\omega}; \omega, 0);$$
$$F_1(p) = \Big\langle \vec{\eta}, \overrightarrow{\Gamma_\lambda} \Big\rangle,$$
$$F_2(p) = \Big\langle \vec{\eta}, \overrightarrow{\Psi_\lambda} \Big\rangle.$$

Firstly we have

$$F_1(p_0) = F_2(p_0) = 0.$$

Secondly, we prove that

$$|J| = \begin{vmatrix} \partial_\lambda F_1 & \partial_y F_1 \\ \partial_\lambda F_2 & \partial_y F_2 \end{vmatrix}_{p=p_0} \neq 0.$$

Indeed, a direct calculation gives that

$$\begin{aligned}\partial_\lambda F_1(p) = \partial_\lambda \Big\langle \vec{\eta}, \overrightarrow{\Gamma_\lambda} \Big\rangle &= \partial_\lambda \Big\langle \vec{u}\big(t, x + y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}}, \overrightarrow{\Gamma_\lambda} \Big\rangle \\ &= \Big\langle \vec{u}\big(t, x + y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}}, \partial_\lambda \overrightarrow{\Gamma_\lambda} \Big\rangle - \Big\langle \partial_\lambda \overrightarrow{\Phi_{\lambda(t)}}, \overrightarrow{\Gamma_\lambda} \Big\rangle.\end{aligned}$$

When $p = p_0$, we observe that $\vec{u}(t, x + y(t)) - \overrightarrow{\Phi_{\lambda(t)}} = 0$, and the first term vanishes. For the second term, we note that $\overrightarrow{\Gamma_\lambda}$ is an odd vector and $\partial_\lambda \overrightarrow{\Phi_{\lambda(t)}}$ is an even vector, so we get

$$\partial_\lambda F_1(p)\Big|_{p=p_0} = 0.$$

A similar computation shows that

$$\partial_y F_1(p)\Big|_{p=p_0} = \left\langle \partial_x \vec{u}(x+y), \overrightarrow{\Gamma_\lambda} \right\rangle\Big|_{p=p_0} = \left\langle \partial_x \overrightarrow{\Phi_\lambda}, \overrightarrow{\Gamma_\lambda} \right\rangle\Big|_{p=p_0} = -\|\phi_\omega\|_{L^2}^2;$$

$$\partial_\lambda F_2(p)\Big|_{p=p_0} = -\left\langle \partial_\lambda \overrightarrow{\Phi_\lambda}, \overrightarrow{\Psi_\lambda} \right\rangle\Big|_{p=p_0} = -\left\langle \partial_\lambda \phi_\lambda, \phi_\lambda \right\rangle\Big|_{p=p_0} = -\frac{1}{2}\partial_\lambda \|\phi_\lambda\|_{L^2}^2\Big|_{p=p_0} = \frac{1}{2\omega}\|\phi_\omega\|_{L^2}^2;$$

$$\partial_y F_1(p)\Big|_{p=p_0} = \left\langle \partial_x \overrightarrow{\Phi_\lambda}, \overrightarrow{\Psi_\lambda} \right\rangle\Big|_{p=p_0} = \int_{\mathbb{R}} \partial_x \phi_\lambda \phi_\lambda \, \mathrm{d}x\Big|_{p=p_0} = 0.$$

Then we find that

$$\begin{vmatrix} \partial_\lambda F_1 & \partial_y F_1 \\ \partial_\lambda F_2 & \partial_y F_2 \end{vmatrix}_{p=p_0} = \frac{1}{2\omega}\|\phi_\omega\|_{L^2}^4 \neq 0.$$

Therefore, the Implicit Function Theorem implies that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\vec{u} \in U_\varepsilon(\overrightarrow{\Phi_\omega})$, there exist unique C^1-functions

$$y : U_\varepsilon(\overrightarrow{\Phi_\omega}) \to \mathbb{R}, \quad \lambda : U_\varepsilon(\overrightarrow{\Phi_\omega}) \to \mathbb{R}^+,$$

such that

$$\left\langle \vec{\eta}, \overrightarrow{\Gamma_\lambda} \right\rangle = \left\langle \vec{\eta}, \overrightarrow{\Psi_\lambda} \right\rangle = 0. \tag{4.5}$$

Furthermore,

$$\begin{pmatrix} \partial_u \lambda & \partial_v \lambda \\ \partial_u y & \partial_v y \end{pmatrix} = J^{-1} \begin{pmatrix} \partial_u F_1 & \partial_v F_1 \\ \partial_u F_2 & \partial_v F_2 \end{pmatrix}.$$

This implies that

$$|\lambda - \omega| \lesssim \|\vec{u} - \overrightarrow{\Phi_\omega}\|_{H^1 \times L^2} < \varepsilon.$$

This finishes the proof of the proposition. □

5. Dynamic of the parameters

In this section, we control the modulation parameters y and λ. The effect of giving a precise control on modulation parameters is to obtain the structure of $I'(t)$ in Section 7. The main result is

Proposition 5.1. *Let $\vec{u} = (u, v)^T$ be the solution of (1.4) with $\vec{u} \in U_\varepsilon(\overrightarrow{\Phi_\omega})$, where ε is the constant obtained in Proposition 4.1. Let y, λ, $\vec{\eta} = (\xi, \eta)^T$ be the parameters and vector obtained in Proposition 4.1. Then*

$$\dot{y} - \lambda = \|\phi_\lambda\|_{L^2}^{-2}\left[Q(\overrightarrow{\Phi_\lambda}) - Q(\overrightarrow{\Phi_\omega})\right] - \|\phi_\lambda\|_{L^2}^{-2}\left[Q(\vec{u}_0) - Q(\overrightarrow{\Phi_\omega})\right] + O\Big(\|\vec{\eta}\|_{H^1\times L^2}^2 + |\lambda - \omega|\|\vec{\eta}\|_{H^1\times L^2}\Big),$$

and

$$\dot{\lambda} = O\big(\|\vec{\eta}\|_{H^1\times L^2}\big).$$

The proof of the proposition is split into the following two lemmas. The first lemma is

Lemma 5.2. *Under the same assumption in Proposition 5.1, then*

$$\dot{y} - \lambda = -\|\phi_\lambda\|_{L^2}^{-2}\langle \eta, \phi_\lambda\rangle + O\big(\|\vec{\eta}\|_{H^1\times L^2}^2 + |\lambda-\omega|\|\vec{\eta}\|_{H^1\times L^2}\big),$$

and

$$\dot{\lambda} = O\big(\|\vec{\eta}\|_{H^1\times L^2}\big).$$

Proof. Recall the definition $\vec{\eta}(t) = \vec{u}\big(t,\cdot+y(t)\big) - \overrightarrow{\Phi_{\lambda(t)}}$ in (4.2), that is,

$$\begin{cases} u(t,x) = \phi_\lambda\big(x-y(t)\big) + \xi\big(t,x-y(t)\big), \\ v(t,x) = -\lambda\phi_\lambda\big(x-y(t)\big) + \eta\big(t,x-y(t)\big). \end{cases} \tag{5.1}$$

Using the first equation of the equivalent system (1.4), we have

$$\dot{\lambda}\partial_\lambda\phi_\lambda - (\dot{y}-\lambda)\partial_x\phi_\lambda = -\dot{\xi} + (\dot{y}-\lambda)\partial_x\xi + \lambda\partial_x\xi + \partial_x\eta. \tag{5.2}$$

We denote γ_λ as the first component of $\overrightarrow{\Gamma_\lambda}$. Now we multiply both sides of equality (5.2) by γ_λ and integrate to obtain

$$\begin{aligned} &\dot{\lambda}\langle\partial_\lambda\phi_\lambda,\gamma_\lambda\rangle - (\dot{y}-\lambda)\langle\partial_x\phi_\lambda,\gamma_\lambda\rangle \\ &\qquad = \langle -\dot{\xi},\gamma_\lambda\rangle + (\dot{y}-\lambda)\langle\partial_x\xi,\gamma_\lambda\rangle + \lambda\langle\partial_x\xi,\gamma_\lambda\rangle + \langle\partial_x\eta,\gamma_\lambda\rangle. \end{aligned} \tag{5.3}$$

We know that $\partial_\lambda\phi_\lambda$ is an even function, and γ_λ is an odd function, so $\langle\partial_\lambda\phi_\lambda,\gamma_\lambda\rangle = 0$. By the orthogonality conditions (4.3), we have

$$\langle\partial_x\xi,\gamma_\lambda\rangle = -\langle\vec{\eta},\overrightarrow{\Psi_\lambda}\rangle = 0,$$

so we get

$$\langle\dot{\xi},\gamma_\lambda\rangle = \partial_t\langle\xi,\gamma_\lambda\rangle - \langle\xi,\partial_t\gamma_\lambda\rangle = \partial_t\langle\vec{\eta},\overrightarrow{\Gamma_\lambda}\rangle - \langle\xi,\partial_t\gamma_\lambda\rangle = -\langle\xi,\partial_t\gamma_\lambda\rangle = -\dot{\lambda}\langle\xi,\partial_\lambda\gamma_\lambda\rangle.$$

Moreover, $\langle\partial_x\phi_\lambda,\gamma_\lambda\rangle = -\langle\phi_\lambda,\partial_x\gamma_\lambda\rangle = -\|\phi_\lambda\|_{L^2}^2$. Thus we simplify equality (5.3) to obtain

$$(\dot{y}-\lambda)\|\phi_\lambda\|_{L^2}^2 - \dot{\lambda}\langle\xi,\partial_\lambda\gamma_\lambda\rangle = -\langle\eta,\phi_\lambda\rangle. \tag{5.4}$$

Next we multiply both sides of equality (5.2) by ϕ_λ and integrate to obtain

$$\begin{aligned} &\dot{\lambda}\langle\partial_\lambda\phi_\lambda,\phi_\lambda\rangle - (\dot{y}-\lambda)\langle\partial_x\phi_\lambda,\phi_\lambda\rangle \\ &\qquad = \langle -\dot{\xi},\phi_\lambda\rangle + (\dot{y}-\lambda)\langle\partial_x\xi,\phi_\lambda\rangle + \lambda\langle\partial_x\xi,\phi_\lambda\rangle + \langle\partial_x\eta,\phi_\lambda\rangle. \end{aligned} \tag{5.5}$$

Now we consider the terms in (5.5) one by one. From Lemma 2.1, we have $\partial_\lambda\|\phi_\lambda\|_{L^2}^2 = -\frac{\|\phi_\lambda\|_{L^2}^2}{\lambda} + O(\lambda-\omega)$, so

$$\dot{\lambda}\langle\partial_\lambda\phi_\lambda,\phi_\lambda\rangle = \frac{1}{2}\dot{\lambda}\partial_\lambda\|\phi_\lambda\|_{L^2}^2 = -\frac{\dot{\lambda}}{2\lambda}\|\phi_\lambda\|_{L^2}^2.$$

The term $-(\dot{y}-\lambda)\langle\partial_x\phi_\lambda,\phi_\lambda\rangle$ vanishes since ϕ_λ is an even function. By the orthogonality conditions (4.3), we have

$$\begin{aligned} \langle\dot{\xi},\phi_\lambda\rangle = \partial_t\langle\xi,\phi_\lambda\rangle - \langle\xi,\partial_t\phi_\lambda\rangle &= \partial_t\langle\vec{\eta},\overrightarrow{\Psi_\lambda}\rangle - \langle\xi,\partial_t\phi_\lambda\rangle \\ &= -\langle\xi,\partial_t\phi_\lambda\rangle = -\dot{\lambda}\langle\xi,\partial_\lambda\phi_\lambda\rangle. \end{aligned}$$

Thus we simplify equality (5.5) to obtain

$$\dot{\lambda}\Big[-\frac{1}{2\lambda}\|\phi_\lambda\|_{L^2}^2-\langle\xi,\partial_\lambda\phi_\lambda\rangle+O(\lambda-\omega)\Big]+(\dot{y}-\lambda)\langle\xi,\partial_x\phi_\lambda\rangle=-\langle\lambda\xi+\eta,\partial_x\phi_\lambda\rangle. \tag{5.6}$$

Since $\overrightarrow{\Psi_\lambda},\overrightarrow{\Gamma_\lambda},\overrightarrow{\Phi_\lambda}$ are smooth functions with exponential decay, combining (5.4) and (5.6), we get

$$\begin{cases}(\dot{y}-\lambda)\|\phi_\lambda\|_{L^2}^2-\dot{\lambda}\langle\xi,\partial_\lambda\gamma_\lambda\rangle+O(\lambda-\omega)=-\langle\eta,\phi_\lambda\rangle,\\ \dot{\lambda}\Big[-\frac{1}{2\lambda}\|\phi_\lambda\|_{L^2}^2-\langle\xi,\partial_\lambda\phi_\lambda\rangle\Big]+(\dot{y}-\lambda)\langle\xi,\partial_x\phi_\lambda\rangle=O(\|\vec{\eta}\|_{H^1\times L^2}).\end{cases} \tag{5.7}$$

We denote

$$A=\begin{pmatrix}-\langle\xi,\partial_\lambda\gamma_\lambda\rangle & \|\phi_\lambda\|_{L^2}^2\\ -\frac{1}{2\lambda}\|\phi_\lambda\|_{L^2}^2-\langle\xi,\partial_\lambda\phi_\lambda\rangle+O(\lambda-\omega) & \langle\xi,\partial_x\phi_\lambda\rangle\end{pmatrix}.$$

Then by a direct calculation, we get

$$\begin{pmatrix}\dot{\lambda}\\ \dot{y}-\lambda\end{pmatrix}=A^{-1}\begin{pmatrix}-\langle\eta,\phi_\lambda\rangle\\ O\big(\|\vec{\eta}\|_{H^1\times L^2}\big)\end{pmatrix}=\begin{pmatrix}O\big(\|\vec{\eta}\|_{H^1\times L^2}\big)\\ -\|\phi_\lambda\|_{L^2}^{-2}\langle\eta,\phi_\lambda\rangle+O\big(\|\vec{\eta}\|_{H^1\times L^2}^2+|\lambda-\omega|\|\vec{\eta}\|_{H^1\times L^2}\big)\end{pmatrix}.$$

This proves the lemma. □

The second lemma we need is the following.

Lemma 5.3. *Under the same assumption in Proposition 5.1, then*

$$\int_{\mathbb{R}}\eta\phi_\lambda\,\mathrm{d}x=\Big[Q(\vec{u}_0)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+\Big[Q\big(\overrightarrow{\Phi_\omega}\big)-Q\big(\overrightarrow{\Phi_\lambda}\big)\Big]+O\big(\|\vec{\eta}\|_{H^1\times L^2}^2\big).$$

Proof. Using equality (5.1) and the expression $Q(\vec{u})=\int_{\mathbb{R}}uv\,\mathrm{d}x$, we have

$$\begin{aligned}Q(\vec{u})&=Q\begin{pmatrix}\phi_\lambda+\xi\\ -\lambda\phi_\lambda+\eta\end{pmatrix}\\ &=\int_{\mathbb{R}}-\lambda\phi_\lambda^2\,\mathrm{d}x-\lambda\int_{\mathbb{R}}\xi\phi_\lambda\,\mathrm{d}x+\int_{\mathbb{R}}\eta\phi_\lambda\,\mathrm{d}x+\int_{\mathbb{R}}\xi\eta\,\mathrm{d}x.\end{aligned}$$

Now we analyse the last equality one by one. By (2.1), we have $Q\big(\overrightarrow{\Phi_\lambda}\big)=\int_{\mathbb{R}}-\lambda\phi_\lambda^2\,\mathrm{d}x$. Recall that we have the orthogonality condition $\Big\langle\vec{\eta},\overrightarrow{\Psi_{\lambda(t)}}\Big\rangle=0$ in (4.3), then

$$-\lambda\int_{\mathbb{R}}\xi\phi_\lambda\,\mathrm{d}x=-\lambda\int_{\mathbb{R}}\vec{\eta}\cdot\overrightarrow{\Psi_{\lambda(t)}}\,\mathrm{d}x=0.$$

The final term gives $\int_{\mathbb{R}}\xi\eta\,\mathrm{d}x=O\big(\|\vec{\eta}\|_{H^1\times L^2}^2\big)$. Therefore,

$$Q(\vec{u})=Q\big(\overrightarrow{\Phi_\lambda}\big)+\int_{\mathbb{R}}\eta\phi_\lambda\,\mathrm{d}x+O(\|\vec{\eta}\|_{H^1\times L^2}^2).$$

From the conservation law of momentum, we know

$$\begin{aligned}\int_{\mathbb{R}}\eta\phi_\lambda\,\mathrm{d}x&=Q(\vec{u})-Q\big(\overrightarrow{\Phi_\lambda}\big)+O\big(\|\vec{\eta}\|_{H^1\times L^2}^2\big)\\ &=\Big[Q(\vec{u}_0)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+\Big[Q\big(\overrightarrow{\Phi_\omega}\big)-Q\big(\overrightarrow{\Phi_\lambda}\big)\Big]+O\big(\|\vec{\eta}\|_{H^1\times L^2}^2\big).\end{aligned}$$

This proves the lemma. □

Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. Combining the estimates obtained in Lemmas 5.2 and 5.3, we have

$$\dot{y}-\lambda=-\|\phi_\lambda\|_{L^2}^{-2}\int_{\mathbb{R}}\eta\phi_\lambda\,\mathrm{d}x+O\Big(\|\vec{\eta}\|_{H^1\times L^2}^2+|\lambda-\omega|\|\vec{\eta}\|_{H^1\times L^2}\Big)$$

$$=\|\phi_\lambda\|_{L^2}^{-2}\Big[Q(\overrightarrow{\Phi_\lambda})-Q(\overrightarrow{\Phi_\omega})\Big]-\|\phi_\lambda\|_{L^2}^{-2}\Big[Q(\vec{u}_0)-Q(\overrightarrow{\Phi_\omega})\Big]+O\Big(\|\vec{\eta}\|_{H^1\times L^2}^2+|\lambda-\omega|\|\vec{\eta}\|_{H^1\times L^2}\Big).$$

This gives the proof of the proposition. □

6. Localized virial identities

The following lemmas are the localized virial identities. One can see [11] for the details of the proof.

Let ν is a H^2-solution of $\partial_x\nu=u$, and

$$I_1(t)=\int_{\mathbb{R}}\nu\partial_t\nu\,\mathrm{d}x.$$

Lemma 6.1. *Let $\vec{u}\in H^1(\mathbb{R})\times L^2(\mathbb{R})$ be the solution of the system* (1.4), *then*

$$I_1'(t)=\|v\|_{L^2}^2-\|u\|_{L^2}^2-\|u_x\|_{L^2}^2+\|u\|_{L^{p+2}}^{p+2}.$$

Let

$$I_2(t)=\int_{\mathbb{R}}\varphi\big(x-y(t)\big)uv\,\mathrm{d}x,$$

then we have the following lemma.

Lemma 6.2. *Let $\varphi\in C^3(\mathbb{R})$, $\vec{u}\in H^1(\mathbb{R})\times L^2(\mathbb{R})$ be the solution of* (1.4), *then*

$$I_2'(t)=-\dot{y}\int_{\mathbb{R}}\varphi'\big(x-y(t)\big)uv\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}}\varphi'\big(x-y(t)\big)\Big(3|u_x|^2+v^2+u^2-\frac{2(p+1)}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x$$
$$+\frac{1}{2}\int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x.$$

7. Proof of the main theorem

This section is devoted to prove our main theorem.

7.1. **Virial identities.**

Let $\varphi(x)$ be a smooth cut–off function, where

$$\varphi(x)=\begin{cases}x, & |x|\le R,\\ 0, & |x|\ge 2R,\end{cases}\tag{7.1}$$

and $0 \le \varphi' \le 1$, $|\varphi'''| \lesssim \frac{1}{R^2}$ for any $x \in \mathbb{R}$. Moreover, we denote

$$I(t) = \Big(\frac{4}{p} - 2\Big) I_1(t) + 2I_2(t).$$

Then we have the following lemma.

Lemma 7.1. *Let $R > 0$, y, λ, $\vec{\eta} = (\xi, \eta)^T$ be the parameters and vector obtained in Proposition 4.1. Then*

$$\begin{aligned} I'(t) =& -2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) - \Big(4\lambda\frac{4-p}{p} + 2\lambda\Big)Q(\vec{u}_0) + \Big(2 - 2\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2 \\ & - 2\Big(\dot{y} - \lambda\Big)Q(\vec{u}_0) + \Big(2 - 2\lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2 + 2\frac{4-p}{p}\|\lambda\xi + \eta\|_{L^2}^2 + R(\vec{u}), \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} R(\vec{u}) = 2\int_{\mathbb{R}} \Big[1 - \varphi'\big(x - y(t)\big)\Big]\Big(\dot{y}uv + \frac{3}{2}u_x^2 + \frac{1}{2}u^2 + \frac{1}{2}v^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x \\ + \int_{\mathbb{R}} \varphi'''\big(x - y(t)\big)u^2\,\mathrm{d}x. \end{aligned} \tag{7.3}$$

Proof. From Lemma 6.2 and the conservation law of momentum, we change the form of $I_2'(t)$ as

$$\begin{aligned} I_2'(t) =& -\dot{y}Q(\vec{u}_0) - \frac{1}{2}\Big[3\|u_x\|_{L^2}^2 + \|u\|_{L^2}^2 + \|v\|_{L^2}^2 - \frac{2(p+1)}{p+2}\|u\|_{L^{p+2}}^{p+2}\Big] + \frac{1}{2}\int \varphi'''\big(x - y(t)\big)u^2\,\mathrm{d}x \\ & + \int_{\mathbb{R}} \Big[1 - \varphi'\big(x - y(t)\big)\Big]\Big(\dot{y}uv + \frac{3}{2}|u_x|^2 + \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x. \end{aligned}$$

Then a direct computation gives

$$\begin{aligned} I'(t) =& \Big(\frac{4}{p} - 2\Big)I_1'(t) + 2I_2'(t) \\ =& -\Big(\frac{4}{p}+1\Big)\|u_x\|_{L^2}^2 + \Big(\frac{4}{p} - 3\Big)\|v\|_{L^2}^2 + \Big(-\frac{4}{p} + 1\Big)\|u\|_{L^2}^2 + \frac{2(p+4)}{p(p+2)}\|u\|_{L^{p+2}}^{p+2} \\ & + 2\int_{\mathbb{R}} \Big[1 - \varphi'\big(x - y(t)\big)\Big]\Big(\dot{y}uv + \frac{3}{2}|u_x|^2 + \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x \\ & + \int_{\mathbb{R}} \varphi'''\big(x - y(t)\big)u^2\,\mathrm{d}x - 2\dot{y}Q(\vec{u}_0). \end{aligned}$$

From the conservation law of energy, we have

$$2E(\vec{u}_0) = \|u_x\|_{L^2}^2 + \|v\|_{L^2}^2 + \|u\|_{L^2}^2 - \frac{2}{p+2}\|u_x\|_{L^{p+2}}^{p+2}.$$

Then

$$\begin{aligned} & -\Big(\frac{4}{p}+1\Big)\|u_x\|_{L^2}^2 + \Big(\frac{4}{p} - 3\Big)\|v\|_{L^2}^2 + \Big(-\frac{4}{p} + 1\Big)\|u\|_{L^2}^2 + \frac{2(p+4)}{p(p+2)}\|u\|_{L^{p+2}}^{p+2} \\ =& -2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\Big[\frac{p}{4-p}\|u\|_{L^2}^2 + \|v\|_{L^2}^2\Big] \\ =& -2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\Big[\lambda^2\|u\|_{L^2}^2 + \|v\|_{L^2}^2\Big] + 2\Big(1 - \lambda^2\frac{4-p}{p}\Big)\|u\|_{L^2}^2 \end{aligned}$$

$$= -2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\|v+\lambda u\|_{L^2}^2 - 4\lambda\frac{4-p}{p}Q(\vec{u}_0) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|u\|_{L^2}^2.$$

By orthogonality condition (4.3) and formula (5.1), we have the following equalities

$$\begin{aligned}\|u\|_{L^2}^2 &= \|\phi_\lambda\|_{L^2}^2 + 2\langle\phi_\lambda,\xi\rangle + \|\xi\|_{L^2}^2\\ &= \|\phi_\lambda\|_{L^2}^2 + 2\langle\overrightarrow{\Psi}_\lambda,\vec{\eta}\rangle + \|\xi\|_{L^2}^2 = \|\phi_\lambda\|_{L^2}^2 + \|\xi\|_{L^2}^2,\\ \|v+\lambda u\|_{L^2}^2 &= \|-\lambda\phi_\lambda+\eta+\lambda\phi_\lambda+\lambda\xi\|_{L^2}^2 = \|\lambda\xi+\eta\|_{L^2}^2.\end{aligned}$$

Hence, using the equalities above, we obtain

$$\begin{aligned}I'(t) =& -2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\|v+\lambda u\|_{L^2}^2 - 4\lambda\frac{4-p}{p}Q(\vec{u}_0) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|u\|_{L^2}^2\\ &+ 2\int_{\mathbb{R}}\Big[1-\varphi'\big(x-y(t)\big)\Big]\Big(\dot{y}uv + \frac{3}{2}u_x^2 + \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x\\ &+ \int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x - 2\dot{y}Q(\vec{u}_0)\\ =& -2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) + \frac{2(4-p)}{p}\|\eta+\lambda\xi\|_{L^2}^2 - 4\lambda\frac{4-p}{p}Q(\vec{u}_0)\\ &+ 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\big(\|\phi_\lambda\|_{L^2}^2 + \|\xi\|_{L^2}^2\big)\\ &+ 2\int_{\mathbb{R}}\Big[1-\varphi'\big(x-y(t)\big)\Big]\Big(\dot{y}uv + \frac{3}{2}u_x^2 + \frac{1}{2}v^2 + \frac{1}{2}u^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x\\ &+ \int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x - 2(\dot{y}-\lambda+\lambda)Q(\vec{u}_0)\\ =& -2\Big(\frac{4}{p}+1\Big)E(\vec{u}_0) - 2\lambda\Big(2\frac{4-p}{p}+1\Big)Q(\vec{u}_0) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2\\ &- 2(\dot{y}-\lambda)Q(\vec{u}_0) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2 + 2\frac{4-p}{p}\|\lambda\xi+\eta\|_{L^2}^2 + R(\vec{u}).\end{aligned}$$

This proves the lemma. □

Now we consider $R(\vec{u})$ in (7.3).

Lemma 7.2. *Let $R(\vec{u})$ be defined in (7.3), then*

$$R(\vec{u}) = O(\|\vec{\eta}\|_{H^1\times L^2}^2 + \frac{1}{R}).$$

Proof. Using the definition of the cut–off function φ in (7.1), we have

$$\begin{aligned}|R(\vec{u})| =& \Bigg|\int_{\{|x-y(t)|>R\}} 2\Big[1-\varphi'\big(x-y(t)\big)\Big]\\ &\cdot\Big(\dot{y}uv + \frac{3}{2}|u_x|^2 + \frac{1}{2}u^2 + \frac{1}{2}v^2 - \frac{p+1}{p+2}|u|^{p+2}\Big)\,\mathrm{d}x + \int_{\mathbb{R}}\varphi'''\big(x-y(t)\big)u^2\,\mathrm{d}x\Bigg|\\ \lesssim& \int_{\{|x-y(t)|>R\}}\Big(1+|\varphi'\big(x-y(t)\big)|\Big)\Big(|\dot{y}||u||v| + |u_x|^2 + u^2 + v^2 + |u|^{p+2}\Big)\,\mathrm{d}x + \frac{1}{R^2}.\end{aligned}$$

By Hölder's inequality, $|\varphi'| \le 1$, and $|\dot{y}| \lesssim 1$ (from Lemma 5.2), we have

$$\begin{aligned} |R(\vec{u})| &\lesssim \int_{\{|x-y(t)|>R\}} \left(|u_x|^2 + u^2 + v^2 + |u|^{p+2}\right) \mathrm{d}x + \frac{1}{R^2} \\ &\lesssim \int_{\{|x|>R\}} \left[(\partial_x\phi_\lambda + \partial_x\xi)^2 + (\phi_\lambda + \xi)^2 + (\lambda\phi_\lambda - \eta)^2 + |\phi_\lambda + \xi|^{p+2}\right] \mathrm{d}x + \frac{1}{R^2}, \end{aligned}$$

where we have used equality (4.2) in the last step. Further, using the property of exponential decay of $\partial_x\phi_\lambda$, we have

$$\int_{\{|x|>R\}} \left(\partial_x\phi_\lambda\right)^2 \mathrm{d}x \le C \int_{\{|x|>R\}} e^{-C|x|} \,\mathrm{d}x \le \frac{C}{R}.$$

Then the Young inequality gives

$$\begin{aligned} &\int_{\{|x|>R\}} (\partial_x\phi_\lambda + \partial_x\xi)^2 \,\mathrm{d}x \\ &\qquad \lesssim \int_{\{|x|>R\}} (\partial_x\phi_\lambda)^2 + (\partial_x\xi)^2 \,\mathrm{d}x \\ &\qquad \lesssim \frac{1}{R} + \|\partial_x\xi\|_{L^2}^2. \end{aligned} \tag{7.4}$$

Using similar method we can prove

$$\int_{\{|x|>R\}} (\phi_\lambda + \xi)^2 \,\mathrm{d}x \le C(\frac{1}{R} + \|\xi\|_{L^2}^2), \tag{7.5}$$

$$\int_{\{|x|>R\}} (\lambda\phi_\lambda - \eta)^2 \,\mathrm{d}x \le C(\frac{1}{R} + \|\eta\|_{L^2}^2), \tag{7.6}$$

$$\int_{\{|x|>R\}} |\phi_\lambda + \xi|^{p+2} \,\mathrm{d}x \le C(\frac{1}{R} + \|\xi\|_{H^1}^2). \tag{7.7}$$

Thus, we combine (7.4)-(7.7) to obtain

$$|R(\vec{u})| \le C(\frac{1}{R} + \|\vec{\eta}\|_{H^1\times L^2}^2).$$

This implies that

$$R(\vec{u}) = O(\|\vec{\eta}\|_{H^1\times L^2}^2 + \frac{1}{R}).$$

This proves the lemma. □

7.2. **Structure of $I'(t)$.** In this subsection, our purpose is to control the difference between u and the modulated solitons, and the modulated scaling parameter. Note that the quantities involved in $I'(t)$ are non–conserved, the main issue is to analyse the quantities in detail. In particular, we structure $I'(t)$ as follows.

Denote

$$\begin{aligned} \rho(\vec{u_0}) =& -2\Big(\frac{4}{p}+1\Big)\Big[E(\vec{u_0}) - E\big(\overrightarrow{\Phi_\omega}\big)\Big] - 2\lambda\Big(2\frac{4-p}{p}+1\Big)\Big[Q(\vec{u_0}) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ &+ 2\|\phi_\lambda\|_{L^2}^{-2} Q(\vec{u_0})\Big[Q(\vec{u_0}) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big], \end{aligned} \tag{7.8}$$

$$h(\lambda) = -2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big) - 2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) + 2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2$$

$$- 2\|\phi_\lambda\|_{L^2}^{-2} Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big], \tag{7.9}$$

$$\begin{aligned}\widetilde{R}(\vec{u}) =& R(\vec{u}) + 2\Big(1 - \lambda^2 \frac{4-p}{p}\Big)\|\xi\|_{L^2}^2 + 2\frac{4-p}{p}\|\lambda\xi + \eta\|_{L^2}^2 \\ & - 2Q(\vec{u}_0)\Big\{(\dot{y} - \lambda) - \frac{1}{\|\phi_\lambda\|_{L^2}^2}\Big[Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ & + \|\phi_\lambda\|_{L^2}^{-2}\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\Big\}. \end{aligned} \tag{7.10}$$

Now we rewrite $I'(t)$ as follows. From the formula below, we remark that there are no one-order terms with respect to $\vec{\eta}$ and λ.

Lemma 7.3.

$$I'(t) = \rho(\vec{u}_0) + h(\lambda) + \widetilde{R}(\vec{u}).$$

Proof. We will make a direct calculation. From (7.2) we know that

$$\begin{aligned} I'(t) =& -2\Big(\frac{4}{p} + 1\Big)E(\vec{u}_0) - 2\lambda\Big(2\frac{4-p}{p} + 1\Big)Q(\vec{u}_0) + 2\Big(1 - \lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2 \\ & - 2(\dot{y} - \lambda)Q(\vec{u}_0) + 2\Big(1 - \lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2 + 2\frac{4-p}{p}\|\lambda\xi + \eta\|_{L^2}^2 + R(\vec{u}) \\ =& -2\Big(\frac{4}{p} + 1\Big)\Big[E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big)\Big] - 2\lambda\Big(2\frac{4-p}{p} + 1\Big)\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ & + 2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ & - 2\Big(\frac{4}{p} + 1\Big)E\big(\overrightarrow{\Phi_\omega}\big) - 2\lambda\Big(2\frac{4-p}{p} + 1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) + 2\Big(1 - \lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2 \\ & - 2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ & + R(\vec{u}) + 2\Big(1 - \lambda^2\frac{4-p}{p}\Big)\|\xi\|_{L^2}^2 + 2\frac{4-p}{p}\|\lambda\xi + \eta\|_{L^2}^2 \\ & - 2Q(\vec{u}_0)\Big\{(\dot{y} - \lambda) - \|\phi_\lambda\|_{L^2}^{-2}\Big[Q\big(\overrightarrow{\Phi_\lambda}\big) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] + \|\phi_\lambda\|_{L^2}^{-2}\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\Big\} \\ =& \rho(\vec{u}_0) + h(\lambda) + \widetilde{R}(\vec{u}). \end{aligned}$$

This completes the proof. □

By Lemma 7.2 and Proposition 5.1, we obtain

$$\widetilde{R}(\vec{u}) = O\big(\|\vec{\eta}\|_{H^1\times L^2}^2 + \frac{1}{R} + |\lambda - \omega|\|\vec{\eta}\|_{H^1\times L^2}\big). \tag{7.11}$$

7.3. **Positivity of the main parts.** As the main parts of $I'(t)$, $\rho(\vec{u}_0)$ and $h(\lambda)$ are considered in this subsection. We shall prove their positivity in the following.

Lemma 7.4. *Let $\vec{u}_0 = (1 + a)\big(\overrightarrow{\Phi_\omega}\big)$, for some small positive constant a. Then*

1) $\rho(\vec{u}_0) \geq C_1 a$, *for some* $C_1 > 0$;

$$2) \qquad h(\lambda) \geq C_2(\lambda-\omega)^2 + O(a(\lambda-\omega)^2) + o((\lambda-\omega)^2), \quad \textit{for some} \quad C_2 > 0.$$

Proof. 1) Firstly, by Taylor's type expansion, we have

$$\begin{aligned} E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big) &= \Big\langle E'\big(\overrightarrow{\Phi_\omega}\big), \vec{u}_0 - \overrightarrow{\Phi_\omega}\Big\rangle + O\big(\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|^2_{H^1\times L^2}\big) \\ &= a\Big\langle E'\big(\overrightarrow{\Phi_\omega}\big), \overrightarrow{\Phi_\omega}\Big\rangle + O(a^2). \end{aligned}$$

Using the expression of $E'\big(\overrightarrow{\Phi_\omega}\big)$ in (2.5) and (1.3), we have

$$\begin{aligned} E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big) &= a\int_{\mathbb{R}} (-\partial_{xx}\phi_\omega + \phi_\omega - \phi_\omega^{p+1}, -\omega\phi_\omega)\cdot \begin{pmatrix} \phi_\omega \\ -\omega\phi_\omega \end{pmatrix} \mathrm{d}x + O(a^2) \\ &= a\int_{\mathbb{R}} (\omega^2\phi_\omega, -\omega\phi_\omega)\cdot \begin{pmatrix} \phi_\omega \\ -\omega\phi_\omega \end{pmatrix} \mathrm{d}x + O(a^2) \\ &= 2a\omega^2\|\phi_\omega\|^2_{L^2} + O(a^2). \end{aligned} \tag{7.12}$$

Next, we compute the term $Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)$,

$$\begin{aligned} Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big) &= \Big\langle Q'\big(\overrightarrow{\Phi_\omega}\big), \vec{u}_0 - \overrightarrow{\Phi_\omega}\Big\rangle + O\big(\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|^2_{H^1\times L^2}\big) \\ &= a\Big\langle Q'\big(\overrightarrow{\Phi_\omega}\big), \overrightarrow{\Phi_\omega}\Big\rangle + O(a^2). \end{aligned}$$

Using the expression of $Q'\big(\overrightarrow{\Phi_\omega}\big)$ in (2.4), we have

$$\begin{aligned} Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big) &= a\int_{\mathbb{R}} (-\omega\phi_\omega, \phi_\omega)\cdot \begin{pmatrix} \phi_\omega \\ -\omega\phi_\omega \end{pmatrix} \mathrm{d}x + O(a^2) \\ &= -2a\omega\|\phi_\omega\|^2_{L^2} + O(a^2). \end{aligned} \tag{7.13}$$

Then we put (7.12) and (7.13) into the expression of $\rho(\vec{u}_0)$ in (7.8) and then obtain

$$\begin{aligned} \rho(\vec{u}_0) =& -2\Big(\frac{4}{p}+1\Big)\Big[E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big)\Big] - 2\lambda\Big(2\frac{4-p}{p}+1\Big)\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ &+ 2\|\phi_\lambda\|^{-2}_{L^2} Q(\vec{u}_0)\Big[Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big] \\ =& -2\Big(\frac{4}{p}+1\Big)\Big[2a\omega^2\|\phi_\omega\|^2_{L^2} + O(a^2)\Big] - 2\lambda\Big(2\frac{4-p}{p}+1\Big)\Big[-2a\omega\|\phi_\omega\|^2_{L^2} + O(a^2)\Big] \\ &+ 2\|\phi_\lambda\|^{-2}_{L^2} Q(\vec{u}_0)\Big[-2a\omega\|\phi_\omega\|^2_{L^2} + O(a^2)\Big] \\ =& -4a\omega^2\Big(\frac{4}{p}+1\Big)\|\phi_\omega\|^2_{L^2} + 4a\omega\lambda\Big(2\frac{4-p}{p}+1\Big)\|\phi_\omega\|^2_{L^2} - 4a\omega Q(\vec{u}_0)\frac{\|\phi_\omega\|^2_{L^2}}{\|\phi_\lambda\|^2_{L^2}} \\ &+ O(a^2). \end{aligned} \tag{7.14}$$

For the term $4a\omega\lambda\Big(2\frac{4-p}{p}+1\Big)\|\phi_\omega\|^2_{L^2}$, we have

$$4a\omega\lambda\Big(2\frac{4-p}{p}+1\Big)\|\phi_\omega\|^2_{L^2} = 4a\omega^2\Big(2\frac{4-p}{p}+1\Big)\|\phi_\omega\|^2_{L^2} + O(a|\lambda-\omega|). \tag{7.15}$$

For the term $-4a\omega Q(\vec{u}_0)\dfrac{\|\phi_\omega\|_{L^2}^2}{\|\phi_\lambda\|_{L^2}^2}$, we use the expression $\phi_\omega(x)=(1-\omega^2)^{\frac{1}{p}}\phi_0\left(\sqrt{1-\omega^2}x\right)$ in (2.2) and Taylor's type expansion again to calculate

$$\begin{aligned}-4a\omega Q(\vec{u}_0)\frac{\|\phi_\omega\|_{L^2}^2}{\|\phi_\lambda\|_{L^2}^2} &= -4a\omega Q(\vec{u}_0)\frac{(1-\omega^2)^{\frac{2}{p}-\frac{1}{2}}\|\phi_0\|_{L^2}^2}{(1-\lambda^2)^{\frac{2}{p}-\frac{1}{2}}\|\phi_0\|_{L^2}^2} = -4a\omega Q(\vec{u}_0)\frac{(1-\omega^2)^{\frac{2}{p}-\frac{1}{2}}}{(1-\lambda^2)^{\frac{2}{p}-\frac{1}{2}}}\\ &= -4a\omega Q(\vec{u}_0)(1-\omega^2)^{\frac{2}{p}-\frac{1}{2}}\Big[(1-\omega^2)^{\frac{1}{2}-\frac{2}{p}}+O(|\lambda-\omega|)\Big]\\ &= -4a\omega Q(\vec{u}_0)+O(a|\lambda-\omega|).\end{aligned}$$

From the definition of $Q(\vec{u})$ in (1.5), we have

$$Q(\vec{u}_0)=Q\Big((1+a)\overrightarrow{\Phi_\omega}\Big)=-\omega\|\phi_\omega\|_{L^2}^2+O(a).$$

Combining the last two estimates, we obtain

$$-4a\omega Q(\vec{u}_0)\frac{\|\phi_\omega\|_{L^2}^2}{\|\phi_\lambda\|_{L^2}^2}=4a\omega^2\|\phi_\omega\|_{L^2}^2+O(a^2)+O(a|\lambda-\omega|). \tag{7.16}$$

Finally we put (7.15) and (7.16) into (7.14) to obtain

$$\begin{aligned}\rho(\vec{u}_0)=&-4a\omega^2\Big(\frac{4}{p}+1\Big)\|\phi_\omega\|_{L^2}^2+4a\omega^2\frac{8-p}{p}\|\phi_\omega\|_{L^2}^2\\ &+4a\omega^2\|\phi_\omega\|_{L^2}^2+O(a|\lambda-\omega|)+O(a^2)\\ =&4a\omega^2\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2+O(a|\lambda-\omega|)+O(a^2).\end{aligned}$$

Choosing a and ε_0 small enough, where ε_0 is the constant in Proposition 4.1, then by (4.4), we obtain the conclusion 1.

2) Recall the definition of $h(\lambda)$ from (7.9),

$$\begin{aligned}h(\lambda)=-2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big)-2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big)+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2\\ -2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big].\end{aligned} \tag{7.17}$$

First, we consider the last term, and claim that

$$\begin{aligned}&-2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\ &\qquad=2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+o\big((\lambda-\omega)^2\big)+O\big(a(\lambda-\omega)^2\big).\end{aligned} \tag{7.18}$$

To prove (7.18), we need the following equalities which can be obtained by the Taylor's type expansion and Lemma 2.1,

$$\begin{aligned}Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)&=\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big)\Big|_{\lambda=\omega}(\lambda-\omega)+O((\lambda-\omega)^2)\\ &=O((\lambda-\omega)^2),\end{aligned} \tag{7.19}$$

$$Q(\vec{u}_0)-Q\big(\overrightarrow{\Phi_\omega}\big)=O(a), \tag{7.20}$$

$$\|\phi_\lambda\|_{L^2}^{-2}-\|\phi_\omega\|_{L^2}^{-2}=O(|\lambda-\omega|). \tag{7.21}$$

Using (7.19)–(7.21), we obtain

$$
\begin{aligned}
&-2\|\phi_\lambda\|_{L^2}^{-2}Q(\vec{u}_0)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\
&\qquad=-2\|\phi_\omega\|_{L^2}^{-2}Q\big(\overrightarrow{\Phi_\omega}\big)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+o((\lambda-\omega)^2)+O\big(a(\lambda-\omega)^2\big).
\end{aligned}
$$

Further, from (2.1) we get

$$
\begin{aligned}
&-2\|\phi_\omega\|_{L^2}^{-2}Q\big(\overrightarrow{\Phi_\omega}\big)\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\
&\qquad=-2\|\phi_\omega\|_{L^2}^{-2}\cdot(-\omega\|\phi_\omega\|_{L^2}^2)\cdot\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]\\
&\qquad=2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big].
\end{aligned}
$$

Thus, we obtain (7.18).

Inserting (7.18) into (7.17), we get

$$
\begin{aligned}
h(\lambda)=&-2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big)-2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big)+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2\\
&+2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big]+o\big((\lambda-\omega)^2\big)+O\big(a(\lambda-\omega)^2\big).
\end{aligned}
$$

Let

$$
\begin{aligned}
h_1(\lambda)=&-2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big)-2\lambda\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big)\\
&+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\|\phi_\lambda\|_{L^2}^2+2\omega\Big[Q\big(\overrightarrow{\Phi_\lambda}\big)-Q\big(\overrightarrow{\Phi_\omega}\big)\Big].
\end{aligned}\tag{7.22}
$$

Then

$$
h(\lambda)=h_1(\lambda)+o\big((\lambda-\omega)^2\big)+O\big(a(\lambda-\omega)^2\big).\tag{7.23}
$$

Now we claim that

$$
h_1(\omega)=0,\quad h_1'(\omega)=0,\quad h_1''(\omega)>0.\tag{7.24}
$$

We prove the claim by the following three steps.

Step 1, $h_1(\omega)=0$.

By the definition of $h_1(\lambda)$, we have

$$
h_1(\omega)=-2\Big(\frac{4}{p}+1\Big)E\big(\overrightarrow{\Phi_\omega}\big)-2\omega\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big)+2\Big(1-\omega^2\frac{4-p}{p}\Big)\|\phi_\omega\|_{L^2}^2.
$$

By (2.1) and $E\big(\overrightarrow{\Phi_\omega}\big)$ in (1.6), we have

$$
\begin{aligned}
h_1(\omega)=&-2\Big(\frac{4}{p}+1\Big)\Big(\frac{1}{2}\int_{\mathbb{R}}\big(|\partial_x\phi_\omega|^2+|\phi_\omega|^2+|-\omega\phi_\omega|^2\big)\,\mathrm{d}x-\frac{1}{p+2}\int_{\mathbb{R}}|\phi_\omega|^{p+2}\,\mathrm{d}x\Big)\\
&-2\omega\Big(2\frac{4-p}{p}+1\Big)\Big(\int_{\mathbb{R}}-\omega\phi_\omega^2\,\mathrm{d}x\Big)+2\Big(1-\omega^2\frac{4-p}{p}\Big)\|\phi_\omega\|_{L^2}^2\\
=&-\frac{8}{p}\omega^2\|\phi_\omega\|_{L^2}^2+2\|\phi_\omega\|_{L^2}^2=0,
\end{aligned}
$$

where we have used $\omega^2=\frac{p}{4}$ in the above. Therefore, we have $h_1(\omega)=0$.

Step 2, $h_1'(\omega)=0$.

Using the definition of $h_1(\lambda)$ in (7.22) again, we have

$$\begin{aligned} h_1'(\lambda) =& -2\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) - 4\lambda\frac{4-p}{p}\|\phi_\lambda\|_{L^2}^2 \\ &+2\Big(1-\lambda^2\frac{4-p}{p}\Big)\partial_\lambda(\|\phi_\lambda\|_{L^2}^2) + 2\omega\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big). \end{aligned} \tag{7.25}$$

By (2.1) and Lemma 2.1, we have

$$\begin{aligned} h_1'(\omega) &= -2\Big(2\frac{4-p}{p}+1\Big)Q\big(\overrightarrow{\Phi_\omega}\big) + 4\frac{4-p}{p}Q\big(\overrightarrow{\Phi_\omega}\big) + 2\Big(1-\omega^2\frac{4-p}{p}\Big)\partial_\lambda(\|\phi_\lambda\|_{L^2}^2)\Big|_{\lambda=\omega} \\ &= -2Q\big(\overrightarrow{\Phi_\omega}\big) + 2\Big(1-\omega^2\frac{4-p}{p}\Big)\partial_\lambda(\|\phi_\lambda\|_{L^2}^2)\Big|_{\lambda=\omega}. \end{aligned} \tag{7.26}$$

Now we compute the term $\partial_\lambda(\|\phi_\lambda\|_{L^2}^2)\Big|_{\lambda=\omega}$. Note that

$$\partial_\lambda Q\big(\overrightarrow{\Phi_\lambda}\big) = \partial_\lambda(-\lambda\|\phi_\lambda\|_{L^2}^2) = -\|\phi_\lambda\|_{L^2}^2 - \lambda\partial_\lambda\|\phi_\lambda\|_{L^2}^2,$$

then Lemma 2.1 gives

$$\partial_\lambda\|\phi_\lambda\|_{L^2}^2\Big|_{\lambda=\omega} = -\frac{1}{\omega}\|\phi_\omega\|_{L^2}^2. \tag{7.27}$$

Taking this result into (7.26), we get

$$\begin{aligned} h_1'(\omega) &= 2\omega\|\phi_\omega\|_{L^2}^2 + 2\Big(1-\omega^2\frac{4-p}{p}\Big)\Big(-\frac{1}{\omega}\|\phi_\omega\|_{L^2}^2\Big) \\ &= \frac{2}{\omega}\Big(\omega^2 - 1 + \omega^2\frac{4-p}{p}\Big)\|\phi_\omega\|_{L^2}^2 \\ &= \frac{2}{\omega}\Big(\frac{4}{p}\omega^2 - 1\Big)\|\phi_\omega\|_{L^2}^2 = 0. \end{aligned}$$

Thus we prove the result $h_1'(\omega) = 0$.

Step 3, $h_1''(\omega) > 0$.

Taking the derivative of (7.25) with respect to λ, we have

$$\begin{aligned} h_1''(\lambda) =& -4\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2 - 8\lambda\frac{4-p}{p}\partial_\lambda\big(\|\phi_\lambda\|_{L^2}^2\big) \\ &+2\Big(1-\frac{4-p}{p}\lambda^2\Big)\partial_\lambda^2\big(\|\phi_\lambda\|_{L^2}^2\big) + 2\omega\partial_\lambda^2 Q\big(\overrightarrow{\Phi_\lambda}\big). \end{aligned}$$

Since

$$\begin{aligned} \partial_\lambda^2 Q\big(\overrightarrow{\Phi_\lambda}\big) &= -\partial_\lambda^2\big(\lambda\|\phi_\lambda\|_{L^2}^2\big) \\ &= -2\partial_\lambda\big(\|\phi_\lambda\|_{L^2}^2\big) - \lambda\partial_\lambda^2\big(\|\phi_\lambda\|_{L^2}^2\big), \end{aligned}$$

we have

$$\begin{aligned} h_1''(\lambda) =& -4\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2 - 4\Big(2\frac{4-p}{p}\lambda+\omega\Big)\partial_\lambda\big(\|\phi_\lambda\|_{L^2}^2\big) \\ &+2\Big(1-\frac{4-p}{p}\lambda^2 - \lambda\omega\Big)\partial_\lambda^2\big(\|\phi_\lambda\|_{L^2}^2\big). \end{aligned}$$

Hence,

$$h_1''(\omega) = -4\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2 - 4\omega\frac{8-p}{p}\partial_\lambda\left(\|\phi_\lambda\|_{L^2}^2\right)\Big|_{\lambda=\omega} + 2\Big(1-\frac{4}{p}\omega^2\Big)\partial_\lambda^2\|\phi_\lambda\|_{L^2}^2\Big|_{\lambda=\omega}.$$

Using (7.27) and $\omega^2 = \frac{p}{4}$, we have

$$h_1''(\omega) = -4\frac{4-p}{p}\|\phi_\omega\|_{L^2}^2 + 4\frac{8-p}{p}\|\phi_\omega\|_{L^2}^2 = \frac{16}{p}\|\phi_\omega\|_{L^2}^2 > 0.$$

Thus we prove the result $h_1''(\omega) > 0$. This gives the claim (7.24).

Using (7.24) and Taylor's type extension, we get

$$\begin{aligned} h_1(\lambda) &= h_1(\omega) + h'(\omega)(\lambda-\omega) + \frac{1}{2}h_1''(\omega)(\lambda-\omega)^2 + o\big((\lambda-\omega)^2\big) \\ &\geq C_2(\lambda-\omega)^2 + o(\lambda-\omega)^2, \end{aligned}$$

where $C_2 = \frac{1}{2}h_1''(\omega) > 0$. Putting this into (7.23), we obtain the conclusion 2. □

Hence, combining Lemma 7.4 and (7.11), we have

$$I'(t) \geq C_1 a + C_2(\lambda-\omega)^2 + O\Big(\|\vec{\eta}\|_{H^1\times L^2}^2 + a(\lambda-\omega)^2 + \frac{1}{R}\Big). \tag{7.28}$$

7.4. **Upper control of $\|\vec{\eta}\|_{H^1\times L^2}$.** From (7.28), to prove the monotonicity of $I'(t)$, we only need to estimate $\|\vec{\eta}\|_{H^1\times L^2}$. In this subsection, we give the following estimate on $\|\vec{\eta}\|_{H^1\times L^2}$.

Lemma 7.5. *Let $\vec{\eta}$ be defined in (4.2), then*

$$\|\vec{\eta}\|_{H^1\times L^2}^2 \lesssim O(a|\lambda-\omega| + a^2) + o\big((\lambda-\omega)^2\big).$$

Proof. Firstly, since $\vec{u} = \Big(\overrightarrow{\Phi_\lambda} + \vec{\eta}\Big)(x-y)$ in (5.1), we have

$$S_\lambda(\vec{u}) - S_\lambda\big(\overrightarrow{\Phi_\lambda}\big) = \Big\langle S_\lambda'\big(\overrightarrow{\Phi_\lambda}\big), \vec{\eta}\Big\rangle + \frac{1}{2}\Big\langle S_\lambda''\big(\overrightarrow{\Phi_\lambda}\big)\vec{\eta}, \vec{\eta}\Big\rangle + o(\|\vec{\eta}\|_{H^1\times L^2}^2).$$

Using $S_\omega'\big(\overrightarrow{\Phi_\omega}\big) = \vec{0}$, we have

$$S_\lambda(\vec{u}) - S_\lambda\big(\overrightarrow{\Phi_\lambda}\big) = \frac{1}{2}\Big\langle S_\lambda''\big(\overrightarrow{\Phi_\lambda}\big)\vec{\eta}, \vec{\eta}\Big\rangle + o(\|\vec{\eta}\|_{H^1\times L^2}^2).$$

Then by the estimate (3.20) in Corollary 3.5, we get

$$S_\lambda(\vec{u}) - S_\lambda\big(\overrightarrow{\Phi_\lambda}\big) \gtrsim \|\vec{\eta}\|_{H^1\times L^2}^2.$$

Secondly, note that

$$S_\lambda(\vec{u}) - S_\lambda\big(\overrightarrow{\Phi_\lambda}\big) = S_\lambda(\vec{u}_0) - S_\lambda\big(\overrightarrow{\Phi_\omega}\big) + S_\lambda\big(\overrightarrow{\Phi_\omega}\big) - S_\lambda\big(\overrightarrow{\Phi_\lambda}\big).$$

By Taylor's type extension, we have

$$\begin{aligned} S_\lambda(\vec{u}_0) - S_\lambda\big(\overrightarrow{\Phi_\omega}\big) &= E(\vec{u}_0) - E\big(\overrightarrow{\Phi_\omega}\big) + \lambda\Big(Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big) \\ &= S_\omega(\vec{u}_0) - S_\omega\big(\overrightarrow{\Phi_\omega}\big) + (\lambda-\omega)\Big(Q(\vec{u}_0) - Q\big(\overrightarrow{\Phi_\omega}\big)\Big) \\ &= \Big\langle S_\omega'\big(\overrightarrow{\Phi_\omega}\big), \vec{u}_0 - \overrightarrow{\Phi_\omega}\Big\rangle + O\Big(\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1\times L^2}^2\Big) + (\lambda-\omega)O\Big(\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1\times L^2}\Big) \\ &= O(a^2 + a|\lambda-\omega|). \end{aligned}$$

By Corollary 2.2, we have

$$S_\lambda(\overrightarrow{\Phi_\omega}) - S_\lambda(\overrightarrow{\Phi_\lambda}) = o((\lambda-\omega)^2).$$

Finally, we get the desired result

$$\begin{aligned}\|\vec{\eta}\|^2_{H^1\times L^2} &\lesssim S_\lambda(\vec{u}) - S_\lambda(\overrightarrow{\Phi_\lambda}) = S_\lambda(\vec{u}_0) - S_\lambda(\overrightarrow{\Phi_\omega}) + S_\lambda(\overrightarrow{\Phi_\omega}) - S_\lambda(\overrightarrow{\Phi_\lambda}) \\ &= O(a|\lambda-\omega| + a^2) + o\big((\lambda-\omega)^2\big).\end{aligned}$$

This completes the proof. □

7.5. **Proof of Theorem 1.2.** As discussion above, we assume that $\vec{u} \in U_\varepsilon(\overrightarrow{\Phi_\omega})$, and thus $|\lambda-\omega| \lesssim \varepsilon$. Firstly, we note that from the definition of $I(t)$ and the Young inequality, we have the time uniform boundedness of $I(t)$,

$$\sup_{t\in\mathbb{R}} I(t) \lesssim R\Big(\|(\overrightarrow{\Phi_\omega})\|^2_{H^1\times L^2} + 1\Big). \tag{7.29}$$

Now we estimate on $I'(t)$. From (7.28) and Lemma 7.5,

$$\begin{aligned}I'(t) &\geq C_1 a + C_2(\lambda-\omega)^2 + O(\|\vec{\eta}\|^2_{H^1\times L^2}) + O\Big(a(\lambda-\omega)^2 + \frac{1}{R}\Big) \\ &\geq \frac{1}{2}C_1 a + C_2(\lambda-\omega)^2 + O(a|\lambda-\omega| + a^2) + o\big((\lambda-\omega)^2\big) + O\Big(\frac{1}{R}\Big).\end{aligned}$$

By (4.4), choosing R satisfying $\frac{1}{R} \leq a^2$, and choosing ε, a_0 small enough, we obtain that for any $a \in (0, a_0)$,

$$\begin{aligned}I'(t) &\geq \frac{1}{2}C_1 a + C_2(\lambda-\omega)^2 + O(a^2 + a|\lambda-\omega|) + o(\lambda-\omega)^2 \\ &\geq \frac{1}{4}C_1 a + \frac{1}{2}C_2(\lambda-\omega)^2.\end{aligned}$$

This implies $I(t) \to +\infty$ when $t \to +\infty$, which is contradicted with (7.29). Hence we prove the instability of the solitary wave $\phi_\omega(x-\omega t)$ and thus give the proof of Theorem 1.2.

References

[1] Bellazzini, J., Ghimenti, M., and Le Coz, S., Multi-solitary waves for the nonlinear Klein-Gordon equation, Comm. Partial Differ. Equ., 39 (2014), 1479–1522.

[2] Bona, J., Sachs, R., Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Comm. Math. Phys., 118 (1988), 15–29.

[3] Boussinesq, J., Théorie des ondes et de remous qui se propagent..., J. Math. Pures Appl., 17 (1872), 55–108.

[4] Comech A., and Pelinovsky, D., Purely nonlinear instability of standing waves with minimal energy, Comm. Pure Appl. Math., 56 (2003), 1565–1607.

[5] Grillakis, M., Shatah, J., and Strauss, W., Stability theorey of solitary waves in the presence of symmetry, I. J. Funct. Anal., 74 (1987), 160–197.

[6] Grillakis, M., Shatah, J., and Strauss, W., Stability theory of solitary waves in the presence of symmetry, II. J. Funct. Anal., 94 (1990), 308–348.

[7] Kwong, M. K., Uniqueness of positive solutions of $\triangle u - u + u^p = 0$ in $\mathbb{R}^n$, Arch. Rational Mech. Anal., 105 (1989), 234–266.

[8] Le Coz, S., Wu, Y.F., Stability of multi-Solitons for the derivative nonlinear Schrödinger equation, Inter. Math. Res. Notices, 2017, RNX013.

[9] Liu, Y., Instability of solitary waves for generalized Boussinesq equations, J. Dynam. Differ. Equs., 5 (1993), 537–558.

[10] Liu, Y., Instability and blow-up of solutions to a generalized Boussinesq equation, SIAM. J. Math. Anal., 26, 6 (1995), 1527–1546.
[11] Liu, Y., Ohta, M., and Todorova, G., Strong instability of solitary waves for nonlinear Klein-Gordon equations and generalized Boussinesq equations, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), 539–548.
[12] Merle, F. and P. Raphaël, P., On universality of blow-up profile for L2 critical nonlinear Schrödinger equation, Invent. Math., 156 (2004), 565–672.
[13] Merle, F. and P. Raphaël, P., The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, Ann. of Math. (2), 161 (2005), 157–222.
[14] Ohta, M., Instability of bound states for abstract nonlinear Schrödinger equations, J. Funct. Anal., 261 (2011), 90–110.
[15] Ohta, M., and Todorova, G., Strong instability of standing waves for the nonlinear Klein-Gordon equation and the Klein-Gordon-Zakharov system, SIAM J. Math. Anal., 38 (2007), 1912–1931.
[16] Strauss, W., Existence of solitary waves in higher dimensions, Comm. Math. Phys., 55, 2 (1977), 149–162.
[17] Weinstein, M., Modulational stability of ground states of nonlinear Schrödinger equations, SIAM J. Math. Anal., 16 (1985), 472–491.
[18] Weinstein, M. I.: Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math., 39 (1986), 51–67.
[19] Wu, Y.F., Instability of the standing waves for the nonlinear Klein-Gordon equations in one dimension, preprint.

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