FACTORIZATION OF GENERALIZED THETA FUNCTIONS REVISITED

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Abstract. This survey is based on my lectures given in last a few years. As a reference, constructions of moduli spaces of parabolic sheaves and generalized parabolic sheaves are provided. By a refinement of the proof of vanishing theorem, we show, without using vanishing theorem, a new observation that $\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ is independent of all of the choices for any smooth curves. The estimate of various codimension and computation of canonical line bundle of moduli space of generalized parabolic sheaves on a reducible curve are provided in Section 6, which is completely new.

1. Introduction

Let C be a smooth projective curve of genus g, $\mathbf{Q}$ be the quotient scheme of quotients $V \otimes \mathcal{O}_C(-N) \to E \to 0$ with

$$\chi(E) = \chi = d + r(1-g)$$

and let $V \otimes \mathcal{O}_{C\times\mathbf{Q}}(-N) \to \mathcal{F} \to 0$ (where $V = \mathbb{C}^{P(N)}$) be the universal quotient on $C \times \mathbf{Q}$. There is an $\mathrm{SL}(V)$-equivariant embeding

$$\mathbf{Q} \hookrightarrow \mathbf{G} = \mathrm{Grass}_{P(m)}(V \otimes \mathrm{H}^0(\mathcal{O}_C(m-N))),$$

and the GIT quotient $\mathcal{U}_C = \mathbf{Q}^{ss}//\mathrm{SL}(V)$ respecting to the polarization

$$\Theta_{\mathbf{Q}^{ss}} := \det R\pi_{\mathbf{Q}^{ss}}(\mathcal{F})^{-k} \otimes \det(\mathcal{F}_y)^{\frac{k\chi}{r}}$$

(where $\mathcal{F}_y = \mathcal{F}|_{\{y\}\times\mathbf{Q}}$) is the so called moduli space of semi-stable rank r vector bundles of degree d on C. When $r|k\chi$, $\Theta_{\mathbf{Q}^{ss}}$ descends to an ample line bundle $\Theta_{\mathcal{U}_C}$ on $\mathcal{U}_C$. When $r=1$, the sections $s \in H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ are nothing but the classical **theta functions of order** k and $\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = k^g$.

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When $r > 1$, the sections $s \in H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ are so called **generalized theta functions of order** k on $\mathcal{U}_C$. It is clearly a very interesting question for mathematicians to find a formula of $\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$, which however was only predicted by **Conformal Field Theory**, the so called Verlinde formula. For example, when $r = 2$,

$$\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = \left(\frac{k}{2}\right)^g \left(\frac{k+2}{2}\right)^{g-1} \sum_{i=0}^{k} \frac{(-1)^{id}}{(\sin\frac{(i+1)\pi}{k+2})^{2g-2}}.$$

According to [1], there are two kinds of approaches for the proof of Verlinde formula: Infinite-dimensional approaches and finite-dimensional approaches (see [1] for an account). Infinite-dimensional appeoach is close to physics, which works for any group G, but the geometry behind it is unclear (at least to me). Finite-dimensional approach depends on well understand of geometry of moduli spaces, but it works only for $r = 2$ (as far as I know).

One of the finite-dimensional approaches is to consider a flat family of projective curves $\mathcal{X} \to T$ such that a fiber $\mathcal{X}_{t_0} := X$ $(t_0 \in T)$ is a connected curve with only one node $x_0 \in X$ and $\mathcal{X}_t$ $(t \in T \setminus \{t_0\})$ are smooth curves with a fiber $\mathcal{X}_{t_1} = C$ $(t_1 \neq t_0)$. Then one can associate a family of moduli spaces $\mathcal{M} \to T$ and a line bundle Θ on $\mathcal{M}$ such that each fiber $\mathcal{M}_t = \mathcal{U}_{\mathcal{X}_t}$ is the moduli space of semi-stable torsion free sheaves on $\mathcal{X}_t$ and $\Theta|_{\mathcal{M}_t} = \Theta_{\mathcal{U}_{\mathcal{X}_t}}$. By degenerating C to an irreducible X, there are two steps to establish a recurrence relation of $D_g(r, d, k) = \dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ in term of g (the genus of C):

(1) (Invariance) $\dim H^0(\mathcal{U}_{\mathcal{X}_t}, \Theta_{\mathcal{U}_{\mathcal{X}_t}})$ are independent of $t \in T$;

(2) (Factorization) Let $\pi : \widetilde{X} \to X$ be the normalization of X, then

$$H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong \bigoplus_{\mu} H^0(\mathcal{U}^{\mu}_{\widetilde{X}}, \Theta_{\mathcal{U}^{\mu}_{\widetilde{X}}}),$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through $0 \leq \mu_r \leq \cdots \leq \mu_1 \leq k-1$, $\mathcal{U}^{\mu}_{\widetilde{X}}$ are moduli spaces of semi-stable parabolic bundles on $\widetilde{X}$ with parabolic structure at $x_i \in \pi^{-1}(x_0) = \{x_1, x_2\}$ determined by μ and $\widetilde{X}$ has genus $g(\widetilde{X}) = g - 1$.

In order to carry throught the induction on g, one has to start with moduli spaces $\mathcal{U}_{\mathcal{X}_t} = \mathcal{U}_{\mathcal{X}_t}(r, d, \omega)$ of semistable parabolic torsion free sheaves E on $\mathcal{X}_t$ of rank r and $\deg(E) = d$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at smooth points $\{x\}_{x\in I} \subset \mathcal{X}_t$, where $\omega = (k, \{\vec{n}(x),\ \vec{a}(x)\}_{x\in I})$ denote the parabolic data. In [9], the factorization theorem as above (2) was proved for $\mathcal{U}_X = \mathcal{U}_X(r, d, \omega)$.

Let $\mathcal{U}_C = \mathcal{U}_C(r, d, \omega)$ be the moduli space of semi-stable parabolic bundles of rank r and degree d on C with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at a finite set $I \subset C$ of points, and

$$D_g(r, d, \omega) = \dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}).$$

If the invariance property that $\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ is independent of C and choices of points $x \in I$ holds (for example, if $H^1(\mathcal{U}_{\mathcal{X}_t}, \Theta_{\mathcal{U}_{\mathcal{X}_t}}) = 0$), we will have the following recurrence relation

$$D_g(r, d, \omega) = \sum_{\mu} D_{g-1}(r, d, \omega^{\mu}) \tag{1.1}$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through $0 \le \mu_r \le \cdots \le \mu_1 < k$ and

$$\omega^{\mu} = (k, \{\vec{n}(x),\ \vec{a}(x)\}_{x\in I\cup\{x_1,x_2\}})$$

with $\vec{n}(x_i)$, $\vec{a}(x_i)$ $(i = 1, 2)$ determined by μ. A vanishing theorem

$$H^1(\mathcal{U}_{\mathcal{X}_t}, \Theta_{\mathcal{U}_{\mathcal{X}_t}}) = 0$$

was proved in [9] when $(r-1)(g-1) + \frac{|I|}{k} \ge 2$, which implies the invariance property for $g \ge 3$.

The recurrence relation (1.1) decreases the genus g, but it increases the number $|I|$ of parabolic points. By degenerating C to an reducible $X = X_1 \cup X_2$, we can establish a recurrence relation for the number of parabolic points if we can prove the invariance property (1) and a factorization (2). In [10], we proved the factorization theorem

$$H^0(\mathcal{U}_{X_1\cup X_2}, \Theta_{\mathcal{U}_{X_1\cup X_2}}) \cong \bigoplus_{\mu} H^0(\mathcal{U}^{\mu}_{X_1}, \Theta_{\mathcal{U}^{\mu}_{X_1}}) \otimes H^0(\mathcal{U}^{\mu}_{X_2}, \Theta_{\mathcal{U}^{\mu}_{X_2}})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through $0 \le \mu_r \le \cdots \le \mu_1 < k$. If

$$H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) = 0$$

holds for $X = X_1 \cup X_2$, fix a partition $I = I_1 \cup I_2$, we have

$$D_g(r, d.\omega) = \sum_{\mu} D_{g_1}(r, d^{\mu}_1, \omega^{\mu}_1) \cdot D_{g_2}(r, d^{\mu}_2, \omega^{\mu}_2), \quad g_1 + g_2 = g \tag{1.2}$$

where $d^{\mu}_1 + d^{\mu}_2 = d$, $\omega^{\mu}_j = (k, \{\vec{n}(x),\ \vec{a}(x)\}_{x\in I_j\cup\{x_j\}})$ $(j = 1, 2)$.

For a projective variety $\hat{M}$ with an ample line bundle $\hat{\mathcal{L}}$, if a reductive group G acts on $\hat{M}$ with respect to the polarization $\hat{\mathcal{L}}$ and assume that $\hat{\mathcal{L}}$ descends to a line bundle $\mathcal{L}$ on GIT quotient $M = \hat{M}^{ss}(\hat{\mathcal{L}})//G$, then

$$H^i(M, \mathcal{L}) = H^i(\hat{M}^{ss}(\hat{\mathcal{L}}), \hat{\mathcal{L}})^{inv.}.$$

If there is another G-variety $\hat{\mathcal{Y}}$ with an G-morphism $p : \hat{\mathcal{Y}} \to \hat{M}$ such that $H^i(\hat{M}, \hat{\mathcal{L}})^{inv.} = H^i(\hat{\mathcal{Y}}, p^*\hat{\mathcal{L}})^{inv.}$, we would be able to show the vanishing theorem $H^i(M, \mathcal{L}) = 0$ by assuming the following statements:

(i) There are line bundles $\hat{\mathcal{L}}_1$, $\hat{\mathcal{L}}_2$ on $\hat{\mathcal{Y}}$ such that $p^*\hat{\mathcal{L}} = \omega_{\hat{\mathcal{Y}}}\otimes\hat{\mathcal{L}}_1\otimes\hat{\mathcal{L}}_2$ (where $\omega_{\hat{\mathcal{Y}}}$ is the canonical line bundle of $\hat{\mathcal{Y}}$) and $\hat{\mathcal{L}}_1$, $\hat{\mathcal{L}}_2$ descend to ample line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on GIT quotient $\mathcal{Y} = \hat{\mathcal{Y}}^{ss}(\hat{\mathcal{L}}_1)//G$;

(ii) If $\psi : \hat{\mathcal{Y}}^{ss}(\hat{\mathcal{L}}_1) \to \mathcal{Y}$ is quotient map, $\omega_{\mathcal{Y}} = (\psi_*\omega_{\hat{\mathcal{Y}}^{ss}(\hat{\mathcal{L}}_1)})^G$;

(iii) $H^i(\hat{M}, \hat{\mathcal{L}})^{inv.} = H^i(\hat{M}^{ss}(\hat{\mathcal{L}}), \hat{\mathcal{L}})^{inv.}$ and

$$H^i(\hat{\mathcal{Y}}, p^*\hat{\mathcal{L}})^{inv.} = H^i(\hat{\mathcal{Y}}^{ss}(\hat{\mathcal{L}}_1), p^*\hat{\mathcal{L}})^{inv.}.$$

The above statements imply $H^i(M, \mathcal{L}) = H^i(\mathcal{Y}, \omega_{\mathcal{Y}} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)$, then Kodaira-type vanishing theorem for $\mathcal{Y}$ do the job. To establish (i), (ii) and (iii), one has to compute canonical bundle and singularities of the moduli spaces, to estimate codimensions of

$$\hat{\mathcal{Y}}^{ss}(\hat{\mathcal{L}}_1) \setminus \hat{\mathcal{Y}}^{s}(\hat{\mathcal{L}}_1), \quad \hat{\mathcal{M}} \setminus \hat{\mathcal{M}}^{ss}(\hat{\mathcal{L}}), \quad \hat{\mathcal{Y}} \setminus \hat{\mathcal{Y}}^{ss}(\hat{\mathcal{L}}_1),$$

which were done in [9] for moduli spaces of parabolic bundles and generalized parabolic sheaves on an irreducible smooth curve, so that $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) = 0$ was only proved for the irreducible nodal curve X of genus $g \geq 3$ in [9]. If $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) = 0$ holds for both irreducible X and reducible X of arbitrary genus, the numbers $D_g(r, d, \omega)$ will satisfy the recurrence relation (1.1) and (1.2) which will imply a formula of $D_g(r, d, \omega)$. However, the vanishing theorem for reducible curve X remains open.

In this survey article, we provide a detail construction of various moduli spaces in Section 2. The theta line bundles $\Theta_{\mathcal{U}_X}$ and the two factorization theorems are reviewed in Section 3. We review firstly the proof of vanishing theorem for smooth curves of $g \geq 2$, then we show, without using the vanishing of $H^1(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$, that the invariance property of $\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ holds for any smooth curve of genus $g \geq 0$ in Section 4 (see Corollary 4.8). Section 5 contains the review of vanishing theorem for irreducible node curves. Section 6 is an attempt to prove, using the same method of Section 5, the vanishing theorem $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) = 0$ for reducible curve $X = X_1 \cup X_2$.

2. Construction of moduli spaces

Let X be an irreducible projective curve of genus g over an algebraically closed field of characteristic zero, which has at most one node x_0. Let I be a finite set of smooth points of X, and E be a coherent sheaf of rank r and degree d on X (the rank $r(E)$ is defined to be dimension of E_ξ at generic point $\xi \in X$, and $d = \chi(E) - r(1 - g)$).

Definition 2.1. By a quasi-parabolic structure on E at a smooth point $x \in X$, we mean a choice of flag of quotients

$$E_x = Q_{l_x+1}(E)_x \twoheadrightarrow Q_{l_x}(E)_x \twoheadrightarrow \cdots\cdots \twoheadrightarrow Q_1(E)_x \twoheadrightarrow Q_0(E)_x = 0$$

of the fibre E_x of E at x (each quotient $Q_i(E)_x \twoheadrightarrow Q_{i-1}(E)_x$ in the flag is not an isomorphism). If, in addition, a sequence of integers called the parabolic weights $0 \le a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) \le k$ are given, we call that E has a parabolic structure at x.

Notice that, let $F_i(E)_x := ker\{E_x \twoheadrightarrow Q_i(E)_x\}$, it is equivalent to give a flag of subspaces of E_x:

$$E_x = F_0(E)_x \supset F_1(E)_x \supset \cdots \supset F_{l_x}(E)_x \supset F_{l_x+1}(E)_x = 0.$$

Let $r_i(x) = \dim(Q_i(E)_x)$, $n_i(x) = \dim(ker\{Q_i(E)_x \twoheadrightarrow Q_{i-1}(E)_x\})$ (or simply defining $n_i(x) = r_i(x) - r_{i-1}(x)$) and

$$\vec{a}(x) : = (a_1(x), a_2(x), \cdots, a_{l_x+1}(x))$$
$$\vec{n}(x) : = (n_1(x), n_2(x), \cdots, n_{l_x+1}(x)).$$

$\vec{a}$ (resp., $\vec{n}$) denotes the map $x \mapsto \vec{a}(x)$ (resp., $x \mapsto \vec{n}(x)$).

Definition 2.2. The parabolic Euler characteristic of E is

$$\mathrm{par}\chi(E) := \chi(E) - \frac{1}{k}\sum_{x\in I}\left(a_{l_x+1}(x)\dim(E^{\tau}_x) - \sum_{i=1}^{l_x+1} a_i(x)n_i(x)\right)$$

where $E^{\tau} \subset E$ is the subsheaf of torsion and $E^{\tau}_x = E^{\tau}|_{\{x\}}$.

Definition 2.3. For any subsheaf $F \subset E$, let $Q_i(E)^F_x \subset Q_i(E)_x$ be the image of F, $n^F_i = \dim(ker\{Q_i(E)^F_x \twoheadrightarrow Q_{i-1}(E)^F_x\})$ and

$$\mathrm{par}\chi(F) := \chi(F) - \frac{1}{k}\sum_{x\in I}\left(a_{l_x+1}(x)\dim(F^{\tau}_x) - \sum_{i=1}^{l_x+1} a_i(x)n^F_i(x)\right).$$

Then E is called semistable (resp., stable) for $(k, \vec{a})$ if for any nontrivial subsheaf $E' \subset E$ such that E/E' is torsion free, one has

$$\mathrm{par}\chi(E') \le \frac{\mathrm{par}\chi(E)}{r} \cdot r(E') \text{ (resp., } <).$$

Remark 2.4. Stable parabolic sheaf must be torsion free. If E is semistable, then E is torsion free outside $x \in I$, the quotient homomorphisms in Definition (2.1) injection E^{τ}_x to $Q_i(E)_x$ $(1 \le i \le l_x)$ for any $x \in I$. Moreover, if $E^{\tau}_x \neq 0$, we must have $a_1(x) = 0$ and $a_{l_x+1}(x) = k$.

Fix a line bundle $\mathcal{O}(1)$ on X of $\deg(\mathcal{O}(1)) = c$, let $\chi = d + r(1-g)$, P denote the polynomial $P(m) = crm + \chi$, $\mathcal{W} = \mathcal{O}(-N) = \mathcal{O}(1)^{-N}$ and $V = \mathbb{C}^{P(N)}$. Consider the Quot scheme

$$\mathrm{Quot}(V\otimes\mathcal{W}, P)(T) = \left\{ \begin{array}{l} T\text{-flat quotients } V\otimes\mathcal{W}\to E\to 0 \text{ over} \\ X\times T \text{ with } \chi(E_t(m)) = P(m)\ (\forall\, t\in T) \end{array} \right\},$$

and let $\mathbf{Q}\subset Quot(V\otimes\mathcal{W}, P)$ be the open set

$$\mathbf{Q}(T) = \left\{ \begin{array}{l} V\otimes\mathcal{W}\to E\to 0, \text{ with } R^1 p_{T*}(E(N)) = 0 \text{ and} \\ V\otimes\mathcal{O}_T\to p_{T*}E(N) \text{ induces an isomorphism} \end{array} \right\}.$$

Choose N large enough so that every semistable parabolic sheaf with Hilbert polynomial P and parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ with weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$ appears as a quotient corresponding to a point of $\mathbf{Q}$. Let $\widetilde{\mathbf{Q}}$ be the closure of $\mathbf{Q}$ in the Quot scheme, $V\otimes\mathcal{W}\to\mathcal{F}\to 0$ be the universal quotient over $X\times\widetilde{\mathbf{Q}}$ and $\mathcal{F}_x$ be the restrication of $\mathcal{F}$ on $\{x\}\times\widetilde{\mathbf{Q}}\cong\widetilde{\mathbf{Q}}$. Let $Flag_{\vec{n}(x)}(\mathcal{F}_x)\to\widetilde{\mathbf{Q}}$ be the relative flag scheme of locally free quotients of type $\vec{n}(x)$, and

$$\mathcal{R} = \underset{x\in I}{\times_{\widetilde{\mathbf{Q}}}} Flag_{\vec{n}(x)}(\mathcal{F}_x)\to\widetilde{\mathbf{Q}}$$

be the product over $\widetilde{\mathbf{Q}}$. A (closed) point $(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I})$ of $\mathcal{R}$ by definition is given by a point $V\otimes\mathcal{W}\xrightarrow{p} E\to 0$ of the Quot scheme, together with the flags of quotients

$$\{E_x \twoheadrightarrow Q_{r_{l_x}(x)}\twoheadrightarrow Q_{r_{l_x-1}(x)}\twoheadrightarrow\cdots\twoheadrightarrow Q_{r_2(x)}\twoheadrightarrow Q_{r_1(x)}\twoheadrightarrow 0\}_{x\in I}$$

where $p_{r_i(x)} : V\otimes\mathcal{W}\xrightarrow{p} E\to E_x\twoheadrightarrow Q_{r_{l_x}(x)}\twoheadrightarrow\cdots\twoheadrightarrow Q_{r_i(x)}$.

For large enough m, we have a $SL(V)$-equivariant embedding

$$\mathcal{R}\hookrightarrow\mathbf{G} = Grass_{P(m)}(V\otimes W_m)\times\mathbf{Flag},$$

where $W_m = H^0(\mathcal{W}(m))$, and $\mathbf{Flag}$ is defined to be

$$\mathbf{Flag} = \prod_{x\in I}\{Grass_{r_1(x)}(V\otimes W_m)\times\cdots\times Grass_{r_{l_x}(x)}(V\otimes W_m)\},$$

which maps a point $(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}) =$

$$(V\otimes\mathcal{W}\xrightarrow{p} E, \{V\otimes\mathcal{W}\xrightarrow{p_{r_1(x)}} Q_{r_1(x)},\ \cdots, V\otimes\mathcal{W}\xrightarrow{p_{r_{l_x}(x)}} Q_{r_{l_x}(x)}\}_{x\in I})$$

of $\mathcal{R}$ to the point $(g, \{g_{r_1(x)}, ..., g_{r_{l_x}(x)}\}_{x\in I}) =$

$$(V\otimes W_m\xrightarrow{g} U, \{V\otimes W_m\xrightarrow{g_{r_1(x)}} U_{r_1(x)}, \cdots, V\otimes W_m\xrightarrow{g_{r_{l_x}(x)}} U_{r_{l_x}(x)}\}_{x\in I})$$

of $\mathbf{G}$, where $g := H^0(p(m))$, $U := H^0(E(m))$, $g_{r_i(x)} := H^0(p_{r_i(x)}(m))$, $U_{r_i(x)} := H^0(Q_{r_i(x)})$ $(i = 1, ..., l_x)$ and $r_i(x) = dim(Q_{r_i(x)})$.

Notation 2.5. Given the polarisation (N large enough) on $\mathbf{G}$:

$$\frac{\ell + kcN}{c(m-N)} \times \prod_{x\in I}\{d_1(x), \cdots, d_{l_x}(x)\}$$

where $d_i(x) = a_{i+1}(x) - a_i(x)$ and ℓ is the rational number satisfying

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) r_i(x) + r\ell = k\chi \tag{2.1}$$

By the general criteria of GIT stability, we have

Proposition 2.6. *A point $(g, \{g_{r_1(x)}, ..., g_{r_{l_x}(x)}\}_{x\in I}) \in \mathbf{G}$ is stable (respectively, semistable) for the action of $SL(V)$, with respect to the above polarisation (we refer to this from now on as GIT-stability), iff for all nontrivial subspaces $H \subset V$ we have (with $h = dimH$)*

$$e(H) := \frac{\ell + kcN}{c(m-N)}(hP(m) - P(N) dim\, g(H \otimes W_m)) +$$
$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)(r_i(x)h - P(N) dim\, g_{r_i(x)}(H \otimes W_m)) < (\le)\, 0.$$

Notation 2.7. Given a point $(p, p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}) \in \mathcal{R}$, and a subsheaf F of E we denote the image of F in $Q_{r_i(x)}$ by $Q^F_{r_i(x)}$. Similarly, given a quotient $E \xrightarrow{T} \mathcal{G} \to 0$, set $Q^{\mathcal{G}}_{r_i(x)} := Q_{r_i(x)}/Im(ker(T))$.

Lemma 2.8. *There exists $M_1(N)$ such that for $m \ge M_1(N)$ the following holds. Suppose $(p, \{p_{r(x)}, p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}) \in \mathcal{R}$ is a point which is GIT-semistable then for all quotients $E \xrightarrow{T} \mathcal{G} \to 0$ we have*

$$h^0(\mathcal{G}(N)) \ge \frac{1}{k}\left(r(\mathcal{G})(\ell + kcN) + \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) h^0(Q^{\mathcal{G}}_{r_i(x)}) \right). \tag{2.2}$$

In particular, $V \to H^0(E(N))$ is an isomorphism and E satisfies the requirements in Remark 2.4.

Proof. The injectivity of $V \xrightarrow{H^0(p(N))} H^0(E(N))$ is easy to see. Let

$$H = ker\{V \xrightarrow{H^0(p(N))} H^0(E(N)) \xrightarrow{H^0(T(N))} H^0(\mathcal{G}(N))\}$$

and $F \subset E$ be the subsheaf generated by H. Since all these F are in a bounded family, $dim\, g(H \otimes W_m) = h^0(F(m)) = \chi(F(m))$ and

$g_{r_i(x)}(H\otimes W_m) = h^0(Q^F_{r_i(x)})$ $(\forall\, x\in I)$ for $m\geq M_1'(N)$. Then, by Proposition 2.6 (with $h=dim(H)$), we have

$$\begin{aligned} e(H) =&(\ell+kcN)(rh-r(F)P(N))+(\ell+kcN)P(N)\frac{h-\chi(F(N))}{c(m-N)} \\ &+\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)\left(r_i(x)h-P(N)h^0(Q^F_{r_i(x)})\right). \end{aligned}$$

By using $h\geq P(N)-h^0(\mathcal{G}(N))$, $r-r(F)\geq r(\mathcal{G})$ and $r_i(x)-h^0(Q^F_{r_i(x)})\geq h^0(Q^{\mathcal{G}}_{r_i(x)})$, we get the inequality

$$\begin{aligned} h^0(\mathcal{G}(N)) \geq&(\ell+kcN)\frac{h-\chi(F(N))}{k(m-N)c}-\frac{e(H)}{kP(N)}+ \\ &\frac{1}{k}\left(r(\mathcal{G})(\ell+kcN)+\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\right). \end{aligned}$$

For given N, the set $\{h-\chi(F(N))\}$ is finite since all these F are in a bounded family. Let $\chi(N)=\min\{h-\chi(F(N))\}$. If $\chi(N)\geq 0$, then

$$h^0(\mathcal{G}(N))\geq\frac{1}{k}\left(r(\mathcal{G})(\ell+kcN)+\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\right)-\frac{e(H)}{kP(N)}.$$

When $\chi(N)<0$, let $M_1(N)>\max\{M_1'(N),-\chi(N)(\ell+kcN)+cN\}$ and $m\geq M_1(N)$. Then, since $e(H)\leq 0$, we have

$$h^0(\mathcal{G}(N))\geq\frac{1}{k}\left(r(\mathcal{G})(\ell+kcN)+\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\right).$$

Now we show that $V\to H^0(E(N))$ is an isomorphism. To see it being surjective, it is enough to show that one can choose N such that $H^1(E(N))=0$ for all such E. If $H^1(E(N))$ is nontrivial, then there is a nontrivial quotient $E(N)\to L\subset\omega_X$ by Serre duality, and thus

$$h^0(\omega_X)\geq h^0(L)\geq N+B,$$

where B is a constant independent of E, we choose N such that $H^1(E(N))=0$ for all GIT-semistable points.

Let $\tau=Tor(E)$, $\mathcal{G}=E/\tau$, note $h^0(\mathcal{G}(N))=P(N)-h^0(\tau)$ and

$$h^0(Q^{\mathcal{G}}_{r_i(x)})=r_i(x)-h^0(Q^{\tau}_{r_i(x)}),$$

then the inequality (2.2) becomes

$$kh^0(\tau)\leq\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^{\tau}_{r_i(x)})\leq\sum_{x\in I}(a_{l_x+1}(x)-a_1(x))h^0(\tau_x)$$

which implies the requirements in Remark 2.4. □

Proposition 2.9. *Suppose* $(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}) \in \mathcal{R}$ *is a point corresponding to a parabolic sheaf* E. *Then* E *is semistable iff for any nontrivial subsheaf* $F \subset E$ *we have*

$$s(F) := \frac{\ell + kcN}{c(m-N)}(\chi(F(N))P(m) - P(N)\chi(F(m)))+$$
$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)(r_i(x)\chi(F(N)) - P(N)h^0(Q^F_{r_i(x)})) \le 0.$$

If $s(F) < 0$ *for any nontrivial* $F \subset E$, *then* E *is stable. Conversely, if* E *is stable, then* $s(F) < 0$ *for any nontrivial subsheaf* $F \subset E$ *except that* $r(F) = r$, $\tau := E/F = 0$ *outside* $x \in I$, $a_{l_x+1}(x) - a_1(x) = k$ *if* $\tau_x \neq 0$, *and* $n^F_1(x) = n_1(x) - h^0(\tau_x)$, $n^F_i(x) = n_i(x)$ $(2 \le i \le l_x + 1)$ *for any* $x \in I$.

Proof. The point corresonding to a quotient $V \otimes \mathcal{W} \xrightarrow{p} E \to 0$ and

$$\{E_x \twoheadrightarrow Q_{r_{l_x}(x)} \twoheadrightarrow Q_{r_{l_x-1}(x)} \twoheadrightarrow \cdots \twoheadrightarrow Q_{r_2(x)} \twoheadrightarrow Q_{r_1(x)} \twoheadrightarrow 0\}_{x\in I}$$

$p_{r_i(x)} : V \otimes \mathcal{W} \xrightarrow{p} E \to E_x \twoheadrightarrow Q_{r_{l_x}(x)} \twoheadrightarrow \cdots \twoheadrightarrow Q_{r_i(x)}$. For $F \subset E$ such that E/F is torsion free, we have the flags of quotient sheaves

$$\{F \twoheadrightarrow F_x \twoheadrightarrow Q^F_{r_{l_x}(x)} \twoheadrightarrow Q^F_{r_{l_x-1}(x)} \twoheadrightarrow \cdots \twoheadrightarrow Q^F_{r_2(x)} \twoheadrightarrow Q^F_{r_1(x)} \twoheadrightarrow 0\}_{x\in I}$$

Let $n^F_i(x) = h^0(Q^F_{r_i(x)}) - h^0(Q^F_{r_{i-1}(x)})$, notice that

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) = r\sum_{x\in I} a_{l_x+1}(x) + \sum_{x\in I} a_{l_x+1}(x)h^0(E^\tau_x)$$
$$- \sum_{x\in I}\sum_{i=1}^{l_x+1} a_i(x)n_i(x)$$

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^F_{r_i(x)}) = r(F)\sum_{x\in I} a_{l_x+1}(x) + \sum_{x\in I} a_{l_x+1}(x)h^0(F^\tau_x)$$
$$- \sum_{x\in I}\sum_{i=1}^{l_x+1} a_i(x)n^F_i(x),$$

$\chi(F(N))P(m)-P(N)\chi(F(m))=c(m-N)(r\chi(F)-r(F)\chi(E))$, then

$$s(F)=\left(r\ell+rkcN+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)\right)\left(\chi(F)-\frac{r(F)}{r}\chi(E)\right)+$$
$$P(N)\left(\frac{r(F)}{r}\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)\ -\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)h^0(Q^F_{r_i(x)})\right)$$
$$=kP(N)\left(\text{par}\chi((F)-\frac{r(F)}{r}\text{par}\chi(E)\right).$$

For any nontrivial subsheaf $F\subset E$, let τ be the torsion of E/F and $F'\subset E$ such that $\tau=F'/F$ and E/F' torsion free. If we write $\tau=\tilde{\tau}+\sum_{x\in I}\tau_x$, note $h^0(\tau_x)+h^0(Q^F_{r_i(x)})-h^0(Q^{F'}_{r_i(x)})\geq 0$, then

$$s(F)-s(F')=-kP(N)h^0(\tilde{\tau})-P(N)\sum_{x\in I}(k-a_{l_x+1}(x)+a_1(x))h^0(\tau_x)$$
$$-P(N)\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(h^0(\tau_x)+h^0(Q^F_{r_i(x)})-h^0(Q^{F'}_{r_i(x)}))\ \leq 0.$$

If E is stable and $s(F)=0$, it is easy to see that the last requirements in the proposition are satisfied. □

Proposition 2.10. *There exists an integer $M_1(N)>0$ such that for $m\geq M_1(N)$ the following is true. If a point*

$$(p,\{p_{r_1(x)},...,p_{r_{l_x}(x)}\}_{x\in I})\in\mathcal{R}$$

is GIT-stable (respectively, GIT-semistable), then the quotient E is a stable (respectively, semistable) parabolic sheaf and $V\to H^0(E(N))$ is an isomorphism.

Proof. If $(p,\{p_{r_1(x)},...,p_{r_{l_x}(x)}\}_{x\in I})\in\mathcal{R}$ is GIT-stable (GIT-semistable), by Lemma 2.8, $V\to H^0(E(N))$ is an isomorphism. For any nontrivial subsheaf $F\subset E$ with E/F torsion free, let $H\subset V$ be the inverse image of $H^0(F(N))$ and $h=dim(H)$, we have (for $m>N$)

$$\chi(F(N))P(m)-P(N)\chi(F(m))\leq hP(m)-P(N)h^0(F(m))$$

for $m>N$ (note $h^1(F(N))\geq h^1(F(m))$). Thus $s(F)\leq e(H)$ since

$$g(H\otimes W_m)\leq h^0(F(m)),\quad g_{r_i(x)}(H\otimes W_m)\leq h^0(Q^F_{r_i(x)})$$

(the inequalities are strict when $h=0$). By Proposition 2.6 and Proposition 2.9, E is stable (respectively, semistable) if the point is GIT stable (respectively, GIT semistable). □

For a semistable parabolic sheaf E of rank r on X, we have, for any subsheaf $F \subset E$, $\chi(F) \le \frac{\chi(E)}{r} r(F) + 2r|I|$. The following elementary lemma should be well-known.

Lemma 2.11. *Let E be a coherent sheaf of rank r on X. If*

$$\chi(F) \le \frac{\chi(E)}{r} r(F) + C, \quad \forall\ F \subset E.$$

Then, for any $F \subset E$ with $H^1(F) \neq 0$, we have

$$h^0(F) \le \frac{\chi(E)}{r}(r(F) - 1) + C + r(F)g.$$

Proof. $H^1(F) \neq 0$ means that we have a nontrivial morphism $F \to \omega_X$. Let F' be the kernel of $F \to \omega_X$, then $h^0(F) \le h^0(F') + g$. If $H^1(F') = 0$, we have $h^0(F) \le \chi(F') + g \le \frac{\chi(E)}{r}(r(F) - 1) + C + g$. If $H^1(F') \neq 0$, by repeating the arguments to F', we get the required inequality. □

Proposition 2.12. *There exist integers $N > 0$ and $M_2(N) > 0$ such that for $m \ge M_2(N)$ the following is true. If a point*

$$(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x \in I}) \in \mathcal{R}$$

corresponds to a semistable parabolic sheaf E, then the point is GIT-semistable. Moreover, if E is a stable parabolic sheaf, then the point is GIT stable except the case $a_{l_x+1}(x) - a_1(x) = k$.

Proof. There is $N_1 > 0$ such that for any $N \ge N_1$ the following is true. For any $V \otimes \mathcal{W} \xrightarrow{p} E \to 0$ with semistable parabolic sheaf E, the induced map $V \to H^0(E(N))$ is an isomorphism.

Let $H \subset V$ be a nontrivial subspace of $\dim(H) = h$ and $F \subset E$ be the sheaf such that $F(N) \subset E(N)$ is generated by H. Since all these F are in a bounded family (for fixed N), $dim\, g(H \otimes W_m) = h^0(F(m)) = \chi(F(m))$, $g_{r_i(x)}(H \otimes W_m) = h^0(Q^F_{r_i(x)})$ ($\forall\ x \in I$) for $m \ge M_1'(N)$ and

$$e(H) = s(F) + \frac{\ell + kcm}{c(m - N)} P(N) \left(h - \chi(F(N))\right).$$

If $H^1(F(N)) = 0$, we have $e(H) \le s(F)$ since $h \le h^0(F(N))$. Then $e(H) \le s(F) \le 0$ by Proposition 2.9 since E is a semistable parabolic sheaf. If $H^1(F(N)) \neq 0$, by Lemma 2.11, we have

$$h^0(F(N)) \le \frac{rcN + \chi}{r}(r(F) - 1) + r(g + 2|I|).$$

Putting $h \leq h^0(F(N))$ and above inequality in the equality

$$e(H) = P(N)\left(kh - (\ell + kcN)r(F) + (\ell + kcN)\frac{h - \chi(F(N))}{c(m-N)}\right) - P(N)\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^F_{r_i(x)}),$$

then, let $C = k|\chi| + r(g + 2|I|)k + |\ell|r$, we have

$$e(H) \leq P(N)\left(-kcN + C + (\ell + kcN)\frac{h - \chi(F(N))}{c(m-N)}\right).$$

Choose an integer $N_2 \geq N_1$ such that $-kcN_2 + C < -1$. Then, for any fixed $N \geq N_2$, there is an integer $M_2(N)$ such that for $m \geq M_2(N)$

$$(\ell + kcN)\frac{h - \chi(F(N))}{c(m-N)} < 1$$

for any $H \subset V$, which implies $e(H) < 0$ and we are done. □

Theorem 2.13. *There exists a seminormal projective variety*

$$\mathcal{U}_X := \mathcal{U}_X(r, d, \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I}),$$

which is the coarse moduli space of s-equivalence classes of semistable parabolic sheaves E of rank r and $\chi(E) = \chi = d + r(1-g)$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$. If X is smooth, then it is normal, with only rational singularities.

Proof. Let $\mathcal{R}^{ss} \subset \mathcal{R}$ be the open set consisting of semistable parabolic sheaves. $\mathcal{U}_X := \mathcal{U}_X(r, \chi, I, k, \vec{a}, \vec{n})$ is defined to be the GIT quotient $\mathcal{R}^{ss}//SL(V)$. The statements about singularities of $\mathcal{U}_X$ are proved in [9]. The case $a_{l_x+1}(x) - a_1(x) = k$ can be covered by the same arguments in [9] where we proved that $\mathcal{H}$ is normal with only rational singularities. □

When X is a reduced projective curve with two smooth irreducible components X_1 and X_2 of genus g_1 and g_2 meeting at only one point x_0 (which is the only node of X), we fix an ample line bundle $\mathcal{O}(1)$ of degree c on X such that $deg(\mathcal{O}(1)|_{X_i}) = c_i > 0$ $(i = 1, 2)$. For any coherent sheaf E, $P(E, n) := \chi(E(n))$ denotes its Hilbert polynomial, which has degree 1. We define the rank of E to be

$$r(E) := \frac{1}{deg(\mathcal{O}(1))} \cdot \lim_{n\to\infty} \frac{P(E,n)}{n}.$$

Let r_i denote the rank of the restriction of E to X_i $(i = 1, 2)$, then

$$P(E, n) = (c_1r_1 + c_2r_2)n + \chi(E), \quad r(E) = \frac{c_1}{c_1 + c_2}r_1 + \frac{c_2}{c_1 + c_2}r_2.$$

We say that E is of rank r on X if $r_1 = r_2 = r$, otherwise it will be said of rank (r_1, r_2).

Fix a finite set $I = I_1 \cup I_2$ of smooth points on X, where $I_i = \{x \in I \,|\, x \in X_i\}$ $(i = 1, 2)$, and parabolic data $\omega = \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I}$ with

$$\ell := \frac{k\chi - \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x)}{r}$$

(recall $d_i(x) = a_{i+1}(x) - a_i(x)$, $r_i(x) = n_1(x) + \cdots + n_i(x)$). Then we will indicate how the same construction gives moduli space of semistable parabolic sheaves on X (see [10] for details). For simplicity, we only state the case that $a_{l_x+1}(x) - a_1(x) < k$ $(\forall\, x \in I)$.

Definition 2.14. For any coherent sheaf F of rank (r_1, r_2), let

$$m(F) := \frac{r(F) - r_1}{k} \sum_{x\in I_1} a_{l_x+1}(x) + \frac{r(F) - r_2}{k} \sum_{x\in I_2} a_{l_x+1}(x),$$

the modified parabolic Euler characteristic and slop of F are

$$\mathrm{par}\chi_m(F) := \mathrm{par}\chi(F) + m(F), \quad \mathrm{par}\mu_m(F) := \frac{\mathrm{par}\chi_m(F)}{r(F)}.$$

A parabolic sheaf E is called semistable (resp. stable) if, for any subsheaf $F \subset E$ such E/F is torsion free, one has, with the induced parabolic structure,

$$\mathrm{par}\chi_m(F) \le \frac{\mathrm{par}\chi_m(E)}{r(E)} r(F) \quad (\mathit{resp.} <).$$

There is a similar $\mathcal{R}$ and a $SL(V)$-equivariant embedding $\mathcal{R} \hookrightarrow \mathbf{G}$. As the same as Notation 2.5, give the polarization on $\mathbf{G}$:

$$\frac{\ell + kcN}{c(m - N)} \times \prod_{x\in I} \{d_1(x), \cdots, d_{l_x}(x)\}.$$

Then we have the same Proposition 2.6, Lemma 2.8, Proposition 2.9 and Lemma 2.11. The modification in the proof of Proposition 2.9 is: for $F \subset E$ of rank (r_1, r_2) such that E/F is torsion free, we have

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) = \ r\sum_{x\in I} a_{l_x+1}(x) - \sum_{x\in I}\sum_{i=1}^{l_x+1} a_i(x)n_i(x),$$

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^F_{r_i(x)}) = r_1\sum_{x\in I_1} a_{l_x+1}(x) + r_2\sum_{x\in I_2} a_{l_x+1}(x) - \sum_{x\in I}\sum_{i=1}^{l_x+1} a_i(x)n^F_i(x),$$

$$s(F) = kP(N)\left(\mathrm{par}\chi_m((F) - \frac{r(F)}{r}\mathrm{par}\chi_m(E)\right).$$

In particular, we have

Proposition 2.15. *There exist integers $N > 0$ and $M_2(N) > 0$ such that for $m \geq M_2(N)$ the following is true. If a point*

$$(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}) \in \mathcal{R}$$

corresponds to a quasi-parabolic sheaf E, then the point is GIT-semistable (resp. GIT-stable) under the above polarization if and only if E is a semistable (resp. stable) parabolic sheaf for the weights $0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k$ ($\forall\ x \in I$).

Theorem 2.16. *There exists a reduced, seminormal projective scheme*

$$\mathcal{U}_X := \mathcal{U}_X(r, d, \mathcal{O}(1), \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I_1\cup I_2})$$

which is the coarse moduli space of s-equivalence classes of semistable parabolic sheaves E of rank r and $\chi(E) = \chi = d + r(1-g)$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$. The moduli space $\mathcal{U}_X$ has at most $r+1$ irreducible components.

Proof. Let $\mathcal{R}^{ss} \subset \mathcal{R}$ be the open set of semi-stable parabolic sheaves. $\mathcal{U}_X := \mathcal{U}_X(r, d, \mathcal{O}(1), \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I_1\cup I_2})$ is defined to be the GIT quotient $\mathcal{R}^{ss}//\mathrm{SL}(V)$. Let $\mathcal{U}_X^0 \subset \mathcal{U}_X$ be the dense open set of locally free sheaves. For any $E \in \mathcal{U}_X^0$, let E_1 and E_2 be the restrictions of E to X_1 and X_2. By the exact sequence

$$0 \to E_1(-x_0) \to E \to E_2 \to 0$$

and semi-stability of E, we have

$$\frac{c_1}{c_1+c_2} par\chi_m(E) \leq par\chi_m(E_1) \leq \frac{c_1}{c_1+c_2} par\chi_m(E) + r,$$

$$\frac{c_2}{c_1+c_2} par\chi_m(E) \leq par\chi_m(E_2) \leq \frac{c_2}{c_1+c_2} par\chi_m(E) + r.$$

For $j = 1, 2$ and $\omega = \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I_1\cup I_2}$, let $\chi_j = \chi(E_j)$ and

$$n_j^\omega = \frac{1}{k}\left(r\frac{c_j}{c_1+c_2}\ell + \sum_{x\in I_j}\sum_{i=1}^{l_x} d_i(x)r_i(x)\right). \tag{2.3}$$

Then the above inequalities can be rewritten as

$$n_1^\omega \leq \chi_1 \leq n_1^\omega + r, \quad n_2^\omega \leq \chi_2 \leq n_2^\omega + r. \tag{2.4}$$

There are at most $r+1$ possible choices of (χ_1, χ_2) satisfying (2.4) and $\chi_1 + \chi_2 = \chi + r$, each of the choices corresponds an irreducible component of $\mathcal{U}_X$. $\square$

Remarks 2.17. (1) If n_j^ω ($j = 1, 2$) are not integers, then there are at most r irreducible components $\mathcal{U}_X^{\chi_1,\chi_2} \subset \mathcal{U}_X$ of $\mathcal{U}_X$ with

$$n_1^\omega < \chi_1 < n_1^\omega + r, \quad n_2^\omega < \chi_2 < n_2^\omega + r \tag{2.5}$$

such that the (dense) open set of parabolic bundles $E \in \mathcal{U}_X^{\chi_1,\chi_2}$ satisfy

$$\chi(E|_{X_1}) = \chi_1, \quad \chi(E|_{X_2}) = \chi_2.$$

For any χ_1, χ_2 satisfying (2.5), let $\mathcal{U}_{X_1}$ (resp. $\mathcal{U}_{X_2}$) be the moduli space of semistable parabolic bundles of rank r and Euler characteristic χ_1 (resp. χ_2), with parabolic structures of type $\{\vec{n}(x)\}_{x\in I_1}$ (resp. $\{\vec{n}(x)\}_{x\in I_2}$) and weights $\{\vec{a}(x)\}_{x\in I_1}$ (resp. $\{\vec{a}(x)\}_{x\in I_2}$) at points $\{x\}_{x\in I_1}$ (resp. $\{x\}_{x\in I_2}$), then $\mathcal{U}_X^{\chi_1,\chi_2}$ is not empty if $\mathcal{U}_{X_j}$ ($j = 1, 2$) are not empty (See Proposition 1.4 of [10]). In fact, $\mathcal{U}_X^{\chi_1,\chi_2}$ contains a stable parabolic bundle if one of $\mathcal{U}_{X_j}$ ($j = 1, 2$) contains a stable parabolic bundle.

(2) Let $E \in \mathcal{U}_X$, for any nontrivial $F \subset E$ of rank (r_1, r_2) such that E/F torsion free, we have

$$\begin{aligned} & kr(F)(par\mu_m(F) - par\mu_m(E)) \\ & = k\chi(F) - \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^F_{r_i(x)}) - r(F)\ell, \end{aligned} \tag{2.6}$$

which implies the following facts: (i) When $\ell = 0$, the moduli spaces $\mathcal{U}_X$ is independent of the choices of $\mathcal{O}(1)$. (ii) When $\ell \neq 0$, we can choose $\mathcal{O}(1)$ such that all the numbers n_1^ω, n_2^ω and $r(F)\ell$ (for all possible $r_1 \neq r_2$) are not integers (we call such $\mathcal{O}(1)$ **a generic polarization**, its existence is an easy excise). Then, for any $E \in \mathcal{U}_X \setminus \mathcal{U}_X^s$ (i.e. non-stable sheaf), the sub-sheaf $F \subset E$ of rank (r_1, r_2) with $par\mu_m(F) = par\mu_m(E)$ must have $r_1 = r_2$.

When X is a connected nodal curve (irreducible or reducible) of genus g, with only one node x_0, let $\pi : \widetilde{X} \to X$ be the normalization and $\pi^{-1}(x_0) = \{x_1, x_2\}$. Then the normalization $\phi : \mathcal{P} \to \mathcal{U}_X$ of $\mathcal{U}_X$ is given by moduli space of generalized parabolic sheaves (GPS) on $\widetilde{X}$.

Recall that a GPS (E, Q) of rank r on $\widetilde{X}$ consists of a sheaf E on $\widetilde{X}$, torsion free of rank r outside $\{x_1, x_2\}$ with parabolic structures at the points of I (we identify I with $\pi^{-1}(I)$) and an r-dimensional quotient

$$E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0.$$

The moduli space $\mathcal{P}$ consists of semistable (E, Q) with additional parabolic structures at the points of I (we identify I with $\pi^{-1}(I)$)

given by the data $\omega = (r, \chi, \{\vec{n}(x), \vec{a}(x)\}_{x\in I}, \mathcal{O}(1), k)$ satisfying

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\widetilde{\ell} = k\widetilde{\chi}$$

where $d_i(x) = a_{i+1}(x) - a_i(x)$, $\widetilde{\chi} = \chi + r$, $\widetilde{\ell} = k + \ell$ and the pullback $\pi^*\mathcal{O}(1)$ is denoted by $\widetilde{\mathcal{O}}(1)$ (See [9] and [10] for details).

Definition 2.18. A GPS (E, Q) is called semistable (resp., stable), if for every nontrivial subsheaf $E' \subset E$ such that E/E' is torsion free outside $\{x_1, x_2\}$, we have, with the induced parabolic structures at points $\{x\}_{x\in I}$,

$$par\chi_m(E') - dim(Q^{E'}) \le r(E')\cdot\frac{par\chi_m(E) - dim(Q)}{r(E)} \quad \text{(resp., <)},$$

where $Q^{E'} = q(E'_{x_1}\oplus E'_{x_2}) \subset Q$.

When X is irreducible, let $\widetilde{P}$ denote the polynomial $\widetilde{P}(m) = crm + \widetilde{\chi}$, $\widetilde{\mathcal{W}} = \widetilde{\mathcal{O}}(-N) = \widetilde{\mathcal{O}}(1)^{-N}$ and $\widetilde{V} = \mathbb{C}^{\widetilde{P}(N)}$. Consider the Quot scheme

$$\mathrm{Quot}(\widetilde{V}\otimes\widetilde{\mathcal{W}}, P)(T) = \left\{\begin{array}{l} T\text{-flat quotients } \widetilde{V}\otimes\widetilde{\mathcal{W}} \to E \to 0 \text{ over} \\ \widetilde{X}\times T \text{ with } \chi(E_t(m)) = \widetilde{P}(m)\ (\forall\, t\in T)\end{array}\right\},$$

and let $\mathbf{Q} \subset Quot(\widetilde{V}\otimes\widetilde{\mathcal{W}}, P)$ be the open set

$$\mathbf{Q}(T) = \left\{\begin{array}{l} \widetilde{V}\otimes\widetilde{\mathcal{W}} \to E \to 0, \text{ with } R^1p_{T*}(E(N)) = 0 \text{ and} \\ \widetilde{V}\otimes\mathcal{O}_T \to p_{T*}E(N) \text{ induces an isomorphism}\end{array}\right\}.$$

Let $\widetilde{\mathbf{Q}}$ be the closure of $\mathbf{Q}$ in the Quot scheme, $\widetilde{V}\otimes\widetilde{\mathcal{W}} \to \widetilde{\mathcal{F}} \to 0$ be the universal quotient over $\widetilde{X}\times\widetilde{\mathbf{Q}}$ and $\widetilde{\mathcal{F}}_x$ be the restriction of $\widetilde{\mathcal{F}}$ on $\{x\}\times\widetilde{\mathbf{Q}} \cong \widetilde{\mathbf{Q}}$. Let $Flag_{\vec{n}(x)}(\widetilde{\mathcal{F}}_x) \to \widetilde{\mathbf{Q}}$ be the relative flag scheme of locally free quotients of type $\vec{n}(x)$, and

$$\widetilde{\mathcal{R}} = \underset{x\in I}{\times_{\widetilde{\mathbf{Q}}}} Flag_{\vec{n}(x)}(\widetilde{\mathcal{F}}_x) \to \widetilde{\mathbf{Q}}, \quad \widetilde{\mathcal{R}'} = \widetilde{\mathcal{R}}\times_{\widetilde{\mathbf{Q}}} Grass_r(\widetilde{\mathcal{F}}_{x_1}\oplus\widetilde{\mathcal{F}}_{x_2}).$$

A (closed) point $(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}, q_s)$ of $\widetilde{\mathcal{R}'}$ by definition is given by a point $\widetilde{V}\otimes\widetilde{\mathcal{W}} \xrightarrow{p} E \to 0$ of the Quot scheme, together with the flags of quotients

$$\{E_x \twoheadrightarrow Q_{r_{l_x}(x)} \twoheadrightarrow Q_{r_{l_x-1}(x)} \twoheadrightarrow \cdots \twoheadrightarrow Q_{r_2(x)} \twoheadrightarrow Q_{r_1(x)} \twoheadrightarrow 0\}_{x\in I}$$

and a r-dimensional quotient $E_{x_1}\oplus E_{x_2} \xrightarrow{q} Q \to 0$, where $p_{r_i(x)}$: $\widetilde{V}\otimes\widetilde{\mathcal{W}} \xrightarrow{p} E \to E_x \twoheadrightarrow Q_{r_{l_x}(x)} \twoheadrightarrow \cdots \twoheadrightarrow Q_{r_i(x)}$ and $q_s : \widetilde{V}\otimes\widetilde{\mathcal{W}} \xrightarrow{p} E \to E_{x_1}\oplus E_{x_2} \xrightarrow{q} Q$. Choose N large enough so that every semistable

GPS (E,Q) with $\chi(E(m)) = \widetilde{P}(m)$ and parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ with weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$ appears as a point of $\widetilde{\mathcal{R}'}$. For large enough m, we have a $SL(\widetilde{V})$-equivariant embedding

$$\widetilde{\mathcal{R}'} \hookrightarrow \mathbf{G}' = Grass_{\widetilde{P}(m)}(\widetilde{V}\otimes W_m) \times \mathbf{Flag} \times Grass_r(\widetilde{V}\otimes W_m),$$

where $W_m = H^0(\widetilde{\mathcal{W}}(m))$, and **Flag** is defined to be

$$\mathbf{Flag} = \prod_{x\in I}\{Grass_{r_1(x)}(\widetilde{V}\otimes W_m)\times\cdots\times Grass_{r_{l_x}(x)}(\widetilde{V}\otimes W_m)\},$$

which maps a point $(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}, q_s) = (\widetilde{V}\otimes\widetilde{\mathcal{W}} \xrightarrow{p} E,$

$$\{\widetilde{V}\otimes\widetilde{\mathcal{W}} \xrightarrow{p_{r_1(x)}} Q_{r_1(x)},\ \cdots, \widetilde{V}\otimes\widetilde{\mathcal{W}} \xrightarrow{p_{r_{l_x}(x)}} Q_{r_{l_x}(x)}\}_{x\in I}, \widetilde{V}\otimes\widetilde{\mathcal{W}} \xrightarrow{q_s} Q)$$

of $\widetilde{\mathcal{R}'}$ to the point $(g, \{g_{r_1(x)}, ..., g_{r_{l_x}(x)}\}_{x\in I}, g_G) = (\widetilde{V}\otimes W_m \xrightarrow{g} U,$

$$\{\widetilde{V}\otimes W_m \xrightarrow{g_{r_1(x)}} U_{r_1(x)},\ \cdots, \widetilde{V}\otimes W_m \xrightarrow{g_{r_{l_x}(x)}} U_{r_{l_x}(x)}\}_{x\in I}, \widetilde{V}\otimes W_m \xrightarrow{g_G} U_r)$$

of $\mathbf{G}'$, where $g := H^0(p(m))$, $U := H^0(E(m))$, $g_{r_i(x)} := H^0(p_{r_i(x)}(m))$, $U_{r_i(x)} := H^0(Q_{r_i(x)})$ $(i = 1, ..., l_x)$, $g_G := H^0(q_s(m))$, $U_r := H^0(Q)$ and $r_i(x) = dim(Q_{r_i(x)})$. Given $\mathbf{G}'$ the polarisation

$$\frac{(\ell + kcN)}{c(m-N)} \times \prod_{x\in I}\{d_1(x), \cdots, d_{l_x}(x)\} \times k.$$

Then, by the general criteria of GIT stability, we have

Proposition 2.19. *A point $(g, \{g_{r_1(x)}, ..., g_{r_{l_x}(x)}\}_{x\in I}, g_G) \in \mathbf{G}'$ is stable (respectively, semistable) for the action of $SL(\widetilde{V})$, with respect to the above polarisation (we refer to this from now on as GIT-stability), iff for all nontrivial subspaces $H \subset \widetilde{V}$ we have (with $h = dimH$)*

$$\begin{aligned} e(H) :=& \frac{\ell + kcN}{c(m-N)}(h\widetilde{P}(m) - \widetilde{P}(N) dim\, g(H\otimes W_m)) + \\ & \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)(r_i(x)h - \widetilde{P}(N) dim\, g_{r_i(x)}(H\otimes W_m)) \\ & + k(rh - \widetilde{P}(N) dim\, g_G(H\otimes W_m)) < (\leq)\, 0. \end{aligned}$$

Lemma 2.20. *There exists $M_1(N)$ such that for $m \geq M_1(N)$ the following holds. Suppose $(p, \{p_{r(x)}, p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}, q_s) \in \widetilde{\mathcal{R}'}$ is GIT-semistable, then for all quotients $E \xrightarrow{T} \mathcal{G} \to 0$ we have*

$$h^0(\mathcal{G}(N)) \geq \frac{1}{k}\left(r(\mathcal{G})(\ell + kcN) + \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) h^0(Q^{\mathcal{G}}_{r_i(x)})\right) + h^0(Q^{\mathcal{G}}).$$

In particular, $\widetilde{V} \to H^0(E(N))$ *is an isomorphism and* E *satisfies the following conditions: (1) the torsion* $\mathrm{Tor}\, E$ *of* E *is supported on* $\{x_1, x_2\}$ *and* $q : (\mathrm{Tor}\, E)_{x_1} \oplus (\mathrm{Tor}\, E)_{x_2} \hookrightarrow Q$*, (2) if* N *is large enough, then* $H^1(E(N)(-x - x_1 - x_2)) = 0$ *for all* E *and* $x \in \widetilde{X}$.

Proof. Let $H = ker\{\widetilde{V} \xrightarrow{H^0(p(N))} H^0(E(N)) \xrightarrow{H^0(T(N))} H^0(\mathcal{G}(N))\}$ and $F \subset E$ be the subsheaf generated by H. Since all these F are in a bounded family, there exists an integer $M_1'(N)$ such that $dim\, g(H \otimes W_m) = h^0(F(m)) = \chi(F(m))$, $g_{r_i(x)}(H \otimes W_m) = h^0(Q^F_{r_i(x)})$ ($\forall\ x \in I$) and $dim\, g_G(H \otimes W_m) = h^0(Q^F)$ for $m \geq M_1'(N)$. Then, by Proposition 2.19 (with $h = dim(H)$), we have

$$e(H) = (\ell + kcN)(rh - r(F)\widetilde{P}(N)) + (\ell + kcN)\widetilde{P}(N)\frac{h - \chi(F(N))}{c(m - N)}$$

$$+ \sum_{x \in I}\sum_{i=1}^{l_x} d_i(x)\left(r_i(x)h - \widetilde{P}(N)h^0(Q^F_{r_i(x)})\right) + k(rh - \widetilde{P}(N)h^0(Q^F)).$$

By using $h \geq \widetilde{P}(N) - h^0(\mathcal{G}(N))$, $r - r(F) \geq r(\mathcal{G})$, $r_i(x) - h^0(Q^F_{r_i(x)}) \geq h^0(Q^{\mathcal{G}}_{r_i(x)})$ and $r - h^0(Q^F) \geq h^0(Q^{\mathcal{G}})$, we get the inequality

$$h^0(\mathcal{G}(N)) \geq (\ell + kcN)\frac{h - \chi(F(N))}{k(m - N)c} - \frac{e(H)}{k\widetilde{P}(N)} + h^0(Q^{\mathcal{G}}) + \frac{1}{k}\left(r(\mathcal{G})(\ell + kcN) + \sum_{x \in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\right).$$

For given N, the set $\{h - \chi(F(N))\}$ is finite since all these F are in a bounded family. Let $\chi(N) = \min\{h - \chi(F(N))\}$. If $\chi(N) \geq 0$, then

$$h^0(\mathcal{G}(N)) \geq \frac{1}{k}\left(r(\mathcal{G})(\ell + kcN) + \sum_{x \in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\right) + h^0(Q^{\mathcal{G}}) - \frac{e(H)}{k\widetilde{P}(N)}.$$

When $\chi(N) < 0$, let $M_1(N) > \max\{M_1'(N), -\chi(N)(\ell + kcN) + cN\}$ and $m \geq M_1(N)$. Then, since $e(H) \leq 0$, we have

$$h^0(\mathcal{G}(N)) \geq \frac{1}{k}\left(r(\mathcal{G})(\ell + kcN) + \sum_{x \in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\right) + h^0(Q^{\mathcal{G}}).$$

Now we show that $\widetilde{V} \to H^0(E(N))$ is an isomorphism. The injectivity of $\widetilde{V} \xrightarrow{H^0(p(N))} H^0(E(N))$ is easy to see. To see it being surjective, it is enough to show that one can choose N such that $H^1(E(N)) = 0$ for all such E. We prove $H^1(E(N)(-x_1 - x_2 - x)) = 0$ for any $x \in \widetilde{X}$. Otherwise, there is a nontrivial quotient $E(N) \to L \subset \omega_{\widetilde{X}}(x_1+x_2+x)$ by Serre duality, and thus

$$h^0(\omega_{\widetilde{X}}(x_1 + x_2 + x)) \geq h^0(L) \geq N + B,$$

where B is a constant independent of E, we choose N such that $H^1(E(N)(-x_1 - x_2 - x)) = 0$ for all GIT-semistable points.

Let $\tau = Tor(E)$, $\mathcal{G} = E/\tau$, note $h^0(\mathcal{G}(N)) = \widetilde{P}(N) - h^0(\tau)$ and

$$h^0(Q^{\mathcal{G}}_{r_i(x)}) = r_i(x) - h^0(Q^{\tau}_{r_i(x)}), \quad h^0(Q^{\mathcal{G}}) = r - h^0(Q^{\tau})$$

then the inequality in Lemma 2.20 becomes

$$\begin{aligned} kh^0(\tau) &\leq kh^0(Q^{\tau}) + \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^{\tau}_{r_i(x)}) \\ &\leq kh^0(Q^{\tau}) + \sum_{x\in I}(a_{l_x+1}(x) - a_1(x))h^0(\tau_x). \end{aligned}$$

Thus $\tau = Tor(E)$ is supported on $\{x_1, x_2\}$ (since $a_{l_x+1}(x) - a_1(x) < k$) and $E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q$ induces injection $\tau_{x_1} \oplus \tau_{x_2} \hookrightarrow Q$. □

Notation 2.21. Let $\mathcal{H} \subset \widetilde{\mathcal{R}'}$ be the subscheme parametrising the generalised parabolic sheaves $E = (E, E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q)$ satisfying the conditions (1) and (2) at the end of Lemma 2.20. Then, if $\widetilde{\mathcal{R}'}^{ss}$ (resp. $\widetilde{\mathcal{R}'}^{s}$) denotes the open set of $\widetilde{\mathcal{R}'}$ consisting of the semistable (resp. stable) GPS, then it is clear that we have open embedding

$$\widetilde{\mathcal{R}'}^{ss} \hookrightarrow \mathcal{H} \hookrightarrow \widetilde{\mathcal{R}'}.$$

Proposition 2.22. *Suppose $(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x\in I}, q_s) \in \mathcal{H}$ is a point corresponding to a GPS (E, Q). Then (E, Q) is stable (resp. semistable) iff for any nontrivial subsheaf $F \subset E$ we have*

$$\begin{aligned} s(F) := &\frac{\ell + kcN}{c(m - N)}(\chi(F(N))\widetilde{P}(m) - \widetilde{P}(N)\chi(F(m)))+ \\ &\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)(r_i(x)\chi(F(N)) - \widetilde{P}(N)h^0(Q^F_{r_i(x)})) \\ &+ k(r\chi(F(N)) - \widetilde{P}(N)h^0(Q^F)) < (resp. \leq) \, 0. \end{aligned}$$

Proof. The point corresonding to a quotient $\widetilde{V} \otimes \widetilde{\mathcal{W}} \xrightarrow{p} E \to 0$ with

$$\{E_x \twoheadrightarrow Q_{r_{l_x}(x)} \twoheadrightarrow Q_{r_{l_x-1}(x)} \twoheadrightarrow \cdots \twoheadrightarrow Q_{r_2(x)} \twoheadrightarrow Q_{r_1(x)} \twoheadrightarrow 0\}_{x\in I}$$

and $E_{x_1}\oplus E_{x_2}\xrightarrow{q} Q\to 0$, where $q_s:\widetilde{V}\otimes\widetilde{\mathcal{W}}\to E_{x_1}\oplus E_{x_2}\xrightarrow{q} Q\to 0$ and $p_{r_i(x)}: V\otimes\mathcal{W}\xrightarrow{p} E\to E_x\twoheadrightarrow Q_{r_{l_x}(x)}\twoheadrightarrow\cdots\twoheadrightarrow Q_{r_i(x)}$. For $F\subset E$ such that E/F is torsion free outside $\{x_1,x_2\}$, we have the flags of quotient sheaves

$$\{F\twoheadrightarrow F_x\twoheadrightarrow Q^F_{r_{l_x}(x)}\twoheadrightarrow Q^F_{r_{l_x-1}(x)}\twoheadrightarrow\cdots\twoheadrightarrow Q^F_{r_2(x)}\twoheadrightarrow Q^F_{r_1(x)}\twoheadrightarrow 0\}_{x\in I}$$

Let $n^F_i(x)=h^0(Q^F_{r_i(x)})-h^0(Q^F_{r_{i-1}(x)})$ and F have rank (r_1,r_2). Then

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x)= r\sum_{x\in I}a_{l_x+1}(x)-\sum_{x\in I}\sum_{i=1}^{l_x+1}a_i(x)n_i(x)$$

$$\begin{aligned}\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^F_{r_i(x)}) =& r_1\sum_{x\in I_1}a_{l_x+1}(x)+r_2\sum_{x\in I_2}a_{l_x+1}(x)\\ &-\sum_{x\in I}\sum_{i=1}^{l_x+1}a_i(x)n^F_i(x).\end{aligned}$$

Thus we have

$$\begin{aligned}s(F)&=k\widetilde{P}(N)\begin{pmatrix}\chi(F)-\frac{1}{k}\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)h^0(Q^F_{r_i(x)})-h^0(Q^F)\\ -\frac{r(F)}{r}\left(\chi(E)-r-\frac{1}{k}\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)\right)\end{pmatrix}\\ &=k\widetilde{P}(N)\left(par\chi_m(F)-\dim(Q^F)-r(F)\frac{par\chi_m(E)-\dim(Q)}{r(E)}\right).\end{aligned}$$

(E,Q) is semi-stable (resp. stable) iff $s(F)\le 0$ (resp. $s(F)<0$) for nontrivial $F\subset E$ such that E/F torsion free outside $\{x_1,x_2\}$.

For any nontrivial subsheaf $F\subset E$, let τ be the torsion of E/F and $F'\subset E$ such that $\tau=F'/F$ and E/F' torsion free. If we write $\tau=\tilde{\tau}+\tau_{x_1}+\tau_{x_2}+\sum_{x\in I}\tau_x$, then

$$\begin{aligned}s(F)-s(F')=&-k\widetilde{P}(N)h^0(\widetilde{\tau})-\widetilde{P}(N)\sum_{x\in I}(k-a_{l_x+1}(x)+a_1(x))h^0(\tau_x)\\ &-\widetilde{P}(N)\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(h^0(\tau_x)+h^0(Q^F_{r_i(x)})-h^0(Q^{F'}_{r_i(x)}))\\ &-k\widetilde{P}(N)(h^0(\tau_{x_1})+h^0(\tau_{x_2})+h^0(Q^F)-h^0(Q^{F'})).\end{aligned}$$

Since $h^0(\tau_x)+h^0(Q^F_{r_i(x)})-h^0(Q^{F'}_{r_i(x)})\ge 0$ and $h^0(\tau_{x_1}\oplus\tau_{x_2})+h^0(Q^F)-h^0(Q^{F'})\ge 0$, we have $s(F)\le s(F')$ and $s(F)<s(F')$ if $\tilde{\tau}+\sum_{x\in I}\tau_x\neq 0$. Thus stability of (E,Q) implies $s(F)<0$ for any nontrivial $F\subset E$. $\square$

Proposition 2.23. *There exist integers N and $M(N) > 0$ such that for $m \geq M(N)$ the following is true. A point*

$$(E, Q) = (p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}, q_s\}_{x \in I}) \in \widetilde{\mathcal{R}'}$$

is GIT-stable (respectively, GIT-semistable) if and only if (E, Q) is a stable (respectively, semistable) GPS such that $\widetilde{V} \to H^0(E(N))$ is an isomorphism and $(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}, q_s\}_{x \in I}) \in \mathcal{H}$.

Proof. If $(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}\}_{x \in I}, q_s) \in \widetilde{\mathcal{R}'}$ is GIT-stable (GIT-semistable), by Lemma 2.20, $\widetilde{V} \to H^0(E(N))$ is an isomorphism and

$$(p, \{p_{r_1(x)}, ..., p_{r_{l_x}(x)}, q_s\}_{x \in I}) \in \mathcal{H}.$$

For any nontrivial subsheaf $F \subset E$ such that E/F is torsion free outside $\{x_1, x_2\}$, let $H \subset \widetilde{V}$ be the inverse image of $H^0(F(N))$ and $h = dim(H)$, note $h^1(F(N)) \geq h^1(F(m))$ when $m > N$, we have

$$\chi(F(N))\widetilde{P}(m) - \widetilde{P}(N)\chi(F(m)) \leq h\widetilde{P}(m) - \widetilde{P}(N)h^0(F(m)).$$

Thus $s(F) \leq e(H)$ since $\dim g(H \otimes W_m) \leq h^0(F(m))$ and

$$\dim g_{r_i(x)}(H \otimes W_m) \leq h^0(Q^F_{r_i(x)}), \quad \dim g_G(H \otimes W_m) \leq h^0(Q^F)$$

(the inequalities are strict when $h = 0$). By Proposition 2.19 and Proposition 2.22, (E, Q) is stable (respectively, semistable) if the point is GIT stable (respectively, GIT semistable).

There is $N_1 > 0$ such that for any $N \geq N_1$ the following is true. For any $\widetilde{V} \otimes \widetilde{\mathcal{W}} \xrightarrow{p} E \to 0$ with semistable GPS (E, Q), the induced map $\widetilde{V} \to H^0(E(N))$ is an isomorphism and $(E, Q) \in \mathcal{H}$.

Let $H \subset \widetilde{V}$ be a nontrivial subspace of $\dim(H) = h$ and $F \subset E$ be the sheaf such that $F(N) \subset E(N)$ is generated by H. Since all these F are in a bounded family (for fixed N), there is a $M_1(N)$ such that

$$\dim g(H \otimes W_m) = h^0(F(m)) = \chi(F(m)), \quad \dim g_G(H \otimes W_m) = h^0(Q^F)$$

and $g_{r_i(x)}(H \otimes W_m) = h^0(Q^F_{r_i(x)})$ ($\forall\, x \in I$) whenever $m \geq M_1(N)$, which imply that

$$e(H) = s(F) + \frac{\ell + kcm}{c(m - N)} \widetilde{P}(N) \left(h - \chi(F(N))\right).$$

If $H^1(F(N)) = 0$, we have $e(H) \leq s(F)$ since $h \leq h^0(F(N))$. Then $e(H) \leq s(F) < (resp. \leq) 0$ by Proposition 2.22 when (E, Q) is stable (resp. semistable). If $H^1(F(N)) \neq 0$, by Lemma 2.11, we have

$$h^0(F(N)) \leq \frac{rcN + \widetilde{\chi}}{r}(r(F) - 1) + A$$

where A is a constant. Putting $h \le h^0(F(N))$ and above inequality in

$$e(H) = \widetilde{P}(N)\left(kh - (\ell + kcN)r(F) + (\ell + kcN)\frac{h - \chi(F(N))}{c(m-N)}\right)$$
$$- \widetilde{P}(N)\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)h^0(Q^F_{r_i(x)}) - k\widetilde{P}(N)h^0(Q^F),$$

then, let $C = k|\chi| + (|A| + |\ell|)r$, we have

$$e(H) \le \widetilde{P}(N)\left(-kcN + C + (\ell + kcN)\frac{h - \chi(F(N))}{c(m-N)}\right).$$

Choose an integer $N_2 \ge N_1$ such that $-kcN_2 + C < -1$. Then, for any fixed $N \ge N_2$, there is an integer $M_2(N)$ such that for $m \ge M_2(N)$

$$(\ell + kcN)\frac{h - \chi(F(N))}{c(m-N)} < 1$$

for any $H \subset V$, which implies $e(H) < 0$ and we are done. □

Theorem 2.24. *When $\widetilde{X}$ is irreducible, there exists a (coarse) moduli space $\mathcal{P}^s$ of stable GPS on $\widetilde{X}$, which is a smooth variety. There is an open immersion $\mathcal{P}^s \hookrightarrow \mathcal{P}$, where $\mathcal{P}$ is the moduli space of s-equivalence classes of semi-stable GPS on $\widetilde{X}$, which is reduced, irreducible and normal projective variety with at most rational singularities.*

Proof. Let $\mathcal{P}^s := \widetilde{\mathcal{R}'}^s//SL(\widetilde{V})$ and $\mathcal{P} := \widetilde{\mathcal{R}'}^{ss}//SL(\widetilde{V})$ be the GIT quotient. When (E, Q) is a stable GPS, E must be torsion free. Thus $\widetilde{\mathcal{R}'}^s$ is a smooth variety, so is $\mathcal{P}^s$. By Proposition 3.2 of [9], $\mathcal{H}$ is reduced, normal with at most rational singularities, so are $\widetilde{\mathcal{R}'}^{ss} \subset \mathcal{H}$ and $\mathcal{P}$. □

The above construction also works for the case when $\widetilde{X} = X_1 \sqcup X_2$ is a disjoint union of two irreducible smooth curves. However, for later applications, we need to use a different quotient space $\widetilde{\mathcal{R}}$. Let χ_1 and χ_2 be integers such that $\chi_1 + \chi_2 - r = \chi$, and fix, for $i = 1, 2$, the polynomials $P_i(m) = c_i rm + \chi_i$ and $\mathcal{W}_i = \mathcal{O}_{X_i}(-N)$ where $\mathcal{O}_{X_i}(1) = \mathcal{O}(1)|_{X_i}$ has degree c_i. Write $V_i = \mathbb{C}^{P_i(N)}$ and consider the Quot schemes $Quot(V_i \otimes \mathcal{W}_i, P_i)$, let $\widetilde{\mathbf{Q}}_i$ be the closure of the open set

$$\mathbf{Q}_i = \left\{ \begin{array}{l} V_i \otimes \mathcal{W}_i \to E_i \to 0, \text{ with } H^1(E_i(N)) = 0 \text{ and} \\ V_i \to H^0(E_i(N)) \text{ induces an isomorphism} \end{array} \right\},$$

we have the universal quotient $V_i\otimes\mathcal{W}_i\to\mathcal{F}^i\to 0$ on $X_i\times\widetilde{\mathbf{Q}}_i$ and the relative flag scheme

$$\mathcal{R}_i=\times_{\substack{\widetilde{\mathbf{Q}}_i\\ x\in I_i}} Flag_{\vec{n}(x)}(\mathcal{F}^i_x)\to\widetilde{\mathbf{Q}}_i.$$

Let $\mathcal{F}=\mathcal{F}^1\oplus\mathcal{F}^2$ denote direct sum of pullbacks of $\mathcal{F}^1$, $\mathcal{F}^2$ on

$$\widetilde{X}\times(\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2)=(X_1\times\widetilde{\mathbf{Q}}_1)\sqcup(X_2\times\widetilde{\mathbf{Q}}_2).$$

Let $\mathcal{E}$ be the pullback of $\mathcal{F}$ to $\widetilde{X}\times(\mathcal{R}_1\times\mathcal{R}_2)$, $\widetilde{V}=V_1\oplus V_2$ and

$$\rho:\widetilde{\mathcal{R}'}:=Grass_r(\mathcal{E}_{x_1}\oplus\mathcal{E}_{x_2})\to\widetilde{\mathcal{R}}:=\mathcal{R}_1\times\mathcal{R}_2\to\widetilde{\mathbf{Q}}:=\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2.$$

Note that $V_1\otimes\mathcal{W}_1\oplus V_2\otimes\mathcal{W}_2\to\mathcal{F}\to 0$ is a $\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2$-flat quotient with Hilbert polynomial $\widetilde{P}(m)=P_1(m)+P_2(m)$ on $\widetilde{X}\times(\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2)$, we have for m large enough a G-equivariant embedding

$$\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2\hookrightarrow Grass_{\widetilde{P}(m)}(V_1\otimes W_1^m\oplus V_2\otimes W_2^m),$$

where $W_i^m=H^0(\mathcal{W}_i(m))$ and $G=(GL(V_1)\times GL(V_2))\cap SL(\widetilde{V})$. Moreover, for large enough m, we have a G-equivariant embedding

$$\widetilde{\mathcal{R}'}\hookrightarrow\mathbf{G}'=Grass_{\widetilde{P}(m)}(\widetilde{V}\otimes W_m)\times\mathbf{Flag}\times Grass_r(\widetilde{V}\otimes W_m)$$

(**Warning** : $\widetilde{V}\otimes W_m:=V_1\otimes W_1^m\oplus V_2\otimes W_2^m$), which maps a point

$$(p=p_1\oplus p_2,\{p_{r_1(x)},...,p_{r_{l_x}(x)}\}_{x\in I},q_s)\in\widetilde{\mathcal{R}'},$$

where $V_i\otimes\mathcal{W}_i\xrightarrow{p_i}E_i\to 0$, $(V_1\otimes\mathcal{W}_1)\oplus(V_2\otimes\mathcal{W}_2)\xrightarrow{p=p_1\oplus p_2}E:=E_1\oplus E_2$ denotes the quotient on $\widetilde{X}=X_1\sqcup X_2$ and

$$\{\ (V_1\otimes\mathcal{W}_1)\oplus(V_2\otimes\mathcal{W}_2)\xrightarrow{p_{r_i(x)}}Q_{r_i(x)}\to 0,\ 1\le i\le l_x\ \}_{x\in I},$$

$(V_1\otimes\mathcal{W}_1)\oplus(V_2\otimes\mathcal{W}_2)\xrightarrow{q_s}Q$ denotes the surjection of sheaves

$$q_s:(V_1\otimes\mathcal{W}_1)\oplus(V_2\otimes\mathcal{W}_2)\to E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q\to 0,$$

to the point $(g,\{g_{r_1(x)},...,g_{r_{l_x}(x)}\}_{x\in I},g_G)=(\widetilde{V}\otimes W_m\xrightarrow{g}U,$

$$\{\widetilde{V}\otimes W_m\xrightarrow{g_{r_1(x)}}U_{r_1(x)},\cdots,\widetilde{V}\otimes W_m\xrightarrow{g_{r_{l_x}(x)}}U_{r_{l_x}(x)}\}_{x\in I},\widetilde{V}\otimes W_m\xrightarrow{g_G}U_r)$$

of $\mathbf{G}'$, where $g:=H^0(p(m))$, $U:=H^0(E(m))$, $g_{r_i(x)}:=H^0(p_{r_i(x)}(m))$, $U_{r_i(x)}:=H^0(Q_{r_i(x)})$ $(i=1,...,l_x)$, $g_G:=H^0(q_s(m))$, $U_r:=H^0(Q)$ and $r_i(x)=dim(Q_{r_i(x)})$. Given $\mathbf{G}'$ the polarisation

$$\frac{\ell+kcN}{c(m-N)}\times\prod_{x\in I}\{d_1(x),\cdots,d_{l_x}(x)\}\times k.$$

Then we have criterion (see Proposition 1.14 and 2.4 of [2])

Proposition 2.25. *A point* $(g, \{g_{r_1(x)}, ..., g_{r_{l_x}(x)}\}_{x\in I}, g_G) \in \mathbf{G}'$ *is stable (semistable) for the action of* G*, with respect to the above polarisation, iff for all nontrivial subspaces* $H \subset \widetilde{V}$*, where* $H = H_1 \oplus H_2$*,* $H_i \subset V_i$ $(i = 1, 2)$*, we have (with* $h = dimH$*,* $\widetilde{H} := H_1 \otimes W_1^m \oplus H_2 \otimes W_2^m$*)*

$$\begin{aligned} e(H) :=& \frac{\ell + kcN}{c(m-N)} \left(\widetilde{P}(m)h - \widetilde{P}(N)dim\, g(\widetilde{H})\right) \\ &+ \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)\left(r_i(x)h - \widetilde{P}(N)dim\, g_{r_i(x)}(\widetilde{H})\right) \\ &+ k\left(rh - \widetilde{P}(N)dim\, g_G(\widetilde{H})\right) < (\le)\, 0. \end{aligned}$$

The Lemma 2.20 and Proposition 2.22 (thus Proposition 2.23) are also true for the case $\widetilde{X} = X_1 \sqcup X_2$. Thus we have

Theorem 2.26. *When* $\widetilde{X} = X_1 \sqcup X_2$*, there exists a (coarse) moduli space* $\mathcal{P}^s$ *of stable GPS on* $\widetilde{X}$*, which is a smooth scheme. There is an open immersion* $\mathcal{P}^s \hookrightarrow \mathcal{P}$*, where* $\mathcal{P}$ *is the moduli space of s-equivalence classes of semi-stable GPS on* $\widetilde{X}$*, which is a disjoint union of at most* $r + 1$ *irreducible, normal projective varieties with at most rational singularities.*

Proof. For any χ_1 and χ_2 satisfying $\chi_1 + \chi_2 = \chi + r$ and

$$n_1^\omega \le \chi_1 \le n_1^\omega + r, \quad n_2^\omega \le \chi_2 \le n_2^\omega + r,$$

let $\mathcal{P}^s_{\chi_1,\chi_2} := \widetilde{\mathcal{R}'}^s//G$, $\mathcal{P}_{\chi_1,\chi_2} := \widetilde{\mathcal{R}'}^{ss}//G$ and

$$\mathcal{P}^s := \bigsqcup_{\chi_1+\chi_2=\chi+r} \mathcal{P}^s_{\chi_1,\chi_2}, \quad \mathcal{P} := \bigsqcup_{\chi_1+\chi_2=\chi+r} \mathcal{P}_{\chi_1,\chi_2}.$$

Then $\mathcal{P}^s_{\chi_1,\chi_2}$ are smooth varieties and $\mathcal{P}_{\chi_1,\chi_2}$ are reduced, irreducible and normal projective varieties with at most rational singularities. □

3. Factorization of generalized theta functions

The moduli spaces $\mathcal{U}_X := \mathcal{U}_X(r, d, \mathcal{O}(1), \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I})$ is independent of the choice of $\mathcal{O}(1)$ when X is irreducible. However, when $X = X_1 \cup X_2$, the moduli spaces $\mathcal{U}_X := \mathcal{U}_X(r, d, \mathcal{O}(1), \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I})$ depends on the choice of $\mathcal{O}(1)$ (more precisely, it only depends on the degree c_i of $\mathcal{O}(1)|_{X_i}$). We will require in this section that

$$\ell := \frac{k\chi - \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x)}{r} \text{ is an integer.} \tag{3.1}$$

When X is irreducible, for any divisor $L = \sum_q \ell_q z_q$ of degree ℓ on X (supported on smooth points), there is an ample line bundle

$$\Theta_{\mathcal{U}_X,\,L} = \Theta(r, d, \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I}, L)$$

on $\mathcal{U}_X$, which is called a theta line bundle on $\mathcal{U}_X$. We are going to define it as follows.

By a family of parabolic sheaves of rank r and Euler characteristic χ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$ parametrized by T, we mean a sheaf $\mathcal{F}$ on $X \times T$, flat over T, and torsion free with rank r and Euler characteristic χ on $X \times \{t\}$ for every $t \in T$, together with, for each $x \in I$, a flag

$$\mathcal{F}_{\{x\}\times T} = \mathcal{Q}_{\{x\}\times T, l_x+1} \twoheadrightarrow \mathcal{Q}_{\{x\}\times T, l_x} \twoheadrightarrow \mathcal{Q}_{\{x\}\times T, l_x-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{\{x\}\times T, 1} \twoheadrightarrow 0$$

of quotients of type $\vec{n}(x)$ and weights $\vec{a}(x)$. We define $\Theta_{\mathcal{F},\,L}$ to be

$$(\det R\pi_T\mathcal{F})^{-k} \otimes \bigotimes_{x\in I}\{\bigotimes_{i=1}^{l_x} \det(\mathcal{Q}_{\{x\}\times T,i})^{d_i(x)}\} \otimes \bigotimes_q \det(\mathcal{F}_{\{z_q\}\times T})^{\ell_q}$$

where π_T is the projection $X \times T \to T$ and $\det R\pi_T\mathcal{F}$ is the determinant of cohomology: $\{\det R\pi_T\mathcal{F}\}_t := \det H^0(X, \mathcal{F}_t) \otimes \det H^1(X, \mathcal{F}_t)^{-1}$. We have the following theorem (see [6] for $r = 2$ and [7] for $r > 2$):

Theorem 3.1. *Let X be irreducible and $L = \sum_q \ell_q z_q$ a divisor of degree ℓ supported on smooth points of X. Then there is an unique ample line bundle $\Theta_{\mathcal{U}_X,\,L} = \Theta(r, d, \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I}, L)$ on $\mathcal{U}_X$ such that*

1. *for any family of parabolic sheaf $\mathcal{F}$ of rank r and degree d parametrised by T, with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$, semistable with respect to the weights $\{\vec{a}(x)\}_{x\in I}$, we have $\phi_T^*\Theta_{\mathcal{U}_X,\,L} = \Theta_{\mathcal{F},\,L}$, where $\phi_T : T \to \mathcal{U}_X$ is the morphism induced by $\mathcal{F}$.*
2. *for any two choices L and L', $\Theta_{\mathcal{U}_X,\,L}$ and $\Theta_{\mathcal{U}_X,\,L'}$ are algebraically equivalent.*

Proof. (1) Let $\mathcal{E}$ be the universal family on $X \times \mathcal{R}^{ss}$, then the line bundle $\Theta_{\mathcal{E},\,L}$ on $\mathcal{R}^{ss}$, which was defined as

$$(\det R\pi_{\mathcal{R}^{ss}}\mathcal{E})^{-k} \otimes \bigotimes_{x\in I}\{\bigotimes_{i=1}^{l_x} \det(\mathcal{Q}_{\{x\}\times\mathcal{R}^{ss},i})^{d_i(x)}\} \otimes \bigotimes_q \det(\mathcal{E}_{\{z_q\}\times\mathcal{R}^{ss}})^{\ell_q},$$

descends to the line bundle $\Theta_{\mathcal{U}_X,\,L}$ on $\mathcal{U}_X$ (see [7] for the detail).

(2) Let $X^0 \subset X$ be the open set of smooth points and $L_0 = L - z$, where z is a point in the support of L. It is enough to show that $\Theta_{\mathcal{U}_X,\,L}$

is algebraically equivalent to $\Theta_{\mathcal{U}_X,\,L_0+y}$ for any $y\in X^0$. To prove it, note that $X^0\times\mathcal{R}^{ss}\to X^0\times\mathcal{U}_X$ is a good quotient and the line bundle

$$\pi^*_{\mathcal{R}^{ss}}(\Theta_{\mathcal{E},\,L}\otimes\det(\mathcal{E}_z)^{-1})\otimes\det(\mathcal{E})$$

descends to a line bundle $\mathcal{L}$ on $X^0\times\mathcal{U}_X$ such that

$$\mathcal{L}|_{\{z\}\times\mathcal{U}_X}=\Theta_{\mathcal{U}_X,\,L}\,,\quad \mathcal{L}|_{\{y\}\times\mathcal{U}_X}=\Theta_{\mathcal{U}_X,\,L_0+y}$$

i.e. $\Theta_{\mathcal{U}_X,\,L}$ and $\Theta_{\mathcal{U}_X,\,L_0+y}$ are algebraically equivalent.

The ampleness of $\Theta_{\mathcal{U}_X,\,L}$ follows the ampleness of $\Theta_{\mathcal{U}_X,\,\ell\cdot y}$, which is the descendant of restriction (on $\mathcal{R}^{ss}$) of the polarization (Notation 2.5) if we choose $\mathcal{O}(1)=\mathcal{O}(cy)$. □

When $X=X_1\cup X_2$, we choose $\mathcal{O}(1)=\mathcal{O}_X(c_1y_1+c_2y_2)$ such that

$$\ell_i=\frac{c_i\ell}{c_1+c_2}\ (i=1,2)\ \textit{are integers.}\tag{3.2}$$

Then the following theorem can be proven similarly (see [10] for the detail).

Theorem 3.2. *Let $X=X_1\cup X_2$ and $L_i=\sum_{q\in X_i}\ell_qz_q$ be a divisor of degree ℓ_i supported on $X_i\setminus\{x_0\}$. Then there is an unique ample line bundle $\Theta_{\mathcal{U}_X,\,L_1+L_2}=\Theta(r,d,\{k,\vec{n}(x),\vec{a}(x)\}_{x\in I_1\cup I_2},L_1+L_2)$ on $\mathcal{U}_X$ such that*

(1) *for any family of parabolic sheaf $\mathcal{F}$ of rank r and degree d parametrised by T, with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$, semistable with respect to the weights $\{\vec{a}(x)\}_{x\in I}$, we have $\phi_T^*\Theta_{\mathcal{U}_X,\,L_1+L_2}=\Theta_{\mathcal{F},\,L_1+L_2}$, where $\phi_T:T\to\mathcal{U}_X$ is the morphism induced by $\mathcal{F}$.*
(2) *for any two choices L_1+L_2, $L_1'+L_2'$, $\Theta_{\mathcal{U}_X,\,L_1+L_2}$ and $\Theta_{\mathcal{U}_X,\,L_1'+L_2'}$ are algebraically equivalent.*

Remarks 3.3. (1) When X is irreducible, the map $E\mapsto E\otimes\mathcal{O}_X(\pm y)$ induces an isomorphism ($\ell\mapsto\ell\pm k$)

$$f:\mathcal{U}_X(r,d,\{k,\vec{n}(x),\vec{a}(x)\}_{x\in I})\to\mathcal{U}_X(r,d\pm r,\{k,\vec{n}(x),\vec{a}(x)\}_{x\in I})$$

such that $\Theta_{\mathcal{U}_X,\,L\pm ky}=f^*\Theta_{\mathcal{U}_X,\,L}$ for the divisor $L=\sum_q\ell_qz_q$ of degree ℓ.

(2) If $\ell\neq0$, for any $L=\sum_{q\in X^0}\ell_qz_q$ of degree ℓ, then $\Theta_{\mathcal{U}_X,\,L}$ is the descendant of restriction (on $\mathcal{R}^{ss}$) of the polarization (Notation 2.5) if we choose $\mathcal{O}(1)=\mathcal{O}(\sum_q\frac{|\ell|\ell_q}{\ell}z_q)$ where $c=|\ell|$.

In the rest of this paper, we will fix a smooth point $y\in X$ (and $y_i\in X_i$ when X is reducible), and choose

$$L=\ell_yy+\sum_{x\in I}\alpha_xx,\qquad L_i=\ell_{y_i}y_i+\sum_{x\in I_i}\alpha_xx\quad(i=1,\,2).$$

This choice determines, when X is irreducible, the theta line bundle

$$\Theta_{\mathcal{U}_X} = \Theta(r, d, \{k, \vec{n}(x), \vec{a}(x), \alpha_x\}_{x\in I}, \ell_y)$$

where $\ell_y + \sum_{x\in I} \alpha_x = \ell$, and it determines, when X is reducible,

$$\Theta_{\mathcal{U}_X} = \Theta(r, d, \{k, \vec{n}(x), \vec{a}(x), \alpha_x\}_{x\in I_1\cup I_2}, \ell_{y_1}, \ell_{y_2})$$

where $\ell_{y_i} + \sum_{x\in I_i} \alpha_x = \ell_i$ $(i = 1, 2)$.

Now we are going to state the factorizations proved in [9] and [10]. Firstly, let X be an irreducible projective curve of genus g, smooth but for one node x_0. Let $\pi : \widetilde{X} \to X$ be the normalization of X, and $\pi^{-1}(x_0) = \{x_1, x_2\}$. Let I be a finite set of smooth points on X and $y \in X$ be a fixed smooth point. Given integers d, k, r, $\{\alpha_x\}_{x\in I}$, ℓ_y,

$$\vec{a}(x) = (a_1(x), a_2(x), \cdots, a_{l_x+1}(x))$$
$$\vec{n}(x) = (n_1(x), n_2(x), \cdots, n_{l_x+1}(x))$$

satisfying $\ell_y + \sum_{x\in I} \alpha_x = \ell$ and

$$0 \le a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k \quad (x \in I).$$

Recall that ℓ is defined by

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\ell = k(d + r(1-g)) = k\chi \tag{3.3}$$

where $d_i(x) = a_{i+1}(x) - a_i(x)$ and $r_i(x) = n_1(x) + \cdots + n_i(x)$.

Let $\mathcal{U}_X$ be the moduli space of (s-equivalence classes of) parabolic torsion free sheaves of rank r and degree d on X, with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$, semistable with respect to the weights $\{\vec{a}(x)\}_{x\in I}$.

For $\mu = (\mu_1, \cdots, \mu_r)$ with $0 \le \mu_r \le \cdots \le \mu_1 \le k-1$, let

$$\{d_i = \mu_{r_i} - \mu_{r_i+1}\}_{1\le i\le l}$$

be the subset of nonzero integers in $\{\mu_i - \mu_{i+1}\}_{i=1,\cdots,r-1}$. We define

$$r_i(x_1) = r_i, \quad d_i(x_1) = d_i, \quad l_{x_1} = l, \quad \alpha_{x_1} = \mu_r$$

$$r_i(x_2) = r - r_{l-i+1}, \quad d_i(x_2) = d_{l-i+1}, \quad l_{x_2} = l, \quad \alpha_{x_2} = k - \mu_1$$

and for $j = 1, 2$, we set

$$\vec{a}(x_j) = \left(\mu_r, \mu_r + d_1(x_j), \cdots, \mu_r + \sum_{i=1}^{l_{x_j}-1} d_i(x_j), \mu_r + \sum_{i=1}^{l_{x_j}} d_i(x_j)\right)$$
$$\vec{n}(x_j) = (r_1(x_j), r_2(x_j) - r_1(x_j), \cdots, r_{l_{x_j}}(x_j) - r_{l_{x_j}-1}(x_j), r - r_{l_{x_j}}(x_j)).$$

Let $\mathcal{U}^{\mu}_{\widetilde{X}}$ be the moduli space of semistable parabolic bundles on $\widetilde{X}$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I\cup\{x_1,x_2\}}$ at points $\{x\}_{x\in I\cup\{x_1,x_2\}}$ and weights $\{\vec{a}(x)\}_{x\in I\cup\{x_1,x_2\}}$, and let

$$\Theta_{\mathcal{U}^{\mu}_{\widetilde{X}}}=\Theta(r,d,\{k,\vec{n}(x),\vec{a}(x),\alpha_x\}_{x\in I\cup\{x_1,x_2\}},\ell_y).$$

Then the following is the so called **Factorization Theorem I**

Theorem 3.4. *There exists a (noncanonical) isomorphism*

$$H^0(\mathcal{U}_X,\Theta_{\mathcal{U}_X})\cong\bigoplus_{\mu}H^0(\mathcal{U}^{\mu}_{\widetilde{X}},\Theta_{\mathcal{U}^{\mu}_{\widetilde{X}}})$$

where $\mu=(\mu_1,\cdots,\mu_r)$ *runs through* $0\le\mu_r\le\cdots\le\mu_1\le k-1$.

When $X=X_1\cup X_2$, $I=I_1\cup I_2$, $\widetilde{X}=X_1\sqcup X_2$ is the disjoint union of smooth projective curves X_1 and X_2. Recall that

$$\Theta_{\mathcal{U}_X}=\Theta(r,d,\{k,\vec{n}(x),\vec{a}(x),\alpha_x\}_{x\in I_1\cup I_2},\ell_{y_1},\ell_{y_2}),$$

where $\ell_{y_i}+\sum_{x\in I_i}\alpha_x=\ell_i$ $(i=1,\,2)$, are the theta line bundles on

$$\mathcal{U}_X=\mathcal{U}_X(r,d,\mathcal{O}(1),\omega).$$

For $\mu=(\mu_1,\cdots,\mu_r)$ with $0\le\mu_r\le\cdots\le\mu_1\le k-1$, we define

$$\chi^{\mu}_1=\frac{1}{k}\left(r\ell_1+\sum_{x\in I_1}\sum_{i=1}^{l_x}d_i(x)r_i(x)\right)+\frac{1}{k}\sum_{i=1}^{r}\mu_i=n^{\omega}_1+\frac{1}{k}\sum_{i=1}^{r}\mu_i$$

$$\chi^{\mu}_2=\frac{1}{k}\left(r\ell_2+\sum_{x\in I_2}\sum_{i=1}^{l_x}d_i(x)r_i(x)\right)+r-\frac{1}{k}\sum_{i=1}^{r}\mu_i=n^{\omega}_2+r-\frac{1}{k}\sum_{i=1}^{r}\mu_i.$$

One can check that the numbers satisfy $(j=1,2)$

$$\sum_{x\in I_j\cup\{x_j\}}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\sum_{x\in I_j\cup\{x_j\}}\alpha_x+r\ell_{y_j}=k\chi^{\mu}_j. \tag{3.4}$$

Let $\omega^{\mu}_j=\{k,\vec{n}(x),\vec{a}(x)\}_{x\in I_j\cup\{x_j\}}$ $(j=1,2)$, $d^{\mu}_j=\chi^{\mu}_j+r(g_j-1)$ and

$$\mathcal{U}^{\mu}_{X_j}:=\mathcal{U}_{X_j}(r,d^{\mu}_j,\omega^{\mu}_j)$$

be the moduli space of s-equivalence classes of semistable parabolic bundles E of rank r on X_j and $\chi(E)=\chi^{\mu}_j$, together with parabolic structures of type $\{\vec{n}(x)\}_{x\in I\cup\{x_j\}}$ and weights $\{\vec{a}(x)\}_{x\in I\cup\{x_j\}}$ at points $\{x\}_{x\in I\cup\{x_j\}}$. We define $\mathcal{U}^{\mu}_{X_j}$ to be empty if χ^{μ}_j is not an integer. Let

$$\Theta_{\mathcal{U}^{\mu}_{X_j}}=\Theta(r,d^{\mu}_j,\{k,\vec{n}(x),\vec{a}(x),\alpha_x\}_{x\in I_j\cup\{x_j\}},\ell_{y_j})$$

then we have **Factorization Theorem II**

Theorem 3.5. *There exists a (noncanonical) isomorphism*

$$H^0(\mathcal{U}_{X_1\cup X_2},\Theta_{\mathcal{U}_{X_1\cup X_2}})\cong\bigoplus_{\mu}H^0(\mathcal{U}^{\mu}_{X_1},\Theta_{\mathcal{U}^{\mu}_{X_1}})\otimes H^0(\mathcal{U}^{\mu}_{X_2},\Theta_{\mathcal{U}^{\mu}_{X_2}})$$

where $\mu=(\mu_1,\cdots,\mu_r)$ *runs through* $0\leq\mu_r\leq\cdots\leq\mu_1\leq k-1$.

4. Invariance of spaces of generalized theta functions

For a smooth projective curve C of genus $g\geq 0$ and a finite set $I_1\subset C$ of points, to compute the dimension of $H^0(\mathcal{U}_C,\Theta_{\mathcal{U}_C})$, we take a family $\{(X_t,I_t)\}_{t\in T}$ of curves with parabolic data such that

$$(X_1,I_1)=(C,I_1)$$

is the curve C with given parabolic data and $(X_0,I_0)=(X,I)$ is an curve X with one node and parabolic data. If dimension of the spaces $H^0(\mathcal{U}_{X_t},\Theta_{\mathcal{U}_{X_t}})$ is invariant, we can reduce, by using **Factorization Theorem I**, the computation of dimension for a genus g curve to the computation of dimension for a genus $g-1$ curve. Then, by the same procedure and using **Factorization Theorem II**, we can decrease the number of parabolic points.

In order to prove the invariance, we proved in [9] that

$$H^1(\mathcal{U}_X,\Theta_{\mathcal{U}_X})=0$$

when X is an irreducible curve of $g\geq 3$ with at most one node (which implies the invariance for $g\geq 3$). We recall in this section the proof of vanishing theorem for smooth curves and remark that our arguments in [9] in fact imply the invariance for any smooth curves $X_t:=\widetilde{X}$.

Let $\widetilde{X}$ be a smooth projective curve of genus $\widetilde{g}$. Fix a line bundle $\mathcal{O}(1)$ on $\widetilde{X}$ of $\deg(\mathcal{O}(1))=c$, let $\widetilde{\chi}=d+r(1-\widetilde{g})$, $\widetilde{P}$ denote the polynomial $\widetilde{P}(m)=crm+\widetilde{\chi}$, $\mathcal{O}_{\widetilde{X}}(-N)=\mathcal{O}(1)^{-N}$ and $V=\mathbb{C}^{\widetilde{P}(N)}$. Let $\widetilde{\mathbf{Q}}$ be the Quot scheme of quotients

$$V\otimes\mathcal{O}_{\widetilde{X}}(-N)\to F\to 0$$

(of rank r and degree d) on $\widetilde{X}$. Thus there is on $\widetilde{X}\times\widetilde{\mathbf{Q}}$ a universal quotient $V\otimes\mathcal{O}_{\widetilde{X}\times\widetilde{\mathbf{Q}}}(-N)\to\mathcal{F}\to 0$. Let $\mathcal{F}_x$ be the sheaf given by restricting $\mathcal{F}$ to $\{x\}\times\widetilde{\mathbf{Q}}$, $Flag_{\vec{n}(x)}(\mathcal{F}_x)\to\widetilde{\mathbf{Q}}$ be the relative flag scheme of type $\vec{n}(x)$ and

$$\widetilde{\mathcal{R}}=\underset{x\in I}{\times_{\widetilde{\mathbf{Q}}}}Flag_{\vec{n}(x)}(\mathcal{F}_x)\to\widetilde{\mathbf{Q}}.$$

Let $\widetilde{\mathcal{R}}_F$ denote open set of locally free quotients and

$$V\otimes\mathcal{O}_{\widetilde{X}\times\widetilde{\mathcal{R}}}(-N)\to\widetilde{\mathcal{F}}\to 0$$

denote pullback of the universal quotient $V\otimes\mathcal{O}_{\widetilde{X}\times\widetilde{\mathbf{Q}}}(-N)\to\mathcal{F}\to 0$. The reductive group $\mathrm{SL}(V)$ acts on $\widetilde{\mathcal{R}}$.

For large enough m, we have a $\mathrm{SL}(V)$-equivariant embedding

$$\widetilde{\mathcal{R}}\hookrightarrow\mathbf{G}=Grass_{\widetilde{P}(m)}(V\otimes W_m)\times\mathbf{Flag},$$

where $W_m=\mathrm{H}^0(\mathcal{O}_{\widetilde{X}}(m))$, and **Flag** is defined to be

$$\mathbf{Flag}=\prod_{x\in I}\{Grass_{r_1(x)}(V\otimes W_m)\times\cdots\times Grass_{r_{l_x}(x)}(V\otimes W_m)\}.$$

For any given data $\omega=\{k,\vec{n}(x),\vec{a}(x)\}_{x\in I}$, $\tilde{\ell}$ is defined by

$$\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\tilde{\ell}=k(d+r(1-\tilde{g})=k\widetilde{\chi},\tag{4.1}$$

ω determines a polarisation (for fixed $\mathcal{O}(1)$) on $\mathbf{G}$:

$$\frac{\widetilde{\ell}+kcN}{c(m-N)}\times\prod_{x\in I}\{d_1(x),\cdots,d_{l_x}(x)\}.$$

The set $\widetilde{\mathcal{R}}^{ss}_\omega\subset\widetilde{\mathcal{R}}_F$ of GIT semistable (resp. stable) points for the $\mathrm{SL}(V)$ action under this polarisation is precisely the set of semistable (resp. stable) parabolic bundles on $\widetilde{X}$ of the type determined by the given data. Its good quotient $\mathcal{U}_{\widetilde{X},\omega}$ is our moduli space and

$$\Theta_{\widetilde{\mathcal{R}}^{ss}_\omega}=(det\,R\pi_{\widetilde{\mathcal{R}}^{ss}_\omega}\widetilde{\mathcal{F}})^{-k}\otimes\bigotimes_{x\in I}\{(det\,\widetilde{\mathcal{F}}_x)^{\alpha_x}\otimes\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\}\otimes(det\,\widetilde{\mathcal{F}}_y)^{\tilde{\ell}_y}$$

where $\tilde{\ell}_y+\sum_{x\in I}\alpha_x=\tilde{\ell}$, descends to an ample line bundle $\Theta_{\mathcal{U}_{\widetilde{X},\omega}}$ on $\mathcal{U}_{\widetilde{X},\omega}$. To prove $H^1(\mathcal{U}_{\widetilde{X},\omega},\Theta_{\mathcal{U}_{\widetilde{X},\omega}})=0$, we need essentially the following codimension estimates:

Proposition 4.1 (Proposition 5.1 of [9]). *Let* $|\mathrm{I}|$ *be the number of parabolic points. Then*

(1) $\mathrm{codim}(\widetilde{\mathcal{R}}^{ss}\setminus\widetilde{\mathcal{R}}^{s})\geq(r-1)(\tilde{g}-1)+\frac{1}{k}|\mathrm{I}|$,
(2) $\mathrm{codim}(\widetilde{\mathcal{R}}_F\setminus\widetilde{\mathcal{R}}^{ss})>(r-1)(\tilde{g}-1)+\frac{1}{k}|\mathrm{I}|$.

Proposition 4.2 (Proposition 2.2 of [9]). *Let* $\omega_{\widetilde{X}}=\mathcal{O}_{\widetilde{X}}(\sum q)$ *and* $\omega_{\widetilde{\mathcal{R}}_F}$ *be the canonical sheaf of* $\widetilde{X}$ *and* $\widetilde{\mathcal{R}}_F$ *respectively. Then*

$$\begin{aligned}\omega^{-1}_{\widetilde{\mathcal{R}}_F}=&(det\,R\pi_{\widetilde{\mathcal{R}}_F}\widetilde{\mathcal{F}})^{-2r}\otimes\bigotimes_{x\in I}\left\{(det\,\widetilde{\mathcal{F}}_x)^{n_{l_x+1}-r}\otimes\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\}\\&\otimes\bigotimes_q(det\,\widetilde{\mathcal{F}}_q)^{1-r}\otimes(det\,\widetilde{\mathcal{F}}_y)^{2\widetilde{\chi}+(r-1)(2\tilde{g}-2)}\otimes\mathrm{Det}^*(\Theta_y^{-2})\end{aligned}$$

where $\mathrm{Det}:\widetilde{\mathcal{R}}_F\to J^d_{\widetilde{X}}$ *is the determinant morphism and* Θ_y *is the theta line bundle on* $J^d_{\widetilde{X}}$.

The following result due to F. Knop is essential in our arguments, whose global form was formulated in [6].

Lemma 4.3 (Lemma 4.17 of [6]). *Let* X *be a normal, Cohen-Macaulay varirty on which a reductive group* G *acts, such that a good quotient* $\pi:X\to Y$ *exists. Suppose that the action is generically free and* $dim\,G=dim\,X-dim\,Y$. *Suppose further that*

(1) *the subset where the action is not free has codimension* ≥ 2,
(2) *for every prime divisor* D *in* X, $\pi(D)$ *has codimension* ≤ 1, *where* D *need not be invariant.*

Then $\omega_Y=(\pi_*\omega_X)^G$ *where* ω_X, ω_Y *are the respective dualizing sheaves.*

Theorem 4.4 (Theorem 5.1 of [9]). *Assume* $(r-1)(\tilde g-1)+\frac{1}{k}|\mathrm{I}|\geq 2$. *Then, for any data* ω *such that* $\tilde\ell\in\mathbb{Z}$, *we have*

$$H^1(\mathcal{U}_{\widetilde{X},\omega},\Theta_{\mathcal{U}_{\widetilde{X},\omega}})=0.$$

Proof. Note that, on good quotient $\mathcal{U}_{\widetilde{X},\omega}$, we always have for any $i\geq 0$

$$H^i(\mathcal{U}_{\widetilde{X},\omega},\Theta_{\mathcal{U}_{\widetilde{X},\omega}})=H^i(\widetilde{\mathcal{R}}^{ss}_\omega,\Theta_{\widetilde{\mathcal{R}}^{ss}_\omega})^{inv}.$$

By the assumption and Proposition 4.1, we have $\mathrm{codim}(\widetilde{\mathcal{R}}_F\setminus\widetilde{\mathcal{R}}^{ss}_\omega)>2$. Thus $H^1(\widetilde{\mathcal{R}}^{ss}_\omega,\Theta_{\widetilde{\mathcal{R}}^{ss}_\omega})^{inv}=H^1(\widetilde{\mathcal{R}}_F,\Theta_{\widetilde{\mathcal{R}}_F})^{inv}$, where

$$\Theta_{\widetilde{\mathcal{R}}_F}=(det\,R\pi_{\widetilde{\mathcal{R}}_F}\widetilde{\mathcal{F}})^{-k}\otimes\bigotimes_{x\in I}\{(det\,\widetilde{\mathcal{F}}_x)^{\alpha_x}\otimes\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\}\otimes(det\,\widetilde{\mathcal{F}}_y)^{\tilde\ell_y}$$

with $\tilde\ell_y+\sum_{x\in I}\alpha_x=\tilde\ell$. Let $J=J^d_{\widetilde{X}}$ be the Jacobian of line bundles of degree d on $\widetilde{X}$, $\mathcal{L}$ the universal line bundle on $\widetilde{X}\times J$ and

$$\Theta_y=\det(R\pi_J\mathcal{L})^{-1}\otimes\mathcal{L}_y^{d+1-\tilde g}.$$

The line bundle $\det(\widetilde{\mathcal{F}})$ on $\widetilde{X}\times\widetilde{\mathcal{R}}_F$ induces (for any data $\bar\omega$)

$$\mathrm{Det}:\widetilde{\mathcal{R}}_F\to J\ ,\quad \mathrm{Det}:\mathcal{U}_{\widetilde{X},\bar\omega}\to J$$

such that $\det R\pi_{\widetilde{\mathcal{R}}_F}det\widetilde{\mathcal{F}}=\mathrm{Det}^*(\det(R\pi_J\mathcal{L}))$. Then we can write

$$\Theta_{\widetilde{\mathcal{R}}_F}\otimes\omega^{-1}_{\widetilde{\mathcal{R}}_F}=\hat\Theta_{\bar\omega}\otimes\mathrm{Det}^*(\Theta_y)^{-2}$$

$$\hat{\Theta}_{\bar\omega} =(det\, R\pi_{\widetilde{\mathcal{R}}_F}\widetilde{\mathcal{F}})^{-\bar{k}}\otimes\bigotimes_{x\in I}\left\{(det\,\widetilde{\mathcal{F}}_x)^{\bar\alpha_x}\otimes\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{\bar{d}_i(x)}\right\}$$
$$\otimes\,(det\widetilde{\mathcal{F}}_y)^{\bar\ell_y}\otimes\bigotimes_q(det\,\widetilde{\mathcal{F}}_q)^{1-r}\otimes(det\widetilde{\mathcal{F}}_y)^{(r-1)(2\tilde{g}-2)}$$

where $\bar{k}=k+2r,\ \bar\alpha_x=\alpha_x+n_{l_x+1}(x)-r,\ \bar\ell_y=2\widetilde{\chi}+\tilde\ell_y$ and

$$\bar{d}_i(x)=d_i(x)+n_i(x)+n_{i+1}(x).$$

Let $\bar\omega = \{\bar{k},\vec{n}(x),\vec{\bar{a}}(x)\}_{x\in I}$ with $\vec{\bar{a}}(x) = (\bar{a}_1(x),\bar{a}_2(x),\cdots,\bar{a}_{l_x+1}(x))$ such that $\bar{d}_i(x)=\bar{a}_{i+1}(x)-\bar{a}_i(x)$ $(i=1,\,2,\,\cdots,\,l_x)$. Let

$$\psi_{\bar\omega}:\widetilde{\mathcal{R}}^{ss}_{\bar\omega}\to\widetilde{\mathcal{R}}^{ss}_{\bar\omega}//\mathrm{SL}(V):=\mathcal{U}_{\widetilde{X}}(r,d,\bar\omega)=\mathcal{U}_{\widetilde{X},\bar\omega},$$

there is an ample line bundle $\Theta_{\bar\omega}$ on $\mathcal{U}_{\widetilde{X},\bar\omega}$ such that $\hat\Theta_{\bar\omega}=\psi^*_{\bar\omega}\Theta_{\bar\omega}$ since

$$\bar\ell:=\frac{\bar{k}\widetilde{\chi}-\sum_{x\in I}\sum_{i=1}^{l_x}\bar{d}_i(x)r_i(x)}{r}=\tilde\ell+2\widetilde{\chi}-r|I|+\sum_{x\in I}n_{l_x+1}(x)$$

is an integer. Then we have $\Theta_{\widetilde{\mathcal{R}}^{ss}_{\bar\omega}}=\psi^*_{\bar\omega}(\Theta_{\bar\omega}\otimes\mathrm{Det}^*(\Theta_y)^{-2})\otimes\omega_{\widetilde{\mathcal{R}}^{ss}_{\bar\omega}}$ and

$$(\psi_{\bar\omega *}\Theta_{\widetilde{\mathcal{R}}^{ss}_{\bar\omega}})^{inv}=(\Theta_{\bar\omega}\otimes\mathrm{Det}^*(\Theta_y)^{-2})\otimes(\psi_{\bar\omega *}\omega_{\widetilde{\mathcal{R}}^{ss}_{\bar\omega}})^{inv}.$$

Since $\mathrm{codim}(\widetilde{\mathcal{R}}^{ss}_{\bar\omega}\setminus\widetilde{\mathcal{R}}^{s}_{\bar\omega})\geq 2$, conditions in Lemma 4.3 are satisfied and

$$(\psi_{\bar\omega *}\omega_{\widetilde{\mathcal{R}}^{ss}_{\bar\omega}})^{inv}=\omega_{\mathcal{U}_{\widetilde{X},\bar\omega}}.$$

Then, since $\Theta_{\bar\omega}\otimes\mathrm{Det}^*(\Theta_y)^{-2}$ is ample by Lemma 5.3 of [9], we have

$$H^1(\mathcal{U}_{\widetilde{X},\omega},\Theta_{\mathcal{U}_{\widetilde{X},\omega}})=H^1(\mathcal{U}_{\widetilde{X},\bar\omega},\Theta_{\bar\omega}\otimes\mathrm{Det}^*(\Theta_y)^{-2}\otimes\omega_{\mathcal{U}_{\widetilde{X},\bar\omega}})=0.$$

□

The idea of the proof is to express $H^1(\mathcal{U}_{\widetilde{X},\omega},\Theta_{\mathcal{U}_{\widetilde{X},\omega}})$ by

$$H^1(M,\mathcal{L}\otimes\omega_M)$$

such that $\mathcal{L}$ is an ample line bundle, where M is another GIT quotient. In this process, we need essentially the equality

$$H^1(\widetilde{\mathcal{R}}^{ss}_\omega,\Theta_{\widetilde{\mathcal{R}}_F})^{inv}=H^1(\widetilde{\mathcal{R}}_F,\Theta_{\widetilde{\mathcal{R}}_F})^{inv}$$

which perhaps holds unconditional. In fact, we have the following

Conjecture 4.5. *For any data ω satisfying* (4.1) *and any $i\geq 0$*

$$H^i(\widetilde{\mathcal{R}}^{ss}_\omega,\Theta_{\widetilde{\mathcal{R}}_F})^{inv}=H^i(\widetilde{\mathcal{R}}_F,\Theta_{\widetilde{\mathcal{R}}_F})^{inv},\tag{4.2}$$

where $\Theta_{\widetilde{\mathcal{R}}_F}$ is the polarization determined by ω.

Then the proof of Theorem 4.4 implies the following

Corollary 4.6. *Assume the Conjecture 4.5 is true. Then, for any data* ω *, we have, for any* $i > 0$,

$$H^i(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) = 0.$$

Proof. For any data $\omega = \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I}$, we choose

$$\omega(I') = \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I\cup I'}$$

such that $(r-1)(\widetilde{g}-1) + \frac{|I\cup I'|}{k+2r} \geq i+2$. Note that the projection

$$p_I : \widetilde{\mathcal{R}}(I') = \times_{\widetilde{\mathbf{Q}}_F \atop x\in I\cup I'} Flag_{\vec{n}(x)}(\mathcal{F}_x) \to \widetilde{\mathcal{R}}_F = \times_{\widetilde{\mathbf{Q}}_F \atop x\in I} Flag_{\vec{n}(x)}(\mathcal{F}_x)$$

is a Flag bundle and SL(V)-invariant. By Conjecture 4.5, we have

$$\begin{aligned}\mathrm{H}^i(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) &= \mathrm{H}^i(\widetilde{\mathcal{R}}^{ss}_{\omega}, \Theta_{\widetilde{\mathcal{R}}_F})^{inv} \\ &= \mathrm{H}^i(\widetilde{\mathcal{R}}_F, \Theta_{\widetilde{\mathcal{R}}_F})^{inv} = \mathrm{H}^i(\widetilde{\mathcal{R}}(I'), p_I^*(\Theta_{\widetilde{\mathcal{R}}_F}))^{inv}.\end{aligned}$$

Write $p_I^*(\Theta_{\widetilde{\mathcal{R}}_F}) \otimes \omega^{-1}_{\widetilde{\mathcal{R}}(I')} := \hat{\Theta}_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2}$, then we have

$$\begin{aligned}\hat{\Theta}_{\bar{\omega}} =& (det\, R\pi_{\widetilde{\mathcal{R}}_F}\widetilde{\mathcal{F}})^{-\bar{k}} \otimes \bigotimes_{x\in I\cup I'} \left\{ (det\, \widetilde{\mathcal{F}}_x)^{\bar{\alpha}_x} \otimes \bigotimes_{i=1}^{l_x} (det\, \mathcal{Q}_{x,i})^{\bar{d}_i(x)} \right\} \\ & \otimes (det\widetilde{\mathcal{F}}_y)^{\bar{\ell}_y} \otimes \bigotimes_q (det\, \widetilde{\mathcal{F}}_q)^{1-r} \otimes (det\widetilde{\mathcal{F}}_y)^{(r-1)(2\tilde{g}-2)}\end{aligned}$$

where $\bar{k} = k+2r$, $\bar{\alpha}_x = \alpha_x + n_{l_x+1}(x) - r$, $\bar{\ell}_y = 2\widetilde{\chi} + \tilde{\ell}_y$ and

$$\bar{d}_i(x) = d_i(x) + n_i(x) + n_{i+1}(x)$$

(we define $\alpha_x = 0$, $d_i(x) = 0$ when $x \in I'$). Let $\bar{\omega} = \{\bar{k}, \vec{n}(x), \vec{\bar{a}}(x)\}_{x\in I\cup I'}$ with $\vec{\bar{a}}(x) = (\bar{a}_1(x), \bar{a}_2(x), \cdots, \bar{a}_{l_x+1}(x))$ such that

$$\bar{d}_i(x) = \bar{a}_{i+1}(x) - \bar{a}_i(x), \quad (i = 1,\, 2,\, \cdots,\, l_x).$$

Let $\widetilde{\mathcal{R}}(I')^{ss}_{\bar{\omega}} \subset \widetilde{\mathcal{R}}(I')$ be the open set of GIT semi-stable points (respect to the polarization defined by $\bar{\omega}$), then

$$\begin{aligned}\mathrm{H}^i(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) &= \mathrm{H}^i(\widetilde{\mathcal{R}}^{ss}_{\omega}, \Theta_{\widetilde{\mathcal{R}}_F})^{inv} = \mathrm{H}^i(\widetilde{\mathcal{R}}_F, \Theta_{\widetilde{\mathcal{R}}_F})^{inv} \\ &= \mathrm{H}^i(\widetilde{\mathcal{R}}(I'), p_I^*(\Theta_{\widetilde{\mathcal{R}}_F}))^{inv} = \mathrm{H}^i(\widetilde{\mathcal{R}}(I')^{ss}, p_I^*(\Theta_{\widetilde{\mathcal{R}}_F}))^{inv}\end{aligned}$$

the last equality holds since, by (2) of Proposition 4.1, we have

$$\mathrm{codim}(\widetilde{\mathcal{R}}(I') \setminus \widetilde{\mathcal{R}}(I)^{ss}_{\bar{\omega}}) > (r-1)(\widetilde{g}-1) + \frac{|I\cup J|}{k+2r} \geq i+2.$$

Let $\psi : \widetilde{\mathcal{R}}(I')^{ss}_{\bar\omega} \to \mathcal{U}_{\widetilde{X},\bar\omega}$ be the good quotient. Then $\hat\Theta_{\bar\omega}$ descends to an ample line bundle $\Theta_{\bar\omega}$ on $\mathcal{U}_{\widetilde{X},\bar\omega}$ and $(\psi_*\omega_{\widetilde{\mathcal{R}}(I')^{ss}_{\bar\omega}})^{inv} = \omega_{\mathcal{U}_{\widetilde{X},\bar\omega}}$ since

$$\operatorname{codim}(\widetilde{\mathcal{R}}(I')^{ss}_{\bar\omega} \setminus \widetilde{\mathcal{R}}(I)^{s}_{\bar\omega}) \geq (r-1)(\widetilde{g}-1) + \frac{|I\cup J|}{k+2r} \geq i+2$$

by (1) of Proposition 4.1. Thus we have

$$\mathrm{H}^i(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) = \mathrm{H}^i(\mathcal{U}_{\widetilde{X},\bar\omega}, \Theta_{\bar\omega}\otimes \mathrm{Det}^*(\Theta_y)^{-2}\otimes\omega_{\mathcal{U}_{\widetilde{X},\bar\omega}}) \tag{4.3}$$

for any $i\geq 0$. In particular, $\mathrm{H}^i(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) = 0$ for $i>0$. □

For $i=0$, Conjecture 4.5 is true according to a general fact

Lemma 4.7 (Lemma 4.15 of [6]). *Let V be a projective scheme on which a reductive group G acts, $\widetilde{\mathcal{L}}$ an ample line bundle linearizing the G-action, and $V^{ss}\subset V$ the open set of semi-stable points. Then, for any open G-invariant (irreducible) normal subscheme $V^{ss}\subset W\subset V$,*

$$\mathrm{H}^0(V^{ss}, \widetilde{\mathcal{L}})^{inv} = \mathrm{H}^0(W, \widetilde{L})^{inv}.$$

Corollary 4.8. *For any data $\omega = \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I})$ such that $\ell\in\mathbb{Z}$, the dimension of*

$$\mathrm{H}^0(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}})$$

is independent of the choices of curve $\widetilde{X}$ and the points $x\in\widetilde{X}$.

Proof. By the above Lemma 4.7 and (4.3), we have

$$\mathrm{H}^0(\mathcal{U}_{\widetilde{X},\omega}, \Theta_{\mathcal{U}_{\widetilde{X},\omega}}) = \mathrm{H}^0(\mathcal{U}_{\widetilde{X},\bar\omega}, \Theta_{\bar\omega}\otimes \mathrm{Det}^*(\Theta_y)^{-2}\otimes\omega_{\mathcal{U}_{\widetilde{X},\bar\omega}}).$$

The dimension of $\mathrm{H}^0(\mathcal{U}_{\widetilde{X},\bar\omega}, \Theta_{\bar\omega}\otimes \mathrm{Det}^*(\Theta_y)^{-2}\otimes\omega_{\mathcal{U}_{\widetilde{X},\bar\omega}})$ is independent of the choices of curve $\widetilde{X}$ and the points $x\in\widetilde{X}$ since

$$\mathrm{H}^i(\mathcal{U}_{\widetilde{X},\bar\omega}, \Theta_{\bar\omega}\otimes \mathrm{Det}^*(\Theta_y)^{-2}\otimes\omega_{\mathcal{U}_{\widetilde{X},\bar\omega}}) = 0$$

for all $i>0$.

□

5. Vanishing theorem for irreducible nodal curves

When curves degenerate to a nodal curve X, the invariance of spaces of generalized theta functions for smooth curves has proved in last section (See Corollary 4.8). To complete the program, we need the vanishing theorem $\mathrm{H}^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) = 0$. Its proof was reduced to prove a vanishing theorem on the normalization $\mathcal{P}$ of $\mathcal{U}_X$.

Let X be a connected nodal curve of genus g, with only one node $x_0\in X$, let $\pi:\widetilde{X}\to X$ be the normalization of X and $\pi^{-1}(x_0) = \{x_1\,,\,x_2\}$. The normalization $\phi:\mathcal{P}\to\mathcal{U}_X$ of $\mathcal{U}_X$ is given by moduli space of semi-stable GPS (E,Q) on $\widetilde{X}$ with additional parabolic structures at the points of I (we identify I with $\pi^{-1}(I)$) given by the data

$$\omega = \{k, \vec{n}(x),\, \vec{a}(x)\}_{x\in I}$$

satisfying

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\widetilde{\ell} = k\widetilde{\chi}$$

where $d_i(x) = a_{i+1}(x) - a_i(x)$, $\widetilde{\chi} = \chi + r$, $\widetilde{\ell} = k + \ell$. Recall that

$$\widetilde{\mathcal{R}}' = Grass_r(\mathcal{F}_{x_1} \oplus \mathcal{F}_{x_2}) \times_{\widetilde{\mathbf{Q}}} \widetilde{\mathcal{R}}$$

with the SL(V)-equivariant embedding

$$\widetilde{\mathcal{R}}' \hookrightarrow \mathbf{G}' = Grass_{\widetilde{P}(m)}(V\otimes W_m) \times \mathbf{Flag} \times Grass_r(V\otimes W_m),$$

where $W_m = H^0(\widetilde{\mathcal{W}}(m))$, and **Flag** is defined to be

$$\mathbf{Flag} = \prod_{x\in I}\{Grass_{r_1(x)}(V\otimes W_m)\times\cdots\times Grass_{r_{l_x}(x)}(V\otimes W_m)\}.$$

On $\mathbf{G}'$, take the polarisation (determined by ω)

$$k \times \frac{(\ell + kcN)}{c(m-N)} \times \prod_{x\in I}\{d_1(x),\cdots, d_{l_x}(x)\}. \tag{5.1}$$

Then, when X is irreducible, $\mathcal{P} := \mathcal{P}_\omega$ is the GIT (good) quotient

$$\psi : \widetilde{\mathcal{R}}'^{ss}_\omega \to \mathcal{P}_\omega := \widetilde{\mathcal{R}}'^{ss}_\omega // \mathrm{SL}(V).$$

There is a open subscheme $\mathcal{H}\subset\widetilde{\mathcal{R}}'$ such that $\widetilde{\mathcal{R}}'^{ss}_\omega \subset \mathcal{H}$ for any data ω (See Notation 2.21), one of the main results proved in [9] and [10] is that $\mathcal{H}$ is reduced, normal and Cohen-Macaulay with only rational singularities (so is $\mathcal{P}$). Thus the Kodaira-type vanishing theorem and Hartogs-type extension theorem for cohomology are applicable.

Let $\rho : \widetilde{\mathcal{R}}' \to \widetilde{\mathcal{R}}$ be the projection, $V\otimes\mathcal{O}_{\widetilde{X}\times\mathcal{H}}(-N) \to \mathcal{E} \to 0$,

$$\{\ \mathcal{E}_{\{x\}\times\mathcal{H}} = \mathcal{Q}_{\{x\}\times\mathcal{H},\, l_x+1} \twoheadrightarrow \mathcal{Q}_{\{x\}\times\mathcal{H},\, l_x} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{\{x\}\times\mathcal{H},1} \twoheadrightarrow 0\ \}_{x\in I}$$

denote pullbacks of universal quotients $V\otimes\mathcal{O}_{\widetilde{X}\times\widetilde{\mathcal{R}}}(-N)\to\widetilde{\mathcal{F}}\to 0$,

$$\{\ \widetilde{\mathcal{F}}_{\{x\}\times\widetilde{R}} = \widetilde{\mathcal{Q}}_{\{x\}\times\widetilde{R},\, l_x+1} \twoheadrightarrow \widetilde{\mathcal{Q}}_{\{x\}\times\widetilde{R},\, l_x} \twoheadrightarrow \cdots \twoheadrightarrow \widetilde{\mathcal{Q}}_{\{x\}\times\widetilde{R},\,1} \twoheadrightarrow 0\ \}_{x\in I}.$$

Then the restriction of polarisation (5.1) to $\mathcal{H}$ is

$$\hat{\Theta}'_{\mathcal{H}} := \det(\mathcal{Q})^k \otimes (\det R\pi_{\mathcal{H}}\mathcal{E}(m))^{\frac{\ell+kcN}{c(m-N)}} \otimes \bigotimes_{x\in I}\left\{\bigotimes_{i=1}^{l_x}\det(\mathcal{Q}_{\{x\}\times\mathcal{H},\,i})^{d_i(x)}\right\}$$

where $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2} \to \mathcal{Q} \to 0$ is the universal quotient on $\mathcal{H}$. If we choose $\mathcal{O}(1) = \mathcal{O}_{\widetilde{X}}(cy)$, note that $\mathcal{O}_{\mathcal{H}} = \det R\pi_{\mathcal{H}}\mathcal{E}(N)$, we have

$$(\det R\pi_{\mathcal{H}}\mathcal{E})^{-1} = (\det \mathcal{E}_y)^{cN}, \quad \det R\pi_{\mathcal{H}}\mathcal{E}(m) = (\det\mathcal{E}_y)^{c(m-N)},$$

$$\hat{\Theta}'_{\mathcal{H}} = \det(\mathcal{Q})^k \otimes (\det R\pi_{\mathcal{H}}\mathcal{E})^{-k} \otimes \bigotimes_{x\in I}\left\{\bigotimes_{i=1}^{l_x} \det(\mathcal{Q}_{\{x\}\times\mathcal{H},\,i})^{d_i(x)}\right\} \otimes (\det\mathcal{E}_y)^{\ell}.$$

We will write $\hat{\Theta}'_{\mathcal{H}} = \eta_y^k \otimes \rho^*\hat{\Theta}_{\widetilde{\mathcal{R}}}$, where $\eta_y = \det(\mathcal{Q}) \otimes \det(\mathcal{E}_y)^{-1}$ and

$$\hat{\Theta}_{\widetilde{\mathcal{R}}} = (\det R\pi_{\widetilde{\mathcal{R}}}\widetilde{\mathcal{F}})^{-k} \otimes \bigotimes_{x\in I}\left\{\bigotimes_{i=1}^{l_x} \det(\widetilde{\mathcal{Q}}_{\{x\}\times\widetilde{\mathcal{R}},\,i})^{d_i(x)}\right\} \otimes (\det\widetilde{\mathcal{F}}_y)^{\widetilde{\ell}}.$$

The universal quotient $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2} \to \mathcal{Q} \to 0$ induces an exact sequence

$$0 \to \mathcal{F}_{\mathcal{H}} \to (\pi \times id_{\mathcal{H}})_*\mathcal{E} \to {}_{x_0}\mathcal{Q} \to 0 \tag{5.2}$$

on $X \times \mathcal{H}$, where $\widetilde{X} \times \mathcal{H} \xrightarrow{\pi\times id_{\mathcal{H}}} X \times \mathcal{H}$. The sheaf $\mathcal{F}_{\widetilde{\mathcal{R}}'^{ss}_{\omega}}$ defines

$$\hat{\phi} : \widetilde{\mathcal{R}}'^{ss}_{\omega} \to \mathcal{U}_X := \mathcal{U}_{X,\,\omega},$$

which induces a morphism $\phi : \mathcal{P} = \widetilde{\mathcal{R}}'^{ss}_{\omega}//\mathrm{SL}(V) \to \mathcal{U}_X$ such that

$$\begin{array}{ccc} \widetilde{\mathcal{R}}'^{ss}_{\omega} & \xrightarrow{\psi} & \mathcal{P} \\ & \searrow^{\hat{\phi}} & \downarrow{\scriptstyle \phi} \\ & & \mathcal{U}_X \end{array}$$

is commutative and $\hat{\phi}^*\Theta_{\mathcal{U}_X} = \hat{\Theta}'_{\widetilde{\mathcal{R}}'^{ss}_{\omega}}$. Thus $\hat{\Theta}'_{\widetilde{\mathcal{R}}'^{ss}_{\omega}}$ descends to an ample line bundle $\Theta_{\mathcal{P}} = \phi^*\Theta_{\mathcal{U}_X}$. In fact, there are more general ample line bundles $\Theta_{\mathcal{P},\,\omega}$ on $\mathcal{P}$, which are the descendants of

$$\begin{aligned}\hat{\Theta}'_{\omega} &= (det\, R\pi_{\widetilde{\mathcal{R}}'}\mathcal{E})^{-k} \otimes \bigotimes_{x\in I}\{(det\,\mathcal{E}_x)^{\alpha_x} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\} \otimes (det\,\mathcal{E}_y)^{\tilde{\ell}_y} \otimes \eta_y^k \\ &= \rho^*\Theta_{\widetilde{\mathcal{R}},\,\omega} \otimes (det\,\mathcal{Q} \otimes det\,\mathcal{E}_y^{-1})^k\end{aligned}$$

such that $\Theta_{\mathcal{P},\,\omega} = \phi^*\Theta_{\mathcal{U}_X,\,\omega}$ where $\widetilde{\ell}_y + \sum_{x\in I}\alpha_x = \widetilde{\ell}$, and $\Theta_{\mathcal{U}_X,\,\omega} = \Theta_{\mathcal{U}_X,L}$ is determined (cf. Theorem 3.1) by the data $\omega = \{k, \vec{n}(x), \vec{a}(x)\}_{x\in I}$ and

$$L = \ell_y y + \sum_{x\in I}\alpha_x x.$$

By Lemma 5.5 of [9], we have injection $\phi^* : \mathrm{H}^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X,\,\omega}) \hookrightarrow \mathrm{H}^1(\mathcal{P}, \Theta_{\mathcal{P},\,\omega})$. Thus it is enough to show $\mathrm{H}^1(\mathcal{P}, \Theta_{\mathcal{P},\,\omega}) = 0$. Let $\mathcal{K}$ be the kernel of

$$V \otimes \mathcal{O}_{\widetilde{X}\times\widetilde{\mathcal{R}}'}(-N) \to \mathcal{E} \to 0,$$

and consider $0 \to \mathcal{K} \to V \otimes \mathcal{O}_{\widetilde{X}\times\mathcal{H}}(-N) \to \mathcal{E} \to 0$. The line bundle $\det(\mathcal{K})^{-1} \otimes \mathcal{O}_{\widetilde{X}\times\mathcal{H}}(-\dim(V)N)$ on $\widetilde{X}\times\mathcal{H}$ defines $\mathrm{Det}_{\mathcal{H}} : \mathcal{H} \to J^d_{\widetilde{X}}$ which induces the determinant morphism (cf. Lemma 5.7 of [9])

$$\mathrm{Det} : \mathcal{P} \to J^d_{\widetilde{X}}. \tag{5.3}$$

Proposition 5.1 (Proposition 3.4 of [9]). *Let* $\omega_{\widetilde{X}} = \mathcal{O}(\sum_q q)$ *and*

$$\Theta_{J^d_{\widetilde{X}}} = (det R\pi_{J^d_{\widetilde{X}}}\mathcal{L})^{-2} \otimes \mathcal{L}^r_{x_1} \otimes \mathcal{L}^r_{x_2} \otimes \mathcal{L}^{2\widetilde{\chi}-2r}_y \otimes \bigotimes_q \mathcal{L}^{r-1}_q$$

where $\mathcal{L}$ *is the universal line bundle on* $\widetilde{X} \times J^d_{\widetilde{X}}$. *Then we have*

$$\omega^{-1}_{\mathcal{H}} = (det\, R\pi_{\mathcal{H}}\mathcal{E})^{-2r}\otimes$$
$$\bigotimes_{x\in I}\left\{(det\,\mathcal{E}_x)^{n_{l_x+1}-r} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\} \otimes (det\,\mathcal{Q})^{2r}$$
$$\otimes (det\,\mathcal{E}_y)^{2\widetilde{\chi}-2r} \otimes \mathrm{Det}^*_{\mathcal{H}}(\Theta^{-1}_{J^d_{\widetilde{X}}}).$$

We will prove $R^1\mathrm{Det}_*(\Theta_{\mathcal{P},\omega}) = 0$ and $H^1(J^d_{\widetilde{X}}, \mathrm{Det}_*\Theta_{\mathcal{P},\omega}) = 0$, which imply $\mathrm{H}^1(\mathcal{P}, \Theta_{\mathcal{P},\omega}) = 0$. To recall the proof of $H^1(J^d_{\widetilde{X}}, \mathrm{Det}_*\Theta_{\mathcal{P},\omega}) = 0$. Let $\widetilde{\mathcal{R}}'_F \subset \widetilde{\mathcal{R}}'$, $\widetilde{\mathcal{R}}_F \subset \widetilde{\mathcal{R}}$ denote open set of locally free quotients, for $\mu = (\mu_1, \cdots, \mu_r)$ with $0 \le \mu_r \le \cdots \le \mu_1 \le k$, let

$$\{d_i = \mu_{r_i} - \mu_{r_i+1}\}_{1\le i\le l}$$

be the subset of nonzero integers in $\{\mu_i - \mu_{i+1}\}_{i=1,\cdots,r-1}$. We define

$$r_i(x_1) = r_i, \quad r_i(x_2) = r - r_{l-i+1}, \quad l_{x_1} = l_{x_2} = l$$

$$\vec{n}(x_j) = (r_1(x_j), r_2(x_j) - r_1(x_j), \cdots, r_{l_{x_j}}(x_j) - r_{l_{x_j}-1}(x_j)),$$

$$\widetilde{\mathcal{R}}^{\mu}_F = \underset{x\in I\cup\{x_1,x_2\}}{\times_{\widetilde{\mathbf{Q}}_F}} Flag_{\vec{n}(x)}(\mathcal{F}_x) \xrightarrow{p^{\mu}} \widetilde{\mathcal{R}}_F = \underset{x\in I}{\times_{\widetilde{\mathbf{Q}}_F}} Flag_{\vec{n}(x)}(\mathcal{F}_x).$$

Then, by Remark 4.2 of [9], we have decomposition (on $\widetilde{\mathcal{R}}_F$)

$$\rho_*(\hat{\Theta}'_{\omega}) = \bigoplus_{\mu} p^{\mu}_*(\hat{\Theta}_{\mu}) \tag{5.4}$$

$\mu = (\mu_1, \cdots, \mu_r)$ runs through integers $0 \le \mu_1 \le \cdots \mu_r \le k$ and

$$\hat{\Theta}_{\mu} = (det\, R\pi_{\widetilde{\mathcal{R}}^{\mu}_F}\widetilde{\mathcal{F}})^{-k}\otimes \bigotimes_{x\in I\cup\{x_1,x_2\}} \{(det\,\widetilde{\mathcal{F}}_x)^{\alpha_x}\otimes\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{d_i(x)}\}\otimes(det\,\widetilde{\mathcal{F}}_y)^{\ell_y}$$

where $r_i(x_1) = r_i$, $d_i(x_1) = d_i$, $l_{x_1} = l$, $\alpha_{x_1} = \mu_r$, $r_i(x_2) = r - r_{l-i+1}$, $d_i(x_2) = d_{l-i+1}$, $l_{x_2} = l$, $\alpha_{x_2} = k - \mu_1$ and for $j = 1, 2$, we set

$$\vec{a}(x_j) = \left(\mu_r, \mu_r + d_1(x_j), \cdots, \mu_r + \sum_{i=1}^{l_{x_j}-1} d_i(x_j), \mu_r + \sum_{i=1}^{l_{x_j}} d_i(x_j)\right).$$

It is easy to check that

$$\sum_{x\in I\cup\{x_1,x_2\}} \sum_{i=1}^{l_x} d_i(x) r_i(x) + r \sum_{x\in I\cup\{x_1,x_2\}} \alpha_x + r\ell_y = k\widetilde{\chi}.$$

For the data $\omega^\mu = \{k, \vec{n}(x), \vec{a}_i(x)\}_{x\in I\cup\{x_1,x_2\}}$, we choose

$$\omega^\mu(I') = \{k, \vec{n}(x), \vec{a}_i(x)\}_{x\in I\cup\{x_1,x_2\}\cup I'}$$

such that $(r-1)(\widetilde{g}-1) + \frac{2+|I\cup I'|}{k+2r} \geq 2$. Note that the projection

$$p_I : \widetilde{\mathcal{R}}^\mu(I') = \widetilde{\mathcal{R}}^\mu_F \times_{\widetilde{\mathbf{Q}}_F} \left(\times_{\substack{\widetilde{\mathbf{Q}}_F \\ x\in I'}} Flag_{\vec{n}(x)}(\widetilde{\mathcal{F}}_x)\right) \to \widetilde{\mathcal{R}}^\mu_F$$

is a SL(V)-invariant Flag bundle, consider the commutative diagram

(5.5)
$$\begin{array}{ccc} \widetilde{\mathcal{R}}^\mu(I') & \xrightarrow{p_I} & \widetilde{\mathcal{R}}^\mu_F \\ & \searrow^{\hat{\mathrm{Det}}^{I'}_\mu} & \downarrow \hat{\mathrm{Det}}_\mu \\ & & J^d_{\widetilde{X}} \end{array}$$

and write $p_I^*(\hat{\Theta}_\mu) \otimes \omega^{-1}_{\widetilde{\mathcal{R}}^\mu(I')} = \hat{\Theta}_{\bar{\omega}_\mu} \otimes (\hat{\mathrm{Det}}^{I'}_\mu)^*(\Theta_y)^{-2}$. Then

$$\begin{aligned} \hat{\Theta}_{\bar{\omega}_\mu} = & (det\, R\pi\widetilde{\mathcal{F}})^{-\bar{k}} \otimes \bigotimes_{x\in I\cup\{x_1,x_2\}\cup I'} \{(det\, \widetilde{\mathcal{F}}_x)^{\bar{\alpha}_x} \otimes \bigotimes_{i=1}^{l_x} (det\, \mathcal{Q}_{x,i})^{\bar{d}_i(x)}\} \\ & \otimes (det\, \widetilde{\mathcal{F}}_y)^{\bar{\ell}_y + (r-1)(2\widetilde{g}-2)} \otimes \bigotimes_q (det \widetilde{\mathcal{F}}_q)^{1-r} \end{aligned}$$

where $\bar{k} = k + 2r$, $\bar{\alpha}_x = \alpha_x + n_{l_x+1}(x) - r$, $\bar{\ell}_y = 2\widetilde{\chi} + \widetilde{\ell}_y$ and

$$\bar{d}_i(x) = d_i(x) + n_i(x) + n_{i+1}(x),$$

$\bar{\omega}_\mu = \{\bar{k}, \vec{n}(x), \vec{\bar{a}}(x)\}_{I\cup\{x_1,x_2\}\cup I'}$ with $\vec{\bar{a}}(x) = (\bar{a}_1(x), \bar{a}_2(x), \cdots, \bar{a}_{l_x+1}(x))$ (note: $\bar{a}_{l_x+1}(x) - \bar{a}_1(x) = \sum_{i=1}^{l_x} \bar{d}_i(x) = a_{l_x+1}(x) - a_1(x) + 2r - n_1(x) - n_{l_x+1}(x) \leq k + 2r - n_1(x) - n_{l_x+1}(x) < \bar{k}$).

Let $\widetilde{\mathcal{R}}^{\mu}(I')^{ss}_{\bar{\omega}_\mu} \subset \widetilde{\mathcal{R}}^{\mu}(I')$ be the open set of GIT semi-stable points (respect to the polarization defined by $\bar{\omega}_\mu$), then

$$\operatorname{codim}(\widetilde{\mathcal{R}}^{\mu}(I')^{ss}_{\bar{\omega}_\mu} \setminus \widetilde{\mathcal{R}}^{\mu}(I')^{s}_{\bar{\omega}_\mu}) \geq (r-1)(\widetilde{g}-1) + \frac{2+|I\cup I'|}{k+2r} \geq 2.$$

Let $\psi : \widetilde{\mathcal{R}}^{\mu}(I')^{ss}_{\bar{\omega}_\mu} \to \mathcal{U}_{\widetilde{X},\bar{\omega}_\mu}$ be the good quotient. Then $\hat{\Theta}_{\bar{\omega}_\mu}$ descends to an ample line bundle $\Theta_{\bar{\omega}_\mu}$ on $\mathcal{U}_{\widetilde{X},\bar{\omega}_\mu}$ and $(\psi_*\omega_{\widetilde{\mathcal{R}}^{\mu}(I)^{ss}_{\bar{\omega}_\mu}})^{inv} = \omega_{\mathcal{U}_{\widetilde{X},\bar{\omega}_\mu}}$.

Lemma 5.2. *Let* $\operatorname{Det}^{I'}_{\mu} : \mathcal{U}_{\widetilde{X},\bar{\omega}_\mu} \to J^d_{\widetilde{X}}$ *be the morphism induced by*

$$\hat{\operatorname{Det}}^{I'}_{\mu} : \widetilde{\mathcal{R}}^{\mu}(I')^{ss}_{\bar{\omega}_\mu} \to J^d_{\widetilde{X}}$$

and $\operatorname{Det} : \mathcal{P} \to J^d_{\widetilde{X}}$ *be the determinant morphism. Then*

$$\operatorname{Det}_*(\Theta_{\mathcal{P},\omega}) = \bigoplus_{\mu} (\operatorname{Det}^{I'}_{\mu})_*(\Theta_{\bar{\omega}_\mu} \otimes (\operatorname{Det}^{I'}_{\mu})^*(\Theta_y)^{-2} \otimes \omega_{\mathcal{U}_{\widetilde{X},\bar{\omega}_\mu}}) \tag{5.6}$$

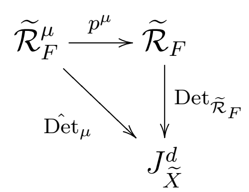
where $\mu = (\mu_1, \cdots, \mu_r)$ *runs through integers* $0 \leq \mu_1 \leq \cdots \mu_r \leq k$. *In particular, we have*

$$H^i(J^d_{\widetilde{X}}, \operatorname{Det}_*\Theta_{\mathcal{P},\omega}) = 0 \quad \forall\ i > 0.$$

Proof. Note $\operatorname{Det}_*(\Theta_{\mathcal{P},\omega}) = \{(\operatorname{Det}_{\widetilde{\mathcal{R}}'^{ss}})_*\hat{\Theta}'_\omega\}^{inv} = \{(\operatorname{Det}_{\widetilde{\mathcal{R}}'_F})_*\hat{\Theta}'_\omega\}^{inv}$ and $(\operatorname{Det}_{\widetilde{\mathcal{R}}'_F})_*\hat{\Theta}'_\omega = (\operatorname{Det}_{\widetilde{\mathcal{R}}_F})_*\rho_*\hat{\Theta}'_\omega$, by the decomposition (5.4), we have

$$(\operatorname{Det}_{\widetilde{\mathcal{R}}'_F})_*\hat{\Theta}'_\omega = \bigoplus_{\mu} (\hat{\operatorname{Det}}_\mu)_*\hat{\Theta}_\mu$$

where $\hat{\operatorname{Det}}_\mu : \widetilde{\mathcal{R}}^{\mu}_F \to J^d_{\widetilde{X}}$ satisfies the commutative diagram



By diagram (5.5) and $p_I^*(\hat{\Theta}_\mu) = \hat{\Theta}_{\bar{\omega}_\mu} \otimes (\hat{\operatorname{Det}}^{I'}_{\mu})^*(\Theta_y)^{-2} \otimes \omega_{\widetilde{\mathcal{R}}^{\mu}(I')}$, we have

$$(\hat{\operatorname{Det}}_\mu)_*\hat{\Theta}_\mu = (\hat{\operatorname{Det}}^{I'}_{\mu})_*(\hat{\Theta}_{\bar{\omega}_\mu} \otimes (\hat{\operatorname{Det}}^{I'}_{\mu})^*(\Theta_y)^{-2} \otimes \omega_{\widetilde{\mathcal{R}}^{\mu}(I')}). \tag{5.7}$$

Recall $\psi : \widetilde{\mathcal{R}}^{\mu}(I')^{ss}_{\bar{\omega}_\mu} \to \mathcal{U}_{\widetilde{X},\bar{\omega}_\mu}$, $\hat{\Theta}_{\bar{\omega}_\mu} = \psi^*\Theta_{\bar{\omega}_\mu}$, $(\psi_*\omega_{\widetilde{\mathcal{R}}^{\mu}(I')^{ss}_{\bar{\omega}_\mu}})^{inv} = \omega_{\mathcal{U}_{\widetilde{X},\bar{\omega}_\mu}}$, then we have the decomposition (5.6). The vanishing result follows the decomposition clearly since $\Theta_{\bar{\omega}_\mu} \otimes (\operatorname{Det}^{I'}_{\mu})^*(\Theta_y)^{-2}$ is ample. $\square$

To prove $R^1\mathrm{Det}_*(\Theta_{\mathcal{P},\omega})=0$, the idea is same with Section 4. Let

$$\widetilde{\mathcal{R}}(I') = \underset{x\in I\cup I'}{\times_{\widetilde{\mathbf{Q}}}} Flag_{\vec{n}(x)}(\mathcal{F}_x) \xrightarrow{p_I} \widetilde{\mathcal{R}} = \underset{x\in I}{\times_{\widetilde{\mathbf{Q}}}} Flag_{\vec{n}(x)}(\mathcal{F}_x),$$

$$\widetilde{\mathcal{R}}'(I') = Grass_r(\mathcal{F}_{x_1}\oplus\mathcal{F}_{x_2})\times_{\widetilde{\mathbf{Q}}}\widetilde{\mathcal{R}}(I') \xrightarrow{p_I} \widetilde{\mathcal{R}}' = Grass_r(\mathcal{F}_{x_1}\oplus\mathcal{F}_{x_2})\times_{\widetilde{\mathbf{Q}}}\widetilde{\mathcal{R}}$$

be the projection, $\mathcal{H}(I')\subset\widetilde{\mathcal{R}}'(I')$, $\mathcal{H}\subset\widetilde{\mathcal{R}}'$ be the open set defined in Notation 2.21. By Proposition 5.1, we have

$$p_I^*(\hat{\Theta}'_\omega)\otimes\omega^{-1}_{\mathcal{H}(I')} = \hat{\Theta}'_{\bar{\omega}}\otimes\mathrm{Det}^*_{\mathcal{H}(I')}(\Theta^{-1}_{J^d_{\widetilde{X}}}) \tag{5.8}$$

with $\bar{\omega}=(d,r,\bar{k},\bar{\ell}_y,\{\bar{\alpha}_x,\,\bar{d}_i(x)\}_{x\in I\cup J,1\le i\le l_x})$ and

$$\hat{\Theta}'_{\bar{\omega}} =(det\, R\pi_{\mathcal{H}(I')}\mathcal{E})^{-\bar{k}}\otimes\bigotimes_{x\in I\cup I'}\{(det\,\mathcal{E}_x)^{\bar{\alpha}_x}\otimes\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{\bar{d}_i(x)}\}$$
$$\otimes\,(det\,\mathcal{E}_y)^{\bar{\ell}_y}\otimes(det\,\mathcal{Q})^{\bar{k}}\otimes(det\,\mathcal{E}_y)^{-\bar{k}}$$

where $\bar{k}=k+2r$, $\bar{\alpha}_x=\alpha_x+n_{l_x+1}(x)-r$, $\bar{\ell}_y=\widetilde{\ell}_y+2\widetilde{\chi}$, and

$$\bar{d}_i(x)=d_i(x)+n_i(x)+n_{i+1}(x).$$

Let $\widetilde{\mathcal{R}}'(I')^{ss}_{\bar{\omega}}\subset\mathcal{H}(I')$ be the open set of GIT semi-stable points (respect to $\bar{\omega}$), $\psi:\widetilde{\mathcal{R}}'(I')^{ss}_{\bar{\omega}}\to\mathcal{P}_{\bar{\omega}}:=\widetilde{\mathcal{R}}'(I)^{ss}_{\bar{\omega}}//\mathrm{SL}(V)$ be the quotient map. There is an ample line bundle $\Theta_{\mathcal{P},\bar{\omega}}$ on $\mathcal{P}_{\bar{\omega}}$ such that $\hat{\Theta}'_{\bar{\omega}}=\psi^*(\Theta_{\mathcal{P},\bar{\omega}})$, and $\omega_{\mathcal{P}_{\bar{\omega}}}=(\psi_*\omega_{\widetilde{\mathcal{R}}'(I')^{ss}_{\bar{\omega}}})^{inv}$ if

$$(r-1)(\widetilde{g}-1)+\frac{|I|+|I'|}{k+2r}\ge 2 \tag{5.9}$$

where we need essentially the estimate of codimension from [9].

Proposition 5.3 (Proposition 5.2 of [9]). *Let $\mathcal{D}^f_1=\hat{\mathcal{D}}_1\cup\hat{\mathcal{D}}^t_1$ and $\mathcal{D}^f_2=\hat{\mathcal{D}}_2\cup\hat{\mathcal{D}}^t_2$, where $\hat{\mathcal{D}}_i\subset\widetilde{\mathcal{R}}'$ is the Zariski closure of $\hat{\mathcal{D}}_{F,1}\subset\widetilde{\mathcal{R}}'_F$ consisting of $(E,Q)\in\widetilde{\mathcal{R}}'_F$ that $E_{x_i}\to Q$ is not an isomorphism, and $\hat{\mathcal{D}}^t_1\subset\widetilde{\mathcal{R}}'$ (rep. $\hat{\mathcal{D}}^t_2\subset\widetilde{\mathcal{R}}'$) consists of $(E,Q)\in\widetilde{\mathcal{R}}'$ such that E is not locally free at x_2 (resp. at x_1). Then*

1. codim$(\mathcal{H}\setminus\widetilde{\mathcal{R}}'^{ss}_\omega)>(r-1)\tilde{g}+\frac{|I|}{k}$.
2. *the complement in $\widetilde{\mathcal{R}}'^{ss}_\omega\setminus\{\mathcal{D}^f_1\cup\mathcal{D}^f_2\}$ of the set $\widetilde{\mathcal{R}}'^{s}_\omega$ of stable points has codimension $\ge(r-1)\tilde{g}+\frac{|I|}{k}$.*

Lemma 5.4. *When $(r-1)\widetilde{g}+\frac{|I|}{k}\ge 2$ and $I'\subset\widetilde{X}\setminus I$ satisfying* (5.9)*,*

$$H^1(\mathcal{P}_\omega,\Theta_{\mathcal{P},\omega})=H^1(\mathcal{P}_{\bar{\omega}},\Theta_{\mathcal{P},\bar{\omega}}\otimes\mathrm{Det}^*_J(\Theta^{-1}_{J^d_{\widetilde{X}}})\otimes\omega_{\mathcal{P}_{\bar{\omega}}}) \tag{5.10}$$

where $\mathrm{Det}_J:\mathcal{P}_{\bar{\omega}}\to J^d_{\widetilde{X}}$ *is induced by* $\mathrm{Det}_{\mathcal{H}(I')}:\mathcal{H}(I')\to J^d_{\widetilde{X}}$.

Proof. By using Proposition 4.1 (1) and Proposition 5.3 (2), we have

$$(\psi_*\omega_{\widetilde{\mathcal{R}}'(I')^{ss}_{\bar\omega}})^{inv} = \omega_{\mathcal{P}_{\bar\omega}}$$

(cf. Lemma 5.6 of [9]). By Proposition 5.3 (1), we have

$$\operatorname{codim}(\mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}_{\omega}) \geq 3, \quad \operatorname{codim}(\mathcal{H}(I') \setminus \widetilde{\mathcal{R}}'(I')^{ss}_{\bar\omega}) \geq 3$$

for any data ω. Thus, by theory of local cohomology, we have

$$\begin{aligned} H^1(\mathcal{P}_\omega, \Theta_{\mathcal{P},\omega}) &= H^1(\widetilde{\mathcal{R}}'^{ss}_\omega, \hat{\Theta}'_\omega)^{inv} = H^1(\mathcal{H}, \hat{\Theta}'_\omega)^{inv} = H^1(\mathcal{H}(I'), p_I^*(\hat{\Theta}'_\omega))^{inv} \\ &= H^1(\mathcal{H}(I'), \hat{\Theta}'_{\bar\omega} \otimes \mathrm{Det}^*_{\mathcal{H}(I')}(\Theta^{-1}_{J^d_{\widetilde{X}}}) \otimes \omega_{\mathcal{H}(I)})^{inv} \\ &= H^1(\widetilde{\mathcal{R}}'(I')^{ss}_{\bar\omega}, \hat{\Theta}'_{\bar\omega} \otimes \mathrm{Det}^*_{\widetilde{\mathcal{R}}'(I')^{ss}_{\bar\omega}}(\Theta^{-1}_{J^d_{\widetilde{X}}}) \otimes \omega_{\widetilde{\mathcal{R}}'(I')^{ss}_{\bar\omega}})^{inv} \\ &= H^1(\widetilde{\mathcal{R}}'(I')^{ss}_{\bar\omega}, \psi^*(\Theta_{\mathcal{P},\bar\omega} \otimes \mathrm{Det}^*_J(\Theta^{-1}_{J^d_{\widetilde{X}}})) \otimes \omega_{\widetilde{\mathcal{R}}'(I')^{ss}_{\bar\omega}})^{inv} \\ &= H^1(\mathcal{P}_{\bar\omega}, \Theta_{\mathcal{P},\bar\omega} \otimes \mathrm{Det}^*_J(\Theta^{-1}_{J^d_{\widetilde{X}}}) \otimes (\psi_*\omega_{\widetilde{\mathcal{R}}'(I')^{ss}_{\bar\omega}})^{inv}) \\ &= H^1(\mathcal{P}_{\bar\omega}, \Theta_{\mathcal{P},\bar\omega} \otimes \mathrm{Det}^*_J(\Theta^{-1}_{J^d_{\widetilde{X}}}) \otimes \omega_{\mathcal{P}_{\bar\omega}}). \end{aligned}$$

□

When X is irreducible, $\Theta_{\mathcal{P},\bar\omega} \otimes \mathrm{Det}^*_J(\Theta^{-1}_{J^d_{\widetilde{X}}})$ may not be an ample line bundle on $\mathcal{P}_{\bar\omega}$. But, for any $L \in J^d_{\widetilde{X}}$, on the fiber $\mathcal{P}^L_\omega = \mathrm{Det}^{-1}(L)$ of

$$\mathrm{Det} : \mathcal{P}_\omega \to J^d_{\widetilde{X}}$$

and the fiber $\mathcal{P}^L_{\bar\omega} = \mathrm{Det}^{-1}_J(L)$ of $\mathrm{Det}_J : \mathcal{P}_{\bar\omega} \to J^d_{\widetilde{X}}$ we have

$$H^1(\mathcal{P}^L_\omega, \Theta^L_{\mathcal{P},\omega}) = H^1(\mathcal{P}^L_{\bar\omega}, \Theta^L_{\mathcal{P},\bar\omega} \otimes \omega_{\mathcal{P}^L_{\bar\omega}}) = 0$$

when $(r-1)(g-1) + \frac{|I|}{k} \geq 2$, which means $\mathrm{R}^1\mathrm{Det}_*(\Theta_{\mathcal{P},\omega}) = 0$.

Theorem 5.5 (Theorem 5.3 of [9])**.** *If X is an irreducible curve of genus g with one node and $(r-1)(g-1) + \frac{|I|}{k} \geq 2$, then*

$$H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X,\omega}) \cong H^1(\mathcal{P}_\omega, \Theta_{\mathcal{P},\omega}) = 0.$$

Remark 5.6. The condition $(r-1)(g-1) + \frac{|I|}{k} \geq 2$ is used only for the proof of $H^1(\widetilde{\mathcal{R}}'^{ss}_\omega, \hat{\Theta}'_\omega)^{inv} = H^1(\mathcal{H}, \hat{\Theta}'_\omega)^{inv}$ in Lemma 5.4, which may hold unconditional. In fact, we conjecture that for any $i \geq 0$ and ω,

$$H^i(\widetilde{\mathcal{R}}'^{ss}_\omega, \hat{\Theta}'_\omega)^{inv} = H^i(\mathcal{H}, \hat{\Theta}'_\omega)^{inv}.$$

If the conjecture is true, $H^i(\mathcal{P}^L_\omega, \Theta^L_{\mathcal{P},\omega}) = 0$ holds unconditional for $i > 0$, which implies that $H^i(\mathcal{P}_\omega, \Theta_{\mathcal{P},\omega}) = 0$ for $i > 0$.

6. GENERALIZED PARABOLIC SHEAVES ON REDUCIBLE NODAL CURVES

A natural idea to prove a vanishing theorem $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X,\omega}) = 0$ for $X = X_1 \cup X_2$ is to extend above method to reducible curves. In this section, we give estimates of various codimension and compute canonical line bundle of moduli space of generalized parabolic sheaves on a reducible curve. However, the estimate is not good enough to prove a vanishing theorem via the method in last section.

Let χ_1 and χ_2 be integers such that $\chi_1 + \chi_2 - r = \chi$, and fix, for $i = 1, 2$, the polynomials $P_i(m) = c_i rm + \chi_i$ and $\mathcal{W}_i = \mathcal{O}_{X_i}(-N)$ where $\mathcal{O}_{X_i}(1) = \mathcal{O}(1)|_{X_i}$ has degree c_i. Write $V_i = \mathbb{C}^{P_i(N)}$ and consider the Quot schemes $Quot(V_i \otimes \mathcal{W}_i, P_i)$, let $\widetilde{\mathbf{Q}}_i$ be the closure of the open set

$$\mathbf{Q}_i = \left\{ \begin{array}{l} V_i \otimes \mathcal{W}_i \to E_i \to 0, \text{ with } H^1(E_i(N)) = 0 \text{ and} \\ V_i \to H^0(E_i(N)) \text{ induces an isomorphism} \end{array} \right\},$$

we have the universal quotient $V_i \otimes \mathcal{W}_i \to \mathcal{F}^i \to 0$ on $X_i \times \widetilde{\mathbf{Q}}_i$ and the relative flag scheme

$$\mathcal{R}_i = \underset{x \in I_i}{\times_{\widetilde{\mathbf{Q}}_i}} Flag_{\vec{n}(x)}(\mathcal{F}^i_x) \to \widetilde{\mathbf{Q}}_i.$$

Let $\mathcal{F} = \mathcal{F}^1 \oplus \mathcal{F}^2$ denote direct sum of pullbacks of $\mathcal{F}^1$, $\mathcal{F}^2$ on

$$\widetilde{X} \times (\widetilde{\mathbf{Q}}_1 \times \widetilde{\mathbf{Q}}_2) = (X_1 \times \widetilde{\mathbf{Q}}_1) \sqcup (X_2 \times \widetilde{\mathbf{Q}}_2).$$

Let $\mathcal{E}$ be the pullback of $\mathcal{F}$ to $\widetilde{X} \times (\mathcal{R}_1 \times \mathcal{R}_2)$, and

$$\rho : \widetilde{\mathcal{R}}' := Grass_r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}) \to \widetilde{\mathcal{R}} := \mathcal{R}_1 \times \mathcal{R}_2 \to \widetilde{\mathbf{Q}} := \widetilde{\mathbf{Q}}_1 \times \widetilde{\mathbf{Q}}_2.$$

When m is large enough, we have a G-equivariant embedding

$$\widetilde{\mathcal{R}}' \hookrightarrow \mathbf{G}' = Grass_{\widetilde{P}(m)}(\widetilde{V} \otimes W_m) \times \mathbf{Flag} \times Grass_r(\widetilde{V} \otimes W_m).$$

For $\omega = (r, \chi_1, \chi_2, \{\vec{n}(x), \vec{a}(x)\}_{x \in I}, \mathcal{O}(1), k)$, give $\mathbf{G}'$ polarization

$$\frac{\ell + kcN}{c(m-N)} \times \prod_{x \in I} \{d_1(x), \cdots, d_{l_x}(x)\} \times k. \tag{6.1}$$

where $I = I_1 \cup I_2$, $d_i(x) = a_{i+1}(x) - a_i(x)$, $r_i(x) = n_1(x) + \cdots + n_i(x)$,

$$\ell = \frac{k\chi - \sum_{x \in I} \sum_{i=1}^{l_x} d_i(x) r_i(x)}{r}.$$

Let $\mathcal{H} \subset \widetilde{\mathcal{R}}'$ be the open set defined in Notation 2.21, $\widetilde{\mathcal{R}}'^{ss}_{\omega} \subset \mathcal{H}$ be the open set of GIT semi-stable points (respect to the polarization). Let

$$\psi : \widetilde{\mathcal{R}}'^{ss}_{\omega} \to \mathcal{P}_{\omega} := \widetilde{\mathcal{R}}'^{ss}_{\omega} // G.$$

If $\mathcal{O}(1)|_{X_j} = \mathcal{O}_{X_j}(c_j y_j)$, the restriction of polarization (6.1) to $\mathcal{H}$ is

$$\hat{\Theta}'_{\mathcal{H}} = \rho^*(\hat{\Theta}_{\mathcal{R}_1} \boxtimes \hat{\Theta}_{\mathcal{R}_2}) \otimes \det(\mathcal{Q})^k$$

where (for $j = 1, 2$, $\pi_{\mathcal{R}_j} : X_j \times \mathcal{R}_j \to \mathcal{R}_j$ is projection) we have

$$\hat{\Theta}_{\mathcal{R}_j} = (\det R\pi_{\mathcal{R}_j}\mathcal{E}^j)^{-k} \otimes \bigotimes_{x \in I_j} \left\{ \bigotimes_{i=1}^{l_x} \det(\mathcal{Q}_{\{x\}\times\mathcal{R}_j, i})^{d_i(x)} \right\} \otimes (\det\mathcal{E}^j_{y_j})^{\frac{c_j \ell}{c_1+c_2}}$$

where we assume that ℓ and $\ell_j := \frac{c_j \ell}{c_1+c_2}$ are integers. The sequence

$$0 \to \mathcal{F} \to (\pi \times id)_*\mathcal{E} \to {}_{x_0}\mathcal{Q} \to 0$$

on $X \times \widetilde{\mathcal{R}}'^{ss}_{\omega}$ defines a morphism $\hat{\phi} : \widetilde{\mathcal{R}}'^{ss}_{\omega} \to \mathcal{U}_X$ such that

$$\hat{\phi}^*(\Theta_{\mathcal{U}_X}) = \det R\pi_{\widetilde{\mathcal{R}}'^{ss}_{\omega}}(\mathcal{F})^{-k} \otimes \bigotimes_{x \in I} \left\{ \bigotimes_{i=1}^{l_x} \det(\mathcal{Q}_{\{x\}\times\widetilde{\mathcal{R}}'^{ss}_{\omega}, i})^{d_i(x)} \right\}$$
$$\otimes (\det\mathcal{F}_{y_1})^{\ell_1} \otimes (\det\mathcal{F}_{y_2})^{\ell_2} = \hat{\Theta}'_{\widetilde{\mathcal{R}}'^{ss}_{\omega}}.$$

Clearly, $\hat{\phi}$ induces a morphism $\phi : \mathcal{P}_\omega \to \mathcal{U}_X$ such that $\hat{\phi} = \phi \cdot \psi$. Thus $\hat{\Theta}'_{\widetilde{\mathcal{R}}'^{ss}_{\omega}}$ descends to an ample line bundle $\Theta_{\mathcal{P}_\omega} = \phi^*(\Theta_{\mathcal{U}_X})$ on $\mathcal{P}_\omega$. Similarly, $\phi^* : H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \hookrightarrow H^1(\mathcal{P}_\omega, \Theta_{\mathcal{P}_\omega})$ is injective. To prove

$$H^1(\mathcal{P}_\omega, \Theta_{\mathcal{P}_\omega}) = 0,$$

we need as before to compute canonical bundle $\omega_{\mathcal{P}_\omega}$ and to estimate the codimension of non-semistable points. However, the situation is slightly different with the case when $\widetilde{X}$ is connected. We firstly figure out some necessary conditions when $(E, Q) \in \widetilde{\mathcal{R}}'^{ss}_{\omega}$.

For $(E, E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0) \in \mathcal{H}$, $F = (F_1, F_2) \subset E = (E_1, E_2)$, let

$$D_m(F) := r(F)\frac{par\chi_m(E) - r}{r} - (par\chi_m(F) - t)$$

$$D(F) := \left(r_1\frac{par\chi(E_1)}{r} - par\chi(F_1)\right) + \left(r_2\frac{par\chi(E_2)}{r} - par\chi(F_2)\right)$$

where $t = \dim(Q^F)$, $Q^F = q(F_{x_1} \oplus F_{x_2}) \subset Q$, $r_i = \mathrm{rk}(F_i)$. Then

$$\begin{aligned} D_m(F) &= D(F) + \frac{(r_1 - r_2)}{r}\left(D_m(E_1) - \dim(Q^{E_1})\right) + t - r_2 \\ &= D(F) + \frac{(r_2 - r_1)}{r}\left(D_m(E_2) - \dim(Q^{E_2})\right) + t - r_1. \end{aligned} \tag{6.2}$$

Lemma 6.1. *For* $(E, Q) \in \widetilde{\mathcal{R}}'^{ss}_{\omega}$, *let* $E_j = E'_j \oplus {}_{x_j}\mathbb{C}^{s_j}$ *and*

$$n^{\omega}_j = \frac{1}{k}\left(r\ell_j + \sum_{x\in I_j}\sum_{i=1}^{l_x} d_i(x)r_i(x)\right) \quad (j = 1, 2).$$

Then, for the fixed $\chi_j := \chi(E_j)$ $(j = 1, 2)$, *we have*

1. $n^{\omega}_j \le \chi_j \le n^{\omega}_j + r$ $\;(j = 1, 2)$,
2. $s_1 \le n^{\omega}_2 + r - \chi_2$, $\; s_2 \le n^{\omega}_1 + r - \chi_1$,
3. *let* $(E, Q) \in \mathcal{H} \setminus \{\mathcal{D}^f_1 \cup \mathcal{D}^f_2\}$ *with* $n^{\omega}_j \le \chi(E_j) \le n^{\omega}_j + r$, *then*
$$E_1 \in \mathcal{R}^{ss}_1,\ E_2 \in \mathcal{R}^{ss}_2 \ \Rightarrow\ (E, Q) \in \widetilde{\mathcal{R}}'^{ss}_{\omega}.$$
Moreover, when $n^{\omega}_1 < \chi_1 < n^{\omega}_1 + r$, *we have* $(E, Q) \in \widetilde{\mathcal{R}}'^{s}_{\omega}$ *if one of* E_1, E_2 *is a stable parabolic bundle,*
4. *let* $(E, Q) \in \mathcal{H} \setminus \{\mathcal{D}^f_1 \cup \mathcal{D}^f_2\}$, *if* $\chi_1 = n^{\omega}_1 + r$ *or* $\chi_1 = n^{\omega}_1$, *then*
$$(E, Q) \in \widetilde{\mathcal{R}}'^{ss}_{\omega} \ \Rightarrow\ E_1 \in \mathcal{R}^{ss}_1,\ E_2 \in \mathcal{R}^{ss}_2.$$

Proof. Note that $\chi_1 + \chi_2 = \chi + r$ and $n^{\omega}_1 + n^{\omega}_2 = \chi$, (1) and (2) are clear by the following formulas $(j = 1, 2)$

$$\chi(E_j) = n^{\omega}_j + \dim(Q^{E_j}) - D_m(E_j)$$
$$\chi(E_1) + s_2 = n^{\omega}_1 + \dim(Q^{E^s_1}) - D_m(E^s_1)$$
$$\chi(E_2) + s_1 = n^{\omega}_2 + \dim(Q^{E^s_2}) - D_m(E^s_2)$$

where $E^s_1 = (E_1, {}_{x_2}\mathbb{C}^{s_2})$, $E^s_2 = ({}_{x_1}\mathbb{C}^{s_1}, E_2)$. The formula (6.2) becomes

$$\begin{aligned} D_m(F) &= D(F) + \frac{r_2 - r_1}{r}(\chi_1 - n^{\omega}_1) + \dim(Q^F) - r_2 \\ &= D(F) + \frac{r_1 - r_2}{r}(\chi_2 - n^{\omega}_2) + \dim(Q^F) - r_1. \end{aligned} \tag{6.3}$$

To prove (3), by (6.3) and $\dim(Q^F) - r_j \ge 0$ $(j = 1, 2)$, we have $D_m(F) \ge 0$ whenever $D(F) \ge 0$. Thus

$$E_1 \in \mathcal{R}^{ss}_1,\ E_2 \in \mathcal{R}^{ss}_2 \ \Rightarrow\ (E, Q) \in \widetilde{\mathcal{R}}'^{ss}_{\omega}.$$

When $n^{\omega}_1 < \chi_1 < n^{\omega}_1 + r$ (which implies $n^{\omega}_2 < \chi_2 < n^{\omega}_2 + r$), we have $D_m(F) > D(F) \ge 0$ if $r_1 \ne r_2$. Thus $(E, Q) \in \widetilde{\mathcal{R}}'^{s}_{\omega}$ if one of E_1, E_2 is a stable parabolic bundle.

To prove (4), if $\chi_1 = n^{\omega}_1 + r$ or $\chi_1 = n^{\omega}_1$, the formula (6.3) becomes

$$D_m(F) = D(F) + \dim(Q^F) - r_1. \tag{6.4}$$

For $F_1 \subset E_1$ of rank r_1, take $F = (F_1, 0) \subset E$ in (6.4), we have

$$D_m(F) = D(F) = r_1\frac{par\chi(E_1)}{r} - par\chi(F_1)$$

which implies that $E_1 \in \mathcal{R}_1^{ss}$ if $(E,Q) \in \widetilde{\mathcal{R}}'^{ss}_\omega$. For $F_2 \subset E_2$ of rank r_2, take $F = (E_1, F_2) \subset E$ in (6.4), we have

$$D_m(F) = D(F) = r_2 \frac{par\chi(E_2)}{r} - par\chi(F_2)$$

which implies that $E_2 \in \mathcal{R}_2^{ss}$ if $(E,Q) \in \widetilde{\mathcal{R}}'^{ss}_\omega$. □

Notation 6.2. For $\omega = (r, \chi_1, \chi_2, \{\vec{n}(x), \vec{a}(x)\}_{x\in I}, \mathcal{O}(1), k)$, let

$$\mathcal{H}^\omega = \left\{ \begin{array}{l} (E,Q) \in \mathcal{H}, \text{ with } n_j^\omega \leq \chi(E_j) = \chi_j \leq n_j^\omega + r \ (j = 1,\, 2), \text{ and} \\ \dim(\mathrm{Tor}(E_1)) \leq n_2^\omega + r - \chi_2, \ \ \dim(\mathrm{Tor}(E_2)) \leq n_1^\omega + r - \chi_1 \end{array} \right\}.$$

Proposition 6.3. *Let $\mathcal{D}_1^f = \hat{\mathcal{D}}_1 \cup \hat{\mathcal{D}}_1^t$ and $\mathcal{D}_2^f = \hat{\mathcal{D}}_2 \cup \hat{\mathcal{D}}_2^t$. Then*

(1) $\mathrm{codim}(\mathcal{H}^\omega \setminus \widetilde{\mathcal{R}}'^{ss}_\omega) > \min\limits_{1\leq i\leq 2} \left\{(r-1)(g_i - \frac{r+3}{4}) + \frac{|I_i|}{k}\right\}$.

(2) $\mathrm{codim}(\widetilde{\mathcal{R}}'^{ss}_\omega \setminus \{\mathcal{D}_1^f \cup \mathcal{D}_2^f\} \setminus \widetilde{\mathcal{R}}'^{s}_\omega) > \min\limits_{1\leq i\leq 2} \left\{(r-1)(g_i - 1) + \frac{|I_i|}{k}\right\}$ *when $n_1^\omega < \chi_1 < n_1^\omega + r$.*

(3) $\mathrm{codim}(\widetilde{\mathcal{R}}'^{ss}_\omega \setminus \{\mathcal{D}_1^f \cup \mathcal{D}_2^f\} \setminus \widetilde{\mathcal{R}}'^{-s}_\omega) \geq \min\limits_{1\leq i\leq 2} \left\{(r-1)(g_i - 1) + \frac{|I_i|}{k}\right\}$ *when $\chi_1 = n_1^\omega$ or $n_1^\omega + r$, where*

$$\widetilde{\mathcal{R}}'^{-s}_\omega := \left\{ \begin{array}{l} (E,Q) \in \widetilde{\mathcal{R}}'^{ss}_\omega \textit{ satisfies } par\mu(F) < par\mu(E) \textit{ for any} \\ \textit{nontrivial } F \subset E \textit{ of rank } (r_1, r_2) \neq (0, r) \textit{ or } (r, 0) \end{array} \right\}.$$

Proof. To prove (1), let $(E,Q) \in \mathcal{H}^\omega \setminus \widetilde{R}'^{ss}_\omega$ with $E = (E_1, E_2)$, then there exists a $F = (F_1, F_2) \subset E$ such that E/F is torsion free and

$$par\chi_m(F) - dim(Q^F) > r(F) \frac{par\chi_m(E) - r}{r}. \tag{6.5}$$

Let $t = dim(Q^F)$, $r_i = rk(F_i)$, $m_i(x) = dim \frac{F_x \cap F_{i-1}(E)_x}{F_x \cap F_i(E)_x}$, $\chi_i = \chi(E_i)$

$$m(F) = \frac{r(F) - r_1}{k} \sum_{x\in I_1} a_{l_x+1}(x) + \frac{r(F) - r_2}{k} \sum_{x\in I_2} a_{l_x+1}(x)$$

where $r(F) = \frac{c_1 r_1 + c_2 r_2}{c_1 + c_2}$. Then we can rewrite (6.5) as

$$\begin{aligned} r\chi(F) - r(F)\chi > & rt - rm(F) + \frac{r(F)}{k} \sum_{x\in I} \sum_{i=1}^{l_x+1} a_i(x) n_i(x) \\ & - \frac{r}{k} \sum_{x\in I} \sum_{i=1}^{l_x+1} a_i(x) m_i(x) \end{aligned} \tag{6.6}$$

$$0 \to F \to E \to E/F := \widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2) \to 0$$

Write $E = E' \oplus {}_{x_1}\mathbb{C}^{s_1} \oplus {}_{x_2}\mathbb{C}^{s_2}$, $F = F' \oplus {}_{x_1}\mathbb{C}^{s_1} \oplus {}_{x_2}\mathbb{C}^{s_2}$ and $F_1 = F_1' \oplus {}_{x_1}\mathbb{C}^{s_1}$, $F_2 = F_2' \oplus {}_{x_2}\mathbb{C}^{s_2}$ where E', F' (thus F_1', F_2') are torsion free sheaves satisfying the exact sequences

$$0 \to F_1' \to E_1' \to \widetilde{F}_1 \to 0, \quad 0 \to F_2' \to E_2' \to \widetilde{F}_2 \to 0.$$

Let $d_i = \deg(F_i')$, $r_i = \mathrm{rk}(F_i')$, $\deg(\widetilde{F}_i) = \chi_i - r(1-g_i) - d_i - s_i$ and

$$P_i(m) = c_i r_i m + d_i + r_i(1-g_i), \quad \widetilde{P}_i(m) = c_i r m + \chi_i - s_i - P_i(m).$$

For $\mathcal{W}_i = \mathcal{O}_{X_i}(-N)$, $V_i = \mathbb{C}^{P_i(N)}$ (resp. $\widetilde{V}_i = \mathbb{C}^{\widetilde{P}_i(N)}$), let

$$Q_i \subset Quot(V_i \otimes \mathcal{W}_i, P_i)$$

(resp. $\widetilde{Q}_i \subset Quot(\widetilde{V}_i \otimes \mathcal{W}_i, \widetilde{P}_i)$) be the open set of locally free quotients F_i' (resp. $\widetilde{F}_i$) with vanishing $H^1(F_i'(N))$ (resp. $H^1(\widetilde{F}_i(N))$) and $F_i'(N)$ (resp. $\widetilde{F}_i(N)$) generated by global sections. Let $\mathcal{F}_i'$ (resp. $\widetilde{F}_i$) be the universal quotient on $X_i \times Q_i$ (resp. on $X_i \times \widetilde{Q}_i$), let $\mathcal{V}_i = Q_i \times \widetilde{Q}_i$ and $\mathcal{G}_i = \widetilde{F}_i^{\vee} \otimes \mathcal{F}_i'$ on $X_i \times \mathcal{V}_i$. Then we have

$$\mathcal{V}_i = \bigcup_{h_i \geq 0} \mathcal{V}_i^{h_i}$$

such that $R^1 f_{i*}(\mathcal{G}_i)$ is locally free of rank h_i on $\mathcal{V}_i^{h_i}$ where $f_i : X_i \times \mathcal{V}_i \to \mathcal{V}_i$ is the projection. Let $P_{h_i} = \mathbb{P}(R^1 f_{i*}(\mathcal{G}_i)^{\vee}) \to \mathcal{V}_i^{h_i}$ be the projective bundle on $\mathcal{V}_i$ and $0 \to \mathcal{F}_i' \otimes \mathcal{O}_{P_{h_i}}(-1) \to \mathcal{E}_i'(h_i) \to \widetilde{\mathcal{F}}_i \to 0$ be the universal extension on $X_i \times P_{h_i}$ (we set $P_{h_i} = \mathcal{V}_i$ and $\mathcal{E}_i'(h_i) = \mathcal{F}_i' \oplus \widetilde{\mathcal{F}}_i$ if $h_i = 0$). For $v_i' = (d_i, r_i, \{m_1(x), \cdots, m_{l_x+1}(x)\}_{x \in I_i}, h_i)$, we can define a variety $X(v_i') \to P_{h_i}$. It parametrises a family of parabolic bundles E_i', which occur as extensions $0 \to F_i' \to E_i' \to \widetilde{F}_i \to 0$ (the extension being split if $h_i = 0$), with parabolic structures at $x \in I_i$ of type $\vec{n}(x) = n_1(x), \cdots, n_{l_x+1}(x))$, whose induced parabolic structures on F_i' are of type $(m_1(x), \cdots, m_{l_x+1}(x))$ (we will forget $m_j(x)$ if it is zero). Let $0 \to \mathcal{F}_i'(-1) \to \mathcal{E}'(v_i') \to \widetilde{\mathcal{F}}_i \to 0$ be the pull back of universal extension to $X_i \times X(v_i')$, $\mathcal{E}(v_i') = \mathcal{E}'(v_i') \oplus {}_{x_i}\mathcal{O}^{s_i}$ and let $F(v_i')$ be the frame bundle of the direct image of $\mathcal{E}(v_i')(N)$ (under the projection $X_i \times X(v_i') \to X(v_i')$). Write $\mathcal{E}(v') := \mathcal{E}(v_1') \oplus \mathcal{E}(v_2')$, we consider

$$G_{v'} := Grass_r(\mathcal{E}(v')_{x_1} \oplus \mathcal{E}(v')_{x_2}) \to F(v_1') \times F(v_2')$$

and define a subvariety of $G_{v'}$ by

$$X(v) := \left\{ \begin{array}{l} (E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0) \in G_{v'}), \;\; ker(q) \cap (\mathbb{C}^{s_1} \oplus \mathbb{C}^{s_2}) = 0, \\ dim(ker(q) \cap (F'_{x_1} \oplus \mathbb{C}^{s_1} \oplus F'_{x_2} \oplus \mathbb{C}^{s_2})) = r_1 + r_2 + s - t \end{array} \right\}.$$

Then $X(v)$ parametrises a family of GPS ($E = E' \oplus {}_{x_1}\mathbb{C}^{s_1} \oplus {}_{x_2}\mathbb{C}^{s_2}$, Q), where $E' = (E_1', E_2')$ occurs as extensions $0 \to F_i' \to E_i' \to \widetilde{F}_i \to 0$ (it is split if $h_i = 0$) with parabolic structures at $x \in I$ of type $\vec{n}(x)$, whose induced parabolic structures on F'_i are of type $(m_1(x), \cdots, m_{l_x+1}(x))$ (we will forget $m_i(x)$ if it is zero), such that ${}_{x_1}\mathbb{C}^{s_1} \oplus {}_{x_2}\mathbb{C}^{s_2} \to Q$ is injective and $\mathrm{rank}(F'_{x_1} \oplus \mathbb{C}^{s_1} \oplus F'_{x_2} \oplus \mathbb{C}^{s_2} \to Q) = t$. There is a morphism $X(v) \to \mathcal{H}^\omega \setminus \widetilde{\mathcal{R}}^{ss}_\omega$ whose image contains (E, Q). Therefore we have a (countable) number of quasi-projective varieties $X(v)$ and morphisms $\varphi_v : X(v) \to \mathcal{H}^\omega \setminus \widetilde{\mathcal{R}}^{ss}_\omega$ such that the union of the images covers $\mathcal{H}^\omega \setminus \widetilde{\mathcal{R}}^{ss}_\omega$.

One computes $\dim F(v'_i) = \dim X(v'_i) + (c_i rN + \chi_i)^2$,

$$\dim X(v'_i) = \begin{cases} \sum_{x\in I_i} \dim X_{v_i(x)} + h_i - 1 + \dim Q_i + \dim \widetilde{Q}_i, & \text{if } h_i \neq 0 \\ \sum_{x\in I_i} \dim X_{v_i(x)} + \dim Q_i + \dim \widetilde{Q}_i & \text{if } h_i = 0 \end{cases}$$

$\dim Q_i + \dim \widetilde{Q}_i = (g_i - 1)(r_i^2 + (r - r_i)^2) + P_i(N)^2 + \widetilde{P}_i(N)^2$ and the dimension of $\mathcal{H}$, $X(v)$ are (let $s = s_1 + s_2$):

$$r^2(g-2) + r^2 + \sum_{i=1}^{2}(c_i rN + \chi_i)^2 + \sum_{x\in I} dim\, Flag_{\vec{n}(x)}(\mathcal{F}_x),$$

$$r(r+s) - (r-t)(r_1 + r_2 + s - t) + \sum_{i=1}^{2}(c_i rN + \chi_i)^2 + \sum_{i=1}^{2} \dim X(v'_i).$$

To estimate the minimum e of fiber dimension of φ_v, note that

$$\dim Aut(E) \geq \dim Aut(E'_1) + \dim Aut(E'_2) + rs + s_1^2 + s_2^2$$

and $0 \to F'_i \to E'_i \to \widetilde{F}_i \to 0$, we have

$$\dim Aut(E'_i) \geq \begin{cases} 1 + h^0(\widetilde{F}_i^\vee \otimes F'_i), & \text{if } h_i \neq 0 \\ 2 + h^0(\widetilde{F}_i^\vee \otimes F'_i) & \text{if } h_i = 0 \end{cases}$$

Define $e(h_i) = 1$ when $h_i \neq 0$ and $e(h_i) = 2$ when $h_i = 0$, then

$$\begin{aligned} e \geq & rs + s_1^2 + s_2^2 + h^0(\widetilde{F}_1^\vee \otimes F'_1) + h^0(\widetilde{F}_2^\vee \otimes F'_2) + e(h_1) \\ & + e(h_2) - 4 + P_1(N)^2 + \widetilde{P}_1(N)^2 + P_2(N)^2 + \widetilde{P}_2(N)^2. \end{aligned}$$

Then the codimension of $\mathcal{H}^\omega \setminus \widetilde{\mathcal{R}}^{ss}_\omega$ is bounded below by

$$\sum_{i=1}^{2} r_i(r - r_i)(g_i - 1) + \sum_{i=1}^{2}(r_i + s_i - t)s_i + (r-t)(r_1 + r_2 - t) +$$

$$r\chi(F) - (r_1\chi_1 + r_2\chi_2) + \sum_{x\in I_1}\sum_{j=1}^{l_x+1}(r_1 - \sum_{i=1}^{j} m_i(x))(n_j(x) - m_j(x))$$

$$+ \sum_{x\in I_2}\sum_{j=1}^{l_x+1}(r_2 - \sum_{i=1}^{j} m_i(x))(n_j(x) - m_j(x)).$$

If $r_1 \geq r_2$, use $\chi_1 + s_2 \leq n_1^\omega + r$ and $\chi_2 = \chi + r - \chi_1$ to get

$$
\begin{aligned}
r\chi(F) - (r_1\chi_1 + r_2\chi_2) \geq{}& r\chi(F) - r(F)\chi + rm(F) - \\
& r_1 r + (r_1 - r_2)s_2 + \frac{r_1 - r(F)}{k}\sum_{x\in I_1}\sum_{i=1}^{l_x+1} a_i(x)n_i(x) \\
& + \frac{r_2 - r(F)}{k}\sum_{x\in I_2}\sum_{i=1}^{l_x+1} a_i(x)n_i(x).
\end{aligned}
\tag{6.7}
$$

Similarly, if $r_2 \geq r_1$, we have

$$
\begin{aligned}
r\chi(F) - (r_1\chi_1 + r_2\chi_2) \geq{}& r\chi(F) - r(F)\chi + rm(F) - \\
& r_2 r + (r_2 - r_1)s_1 + \frac{r_1 - r(F)}{k}\sum_{x\in I_1}\sum_{i=1}^{l_x+1} a_i(x)n_i(x) \\
& + \frac{r_2 - r(F)}{k}\sum_{x\in I_2}\sum_{i=1}^{l_x+1} a_i(x)n_i(x).
\end{aligned}
\tag{6.8}
$$

By using of the inequalities (6.6), (6.7) and (6.8), we have

$$
\begin{aligned}
codim(\mathcal{H}^\omega \setminus \widetilde{\mathcal{R}}'^{ss}_\omega) >{}& \sum_{i=1}^{2} r_i(r - r_i)(g_i - 1) + (\max\{r_1, r_2\} - t)s \\
& + s_1^2 + s_2^2 + r\cdot\min\{r_1, r_2\} - t(r_1 + r_2 - t) \\
& + \sum_{x\in I_1}\left\{\begin{array}{l} \displaystyle\sum_{j=1}^{l_x+1}(r_1 - \sum_{i=1}^{j} m_i(x))(n_j(x) - m_j(x)) \\ \displaystyle + \sum_{j=1}^{l_x+1}(r_1 n_j(x) - rm_j(x))\frac{a_j(x)}{k} \end{array}\right\} \\
& + \sum_{x\in I_2}\left\{\begin{array}{l} \displaystyle\sum_{j=1}^{l_x+1}(r_2 - \sum_{i=1}^{j} m_i(x))(n_j(x) - m_j(x)) \\ \displaystyle + \sum_{j=1}^{l_x+1}(r_1 n_j(x) - rm_j(x))\frac{a_j(x)}{k} \end{array}\right\},
\end{aligned}
$$

where $s = s_1 + s_2$. Let $f(r_1, r_2, s_1, s_2, t)$ denote

$$(\max\{r_1, r_2\} - t)s + s_1^2 + s_2^2 + r \cdot \min\{r_1, r_2\} - t(r_1 + r_2 - t) =$$
$$(t - \frac{r_1 + r_2 + s}{2})^2 + \frac{2(s_1^2 + s_2^2) + (s_1 - s_2)^2}{4} + \frac{\max\{r_1, r_2\} - \min\{r_1, r_2\}}{2}s$$
$$+ \min\{r_1, r_2\}(r - \max\{r_1, r_2\}) - \frac{(r_1 - r_2)^2}{4}.$$

When $r_1 = r_2$, it is clear that $f(r_1, r_2, s_1, s_2, t) \geq r_1(r - r_1)$ and we have

$$codim(\mathcal{H}^\omega \setminus \widetilde{\mathcal{R}}'^{ss}_\omega) > r_1(r - r_1)(g - 1) + \frac{|I|}{k}.$$

In general, we have only $f(r_1, r_2, s_1, s_2, t) \geq -\frac{(r-1)^2}{4}$ and

$$\mathrm{codim}(\mathcal{H}^\omega \setminus \widetilde{\mathcal{R}}'^{ss}_\omega) > \min_{1 \leq i \leq 2} \left\{(r - 1)(g_i - \frac{r + 3}{4}) + \frac{|I_i|}{k}\right\}.$$

To prove (2), note $s_1 = s_2 = 0$, $\max\{r_1, r_2\} \leq t$ for $(E, Q) \in \widetilde{\mathcal{R}}'^{ss}_\omega \setminus \{\mathcal{D}_1^f \cup \mathcal{D}_2^f\}$, we have $f(r_1, r_2, s_1, s_2, t) = r \cdot \min\{r_1, r_2\} + t(t - r_1 - r_2) \geq 0$. Then, when $n_1^\omega < \chi_1 < n_1^\omega + r$, which implies $(r_1, r_2) \neq (r, 0)$, $(0, r)$,

$$\mathrm{codim}(\widetilde{\mathcal{R}}'^{ss}_\omega \setminus \{\mathcal{D}_1^f \cup \mathcal{D}_2^f\} \setminus \widetilde{\mathcal{R}}'^{s}_\omega) > \min_{1 \leq i \leq 2} \left\{(r - 1)(g_i - 1) + \frac{|I_i|}{k}\right\}.$$

The assertion (3) follows the same arguments of (2) and the definition of $\widetilde{\mathcal{R}}'^{-s}_\omega$. In fact, $\widetilde{\mathcal{R}}'^{-s}_\omega = \rho^{-1}(\mathcal{R}_1^s \times \mathcal{R}_2^s)$ by Lemma 6.1 (4), where

$$\rho : \widetilde{\mathcal{R}}'^{ss}_\omega \setminus \{\mathcal{D}_1^f \cup \mathcal{D}_2^f\} \to \mathcal{R}_1^{ss} \times \mathcal{R}_2^{ss}.$$

□

The schemes $\mathcal{H}$ and $\mathcal{P}$ are Gorenstein, so they have canonical sheaves. To compute the canonical sheaves $\omega_{\mathcal{H}}$ and $\omega_{\mathcal{P}}$, let

$$(6.9) \qquad 0 \to \mathcal{K}^j \to V_j \otimes \mathcal{O}_{X_j \times \mathcal{R}_j}(-N) \to \mathcal{E}^j \to 0 \quad (j = 1,\, 2)$$

be the universal quotient on $X_j \times \mathcal{R}_j$ ($\mathcal{K}^j$ are in fact locally free), and

$$\omega^{-1}_{\mathcal{R}_j} = (det\, R\pi_{\mathcal{R}_j}\mathcal{E}^j)^{-2r} \otimes \bigotimes_{x \in I_j} \left\{(det\, \mathcal{E}^j_x)^{n_{l_x+1}(x) - r} \otimes \bigotimes_{i=1}^{l_x} (det\, \mathcal{Q}_{x,i})^{n_i(x) + n_{i+1}(x)}\right\}$$
$$\otimes \bigotimes_{q \in X_j} (det\, \mathcal{E}^j_q)^{1-r} \otimes (det\, R\pi_{\mathcal{R}_j} det \mathcal{E}^j)^2$$

where $\omega_{X_j} = \mathcal{O}_{X_j}(\sum_{q \in X_j} q)$. Let $\hat{\mathrm{Det}}_j : \mathcal{R}_j \to J^{d_j}_{X_j}$, where $d_j = \chi_j + r(g_j - 1)$, be defined by $\mathrm{det}\mathcal{E}^j := (\mathrm{det}\mathcal{K}^j)^{-1} \otimes \mathcal{O}_{X_j \times \mathcal{R}_j}(-P_j(N)N)$, let

$\mathcal{L}_j$ be a universal line bundle on $X_j \times J^{d_j}_{X_j}$ and

$$\Theta_{J^{d_j}_{X_j}} = (\det R\pi_{J^{d_j}_{X_j}}\mathcal{L}_j)^{-2} \otimes (\mathcal{L}_j)^r_{x_j} \otimes \bigotimes_{q\in X_j}(\mathcal{L}_j)^{r-1}_q \otimes (\mathcal{L}_j)^{2\chi_j - r}_{y_j} \tag{6.10}$$

(which are independent of the choices of $\mathcal{L}_j$). Let

$$\hat{\mathrm{Det}}_{\widetilde{\mathcal{R}}} := (\hat{\mathrm{Det}}_1, \hat{\mathrm{Det}}_2) : \widetilde{\mathcal{R}} = \mathcal{R}_1 \times \mathcal{R}_2 \to J^d_{\widetilde{X}} := J^{d_1}_{X_1} \times J^{d_2}_{X_2},$$

which induces $\hat{\mathrm{Det}}_{\mathcal{H}} : \mathcal{H} \to J^d_{\widetilde{X}}$ and $\mathrm{Det} : \mathcal{P}_\omega \to J^d_{\widetilde{X}}$ such that

$$\begin{array}{ccc} \mathcal{H} & & \\ \downarrow{\scriptstyle \rho} & \searrow^{\hat{\mathrm{Det}}_{\mathcal{H}}} & \\ \widetilde{\mathcal{R}} & \xrightarrow[\hat{\mathrm{Det}}_{\widetilde{\mathcal{R}}}]{} & J^d_{\widetilde{X}} \end{array} \qquad \begin{array}{ccc} \widetilde{\mathcal{R}}'^{ss}_\omega & & \\ \downarrow{\scriptstyle \psi} & \searrow^{\hat{\mathrm{Det}}_{\widetilde{\mathcal{R}}'^{ss}_\omega}} & \\ \mathcal{P}_\omega & \xrightarrow[\mathrm{Det}]{} & J^d_{\widetilde{X}} \end{array}$$

are commutative. Let $\Theta_{J^d_{\widetilde{X}}} = p_1^*\Theta_{J^{d_1}_{X_1}} \otimes p_2^*\Theta_{J^{d_2}_{X_2}}$ (where $p_j : J^d_{\widetilde{X}} := J^{d_1}_{X_1} \times J^{d_2}_{X_2} \to J^{d_j}_{X_j}$ are projections). Then similar arguments of [9] give

Proposition 6.4. *Let $\rho : \mathcal{H} \to \widetilde{\mathcal{R}} := \mathcal{R}_1 \times \mathcal{R}_2$ and $\mathcal{E}^1_{x_1} \oplus \mathcal{E}^2_{x_2} \to \mathcal{Q} \to 0$ be the universal quotient on $\mathcal{H}$. Then*

$$\begin{aligned} \omega^{-1}_{\mathcal{H}} = \rho^*(\omega^{-1}_{\mathcal{R}_1} \otimes \omega^{-1}_{\mathcal{R}_2}) \otimes (det\mathcal{Q})^{2r} \otimes (det\mathcal{K}^1_{x_1})^r \otimes (det\mathcal{K}^2_{x_2})^r = \\ (detR\pi_{\mathcal{H}}\mathcal{E})^{-2r} \otimes \bigotimes_{x\in I}\left\{(det\,\mathcal{E}_x)^{-r l_x(x)} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\} \\ \otimes (det\mathcal{Q})^{2r} \otimes \bigotimes_{j=1}^{2}(det\mathcal{E}_{y_j})^{2\chi_j - r} \otimes \hat{\mathrm{Det}}^*_{\mathcal{H}}(\Theta^{-1}_{J^d_{\widetilde{X}}}) = \hat{\Theta}'_{\omega^c} \otimes \hat{\mathrm{Det}}^*_{\mathcal{H}}(\Theta^{-1}_{J^d_{\widetilde{X}}}) \end{aligned}$$

where

$$\begin{aligned} \hat{\Theta}'_{\omega^c} =& (det\,R\pi_{\mathcal{H}}\mathcal{E})^{-2r} \otimes \bigotimes_{x\in I}\left\{\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\} \otimes det(\mathcal{Q})^{2r} \\ & \otimes (\det\mathcal{E}_{y_1})^{2\chi_1 - r} \otimes (\det\mathcal{E}_{y_2})^{2\chi_2 - r} \otimes \bigotimes_{x\in I}(det\,\mathcal{E}_x)^{-r l_x(x)}. \end{aligned}$$

Let $J_i \subset X_i \setminus (I_i \cup \{x_i\})$ be a subset, $J = J_1 \cup J_2$ and

$$\mathcal{R}(J)_i = \underset{x\in I_i\cup J_i}{\times_{\widetilde{\mathbf{Q}}_i}} Flag_{\vec{n}(x)}(\mathcal{F}^i_x) \to \widetilde{\mathbf{Q}}_i,$$

$\widetilde{\mathcal{R}}(J)=\mathcal{R}(J)_1\times\mathcal{R}(J)_2\xrightarrow{p_J}\widetilde{\mathcal{R}}=\mathcal{R}_1\times\mathcal{R}_2$ be the projection. Consider

$$\begin{array}{ccc}\widetilde{\mathcal{R}}(J)' & \xrightarrow{p_J} & \widetilde{\mathcal{R}}' \\ {\scriptstyle\rho}\big\downarrow & & {\scriptstyle\rho}\big\downarrow \\ \widetilde{\mathcal{R}}(J) & \xrightarrow{p_J} & \widetilde{\mathcal{R}}\end{array}$$

and $\mathcal{H}(J):=p_J^{-1}(\mathcal{H})\xrightarrow{p_J}\mathcal{H}$. Then, by Proposition 6.4, we have

$$\omega^{-1}_{\mathcal{H}(J)}=\hat{\Theta}'_{\omega^c(J)}\otimes\hat{\mathrm{Det}}^*_{\mathcal{H}(J)}(\Theta^{-1}_{J^d_{\widetilde{X}}}), \tag{6.11}$$

where

$$\begin{aligned}\hat{\Theta}'_{\omega^c(J)}=&(det\,R\pi_{\mathcal{H}(J)}\mathcal{E})^{-2r}\otimes\bigotimes_{x\in I\cup J}\left\{\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\}\otimes det(\mathcal{Q})^{2r}\\ &\otimes(\det\mathcal{E}_{y_1})^{2\chi_1-r}\otimes(\det\mathcal{E}_{y_2})^{2\chi_2-r}\otimes\bigotimes_{x\in I\cup J}(det\,\mathcal{E}_x)^{-r_{l_x}(x)}.\end{aligned}$$

Let $\omega^c(J)=(r,\chi_1,\chi_2,\{\{n_i(x)\}_{1\le i\le l_x+1},\,\{d^c_i(x)\}_{1\le i\le l_x}\}_{x\in I\cup J},\mathcal{O}(1),k^c)$ where $k^c=2r$, $d^c_i(x)=n_i(x)+n_{i+1}(x)$, let $\ell^c_j=2\chi_j-r-\sum_{x\in I_j\cup J_j}r_{l_x}(x)$ and $\ell^c=\ell^c_1+\ell^c_2=2\chi-\sum_{x\in I\cup J}r_{l_x}(x)$. Then

$$\sum_{x\in I\cup J}\sum_{i=1}^{l_x}d^c_i(x)r_i(x)+r\ell^c=k^c\chi.$$

The type $\{\vec{n}(x)\}_{x\in J}$ of flags at $x\in J$ will be chosen to satisfy

$$\ell^c_1=\frac{c_1}{c_1+c_2}\ell^c \tag{6.12}$$

which is equivalent to the following condition

$$\begin{aligned}&c_1\sum_{x\in J_2}r_{l_x}(x)-c_2\sum_{x\in J_1}r_{l_x}(x)=\\ &c_1\left(2\chi_2-r-\sum_{x\in I_2}r_{l_x}(x)\right)-c_2\left(2\chi_1-r-\sum_{x\in I_1}r_{l_x}(x)\right).\end{aligned} \tag{6.13}$$

The choices of $\{\vec{n}(x)\}_{x\in J}$ satisfying (6.12) for arbitrary large $|J_1|$ and $|J_2|$ are possible since the equation (6.13) has arbitrary large integer solutions. In this case, the line bundle $\hat{\Theta}'_{\omega^c(J)}$ is (algebraically) equivalent to the restriction (on $\mathcal{H}(J)$) of the following polarization

$$\frac{\ell^c+k^c cN}{c(m-N)}\times\prod_{x\in I\cap J}\{d^c_1(x),\cdots,d^c_{l_x}(x)\}\times k^c.$$

On the other hand, it is easy to compute that $n_j^{\omega^c(J)}=\chi_j-\frac{r}{2}$, thus

$$n_j^{\omega^c(J)}<\chi_j<n_j^{\omega^c(J)}+r \qquad (j=1,\ 2).$$

Moreover, for any polarization (6.1) (determined by ω), let $\hat{\Theta}'_{\mathcal{H}}$ be its restriction to $\mathcal{H}$. Then we can write

$$p_J^*(\hat{\Theta}'_{\mathcal{H}})=\omega_{\mathcal{H}(J)}\otimes\hat{\Theta}'_{\bar\omega}\otimes\hat{\mathrm{Det}}^*_{\mathcal{H}(J)}(\Theta^{-1}_{J^d_{\widetilde X}}),$$

where $\bar\omega=(r,\chi_1,\chi_2,\{\{n_i(x)\}_{1\le i\le l_x+1},\ \{\bar d_i(x)\}_{1\le i\le l_x}\}_{x\in I\cup J},\mathcal{O}(1),\bar k)$,

$$\begin{aligned}\hat{\Theta}'_{\bar\omega}=&(det\,R\pi_{\mathcal{H}(J)}\mathcal{E})^{-\bar k}\otimes\bigotimes_{x\in I\cup J}\left\{\bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{\bar d_i(x)}\right\}\otimes det(\mathcal{Q})^{\bar k}\\ &\otimes(\det\mathcal{E}_{y_1})^{\ell_1+2\chi_1-r}\otimes(\det\mathcal{E}_{y_2})^{\ell_2+2\chi_2-r}\otimes\bigotimes_{x\in I\cup J}(det\,\mathcal{E}_x)^{-r_{l_x}(x)},\end{aligned}$$

$\bar k=k+2r$, $\bar d_i(x)=d_i(x)+n_i(x)+n_{i+1}(x)$ ($d_i(x)=0$ for $x\in J$). Let

$$\bar\ell_j=\ell_j+2\chi_j-r-\sum_{x\in I_j\cup J_j}r_{l_x}(x)=\ell_j+\ell_j^c,$$

$$\bar\ell:=\bar\ell_1+\bar\ell_2=\ell+2\chi-\sum_{x\in I\cup J}r_{l_x}(x)=\ell+\ell^c.$$

Then it is easy to see that $\bar\ell_j=\frac{c_j}{c_1+c_2}\bar\ell$ (by (6.12)),

$$\sum_{x\in I\cup J}\sum_{i=1}^{l_x}\bar d_i(x)r_i(x)+r\bar\ell=\bar k\chi,$$

and $\hat{\Theta}'_{\bar\omega}$ is (algebraically) equivalent to the restriction of polarization determined by $\bar\omega$. The condition (6.12) implies the following identities

$$2r(\chi_j-n_j^{\bar\omega})=r^2+k(n_j^{\bar\omega}-n_j^{\omega})\quad (j=1,2). \tag{6.14}$$

Lemma 6.5. *For any $(E,Q)\in\mathcal{H}(J)$, we have $n_j^{\bar\omega}\le\chi_j\le n_j^{\bar\omega}+r$ (which is the necessary condition that $\widetilde{\mathcal{R}}(J)'^{ss}_{\bar\omega}\neq\emptyset$).*

Proof. If $n_1^{\bar\omega}\ge n_1^{\omega}$, by (6.14), we have $n_1^{\bar\omega}<\chi_1\le n_1^{\omega}+r\le n_1^{\bar\omega}+r$, which implies $n_2^{\bar\omega}\le\chi_2<n_2^{\bar\omega}+r$. If $n_1^{\bar\omega}<n_1^{\omega}$, by $n_1^{\bar\omega}+n_2^{\bar\omega}=\chi=n_1^{\omega}+n_2^{\omega}$, we have $n_2^{\bar\omega}>n_2^{\omega}$ which implies $n_2^{\bar\omega}<\chi_2\le n_2^{\omega}+r<n_2^{\bar\omega}+r$ by (6.14) again (thus $n_1^{\bar\omega}<\chi_1<n_1^{\bar\omega}+r$). □

To prove $H^1(\mathcal{P}_\omega, \Theta_{\mathcal{P}_\omega}) = 0$ via the same method of Section 5, even if we assume that $\min_{1\le i\le 2}\left\{(r-1)(g_i - \frac{r+3}{4}) + \frac{|I_i|}{k}\right\} \ge 3$, we only have

$$\begin{aligned} H^1(\mathcal{P}_\omega, \Theta_{\mathcal{P}_\omega}) :&= H^1(\widetilde{\mathcal{R}}'^{ss}_\omega, \hat{\Theta}'_{\widetilde{\mathcal{R}}'^{ss}_\omega})^{inv.} = H^1(\mathcal{H}^\omega, \hat{\Theta}'_{\mathcal{H}})^{inv.} \\ &= H^1(p_J^{-1}(\mathcal{H}^\omega), p_J^*(\hat{\Theta}'_{\mathcal{H}}))^{inv.} \\ &= H^1(p_J^{-1}(\mathcal{H}^\omega), \omega_{\mathcal{H}(J)} \otimes \hat{\Theta}'_{\bar\omega} \otimes \hat{\mathrm{Det}}^*_{\mathcal{H}(J)}(\Theta^{-1}_{J^d_{\widetilde X}}))^{inv.}. \end{aligned}$$

If $p_J^{-1}(\mathcal{H}^\omega) = \mathcal{H}(J)^{\bar\omega}$, we would have (choosing $|J_1|$, $|J_2|$ large enough)

$$\begin{aligned} &H^1(p_J^{-1}(\mathcal{H}^\omega), \omega_{\mathcal{H}(J)} \otimes \hat{\Theta}'_{\bar\omega} \otimes \hat{\mathrm{Det}}^*_{\mathcal{H}(J)}(\Theta^{-1}_{J^d_{\widetilde X}}))^{inv.} \\ &= H^1(\mathcal{P}_{\bar\omega}, \omega_{\mathcal{P}_{\bar\omega}} \otimes \Theta_{\mathcal{P}_{\bar\omega}} \otimes \mathrm{Det}^*_{\mathcal{P}_{\bar\omega}}(\Theta^{-1}_{J^d_{\widetilde X}})) \end{aligned}$$

which vanishes by Kodaira-type theorem and the following lemma.

Lemma 6.6. *When $X = X_1 \cup X_2$ with node x_0, the line bundle*

$$\Theta_{\mathcal{P}_{\bar\omega}} \otimes \mathrm{Det}^*_{\mathcal{P}_{\bar\omega}}(\Theta^{-1}_{J^d_{\widetilde X}})$$

on $\mathcal{P}_{\bar\omega}$ is ample if $\bar k > 2r$.

Proof. When $X = X_1 \cup X_2$, the moduli space $\mathcal{P}_{\bar\omega}$ is a disjoint union of

$$\{\mathcal{P}_{d_1,d_2}\}_{d_1+d_2=d}.$$

It is enough to consider $\mathcal{P}_{\bar\omega} = \mathcal{P}_{d_1,d_2}$, thus we the flat morphism

$$\mathrm{Det} : \mathcal{P}_{\bar\omega} \to J^d_{\widetilde X} = J^{d_1}_{X_1} \times J^{d_2}_{X_2} = J^d_X$$

and $J^0_{\widetilde X} = J^0_{X_1} \times J^0_{X_2} = J^0_X$ acts on $\mathcal{P}_{\bar\omega}$ by

$$((E,Q),\mathcal{N}) \mapsto (E \otimes \pi^*\mathcal{N}, Q \otimes \mathcal{N}_{x_0}).$$

Let $\mathcal{P}^L_{\bar\omega} = \mathrm{Det}^{-1}_{\mathcal{P}_{\bar\omega}}(L)$ (which is unirational), consider the morphism

$$f : \mathcal{P}^L_{\bar\omega} \times J^0_X \to \mathcal{P}_{\bar\omega}.$$

Then it is enough to check the ampleness of

$$f^*(\Theta_{\mathcal{P}_{\bar\omega}} \otimes \mathrm{Det}^*_{\mathcal{P}_{\bar\omega}}(\Theta^{-1}_{J^d_{\widetilde X}}))|_{\{(E,Q)\}\times J^0_X}, \quad f^*(\Theta_{\mathcal{P}_{\bar\omega}} \otimes \mathrm{Det}^*_{\mathcal{P}_{\bar\omega}}(\Theta^{-1}_{J^d_{\widetilde X}}))|_{\mathcal{P}^L_{\bar\omega}\times\{\mathcal{N}\}}.$$

It is clearly that $f^*(\Theta_{\mathcal{P}_{\bar\omega}} \otimes \mathrm{Det}^*_{\mathcal{P}_{\bar\omega}}(\Theta^{-1}_{J^d_{\widetilde X}}))|_{\mathcal{P}^L_{\bar\omega}\times\{\mathcal{N}\}}$ is ample, and

$$f^*(\Theta_{\mathcal{P}_{\bar\omega}} \otimes \mathrm{Det}^*_{\mathcal{P}_{\bar\omega}}(\Theta^{-1}_{J^d_{\widetilde X}}))|_{\{(E,Q)\}\times J^0_X} = M_1 \otimes M_2$$

where $M_1 = f_1^*(\Theta_{\mathcal{P}_{\bar\omega}})$, $M_2 = f_2^*(\Theta^{-1}_{J^d_{\widetilde X}})$, $f_1 : J^0_X \to \mathcal{P}_{\bar\omega}$, $f_2 : J^0_X \to J^d_X$,

$$f_1(\mathcal{N}) = (E \otimes \pi^*\mathcal{N}, Q \otimes \mathcal{N}_{x_0}), \quad f_2(L_0) = L_0^r \otimes L.$$

Then M_1 (resp. M_2) is algebraically equivalent to $\Theta_y^{r\bar{k}}$ (resp. $\Theta_y^{-2r^2}$) (see Lemma 5.3 of [9] for details). Thus $M_1 \otimes M_2$ is algebraically equivalent to $\Theta_y^{r\bar{k}-2r^2}$, which is ample when $\bar{k} > 2r$. □

Remarks 6.7. (1) The equality $p_J^{-1}(\mathcal{H}^{\omega}) = \mathcal{H}(J)^{\bar{\omega}}$ is equivalent to the statement that for any $(E,Q) \in \mathcal{H}(J)$ with torsion τ_i at x_i we have

$$\tau_i \le n_j^{\omega} + r - \chi_j \ (j \neq i) \ \Leftrightarrow \ \tau_i \le n_j^{\bar{\omega}} + r - \chi_j \ (j \neq i) \tag{6.15}$$

which may not be true unfortunately. (2) The proof of Proposition 6.3 in fact implies the following estimate

$$\operatorname{codim}(\mathcal{H} \setminus \widetilde{\mathcal{R}}_{\omega}'^{-ss}) > \min_{1\le i\le 2}\left\{(r-1)(g_i - \frac{r+3}{4}) + \frac{|I_i|}{k}\right\} \tag{6.16}$$

where the open set $\widetilde{\mathcal{R}}_{\omega}'^{-ss} \subset \mathcal{H}$ satisfying $\widetilde{\mathcal{R}}_{\omega}'^{-ss} \supset \widetilde{\mathcal{R}}_{\omega}'^{ss}$ is defined to be

$$\widetilde{\mathcal{R}}_{\omega}'^{-ss} := \left\{\begin{array}{l} (E,Q) \in \mathcal{H} \, \text{satisfies } \ par\mu(F) \le par\mu(E) \text{ for any} \\ \text{nontrivial } F \subset E \text{ of rank } (r_1,r_2) \neq (0,r) \text{ or } (r,0) \end{array}\right\}.$$

We end up by some comments about quantization conjecture of Guillemin-Sternberg. Let M be a projective variety with an action of a reductive group G and an ample L linearizing the action of G. If $M_L^{ss} \subset M$ is the open set of GIT semistable points, then the so called quantization conjecture of Guillemin-Sternberg predict that

$$H^i(M,L)^{inv.} = H^i(M_L^{ss},L)^{inv.} \tag{6.17}$$

which was proved when M is projective and has at most rational singularities (see [12], [13] and [14]). There is an example in [12] showing the failure of (6.17) when M has worse singularities. However, for the applications of studying moduli spaces in algebraic geometry, M is in general a locally closed subvariety of Quotient schemes or Hilbert schemes (for example, $M = \widetilde{\mathcal{R}}_F$, $\mathcal{H}$ in this article, which are quasi-projective and have at most rational singularities). Thus the following question seems natural and important for application.

Question 6.8. Let M be a normal, projective variety with action by a reductive group G. If $M_0 \subset M$ is an G-invariant open set such that $M_L^{ss} \subset M_0$ for any ample linearization L. Does the equality

$$H^i(M_0,L)^{inv.} = H^i(M_L^{ss},L)^{inv.}$$

holds for any $i \ge 0$?

If the question has an affirmative answer, conjecture in Remark 5.6 and Conjecture 4.5 will hold, which imply $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X,\omega}) = 0$ for any irreducible X with one node and any data ω (see Remark

5.6). However, the affirmative answer of Question 6.8 seems not imply $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X,\omega}) = 0$ for reducible $X = X_1 \cup X_2$.

Let $q_L : M_L^{ss} \to \mathcal{M}_L := M_L^{ss}//G$ be the GIT quotient and assume that L descends to a line bundle $\mathcal{L}$ (i.e. L is the pullback of $\mathcal{L}$). One of the general strategy of proving $H^i(\mathcal{M}_L, \mathcal{L}) = 0$ is to use equalities

$$H^i(\mathcal{M}_L, \mathcal{L}) = H^i(M_L^{ss}, L)^{inv.} = H^i(M_0, L)^{inv.}$$

where the first equality holds by definition and the second holds by the affirmative answer of Question 6.8. Then one can write (on M_0)

$$L = \omega_{M_0} \otimes L', \quad L' = \omega_{M_0}^{-1} \otimes L$$

where ω_{M_0} is the canonical bundle of M_0. Let $q_{L'} : M_{L'}^{ss} \to \mathcal{M}_{L'}$ be the GIT quotient and L' descend to $\mathcal{L}'$. Assume that

$$H^i(M_0, L)^{inv.} = H^i(M_{L'}^{ss}, L)^{inv.}, \quad \omega_{M_0} = q_{L'}^*(\omega_{\mathcal{M}_{L'}}). \tag{6.18}$$

Then $H^i(\mathcal{M}_L, \mathcal{L}) = H^i(\mathcal{M}_{L'}, \omega_{\mathcal{M}_{L'}} \otimes \mathcal{L}') = 0$ ($\forall\ i > 0$). Assumption (6.18) does not hold in general, which need a good estimate of codimension of $M_0 \setminus M_{L'}^{ss}$ and $M_{L'}^{ss} \setminus M_{L'}^{s}$. It is the reason that this strategy does not work for reducible $X = X_1 \cup X_2$ since we do not have a good estimate of codimension of $\mathcal{H} \setminus \widetilde{\mathcal{R}}'^{ss}_{\omega}$ (we have only an estimate of codim$(\mathcal{H}^{\omega} \setminus \widetilde{\mathcal{R}}'^{ss}_{\omega})$). However, we will prove vanishing theorems in a forthcoming article [11] for all of these moduli spaces by a method of modulo p reduction, which essentially needs the estimates of codimension and computation of canonical bundles.

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