Harnack and Super Poincaré Inequalities for Generalized Cox-Ingersoll-Ross Model *

Xing Huang [a)], **Fei Zhao**[a)†]

a) Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

December 22, 2019

Abstract

In this paper, Wang's Harnack inequalities and super Poincaré inequality for generalized Cox-Ingersoll-Ross model are obtained. Since the noise is degenerate, the intrinsic metric has been introduced to construct the coupling by change of measure. By using isoperimetric constant, some optimal estimate of the rate function in the super Poincaré inequality for the associated Dirichlet form is also obtained.

AMS subject Classification: 60H10, 60H15.
Keywords: Coupling by change of measure; Harnack inequality; Isoperimetric constant; Super Poincaré inequality; Generalized Cox-Ingersoll-Ross model

1 Introduction

The SDE

$$\mathrm{d}X_t = (\alpha - \delta X_t)\mathrm{d}t + \sqrt{X_t}\mathrm{d}B_t, \quad X_0 > 0, \tag{1.1}$$

which is called CIR (Cox-Ingersoll-Ross) model [6, Section 4.6], is used to characterize the evolution of the interest rate in finance. In [1, 2, 3, 7, 11, 14, 15, 23], the authors investigate the convergence rate of various numerical schemes of (1.1), see also [5] for distribution dependent SDEs with Hölder continuous diffusion coefficients. Zhang and Zheng [24] obtain the Harnack inequality and super Poincaré for (1.1). See [8, 9, 10] for more introductions on (1.1).

*Supported in part by NNSFC (11801406).
†Corresponding author: Fei Zhao, zhaofeitju@163.com

In this paper, we consider stochastic differential equations on $[0,\infty)$:

$$\mathrm{d}X_t = (\alpha - \delta X_t)\mathrm{d}t + X_t^h \mathrm{d}B_t, \tag{1.2}$$

with constant $\frac{1}{2} < h < 1$, $\alpha, \delta > 0$, and B_t a is one-dimensional Brownian motion on some complete filtration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. We call (1.2) a generalized CIR model. By [12, 13], (1.2) has a unique non-negative strong solution for any positive initial value.

Compared with SDE (1.1), the diffusion in (1.2) has stronger degeneration on 0 due to $h > \frac{1}{2}$, which leads to worse regularity of the solution. Thus, the Harnack inequality for the semigroup associated to (1.2) is non-trivial.

Wang [18] introduced coupling by change of measure to establish Harnack inequality in the SDEs with non-degenerate diffusion coefficients, see [4, 19, 22] for more models. Wang [17, Section 3] also gives some conditions to obtain super Poincaré inequality, which was firstly introduced in [21] to characterize the essential spectrum. Zhang and Zheng [24] obtained the functional inequalities of (1.1) under some reasonable conditions.

In this paper, we will prove the Harnack and super Poincaré inequality for (1.2), which cover the results in [24] where h is assumed to be $\frac{1}{2}$.

The paper is organized as follows: In Section 2, we give main results on Harnack and super Poincaré inequality, which will be proved in Section 4 and Section 5 respectively; In Section 3, we give some lemmas which will be used in the sequel.

2 Main Results

2.1 Harnack Inequality and Gradient Estimate

As we know, the intrinsic metric associate to the generator of (1.2) is defined as

$$\rho(s,t) = \int_{s\wedge t}^{s\vee t} \frac{\mathrm{d}r}{r^h}, \quad s,t \in [0,\infty). \tag{2.1}$$

For any $f \in C^1([0,\infty))$, $x \in [0,\infty)$, define

$$\nabla^h f(x) := \lim_{y\to x} \frac{f(y) - f(x)}{\frac{1}{1-h}y^{1-h} - \frac{1}{1-h}x^{1-h}} = \frac{f'(x)}{x^{-h}} = x^h f'(x).$$

∇^h is called the intrinsic gradient. Obviously, we have

$$|\nabla^h f(x)| = \lim_{\rho(y,x)\to 0} \frac{|f(y) - f(x)|}{\rho(y,x)} = x^h |f'(x)|.$$

The following theorem gives Wang type Harnack inequality and gradient estimate ∇^h for the Markov semigroup P_t associated with (1.2):

$$P_t f(x) = \mathbb{E} f(X_t^x), \quad t \geq 0, x \in [0,\infty), f \in \mathscr{B}_b([0,\infty)),$$

where X_t^x is the unique solution X_t^x to (1.2) starting at x.

Theorem 2.1. *Assume $\frac{1}{2} < h < 1$ and $\alpha \geq \frac{h}{2}$. Then the following assertions hold.*

(1) The Harnack inequality holds, i.e. for any $T > 0$, $p > 1$ and x, $y \in [0, \infty)$, it holds

$$(P_T f)^p(y) \leq P_T f^p(x) \exp\left[\frac{p(\delta - \frac{h}{2})(y^{1-h} - x^{1-h})^2}{(p-1)(1-h)(\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})T} - 1)}\right], \ f \in \mathscr{B}_b^+([0, \infty)).$$

Moreover, for any $f \in \mathscr{B}_b^+([0, \infty))$ with $f > 0$, the Log-Harnack inequality

$$P_T \log f(y) \leq \log P_T f(x) + \frac{(\delta - \frac{h}{2})(y^{1-h} - x^{1-h})^2}{(1-h)(\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})T} - 1)}$$

holds.

(2) For any $f \in C_b^1([0, \infty)$, the estimate of the intrinsic gradient holds:

$$|\nabla^h P_T f(x)| \leq \mathrm{e}^{-(1-h)(\delta-\frac{h}{2})T} P_T |\nabla^h f|(x), \quad T > 0, x \in [0, \infty).$$

Remark 2.2. *This type inequality was introduced in [16] to characterize the hypercontractivity of diffusion semigroups on Riemannian manifolds, and has been intensively extended and applied to various different models of SDEs and SPDEs, see [18] for a general theory on this inequality and applications.*

2.2 Super Poincaré inequality

We firstly introduce some notations:

$$C(x) = \int_1^x \frac{\alpha - \delta y}{y^{2h}} \mathrm{d}y = \frac{2\alpha}{1-2h} x^{1-2h} - \frac{\delta}{1-h} x^{2-2h} - \left(\frac{2\alpha}{1-2h} - \frac{\delta}{1-h}\right), \quad x > 0,$$
$$\Gamma_0 = 2\mathrm{e}^{-\left(\frac{2\alpha}{1-2h} - \frac{\delta}{1-h}\right)},$$
$$Z = \int_0^\infty \frac{\mathrm{e}^{C(x)}}{\frac{1}{2} x^{2h}} \mathrm{d}x = \Gamma_0 \int_0^\infty x^{-2h} \mathrm{e}^{\frac{2\alpha}{1-2h} x^{1-2h} - \frac{\delta}{1-h} x^{2-2h}} \mathrm{d}x,$$

and

$$\mu(\mathrm{d}x) = \frac{\Gamma_0 x^{-2h} \mathrm{e}^{\frac{2\alpha}{1-2h} x^{1-2h} - \frac{\delta}{1-h} x^{2-2h}}}{Z} \mathrm{d}x =: \eta(x) \mathrm{d}x.$$

Consider second-ordered differential operator on $L^2(\mu)$:

$$L = \frac{1}{2} x^{2h} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\alpha - \delta x) \frac{\mathrm{d}}{\mathrm{d}x}.$$

Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the associated Dirichlet form to L on $L^2(\mu)$. In particular, we have

$$\mathcal{E}(f, f) = \frac{1}{2} \int_0^\infty x^{2h} [f'(x)]^2 \mu(\mathrm{d}x), \quad f \in C_0^1([0, \infty)).$$

Let ρ be defined in (2.1), then we have

$$\rho(x,y)=\int_{x\wedge y}^{x\vee y}\frac{\mathrm{d}r}{r^h}=\frac{1}{1-h}|y^{1-h}-x^{1-h}|,\quad x,y\in[0,\infty).$$

For any open set $D\subset[0,\infty)$, the boundary measure of D induced by μ is defined as

$$\mu_\partial(\partial D):=\lim_{\varepsilon\to 0}\frac{\mu(D_\varepsilon)-\mu(D)}{\varepsilon}$$

with $D_\varepsilon=\{x\in[0,\infty)|\rho(x,D)\le\varepsilon\}$.

The isoperimetric constant is defined as

$$k(r):=\inf_{\mu(D)\le r}\frac{\mu_\partial(\partial D)}{\mu(D)},\quad r>0.$$

Theorem 2.3. *Assume $\frac{1}{2}<h<1$. Then the following assertions hold.*

(1) The super Poincaré inequality

$$\mu(f^2)\le r\mathcal{E}(f,f)+\beta(r)\mu(|f|)^2,\quad r>0, f\in\mathcal{D}(\mathcal{E}) \tag{2.2}$$

holds for $\beta(r)=\frac{4}{k^{-1}(2\sqrt{2}r^{-\frac{1}{2}})}$ with $k^{-1}(r)=\sup\{s\ge 0,k(s)>r)\}$.

(2) Moreover, there exists constants $c,r_0>0$ such that $k(r)\ge c(-\log r)^{\frac{1}{2}}$ for any $r\in(0,r_0)$. Thus, (2.2) holds with $\beta(r)=\mathrm{e}^{C(1+r^{-1})}$ for some constant $C>0$.

(3) Finally, $\beta(r)$ in (2) is optimal in the following sense: the super Poincaré inequality can not hold for any $\beta(r)=\mathrm{e}^{C(1+r^{-\lambda})}$ with $0<\lambda<1$ and some constant $C>0$.

Remark 2.4. *The super Poincaré inequality was introduced in [21] to characterize the absence of essential spectrum of Markov generators. A general theory on this type inequality and applications has been summarized in [17].*

3 Some Preparations

In this section, we give two important lemmas which will be used in the proof of Theorem 2.1.

Lemma 3.1. *Assume $\frac{1}{2}<h<1$ and SDE*

$$\mathrm{d}X_t=b(X_t)\mathrm{d}t+X_t^h\mathrm{d}B_t,\quad X_0=x>0$$

has a non-explosive and non-negative solution X_t. Here, $b:[0,\infty)\to\mathbb{R}$ is locally bounded and continuous at point 0, $b(0)>0$. Then $\mathbb{P}$-a.s.

$$\int_0^\infty I_{\{0\}}(X_t)\mathrm{d}t=0.$$

Proof. For any $n \geq 1$, construct function $\varphi_n : [0, \infty) \to \mathbb{R}$ as follows

$$\varphi_n(x) = \begin{cases} \frac{1}{2(h+1)n}, & x \geq \frac{1}{n}, \\ \frac{n^{2h+1}}{2(h+1)}(\frac{1}{n} - x)^{2(h+1)} + \frac{1}{2(h+1)n}, & 0 \leq x < \frac{1}{n}. \end{cases}$$

It is not difficult to see

$$|\varphi_n(x)| \leq \frac{1}{h+1}, \quad |\varphi_n'(x)| \leq 1, \quad |\varphi_n''(x) \cdot x^{2h}| = \frac{2h+1}{n^{2h-1}} \leq 2h+1, \tag{3.1}$$

and

$$\lim_{n\to\infty} \varphi_n(x) = 0, \quad \lim_{n\to\infty} \varphi_n'(x) = -I_{\{0\}}(x), \quad \lim_{n\to\infty} |\varphi_n''(x) \cdot x^{2h}| = 0. \tag{3.2}$$

Letting τ_m=inf $\{ t \geq 0$: $X_t \geq m\}$, since X_t is non-explosive, then we have $\tau_\infty := \lim_{m\to\infty} \tau_m = \infty$. Applying Itô's formula to $\varphi_n(X_t)$, we arrive at

$$\mathrm{d}\varphi_n(X_t) = \varphi_n'(X_t)b(X_t)\mathrm{d}t + \frac{1}{2}\varphi_n''(X_t)X_t^{2h}\mathrm{d}t + \varphi_n'(X_t)X_t^h\mathrm{d}B_t. \tag{3.3}$$

This implies

$$\begin{aligned} \varphi_n(X_{t\wedge\tau_m}) = \varphi_n(x) &+ \int_0^{t\wedge\tau_m} \varphi_n'(X_s)b(X_s)\mathrm{d}s \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_m} \varphi_n''(X_s)X_s^{2h}\mathrm{d}s + \int_0^{t\wedge\tau_m} \varphi_n'(X_s)X_s^h\mathrm{d}B_s. \end{aligned} \tag{3.4}$$

Combining the definition of τ_m and taking expectations in (3.4), we obtain

$$\mathbb{E}\big(\varphi_n(X_{t\wedge\tau_m})\big) - \varphi_n(x) = \mathbb{E}\int_0^{t\wedge\tau_m} \varphi_n'(X_s)b(X_s)\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_m} \varphi_n''(X_s)X_s^{2h}\mathrm{d}s.$$

Thus, (3.1)-(3.2) and dominated convergence theorem yield

$$\lim_{n\to\infty} \mathbb{E}\int_0^{t\wedge\tau_m} \varphi_n'(X_s)b(X_s)\mathrm{d}s = 0. \tag{3.5}$$

Since b is locally bounded, there exists $M > 0$ such that

$$\big|\varphi_n'(x)b(x)\big| \leq \sup_{x\in[0,\frac{1}{n})} \big|b(x)\big| \leq \sup_{x\in[0,1)} \big|b(x)\big| \leq M.$$

Mover, it is clear

$$\lim_{n\to\infty} \varphi_n'(x)b(x) = -I_{\{0\}}(x)b(0).$$

So, this together with dominated convergence theorem and (3.5) implies

$$\lim_{n\to\infty} \mathbb{E}\int_0^{t\wedge\tau_m} \varphi_n'(X_s)b(X_s)\mathrm{d}s = \mathbb{E}\int_0^{t\wedge\tau_m} I_{\{0\}}(X_s)b(0)\mathrm{d}s = 0,$$

which yields $\mathbb{E}\int_0^{t\wedge\tau_m} I_{\{0\}}(X_s)\mathrm{d}s = 0$ due to $b(0) > 0$. Letting firstly t goes to ∞ and then m tends to ∞, we have $\mathbb{E}\int_0^{\infty} I_{\{0\}}(X_s)\mathrm{d}s = 0$. Thus, we have $\mathbb{P}$-a.s. $\int_0^{\infty} I_{\{0\}}(X_s)\mathrm{d}s = 0$. □

Lemma 3.2. *Let $\frac{1}{2} < h < 1$ and $\alpha \geq \frac{h}{2}$. Then for any x, $y \in [0, \infty)$ with $x < y$, we have*

$$\alpha\left(\frac{1}{y^h} - \frac{1}{x^h}\right) + \frac{h}{2}\left(\frac{1}{x^{1-h}} - \frac{1}{y^{1-h}} + x^{1-h} - y^{1-h}\right) \leq 0. \tag{3.6}$$

Proof. We divide the proof into two cases.

(1) Case 1: $0 \leq x < 1$.

(i) $y < 1$. Consider function $w(z) = \frac{1}{z^{1-h}} - \frac{1}{z^h}$, $z > 0$. The derivative of w is

$$w'(z) = \frac{h - (1-h)z^{2h-1}}{z^{1+h}}.$$

Letting

$$z_0 = \left(\frac{h}{1-h}\right)^{\frac{1}{2h-1}},$$

then we have $w'(z_0) = 0$. Noting that $z_0 > 1$ due to $h \in (\frac{1}{2}, 1)$, w is strictly increasing on $[0, 1)$. Since $0 \leq x < y \leq 1$, we obtain $w(x) < w(y)$, i.e.

$$\frac{1}{x^{1-h}} - \frac{1}{y^{1-h}} + \frac{1}{y^h} - \frac{1}{x^h} < 0.$$

This together with $\frac{1}{2} < h < 1$, $\alpha \geq \frac{h}{2}$ and $x < y$ implies

$$\begin{aligned}&\alpha\left(\frac{1}{y^h} - \frac{1}{x^h}\right) + \frac{h}{2}\left(\frac{1}{x^{1-h}} - \frac{1}{y^{1-h}} + x^{1-h} - y^{1-h}\right)\\ &= \left(\alpha - \frac{h}{2}\right)\left(\frac{1}{y^h} - \frac{1}{x^h}\right) + \frac{h}{2}\left(\frac{1}{x^{1-h}} - \frac{1}{y^{1-h}} + \frac{1}{y^h} - \frac{1}{x^h}\right) + \frac{h}{2}\left(x^{1-h} - y^{1-h}\right) < 0.\end{aligned}$$

(ii) $y > 1$. Since $\frac{1}{2} < h < 1$, we have $\frac{1}{y^h} < \frac{1}{y^{1-h}}$. By the same reason, it holds $\frac{1}{x^h} > \frac{1}{x^{1-h}}$ due to $x < 1$. Thus,

$$\frac{1}{x^{1-h}} - \frac{1}{y^{1-h}} + \frac{1}{y^h} - \frac{1}{x^h} < 0.$$

Again thanks to $\frac{1}{2} < h < 1$, $\alpha \geq \frac{h}{2}$ and $x < y$, we obtain

$$\begin{aligned}&\alpha\left(\frac{1}{y^h} - \frac{1}{x^h}\right) + \frac{h}{2}\left(\frac{1}{x^{1-h}} - \frac{1}{y^{1-h}} + x^{1-h} - y^{1-h}\right)\\ &= \left(\alpha - \frac{h}{2}\right)\left(\frac{1}{y^h} - \frac{1}{x^h}\right) + \frac{h}{2}\left(\frac{1}{x^{1-h}} - \frac{1}{y^{1-h}} + \frac{1}{y^h} - \frac{1}{x^h}\right) + \frac{h}{2}\left(x^{1-h} - y^{1-h}\right) < 0.\end{aligned}$$

(2) Case 2: $x \geq 1$. Firstly, we have $y > 1$ due to $x < y$. So, we get from $\frac{1}{2} < h < 1$ and $x < y$ that

$$\begin{aligned}
&\alpha\left(\frac{1}{y^h} - \frac{1}{x^h}\right) + \frac{h}{2}\left(\frac{1}{x^{1-h}} - \frac{1}{y^{1-h}} + x^{1-h} - y^{1-h}\right) \\
&\leq \frac{h}{2}\left(\frac{y^{1-h} - x^{1-h}}{x^{1-h}y^{1-h}} + \left(x^{1-h} - y^{1-h}\right)\right) \\
&\leq \frac{h}{2}\left(y^{1-h} - x^{1-h} + \left(x^{1-h} - y^{1-h}\right)\right) = 0.
\end{aligned}$$

Thus, we complete the proof. □

With the above two lemmas in hand, we finish the proof of Theorem 2.1 below.

4 Proof of Theorem 2.1

We use the coupling by change of measure to derive the Harnack inequality.

Proof of Theorem 2.1. (1) Fix $T > 0$. For any $x, y \in [0, \infty)$, without loss of generality, we may assume that $y > x$. Let X_t solve (1.2) with $X_0 = x$, and Y_t solve the equation

$$\mathrm{d}Y_t = (\alpha - \delta Y_t)\mathrm{d}t + Y_t^h \mathrm{d}B_t - I_{[0,\tau)}\xi(t)Y_t^h \mathrm{d}t \tag{4.1}$$

with $Y_0 = y$, here

$$\xi(t) := \frac{2(\delta - \frac{h}{2})(y^{1-h} - x^{1-h})\mathrm{e}^{(1-h)(\delta - \frac{h}{2})t}}{(\mathrm{e}^{2(1-h)(\delta - \frac{h}{2})T} - 1)}, \quad t \geq 0,$$

and $\tau := \inf\{t \geq 0 : X_t = Y_t\}$, which is the coupling time. Let $Y_t = X_t$ for $t \geq \tau$. We will prove $\tau < T$.

For any $\varepsilon > 0$, let

$$\rho_\varepsilon(s, t) = \int_{s \wedge t}^{s \vee t} \frac{\mathrm{d}r}{(r + \varepsilon)^h}, \quad s, t \in [0, \infty).$$

Applying Itô's formula to $\rho_\varepsilon(X_t, Y_t)$, we have

$$\begin{aligned}
&\mathrm{d}\rho_\varepsilon(X_t, Y_t) \\
&= \frac{\partial \rho_\varepsilon(X_t, Y_t)}{\partial x}\mathrm{d}X_t + \frac{\partial \rho_\varepsilon(X_t, Y_t)}{\partial y}\mathrm{d}Y_t + \frac{1}{2}\frac{\partial^2 \rho_\varepsilon(X_t, Y_t)}{\partial x^2}\mathrm{d}\langle X\rangle_t + \frac{1}{2}\frac{\partial^2 \rho_\varepsilon(X_t, Y_t)}{\partial y^2}\mathrm{d}\langle Y\rangle_t \\
&+ \frac{\partial^2 \rho_\varepsilon(X_t, Y_t)}{\partial x \partial y}\mathrm{d}\langle X, Y\rangle_t \\
&= -\frac{\mathrm{d}X_t}{(X_t + \varepsilon)^h} + \frac{\mathrm{d}Y_t}{(Y_t + \varepsilon)^h} + \frac{hX_t^{2h}\mathrm{d}t}{2(X_t + \varepsilon)^{h+1}} - \frac{hY_t^{2h}\mathrm{d}t}{2(Y_t + \varepsilon)^{h+1}} \\
&= \left(\frac{Y_t^h}{(Y_t + \varepsilon)^h} - \frac{X_t^h}{(X_t + \varepsilon)^h}\right)\mathrm{d}B_t - \delta\left((Y_t + \varepsilon)^{1-h} - (X_t + \varepsilon)^{1-h}\right)\mathrm{d}t - \frac{\xi(t)Y_t^h}{(Y_t + \varepsilon)^h}\mathrm{d}t \\
&+ (\delta\varepsilon + \alpha)\left(\frac{1}{(Y_t + \varepsilon)^h} - \frac{1}{(X_t + \varepsilon)^h}\right)\mathrm{d}t + \frac{h}{2}\left(\frac{X_t^{2h}}{(X_t + \varepsilon)^{h+1}} - \frac{Y_t^{2h}}{(Y_t + \varepsilon)^{h+1}}\right)\mathrm{d}t, \ t < \tau.
\end{aligned} \tag{4.2}$$

Combining the definition of ρ_ε, we arrive at

$$
\begin{aligned}
&\mathrm{d}\left[\frac{1}{1-h}\left((Y_t+\varepsilon)^{1-h}-(X_t+\varepsilon)^{1-h}\right)\right]\\
&=\left(\frac{Y_t^h}{(Y_t+\varepsilon)^h}-\frac{X_t^h}{(X_t+\varepsilon)^h}\right)\mathrm{d}B_t-\left(\delta-\frac{h}{2}\right)\left((Y_t+\varepsilon)^{1-h}-(X_t+\varepsilon)^{1-h}\right)\mathrm{d}t\\
&-\frac{\xi(t)Y_t^h}{(Y_t+\varepsilon)^h}\mathrm{d}t+M(X_t,Y_t,\varepsilon)\mathrm{d}t,\quad t<\tau,
\end{aligned}\tag{4.3}
$$

where

$$
\begin{aligned}
&M(X_t,Y_t,\varepsilon)=(\delta\varepsilon+\alpha)\left(\frac{1}{(Y_t+\varepsilon)^h}-\frac{1}{(X_t+\varepsilon)^h}\right)\\
&+\frac{h}{2}\left[\left(\frac{X_t^{2h}}{(X_t+\varepsilon)^{h+1}}-\frac{Y_t^{2h}}{(Y_t+\varepsilon)^{h+1}}\right)\right]-\frac{h}{2}\left((Y_t+\varepsilon)^{1-h}-(X_t+\varepsilon)^{1-h}\right).
\end{aligned}
$$

It follows from (4.3) that

$$
\begin{aligned}
&\frac{1}{1-h}\mathrm{e}^{(1-h)(\delta-\frac{h}{2})(\tau\wedge T)}\left((Y_{\tau\wedge T}+\varepsilon)^{1-h}-(X_{\tau\wedge T}+\varepsilon)^{1-h}\right)+\int_0^{\tau\wedge T}\frac{\mathrm{e}^{(1-h)(\delta-\frac{h}{2})t}\xi(t)Y_t^h}{(Y_t+\varepsilon)^h}\mathrm{d}t\\
&=\frac{1}{1-h}\left((y+\varepsilon)^{1-h}-(x+\varepsilon)^{1-h}\right)+\int_0^{\tau\wedge T}\mathrm{e}^{(1-h)(\delta-\frac{h}{2})t}\left(\frac{Y_t^h}{(Y_t+\varepsilon)^h}-\frac{X_t^h}{(X_t+\varepsilon)^h}\right)\mathrm{d}B_t\\
&+\int_0^{\tau\wedge T}\mathrm{e}^{(1-h)(\delta-\frac{h}{2})t}M(X_t,Y_t,\varepsilon)\mathrm{d}t.
\end{aligned}\tag{4.4}
$$

Let

$$
\begin{aligned}
I_1:&=\lim_{\varepsilon\to 0}\mathbb{E}\left|\int_0^{\tau\wedge T}\mathrm{e}^{(1-h)(\delta-\frac{h}{2})t}\Big(\frac{Y_t^h}{(Y_t+\varepsilon)^h}-\frac{X_t^h}{(X_t+\varepsilon)^h}\Big)\mathrm{d}B_t\right|^2\\
&=\lim_{\varepsilon\to 0}\mathbb{E}\int_0^{\tau\wedge T}\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})t}\Big(\frac{Y_t^h}{(Y_t+\varepsilon)^h}-\frac{X_t^h}{(X_t+\varepsilon)^h}\Big)^2\mathrm{d}t,
\end{aligned}
$$

By Lemma 3.1 and $Y_s\geq X_s$, we have $\mathbb{P}$-a.s

$$\int_0^\infty I_{\{X_s=0\}}\mathrm{d}s=0,\quad \int_0^\infty I_{\{Y_s=0\}}\mathrm{d}s=0.$$

This implies

$$I_1\leq\mathbb{E}\left(\int_0^T\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})t}(I_{\{Y_t\neq 0\}}-I_{\{X_t\neq 0\}})^2\mathrm{d}t\right)=0.$$

Since X and Y are continuous, by dominated convergence theorem and Lemma 3.2, we obtain

$$\lim_{\varepsilon\to 0}\int_0^{\tau\wedge T}\mathrm{e}^{(1-h)(\delta-\frac{h}{2})t}M(X_t,Y_t,\varepsilon)\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^{\tau\wedge T} \mathrm{e}^{(1-h)(\delta-\frac{h}{2})t} \lim_{\varepsilon\to 0} M(X_t, Y_t, \varepsilon)\mathrm{d}t \\
&= \int_0^{\tau\wedge T} \left(\alpha\left(\frac{1}{Y_t^h} - \frac{1}{X_t^h}\right) + \frac{h}{2}\left[\left(\frac{X_t^{2h}}{X_t^{h+1}} - \frac{Y_t^{2h}}{Y_t^{h+1}}\right) - \left(Y_t^{1-h} - X_t^{1-h}\right)\right]\right)\mathrm{d}t \\
&\le 0.
\end{aligned}
$$

Thus, letting ε go to 0 in (4.4), it holds $\mathbb{P}$-a.s.

$$
\int_0^{\tau\wedge T} \mathrm{e}^{(1-h)(\delta-\frac{h}{2})t}\xi(t)\mathrm{d}t + \frac{1}{1-h}\mathrm{e}^{(1-h)(\delta-\frac{h}{2})(\tau\wedge T)}(Y_{\tau\wedge T}^{1-h} - X_{\tau\wedge T}^{1-h}) \le \frac{1}{1-h}(y^{1-h} - x^{1-h}). \tag{4.5}
$$

On the other hand, by the definition of $\xi(t)$, it is easy to see

$$
\int_0^{\tau\wedge T} \mathrm{e}^{(1-h)(\delta-\frac{h}{2})t}\xi(t)\mathrm{d}t = \frac{(y^{1-h} - x^{1-h})(\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})(\tau\wedge T)} - 1)}{(1-h)(\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})T} - 1)}. \tag{4.6}
$$

This and (4.5) imply $\mathbb{P}(\tau > T) = 0$. In fact, if $\mathbb{P}(\tau > T) > 0$, considering (4.5) on the set $\{\tau > T\}$, we have

$$
\frac{1}{1-h}(y^{1-h} - x^{1-h}) + \frac{1}{1-h}\mathrm{e}^{(1-h)(\delta-\frac{h}{2})T}(Y_T^{1-h} - X_T^{1-h}) \le \frac{1}{1-h}(y^{1-h} - x^{1-h}).
$$

This is impossible, and $\mathbb{P}(\tau \le T) = 1$.

Let

$$
R = \exp\left[\int_0^\tau \xi(t)\mathrm{d}B_t - \frac{1}{2}\int_0^\tau \xi^2(t)\mathrm{d}t\right].
$$

By Girsanov's theorem, under the probability $\mathrm{d}\mathbb{Q} := R\mathrm{d}\mathbb{P}$, the process

$$
\widetilde{B_t} = B_t - \int_0^t 1_{[0,\tau)}(s)\xi(s)\mathrm{d}s, \quad t \ge 0
$$

is a one-dimensional Brownian motion. Rewrite the equation for Y_t as

$$
\mathrm{d}Y_t = (\alpha - \delta Y_t)\mathrm{d}t + Y_t^h\mathrm{d}\widetilde{B_t}, \quad Y_0 = y.
$$

We see that the distribution of Y under $\mathbb{Q}$ coincides with that of X^y under $\mathbb{P}$. Moreover, $\mathbb{Q}$-a.s. $X_T = Y_T$. Thus,

$$
P_T f(y) = \mathbb{E}^{\mathbb{Q}}(f(Y_T)) = \mathbb{E}^{\mathbb{Q}}(f(X_T)) = \mathbb{E}(Rf(X_T)).
$$

By Hölder's inequality, we have

$$
(P_T f(y))^p \le (\mathbb{E}(R^{p/(p-1)}))^{p-1} \cdot \mathbb{E}(f^p(X_T)) = P_T f^p(x) \cdot (\mathbb{E}(R^{p/(p-1)}))^{p-1}. \tag{4.7}
$$

On the other hand, from the definition of R and $\xi(t)$, we arrive at

$$
\mathbb{E}(R^{p/(p-1)}) \le \exp\left[\frac{p}{2(p-1)^2}\int_0^T \xi^2(t)\mathrm{d}t\right]
$$

$$
\begin{aligned}
&\times \mathbb{E}\left(\exp\left[\frac{p}{p-1}\int_0^\tau \xi(t)\mathrm{d}B_t - \frac{p^2}{2(p-1)^2}\int_0^\tau \xi^2(t)\mathrm{d}t\right]\right)\\
&\le \exp\left[\frac{p}{2(p-1)^2}\int_0^T \xi^2(t)\mathrm{d}t\right]\\
&= \exp\left[\frac{p(\delta-\frac{h}{2})(y^{1-h}-x^{1-h})^2}{(p-1)^2(1-h)(\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})T}-1)}\right].
\end{aligned}
$$

This together with (4.7) yields

$$
(P_T f(y))^p \le (P_T f^p(x)) \exp\left[\frac{p(\delta-\frac{h}{2})(y^{1-h}-x^{1-h})^2}{(p-1)(1-h)(\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})T}-1)}\right], \quad f \in \mathscr{B}_b^+([0,\infty)).
$$

Similarly, we have

$$
P_T \log f(y) = \mathbb{E}^{\mathbb{Q}}(\log f(Y_T)) = \mathbb{E}^{\mathbb{Q}}(\log f(X_T)) = \mathbb{E}(R\log f(X_T)).
$$

Young's inequality implies

$$
\mathbb{E}(R\log f(X_T)) \le \mathbb{E}(R\log R) + \log\ \mathbb{E}(f(X_T)) = \mathbb{E}(R\log R) + \log(P_T f(x)).
$$

It is not difficult to see that

$$
\begin{aligned}
&\mathbb{E}(R\log R) = \mathbb{E}^{\mathbb{Q}}\log R\\
&= \mathbb{E}^{\mathbb{Q}}\left(\int_0^\tau \xi(t)\mathrm{d}B_t - \frac{1}{2}\int_0^\tau \xi^2(t)\mathrm{d}t\right)\\
&= \mathbb{E}^{\mathbb{Q}}\left(\int_0^\tau \xi(t)\mathrm{d}\widetilde{B_t} + \int_0^\tau \xi^2(t)\mathrm{d}t - \frac{1}{2}\int_0^\tau \xi^2(t)\mathrm{d}t\right)\\
&= \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left(\int_0^\tau \xi^2(t)\mathrm{d}t\right) \le \frac{(\delta-\frac{h}{2})(y^{1-h}-x^{1-h})^2}{(1-h)(\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})T}-1)}.
\end{aligned}
$$

Thus, the log-Harnack inequality holds, i.e.

$$
P_T \log f(y) \le \log P_T f(x)) + \frac{(\delta-\frac{h}{2})(y^{1-h}-x^{1-h})^2}{(1-h)(\mathrm{e}^{2(1-h)(\delta-\frac{h}{2})T}-1)}, \quad f>0, f \in \mathscr{B}_b^+([0,\infty)).
$$

(2) Repeat the proof of (1) with $\xi(t) = 0$ and $\tau = \infty$. From (4.5), we arrive at

$$
\mathrm{e}^{(1-h)(\delta-\frac{h}{2})T}\frac{1}{1-h}(Y_T^{1-h} - X_T^{1-h}) \le \frac{1}{1-h}(y^{1-h}-x^{1-h}), \tag{4.8}
$$

which means

$$
\rho(X_T^y, X_T^x) \le \mathrm{e}^{-(1-h)(\delta-\frac{h}{2})T}\rho(y,x). \tag{4.9}
$$

Thus, for any $f \in C_b^1([0,\infty)$, we have

$$\begin{aligned}
|\nabla^h P_T f(x)| &= \lim_{\rho(y,x)\to 0} \frac{|P_T f(y) - P_T f(x)|}{\rho(y,x)} \\
&= \lim_{\rho(y,x)\to 0} \frac{|\mathbb{E} f(X_T^y) - \mathbb{E} f(X_T^x)|}{\rho(y,x)} \\
&= \lim_{\rho(y,x)\to 0} \frac{|\mathbb{E} f(X_T^y) - \mathbb{E} f(X_T^x)|}{\rho(X_T^y, X_T^x)} \frac{\rho(X_T^y, X_T^x)}{\rho(y,x)} \\
&\leq \mathrm{e}^{-(1-h)(\delta - \frac{h}{2})T} P_T |\nabla^h f|(x).
\end{aligned}$$

□

Remark 4.1. *In [24], i.e. $h = \frac{1}{2}$, as ε goes to 0, the first and second term in $M(X_t, Y_t, \varepsilon)$ can be non-positive if $\alpha \geq \frac{1}{4}$. However, it does not hold when $h \in (\frac{1}{2}, 1)$, and this is why we construct $M(X_t, Y_t, \varepsilon)$ as in the proof of Theorem 2.1.*

5 Proof of Theorem 2.3

In this section, we use isoperimetric constant to derive the super Poincaré inequality.

Lemma 5.1. *There exists a small enough constant $r_0 \in (0,1)$ such that for any $x_1, x_2 > 0$ satisfying $\mu((0,x_1)) = \mu((x_2,\infty)) \leq r_0$, it holds*

$$\mu_\partial(\partial(0,x_1)) > \mu_\partial(\partial(x_2,\infty)).$$

Proof. By the definition of μ_∂, we have

$$\begin{aligned}
\mu_\partial((0,x)) &= \lim_{\varepsilon\to 0} \frac{\mu\left(\{y : 0 < \frac{1}{1-h}(y^{1-h} - x^{1-h}) \leq \varepsilon\}\right)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\int_x^{[x^{1-h}+(1-h)\varepsilon]^{\frac{1}{1-h}}} \eta(y)\mathrm{d}y}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\eta(x)\left\{[x^{1-h} + (1-h)\varepsilon]^{\frac{1}{1-h}} - x\right\}}{\varepsilon} \\
&= x^h \eta(x) = \frac{\Gamma_0 x^{-h} \mathrm{e}^{\frac{2\alpha}{1-2h} x^{1-2h} - \frac{\delta}{1-h} x^{2-2h}}}{Z}.
\end{aligned}$$

Similarly, we arrive at

$$\begin{aligned}\mu_\partial((x,\infty)) &= \lim_{\varepsilon\to 0}\frac{\mu\left(\{y:-\varepsilon\le \frac{1}{1-h}(y^{1-h}-x^{1-h})<0\}\right)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{\int_{[x^{1-h}-(1-h)\varepsilon]^{\frac{1}{1-h}}}^{x}\eta(y)\mathrm{d}y}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{\eta(x)\left\{x-[x^{1-h}-(1-h)\varepsilon]^{\frac{1}{1-h}}\right\}}{\varepsilon}\\ &= x^h\eta(x)=\frac{\Gamma_0 x^{-h}\mathrm{e}^{\frac{2\alpha}{1-2h}x^{1-2h}-\frac{\delta}{1-h}x^{2-2h}}}{Z}.\end{aligned}$$

Letting $(x^h\eta(x))'=0$, we get

$$2\alpha=2\delta x+hx^{2h-1}.$$

Since $h\in(\frac{1}{2},1)$, there exists x_0 such that $x^h\eta(x)$ is strictly increasing on $(0,x_0)$ and strictly decreasing on (x_0,∞).

Letting $r>0$ be small enough, take $x_1(r),x_2(r)\in[0,\infty)$ such that

$$\mu((0,x_1(r)))=\mu((x_2(r),\infty))=r.$$

It is clear

$$\lim_{r\to 0}x_1(r)=0,\quad \lim_{r\to 0}x_2(r)=\infty. \tag{5.1}$$

Moreover, by L'Hopital's rule, we have

$$\lim_{r\to 0}\frac{\int_0^{x_1(r)}s^{-2h}\mathrm{e}^{\frac{2\alpha}{1-2h}s^{1-2h}-\frac{\delta}{1-h}s^{2-2h}}\mathrm{d}s}{\frac{1}{2\alpha}\mathrm{e}^{\frac{2\alpha}{1-2h}(x_1(r))^{1-2h}}}=1,$$

and

$$\lim_{r\to 0}\frac{\int_{x_2(r)}^{\infty}s^{-2h}\mathrm{e}^{\frac{2\alpha}{1-2h}s^{1-2h}-\frac{\delta}{1-h}s^{2-2h}}\mathrm{d}s}{\frac{1}{2\delta}(x_2(r))^{-1}\mathrm{e}^{-\frac{\delta}{1-h}(x_2(r))^{2-2h}}}=1. \tag{5.2}$$

Thus, it holds

$$\begin{aligned}1 &= \lim_{r\to 0}\frac{\frac{1}{2\alpha}\mathrm{e}^{\frac{2\alpha}{1-2h}(x_1(r))^{1-2h}}}{\frac{1}{2\delta}(x_2(r))^{-1}\mathrm{e}^{-\frac{\delta}{1-h}(x_2(r))^{2-2h}}}\\ &= \frac{\delta}{\alpha}\lim_{r\to 0}\frac{\mathrm{e}^{\frac{2\alpha}{1-2h}(x_1(r)^{-1})^{2h-1}}}{(x_2(r))^{-1}\mathrm{e}^{-\frac{\delta}{1-h}(x_2(r))^{2-2h}}}\\ &= \frac{\delta}{\alpha}\lim_{r\to 0}\mathrm{e}^{-\frac{2\alpha}{2h-1}(x_1(r)^{-1})^{2h-1}+\frac{\delta}{1-h}(x_2(r))^{2-2h}+\log(x_2(r))}.\end{aligned}$$

This means

$$\lim_{r\to 0}\left\{-\frac{2\alpha}{2h-1}(x_1(r)^{-1})^{2h-1}+\frac{\delta}{1-h}(x_2(r))^{2-2h}+\log(x_2(r))\right\}=\log\frac{\alpha}{\delta}.$$

Thus, (5.1) yields

$$\lim_{r\to 0}\left\{-\frac{2\alpha}{2h-1}\frac{(x_1(r)^{-1})^{2h-1}}{(x_2(r))^{\frac{(1-h)(2h-1)}{h}}}+\frac{\delta}{1-h}\frac{(x_2(r))^{2-2h}}{(x_2(r))^{\frac{(1-h)(2h-1)}{h}}}\right\}=0.$$

Since

$$\lim_{r\to 0}\frac{(x_2(r))^{2-2h}}{(x_2(r))^{\frac{(1-h)(2h-1)}{h}}}=\lim_{r\to 0}(x_2(r))^{\frac{1-h}{h}}=\infty,$$

we have

$$\lim_{r\to 0}\frac{(x_1(r)^{-1})^{2h-1}}{(x_2(r))^{\frac{(1-h)(2h-1)}{h}}}=\infty.$$

This together with (5.1) and the representation of $\mu_\partial((0,x))$ and $\mu_\partial((x,\infty))$ implies

$$\begin{aligned}
&\lim_{r\to 0}\frac{\mu_\partial(\partial(0,x_1(r)))}{\mu_\partial(\partial(x_2(r),\infty))}\\
&=\lim_{r\to 0}\frac{(x_1(r))^{-h}\mathrm{e}^{\frac{2\alpha}{1-2h}(x_1(r))^{1-2h}-\frac{\delta}{1-h}(x_1(r))^{2-2h}}}{(x_2(r))^{-h}\mathrm{e}^{\frac{2\alpha}{1-2h}(x_2(r))^{1-2h}-\frac{\delta}{1-h}(x_2(r))^{2-2h}}}\\
&=\lim_{r\to 0}\frac{(x_1(r))^{-h}\mathrm{e}^{\frac{2\alpha}{1-2h}(x_1(r))^{1-2h}}}{(x_2(r))^{-h}\mathrm{e}^{-\frac{\delta}{1-h}(x_2(r))^{2-2h}}}\\
&=\frac{\alpha}{\delta}\lim_{r\to 0}\frac{((x_1(r))^{-1})^{h}}{(x_2(r))^{1-h}}\\
&=\frac{\alpha}{\delta}\lim_{r\to 0}\left(\frac{(x_1(r)^{-1})^{2h-1}}{(x_2(r))^{\frac{(1-h)(2h-1)}{h}}}\right)^{\frac{h}{2h-1}}=\infty.
\end{aligned}$$

So, there exists $r_0>0$ such that for any $x_1,x_2\in[0,\infty)$ satisfying $\mu((0,x_1))=\mu((x_2,\infty))\le r_0$, it holds

$$\mu_\partial(\partial(0,x_1))>\mu_\partial(\partial(x_2,\infty)). \tag{5.3}$$

Thus, we complete the proof. □

Proof of Theorem 2.3. (1) Firstly, we prove that there exists small enough $\bar r_0>0$ such that for any $r\in(0,\bar r_0)$, $k(r)$ can only get the lower bound on the set (x,∞) with $\mu((x,\infty))\le r$.

Let $\bar r_0=\frac{1}{2}\{\mu(0,x_0)\wedge\mu(x_0,\infty)\}\wedge r_0$ with r_0 introduced in Lemma 5.1. Fix $r\in(0,\bar r_0)$. For any open set $A\subset[0,\infty)$ with $\mu(A)=r$, let $A_1:=A\cap(0,x_0)$ and $A_2:=A\cap(x_0,\infty)$. Then $\mu(A_1)\le\frac{1}{2}\mu(0,x_0)$ and $\mu(A_2)\le\frac{1}{2}\mu(x_0,\infty)$. Let $x_2=\inf\{x:x\in A_2\}$ and $x_1=\sup\{x:$

$x \in A_1\}$. Take $\bar{x}_1 \le x_1$ and $\bar{x}_2 \ge x_2$ such that $\mu((0,\bar{x}_1)) = \mu(A_1)$ and $\mu((\bar{x}_2,\infty)) = \mu(A_2)$. Since $x^h\eta(x)$ is strictly increasing on $(0, x_0)$ and strictly decreasing on (x_0,∞), we have

$$\mu_\partial(\partial A_1) \ge \mu_\partial(\partial(0, x_1)) \ge \mu_\partial(\partial(0,\bar{x}_1))$$

and

$$\mu_\partial(\partial A_2) \ge \mu_\partial(\partial(x_2,\infty)) \ge \mu_\partial(\partial(\bar{x}_2,\infty)).$$

This yields

$$\frac{\mu_\partial(\partial A)}{\mu(A)} \ge \frac{\mu_\partial(\partial((0,\bar{x}_1)\cup(\bar{x}_2,\infty)))}{\mu((0,\bar{x}_1)\cup(\bar{x}_2,\infty))}.$$

For any $y_1, y_2 \in (0,\infty)$ satisfying $\mu((0, y_1)) + \mu((y_2,\infty)) = r$, define

$$\varphi(y_1, y_2) := \mu_\partial(\partial((0, y_1)\cup(y_2,\infty))) = y_1^h\eta(y_1) + y_2^h\eta(y_2).$$

Next, we show that

$$\varphi(0, x) = \inf\{\varphi(y_1, y_2) : \mu((0, y_1)) + \mu((y_2,\infty)) = r\},$$

here, $\mu((x,\infty)) = r$. In fact, from $\mu((0, y_1)) + \mu((y_2,\infty)) = r$, there exists a function ϕ such that $y_2 = \phi(y_1)$ and $\phi'(y_1) = \frac{\eta(y_1)}{\eta(y_2)}$. Thus, we obtain

$$\varphi(y_1, y_2) = \varphi(y_1, \phi(y_1)) =: \Phi(y_1).$$

By the representation of $\eta(s)$, we have

$$\begin{aligned}
\Phi'(y_1) &= \frac{\partial\varphi}{\partial y_1}(y_1, y_2) + \frac{\partial\varphi}{\partial y_2}(y_1, y_2)\phi'(y_1)\\
&= y_1^h\eta'(y_1) + hy_1^{h-1}\eta(y_1) + (y_2^h\eta'(y_2) + hy_2^{h-1}\eta(y_2))\phi'(y_1)\\
&= y_1^h\eta'(y_1) + hy_1^{h-1}\eta(y_1) + (y_2^h\eta'(y_2) + hy_2^{h-1}\eta(y_2))\frac{\eta(y_1)}{\eta(y_2)}\\
&= \frac{\Gamma_0}{Z}\mathrm{e}^{\frac{2\alpha}{1-2h}y_1^{1-2h}-\frac{\delta}{1-h}y_1^{2-2h}}\Bigg(y_1^h(-2hy_1^{-2h-1} + y_1^{-2h}2\alpha y_1^{-2h} - y_1^{-2h}2\delta y_1^{1-2h}) + hy_1^{h-1}y_1^{-2h}\\
&\quad + y_2^h\frac{y_1^{-2h}}{y_2^{-2h}}(-2hy_2^{-2h-1} + y_2^{-2h}2\alpha y_2^{-2h} - y_2^{-2h}2\delta y_2^{1-2h}) + hy_2^{h-1}y_1^{-2h}\Bigg)\\
&= \eta(y_1)\Bigg(-h(y_1^{h-1} + y_2^{h-1}) + 2\alpha(y_1^{-h} + y_2^{-h}) - 2\delta(y_1^{1-h} + y_2^{1-h})\Bigg).
\end{aligned}$$

Since $h \in (\frac{1}{2}, 1)$ and $\alpha, \delta > 0$, there exists a small enough constant $r_1 > 0$ such that $\Phi'(y_1) > 0$ when $y_1 \in (0, r_1)$, and there exists a big enough constant $r_2 > 0$ such that $\Phi'(y_1) < 0$ when $y_2 \in (r_2,\infty)$. Thus, φ can only take minimum on (y_2,∞) with $\mu((y_2,\infty)) = r$ or on $(0, y_1)$ with $\mu((0, y_1)) = r$. By (5.3), φ take minimum on (y_2,∞) with $\mu((y_2,\infty)) = r$. Thus, we obtain

$$\frac{\mu_\partial(\partial A)}{\mu(A)} \ge \frac{\mu_\partial(\partial((0,\bar{x}_1)\cup(\bar{x}_2,\infty)))}{\mu((0,\bar{x}_1)\cup(\bar{x}_2,\infty))} \ge \frac{\mu_\partial(\partial(x,\infty))}{\mu((x,\infty))},$$

here, $\mu((x,\infty))=r$. Thus, we have

$$k(r)=\inf_{\{x|\mu((x,\infty))\le r\}}\frac{x^h\eta(x)}{\mu((x,\infty))}.$$

Take $x_r>0$ such that $\mu((x_r,\infty))=r$. Then we have $\lim_{r\to 0}x_r=\infty$. By (5.2), we have

$$\lim_{r\to 0}\frac{x_r^{2h-1}\eta(x_r)}{\mu((x_r,\infty))}=2\delta.$$

This implies

$$\lim_{r\to 0}k(r)=\lim_{r\to 0}\frac{x_r^h\eta(x_r)}{\mu((x_r,\infty))}=\lim_{r\to 0}\frac{x_r^{1-h}x_r^{2h-1}\eta(x_r)}{\mu((x_r,\infty))}=\lim_{r\to 0}x_r^{1-h}=\infty. \tag{5.4}$$

According to [17, Theorem 3.4.16], the super Poincaré inequality holds for

$$\beta(r)=\frac{4}{k^{-1}(2\sqrt{2}r^{-\frac{1}{2}})},\quad r>0.$$

(2) It follows form (5.2) that

$$\lim_{r\to 0}\frac{\mu((x_r,\infty))}{\frac{\Gamma_0 x_r^{-1}\mathrm{e}^{\frac{2\alpha}{1-2h}x_r^{1-2h}-\frac{\delta}{1-h}x_r^{2-2h}}}{2Z\delta}}=1,$$

which implies

$$\lim_{r\to 0}\frac{\mathrm{e}^{\log r}}{\mathrm{e}^{-\frac{\delta}{1-h}x_r^{2-2h}-\log x_r}}=\lim_{r\to 0}\mathrm{e}^{\log r+\frac{\delta}{1-h}x_r^{2-2h}+\log x_r}=\frac{\Gamma_0}{2Z\delta}.$$

This yields

$$\lim_{r\to 0}\{\log r+\frac{\delta}{1-h}x_r^{2-2h}+\log x_r\}=\log\frac{\Gamma_0}{2Z\delta}.$$

Since $h\in(\frac{1}{2},1)$, we obtain

$$\lim_{r\to 0}\frac{\sqrt{\log r^{-1}}}{x_r^{1-h}}=\sqrt{\frac{\delta}{1-h}}.$$

Combining this with (5.4), we arrive at

$$\lim_{r\to 0}\frac{k(r)}{\sqrt{\frac{1-h}{\delta}}\sqrt{\log r^{-1}}}=\lim_{r\to 0}\frac{x_r^{1-h}}{\sqrt{\frac{1-h}{\delta}}\sqrt{\log r^{-1}}}=1.$$

Thus, there exist constants $r_0>0$ and $c>0$ such that $k(r)\ge c[-\log r]^{\frac{1}{2}}$ for $r\in(0.r_0)$. According to [17, Corollary 3.4.17] with $\delta=1$, (2.2) holds with $\beta(r)=\mathrm{e}^{C(1+r^{-1})}$ for some constant $C>0$.

(3) Let $\rho(0,x)=\frac{1}{1-h}x^{1-h}$, then $\rho(0,\cdot)\in\mathcal{D}(\mathcal{E})$. Set $h_n=\rho(0,\cdot)\wedge n$. For any $g\in\mathcal{D}(\mathcal{E})$ with $\mu(|g|)\leq 1$, we have

$$\begin{aligned}&\mathcal{E}(h_ng,h_n)-\frac{1}{2}\mathcal{E}(h_n^2,g)\\&=\frac{1}{2}\int_0^\infty x^{2h}(h_ng)'(x)h_n'(x)\mu(\mathrm{d}x)-\frac{1}{4}\int_0^\infty x^{2h}(h_n^2)'(x)g'(x)\mu(\mathrm{d}x)\\&\leq\frac{1}{2}\int_0^{(n(1-h))^{\frac{1}{1-h}}}x^{2h}(h_n')^2(x)g(x)\mu(\mathrm{d}x)\leq\frac{1}{2}\mu(|g|)\leq\frac{1}{2}.\end{aligned}$$

So by [17, Definition 1.2.1], $L_{\mathcal{E}}(\rho(0,\cdot))\leq 1$.

However, for any $\lambda\in(\frac{1}{2},1)$ and $\varepsilon>0$, we have

$$\mu\{\exp\{\varepsilon\rho(0,\cdot)^{\frac{2\lambda}{2\lambda-1}}\}\}=\frac{\Gamma_0}{Z}\int_0^\infty x^{-2h}\mathrm{e}^{\frac{2\alpha}{1-2h}x^{1-2h}-\frac{\delta}{1-h}x^{2-2h}+\varepsilon(\frac{1}{1-h})^{\frac{2\lambda}{2\lambda-1}}x^{\frac{2\lambda(1-h)}{2\lambda-1}}}=\infty,$$

here, in the last display, we have used $\frac{2\lambda}{2\lambda-1}>2$ for any $\lambda\in(\frac{1}{2},1)$. By [17, Corollary 3.3.22], the super Poincaré inequality (2.2) does not hold with $\beta(r)=\mathrm{e}^{C(1+r^{-\lambda})}$ for $\frac{1}{2}<\lambda<1$. Similarly, we can show $\mu(\exp[\exp(\varepsilon\rho(0,\cdot))])=\|\rho(0,\cdot)\|_\infty=\infty$. Again by [17, Corollary 3.3.22], (2.2) does not hold with $\beta(r)=\mathrm{e}^{C(1+r^{-\lambda})}$ for $0<\lambda\leq\frac{1}{2}$. Thus, we finish the proof. $\square$

Acknowledgement. The authors would like to thank Professor Feng-Yu Wang and Shao-Qin Zhang for corrections and helpful comments.

References

[1] Alfonsi, A., *On the discretization schemes for the CIR (and Bessel squared) processes,* Monte Carlo Methods and Applications, 11(2005), 355-384.

[2] Alfonsi, A., *High order discretization schemes for the CIR process: Application to affine term structure and Heston models,* Mathematics of Computation, 79(2010), 209-237.

[3] Alfonsi, A., *Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process,* Statistics and Probability Letters, 83(2013), 602-607.

[4] Arnaudon, M., Thalmaier, A., Wang, F.-Y., *Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below,* Bull. Sci. Math., 130(2006), 223-233.

[5] Bao, J., Huang, X., *Approximations of Mckean-Vlasov SDEs with Irregular Coefficients,* arXiv:1905.08522.

[6] Cairns, A. J. G., *Interest rate models: an introduction,* Princeton University Press, 2004.

[7] Chassagneux, J. F., Jacquler, A., Mihaylov, I., *An Explicit Euler Scheme with Strong Rate of Convergence for Financial SDEs with Non-Lipschitz Coefficients,* Siam Journal on Financial Mathematics, 7(2016), 993-1021.

[8] Cox, J. C., Ingersoll, J. E., Ross, S. A., *A theory of the term structure of interest rates,* Econometrica, 53(1985), 385-407.

[9] Cox, J. C., Ross, S. A., *An intertemporal general equilibrium model of asset prices,* Econometrica, 53(1985), 363-384.

[10] Chou, C. S., Lin, H. J., *Some properties of CIR processes,* Stochastic Analysis and Applications, 24(2006), 901-912.

[11] Gyöngy, I., Rásonyi, M., A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients, Stoch. Proc. Appl., 121(2011), 2189–2200.

[12] Ikeda, N., Watanabe, S., *Stochastic differential equations and diffusion processes,* 2nd ed. Amsterdam: North Holland, 1989.

[13] Karatzas, I., Shreve, S. E., *Brownian motion and stochastic calculus,* 2nd edition, corrected 6th printing. Springer, 2000.

[14] Stamatiou, I., S., *An explicit positivity preserving numerical scheme for CIR/CEV type delay models with jump,* Journal of Computational and Applied Mathematics, 360(2019), 78-98.

[15] Wu, F., Mao, X., R., Chen, k., *The Cox-Ingersoll-Ross model with delay and strong convergence of its Euler-Maruyama approximate solutions,* Applied Numerical Mathematics, 59(2009), 2641-2658.

[16] Wang, F.-Y., *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds,* Probab. Theory Related Fields, 109(1997), 417-424.

[17] Wang, F.-Y., *Functional inequalities, Markov semigroups and Spectral theory,* Beijing: Science Press, 2005.

[18] Wang, F.-Y., *Harnack Inequality and Applications for Stochastic Partial Differential Equations,* Springer, New York, 2013.

[19] Wang, F.-Y., *Harnack inequality and applications for stochastic generalized porous media equations,* Ann Probab., 35(2007), 1333-1350.

[20] Wang, F.-Y., *Harnack inequality on manifolds with boundary and applications,* J. Math. Pures Appl., 94(2010), 304-321.

[21] Wang, F.-Y., *Functional Inequalities for Empty Essential Spectrum,* J. Funct. Anal., 170(2000), 219-245.

[22] Wang, F.-Y., Yuan, C., *Harnack inequality for functional SDEs with multiplicative noise and applications,* Stoch. Proc. Appl., 121(2011), 2692-2710.

[23] Yang, X., Wang, X., J., *A transformed jump-adapted backward Euler method for jump-extended CIR and CEV models,* Numerical Algorithms, 74(2017), 39-57.

[24] Zhang, S.-Q., Zheng, Y., *Functional Inequality and Spectrum Structure for CIR Model, in Chinese,* Journal of Beijing Normal University (Natural Science), 54(2018), 572-582.