On Lattice Coverings by Simplices

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Abstract. By studying the volume of a generalized difference body, this paper presents the first nontrivial lower bound for the lattice covering density by n-dimensional simplices.

1. Introduction

More than 2,300 years ago, Aristotle (384-322 BCE) claimed that the regular tetrahedra can fill the whole space. In the modern terms, regular tetrahedra of given size can form a tiling of the three-dimensional Euclidean space $\mathbb{E}^3$. In other words, they can form both a packing and a covering in $\mathbb{E}^3$ simultaneously. If this were true, both the density of the densest packing by congruent regular tetrahedra and the density of the thinnest covering of $\mathbb{E}^3$ by congruent regular tetrahedra would be one. Unfortunately, Aristotle is wrong and such a tiling is impossible. Aristotle's mistake was discovered in the fifteenth century by Regiomontanus (see [23]). Then, one may ask two natural questions: *What is the density of the densest packing by congruent regular tetrahedra* and *what is the density of the thinnest covering of* $\mathbb{E}^3$ *by congruent regular tetrahedra*?

As a part of his 18th mathematical problems, D. Hilbert [21] wrote:"*I point out the following question, related to the preceding one, and important to number theory and perhaps sometimes useful to physics and chemistry: How can one arrange most densely in space an infinite number of equal solids of given form, e.g., spheres with given radii or regular tetrahedra with given edges* (*or in prescribed position*)*, that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as great as possible*?" Since then, many mathematicians made contributions (mistakes as well) to tetrahedra packings. For the complicated history, we refer to [23].

Covering, in certain sense, is a counterpart of packing. Let K denote a convex body in $\mathbb{E}^n$ and let C denote a centrally symmetric one. In particular, let B_n, T_n and W_n denote the n-dimensional unit ball, the n-dimensional regular simplex with unit edges, and the n-dimensional unit cube $\{\mathbf{x} : \ 0 \leq |x_i| \leq \frac{1}{2}\}$, respectively. We call $\mathcal{K} = \{K_i : \ K_i \text{ are congruent to } K\}$ a covering of $\mathbb{E}^n$ if $\bigcup_{K_i \in \mathcal{K}} K_i = \mathbb{E}^n$. For such a $\mathcal{K}$ we define an density

$$\theta(\mathcal{K}) = \liminf_{\ell \to \infty} \frac{\mathrm{vol}(\mathcal{K} \cap \ell W_n)}{\mathrm{vol}(\ell W_n)}.$$

Then, we define the *congruent covering density*, the *translative covering density* and the *lattice covering density* of K respectively as

$$\theta^c(K) = \min_{\mathcal{K}} \{\theta(\mathcal{K}) : \mathcal{K} \ \text{ a general covering}\},$$

$$\theta^t(K) = \min_{\mathcal{K}} \{\theta(\mathcal{K}) : \mathcal{K} \text{ uses translates of } K\}$$

and

$$\theta^l(K) = \min_{\mathcal{K}} \{\theta(\mathcal{K}) : \mathcal{K} \text{ is a lattice covering}\}.$$

In fact, for $\theta^c(K)$, $\theta^t(K)$ and $\theta^l(K)$, the unit cube W_n in the definition of $\theta(\mathcal{K})$ can be replaced by any other fixed convex body. In addition, both $\theta^t(K)$ and $\theta^l(K)$ are invariant under non-singular affine linear transformations. Clearly, for these numbers we have

$$1 \leq \theta^c(K) \leq \theta^t(K) \leq \theta^l(K).$$

Let Λ be a lattice with determinant $\det(\Lambda)$, and let $\mathcal{L}$ denote the family of all lattices Λ such that $K + \Lambda$ is a covering of $\mathbb{E}^n$. Then $\theta^l(K)$ can be reformulated as

$$\theta^l(K) = \min_{\Lambda \in \mathcal{L}} \frac{\mathrm{vol}(K)}{\det(\Lambda)}.$$

In 1939, Kerschner [19] proved

$$\theta^c(B_2) = \theta^t(B_2) = \theta^l(B_2) = \frac{2\pi}{\sqrt{27}}.$$

In 1946 and 1950, L. Fejes Tóth [12] and [13] proved that

$$\theta^t(C) = \theta^l(C) \leq \frac{2\pi}{\sqrt{27}}$$

holds for all two-dimensional centrally symmetric convex domains, where equality is attained precisely for the ellipses. In 1950, Fáry [9] proved that $\theta^l(K) \leq 3/2$ holds for all two-dimensional convex domains and the equality holds if and only if K is a triangle. It is trivial that $\theta^c(T_2) = 1$. However, the fact $\theta^t(T_2) = 3/2$ was proved only in 2010 by Januszewski [22]. Even in the plane, the following basic problems are still open (see p.19 of [5]):

Conjecture 1. *For every two-dimensional centrally symmetric convex domain C we have*

$$\theta^c(C) = \theta^l(C).$$

Conjecture 2. *For every two-dimensional convex domain K we have*

$$\theta^t(K) = \theta^l(K).$$

In $\mathbb{E}^3$, our knowledge about $\theta^c(K)$, $\theta^t(K)$ and $\theta^l(K)$ is very limited. In fact, except the five types of parallelohedra P which can tile the whole space and therefore $\theta^c(P) = \theta^t(P) = \theta^l(P) = 1$ (see [10]), the only known exact result is

$$\theta^l(B_3) = \frac{5\sqrt{5}\pi}{24} = 1.463503\ldots,$$

which was first established by Bambah [1] in 1954 (different proofs were discovered by Barnes [2] and Few [14]). About 2000, a particular lattice tiling was independently discovered by [15] and [8] which implies

$$\theta^l(T_3) \leq \frac{125}{63}.$$

In 2006, Conway and Torquato [6] discovered a tetrahedra covering which implies

$$\theta^c(T_3) \leq \frac{9}{8}.$$

In n-dimensional space, through the works of Bambah, Coxeter, Davenport, Erdös, Few, Watson and in particular Rogers, we know that

$$\theta^t(K) \leq n \log n + n \log\log n + 5n,$$

$$\theta^l(K) \leq n^{\log_2 \log_e n + c},$$

and

$$\frac{n}{e\sqrt{e}} \ll \theta^t(B_n) \leq \theta^l(B_n) \leq c \cdot n(\log_e n)^{\frac{1}{2}\log_2 2\pi e}.$$

In this paper, we prove the following results:

Theorem 1. *For any pair of positive numbers k and m, we have*

$$\frac{\mathrm{vol}(kT_n - mT_n)}{\mathrm{vol}(T_n)} = \sum_{i=0}^{n} \binom{n}{i}^2 k^i m^{n-i}.$$

Theorem 2. *When $n \geq 3$, we have*

$$\theta^l(T_n) \geq 1 + \frac{1}{2^{3n+7}}.$$

2. Generalized Difference Bodies

In 1904, to study lattice packing of convex bodies, Minkowski [25] introduced the *difference body* $D(K)$ of K. Namely,

$$D(K) = \{\mathbf{x}_1 - \mathbf{x}_2 : \ \mathbf{x}_i \in K\}.$$

In 1920, Blaschke [4] asked for bounds for the volume of $D(K)$ in terms of the volume of K. Through the works of Blaschke, Bonnesen, Estermann, Fenchel, Rademacher, Süss and in particular the surprising work of Rogers and Shephard [27] (also see [26]), we have

$$2^n \leq \frac{\mathrm{vol}(D(K))}{\mathrm{vol}(K)} \leq \binom{2n}{n},$$

where the lower bound can be attained if and only if K is centrally symmetric, and the upper bound can be attained if and only if K is a simplex.

Let λ be a positive number, to generalize Blaschke's problem, it is natural to ask for bounds for

$$\frac{\mathrm{vol}(K - \lambda K)}{\mathrm{vol}(K)}.$$

By the *Brun-Minkowski inequality* it follows that

$$\frac{\mathrm{vol}(K - \lambda K)}{\mathrm{vol}(K)} \geq (1+\lambda)^n,$$

where the equality holds if and only if K is centrally symmetric. For the upper bounds, it turns out to be challenging.

Theorem 1. *Let T_n denote an n-dimensional simplex, then we have*

$$\frac{\mathrm{vol}(\mu T_n - \nu T_n)}{\mathrm{vol}(T_n)} = \sum_{i=0}^{n} \binom{n}{i}^2 \mu^i \nu^{n-i}.$$

Proof. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ denote a standard basis of $\mathbb{E}^n$. Let σ be a nonsingular linear transformation from $\mathbb{E}^n$ to $\mathbb{E}^n$. For any pair of convex bodies K_1 and K_2, both contain the origin $\mathbf{o}$, we have

$$\sigma(K_1 + K_2) = \sigma(K_1) + \sigma(K_2).$$

Therefore, without loss of generality, we assume that

$$T_n = \left\{ (x_1, x_2, \ldots, x_n) : \ x_i \geq 0, \ \sum x_i \leq 1 \right\}.$$

In other words, $T_n = \text{conv}\{\mathbf{o}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

Let F_i denote an i-dimensional face of T_n which contains the origin. Clearly,

$$F_i = \text{conv}\{\mathbf{o}, \mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \ldots, \mathbf{e}_{j_i}\}$$

holds for i different base vectors and T_n has $\binom{n}{i}$ such faces. For convenience, we enumerate all such faces as $F_{i,j}$, where $j = 1, 2, \ldots, \binom{n}{i}$, and denote the $(n-i)$-dimensional face of T_n containing $\mathbf{o}$ and orthogonal to $F_{i,j}$ by $F^*_{i,j}$.

Then, one can deduce that

$$\mu T_n - \nu T_n = \bigcup_{i=0}^{n} \bigcup_{j=1}^{\binom{n}{i}} \left(\mu F_{i,j} - \nu F^*_{i,j} \right),$$

$$\text{int}\left(\mu F_{i_1,j_1} - \nu F^*_{i_1,j_1} \right) \bigcap \text{int}\left(\mu F_{i_2,j_2} - \nu F^*_{i_2,j_2} \right) = \emptyset$$

holds for all $(i_1, j_1) \neq (i_2, j_2)$, and

$$\text{vol}\left(\mu F_{i,j} - \nu F^*_{i,j} \right) = \frac{1}{i!} \cdot \frac{1}{(n-i)!} \cdot \mu^i \nu^{n-i}.$$

Therefore, we have

$$\begin{aligned} \frac{\text{vol}(\mu T_n - \nu T_n)}{\text{vol}(T_n)} &= n! \sum_{i=0}^{n} \binom{n}{i} \frac{1}{i!} \cdot \frac{1}{(n-i)!} \cdot \mu^i \nu^{n-i} \\ &= \sum_{i=0}^{n} \binom{n}{i}^2 \mu^i \nu^{n-i}. \end{aligned}$$

The theorem is proved. □

Remark 1. Rogers and Shephard [27] did suggest a mean to compute the volume of $D(T_n)$. Our proof here is different from their argument.

Conjecture 1. For every n-dimensional convex body K we have

$$\frac{\text{vol}(\mu K - \nu K)}{\text{vol}(K)} \leq \sum_{i=0}^{n} \binom{n}{i}^2 \mu^i \nu^{n-i},$$

where the equality holds if and only if K is a simplex.

Remark 2. As a special case of Minkowski's theorem on mixed volumes, for any fixed n-dimensional convex body K we have

$$\text{vol}(K - \lambda K) = \sum_{i=0}^{n} \binom{n}{i} W_i(K, -K) \cdot \lambda^i,$$

where $W_i(K, -K)$ are constants determined by K. It was conjectured by Godbersen [17] and Makai jr. [24] (see p.412 of Schneider [29]) that

$$W_i(K, -K) \leq \binom{n}{i} \text{vol}(K),$$

where the equality holds if and only if K is a simplex. Clearly, Godbersen and Makai's conjecture implies Conjecture 1.

3. Lattice Coverings by Simplices

Assume that $K+\Lambda$ is a lattice covering of $\mathbb{E}^n$. Let $\alpha(K,\Lambda)$ denote its *star number* and let $\theta(K,\Lambda)$ denote its density. In other words, $\alpha(K,\Lambda)$ is the number of the lattice points $\mathbf{u}\in\Lambda\setminus\{\mathbf{o}\}$ such that $K\cap(K+\mathbf{u})\neq\emptyset$, and $\theta(K,\Lambda)=\mathrm{vol}(K)/\det(\Lambda)$.

To show Theorem 2, we need two basic lemmas. Namely,

Lemma 1 (Hadwiger [20]**, see p.283 of** [18]**).** *Let $K+\Lambda$ be a lattice covering of $\mathbb{E}^n$. Then we have*

$$\frac{\mathrm{vol}(2K-K)}{\mathrm{vol}(K)}\cdot\theta(K,\Lambda)\geq\alpha(K,\Lambda).$$

Lemma 2 (Rogers and Shephard [27]**).** *An n-dimensional convex body K is a simplex if and only if, for any $\mathbf{x}\in\mathrm{int}(D(K))$, the intersection $K\cap(K+\mathbf{x})$ is positively homothetic to K.*

Let $K+\Lambda$ be a lattice covering and let K^j denote the subset of K such that every point $\mathbf{x}\in K^j$ is covered by exact j translates in $K+\Lambda$. We have the following basic result.

Lemma 3. *If $K+\Lambda$ is a covering of $\mathbb{E}^n$, we have*

$$\theta(K,\Lambda)=\frac{\mathrm{vol}(K)}{\sum\frac{1}{j}\mathrm{vol}(K^j)}=\frac{\mathrm{vol}(K)}{\mathrm{vol}(K)-\sum\frac{j-1}{j}\mathrm{vol}(K^j)}.$$

Proof. Let $K+\Lambda$ be a lattice covering of $\mathbb{E}^n$ with density $\theta(K,\Lambda)$. Let ℓ be a large positive number, let ℓW_n be a big cube with edge length ℓ, and let $p(\ell)$ denote the number of the lattice points in ℓW_n. Clearly we have

$$\theta(K,\Lambda)=\lim_{\ell\to\infty}\frac{p(\ell)\cdot\mathrm{vol}(K)}{\mathrm{vol}(\ell W_n)}. \qquad (1)$$

Let $\mathbf{x}$ be a point in $\mathbb{E}^n$ and let $\mathbf{u}$ be a lattice point. We attach a mass density

$$\delta(\mathbf{x},K+\mathbf{u})=\frac{1}{j}$$

to $\mathbf{x}$ with respect to $K+\mathbf{u}$ if $\mathbf{x}\in K+\mathbf{u}$ and $\mathbf{x}$ belongs to exact j different translates of K in the lattice covering. If $\mathbf{x}\notin K+\mathbf{u}$, we define $\delta(\mathbf{x},K+\mathbf{u})=0$. Then the total mass density $\delta(\mathbf{x})$ at $\mathbf{x}$ is

$$\delta(\mathbf{x})=\sum_{\mathbf{u}\in\Lambda}\delta(\mathbf{x},K+\mathbf{u})=j\cdot\frac{1}{j}=1.$$

Therefore we have

$$\begin{aligned}
\mathrm{vol}(\ell W_n) &= \int_{\ell W_n} \delta(\mathbf{x}) d\mathbf{x} \\
&= \int_{\ell W_n} \sum_{\mathbf{u}\in\Lambda} \delta(\mathbf{x}, K+\mathbf{u}) d\mathbf{x} \\
&= \sum_{\mathbf{u}\in\Lambda} \int_{\ell W_n} \delta(\mathbf{x}, K+\mathbf{u}) d\mathbf{x} \\
&= (1+o(1))\cdot p(\ell) \int_{\mathbb{E}^n} \delta(\mathbf{x}, K) d\mathbf{x} \\
&= (1+o(1))\cdot p(\ell) \int_{K} \delta(\mathbf{x}, K) d\mathbf{x} \\
&= (1+o(1))\cdot p(\ell) \sum \frac{1}{j}\mathrm{vol}(K^j). \qquad (2)
\end{aligned}$$

By (1) and (2), the lemma follows. □

Proof of Theorem 2. For convenience, without loss of generality, we assume that T_n is a regular simplex with unit edges in $\mathbb{E}^n$. We consider two cases.

Case 1. $\alpha(T_n, \Lambda) \geq 2^{3n+1}$.

As a corollary of Theorem 1, we get

$$\frac{\mathrm{vol}(2T_n - T_n)}{\mathrm{vol}(T_n)} = \sum_{i=0}^{n} \binom{n}{i}^2 2^i \leq 2^n \binom{n}{[n/2]}^2 \leq 2^{3n}. \qquad (3)$$

Therefore, by Lemma 1 we have

$$\theta(T_n, \Lambda) \geq \frac{\alpha(T_n, \Lambda)}{2^{3n}} \geq 2.$$

Case 2. $\alpha(T_n, \Lambda) \leq 2^{3n+1}$.

Let $\partial(K)$ denote the boundary of K, and let $\overline{\mathrm{vol}}(X)$ denote the $(n-1)$-dimensional measure of a set X in $\mathbb{E}^n$.

Assume that T_n is intersected by $T_n + \mathbf{u}_1$, $T_n + \mathbf{u}_2$, …, $T_n + \mathbf{u}_m$, where $m = \alpha(T_n, \Lambda)$. Then, we have

$$\partial(T_n) = \bigcup_{i=1}^{m} \left(\partial(T_n) \cap (T_n + \mathbf{u}_i)\right)$$

and therefore

$$\overline{\mathrm{vol}}\left(\partial(T_n) \cap (T_n + \mathbf{u}_k)\right) \geq \frac{1}{m}\overline{\mathrm{vol}}(\partial(T_n))$$

holds at least for one of these translates.

By Lemma 2, we know that $T_n \cap (T_n + \mathbf{u}_k)$ is homothetic to T_n. Assuming that

$$T_n \cap (T_n + \mathbf{u}_k) = \lambda T_n + \mathbf{y}$$

holds for some suitable positive number λ and a point $\mathbf{y}$, one can deduce that

$$n \cdot \lambda^{n-1} \cdot \mathrm{vol}(T_{n-1}) \geq \frac{1}{m} \cdot (n+1) \cdot \mathrm{vol}(T_{n-1}),$$

$$\lambda \geq \left(\frac{n+1}{mn}\right)^{\frac{1}{n-1}},$$

$$\mathrm{vol}\left(T_n \cap (T_n + \mathbf{u}_k)\right) \geq \left(\frac{n+1}{mn}\right)^{\frac{n}{n-1}} \mathrm{vol}(T_n),$$

and therefore, when $n \geq 3$,

$$\begin{aligned}
\theta(T_n, \Lambda) &= \frac{\mathrm{vol}(T_n)}{\mathrm{vol}(T_n) - \sum \frac{j-1}{j}\mathrm{vol}(T_n^j)} \\
&\geq \frac{\mathrm{vol}(T_n)}{\mathrm{vol}(T_n) - \frac{1}{2}\sum \mathrm{vol}(T_n^j)} \\
&\geq \frac{\mathrm{vol}(T_n)}{\mathrm{vol}(T_n) - \frac{1}{2}\mathrm{vol}(T_n \cap (T_n + \mathbf{u}_k))} \\
&\geq \frac{1}{1 - \frac{1}{2}\left(\frac{n+1}{mn}\right)^{\frac{n}{n-1}}} \\
&\geq \frac{1}{1 - 2^{-\frac{(3n+1)n}{n-1}-1}} \\
&\geq 1 + \frac{1}{2^{3n+7}}.
\end{aligned}$$

As a conclusion of the two cases, Theorem 2 is proved. □

Remark 3. By careful estimation, the lower bound can be further slightly improved.

Acknowledgements. This work is supported by 973 Programs 2013CB834201 and 2011CB302401, the National Science Foundation of China (No.11071003), and the Chang Jiang Scholars Program of China.

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